

STOCHASTIC STABILITY OF POLLICOTT–RUELLE RESONANCES

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ABSTRACT. Pollicott–Ruelle resonances for chaotic flows are the characteristic frequencies of correlations. They are typically defined as eigenvalues of the generator of the flow acting on specially designed functional spaces. We show that these resonances can be computed as viscosity limits of eigenvalues of second order elliptic operators. These eigenvalues are the characteristic frequencies of correlations for a stochastically perturbed flow.

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider an Anosov flow $\varphi_t = e^{tV}$ on a compact manifold X . For the Laplacian $\Delta_g \leq 0$ with respect to some metric on X , we define

$$P_\varepsilon = \frac{1}{i}V + i\varepsilon\Delta_g, \quad (1.1)$$

For $\varepsilon \neq 0$ this operator is elliptic and hence has a discrete $L^2(X)$ -spectrum $\{\lambda_j(\varepsilon)\}_{j=0}^\infty$. However, for $\varepsilon = 0$ most of the L^2 spectrum is not discrete.

Following the seminal work of Ruelle [Ru] and Pollicott [Po], many authors investigated the discrete spectrum of P_0 acting on specially designed *anisotropic Sobolev spaces* and the role of that spectrum in the expansion of correlations – see Blank–Keller–Liverani [BKL], Baladi–Tsuji [BaTs], Faure–Sjöstrand [FaSj], Faure–Tsuji [FaTs1, FaTs2], Gouëzel–Liverani [GoLi1], Liverani [Li2], Tsuji [Ts1, Ts2] and references given there. We review a yet another approach based on [DyZw1] in §3. These complex eigenvalues of P_0 , $\{\lambda_j\}_{j=0}^\infty$, are called *Pollicott–Ruelle resonances*. For perspectives on physical manifestations of these resonances see for instance Gaspard–Ramirez [GaRa] or Chekroun et al [CNKMG].

The purpose of this note is to show that Pollicott–Ruelle resonances can be defined as limits of $\lambda_j(\varepsilon)$ as $\varepsilon \rightarrow 0+$. This can be considered a *stochastic stability* of resonances:

Theorem 1. *Let P_ε be given by (1.1) and let $\{\lambda_j(\varepsilon)\}_{j=0}^\infty$ be the set of its L^2 -eigenvalues. If $\{\lambda_j\}_{j=0}^\infty$ is the set of the Pollicott–Ruelle resonances of the flow φ_t , then*

$$\lambda_j(\varepsilon) \longrightarrow \lambda_j, \quad \varepsilon \rightarrow 0+,$$

with convergence uniform for λ_j in a compact set.

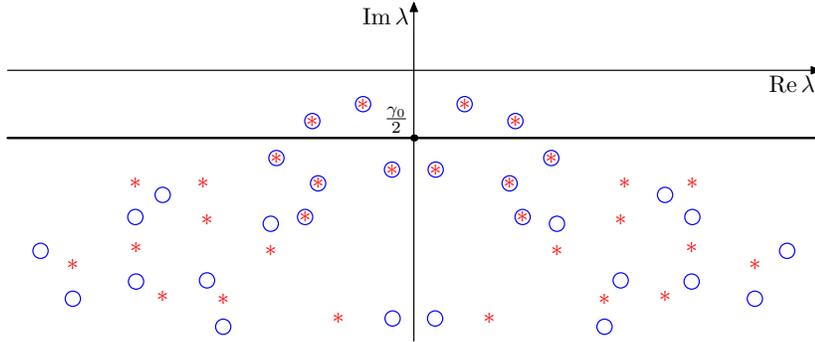


FIGURE 1. A schematic presentation of the results in Theorems 1 and 2. Pollicott–Ruelle resonances of the generator of the flow V (denoted by red asterisks) are approximated by the eigenvalues of $V/i + i\varepsilon\Delta_g$ (denoted by blue circles) uniformly on compact sets. The asymptotic resonance free strip is uniform with respect to ε .

The nature of convergence is much more precise – see §5. In particular the spectral projections depend smoothly on $\varepsilon \in [0, \varepsilon_0]$ where ε_0 depends on the compact set. Also, when λ_j is a simple resonance then for ε sufficiently small the map $\varepsilon \mapsto \lambda_j(\varepsilon)$ is smooth all the way up to $\varepsilon = 0$. As explained in the next paragraph $\lambda_j(\varepsilon) \rightarrow \bar{\lambda}_j$ when $\varepsilon \rightarrow 0-$. We also note the symmetry of $\lambda_j(\varepsilon)$'s with respect to the imaginary axis (see Fig. 1). That follows from the fact that $\overline{P_\varepsilon u} = -P_\varepsilon \bar{u}$ and thus

$$\overline{(P_\varepsilon - \lambda)^{-1}u} = -(P_\varepsilon + \bar{\lambda})^{-1}\bar{u}. \quad (1.2)$$

The proof of Theorem 1 relies on the fact that $P_\varepsilon - \lambda$ is a Fredholm operator on the same anisotropic Sobolev spaces as $P_0 - \lambda$, in a way which is controlled uniformly as $\varepsilon \rightarrow 0+$. This Fredholm property is established by the same methods as those used in [DyZw1] for the case of $\varepsilon = 0$. The key feature of the damping term $i\varepsilon\Delta_g$ is that its imaginary part is nonpositive and thus the propagation of singularities theorem of Duistermaat–Hörmander (see (2.10)) still applies in the forward time direction. For $\varepsilon < 0$, the damping term is nonnegative and propagation of singularities applies in the negative time direction, which means that we have to consider the dual anisotropic Sobolev spaces $H_{-sG(h)}$ and the spectrum of P_0 on these spaces is given by $\{\bar{\lambda}_j\}$.

We remark that all the results of this paper are valid for the operators acting on sections of vector bundles arising in dynamical systems – see [DyZw1]. We consider the scalar case to make the notation, which is all that is affected, simpler.

Previously, stability of Pollicott–Ruelle resonances has been established for Anosov maps, $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$, [BKL],[Li2], following a very general argument of Keller–Liverani [KeLi]. In that case the Koopman operator $f^* : C^\infty(\mathbb{T}^d) \rightarrow C^\infty(\mathbb{T}^d)$ is replaced by a “noisy propagator” $G_\varepsilon \circ f^*$, where $G_\varepsilon u = g_\varepsilon * u$, $g_\varepsilon \rightarrow \delta_0$, $\varepsilon \rightarrow 0$. For general Anosov

maps on compact manifolds a semiclassical proof was given by Faure–Roy–Sjöstrand [FRS, Theorem 5]. Further refinements concerning dependence on ε can be found in [GoLi1, §8] and interesting applications were obtained by Gouëzel–Liverani [GoLi2] and Fannjiang–Nonnenmacher–Wołowski [FNW]. For a physics perspective on this see for instance Blum–Agam [BlAg] and Venegeroles [Ve].

For flows, Butterley–Liverani [BuLi1],[BuLi2] showed that if a vector field depends smoothly on a parameter, then the spectrum of the transfer operator associated to the weight corresponding to the SRB measure is smooth in that parameter. Constantin–Kiselev–Ryzhik–Zlatoš [CKRZ] established that solutions to the heat equation with a large transport term equidistribute after arbitrarily small times if and only if the flow corresponding to the transport term is mixing; this can be viewed as an analogue of our work for the L^2 spectrum on the real line instead of resonances.

A dynamical interpretation of $\lambda_j(\varepsilon)$'s can be formulated as follows. In terms of the operator P_0 the flow, $x(t) := \varphi_{-t}(x(0))$, is given by

$$e^{-itP_0} f(x) = f(x(t)), \quad \dot{x}(t) = -V_{x(t)}, \quad x(0) = x.$$

For $\varepsilon > 0$ the evolution equation is replaced by the Langevin equation:

$$e^{-itP_\varepsilon} f(x) = \mathbb{E}[f(x(t))], \quad \dot{x}(t) = -V_{x(t)} + \sqrt{2\varepsilon}\dot{B}(t), \quad x(0) = x,$$

where $B(t)$ is the Brownian motion corresponding to the metric g on X (presented here in an informal way; see [El]). This explains why considering P_ε corresponds to a stochastic perturbation of the deterministic flow. See also Kifer [Ki] for other perspective on random perturbations of dynamical systems.

We also remark that a result similar to Theorem 1 is valid for *scattering resonances*: for $V \in L_c^\infty(\mathbb{R}^n; \mathbb{C})$ (and in greater generality) they appear as limits of eigenvalues of $-\Delta + V(x) - i\varepsilon|x|^2$ when $\varepsilon \rightarrow 0+$, see [Zw2]. The proof is based on the method of complex scaling and is technically very different than the one presented here. The result however is exactly analogous with spacial infinity, $|x| \rightarrow \infty$, replacing the momentum infinity, $|\xi| \rightarrow \infty$.

Pretending that spectrum of P_ε is semisimple (algebraic multiplicities are equal to geometric multiplicities – see §5 for the general statement), the relation to the eigenvalues $\lambda_j(\varepsilon)$ comes from considering long time behaviour: for any $f \in C^\infty(X)$, and $t > 0$,

$$e^{-itP_\varepsilon} f(x) = \sum_{\operatorname{Im} \lambda_j > -A} e^{-it\lambda_j(\varepsilon)} u_j^\varepsilon(x) \int_X v_j^\varepsilon(y) f(y) d\operatorname{vol}_g(y) + \mathcal{O}_f(e^{-tA})_{C^\infty}, \quad (1.3)$$

where $u_j^\varepsilon, v_j^\varepsilon \in C^\infty(X)$ are the eigenfunctions of P_ε and P_ε^t corresponding to $\lambda_j(\varepsilon)$. We note that there are no convergence problems as the number of $\lambda_j(\varepsilon)$ with imaginary

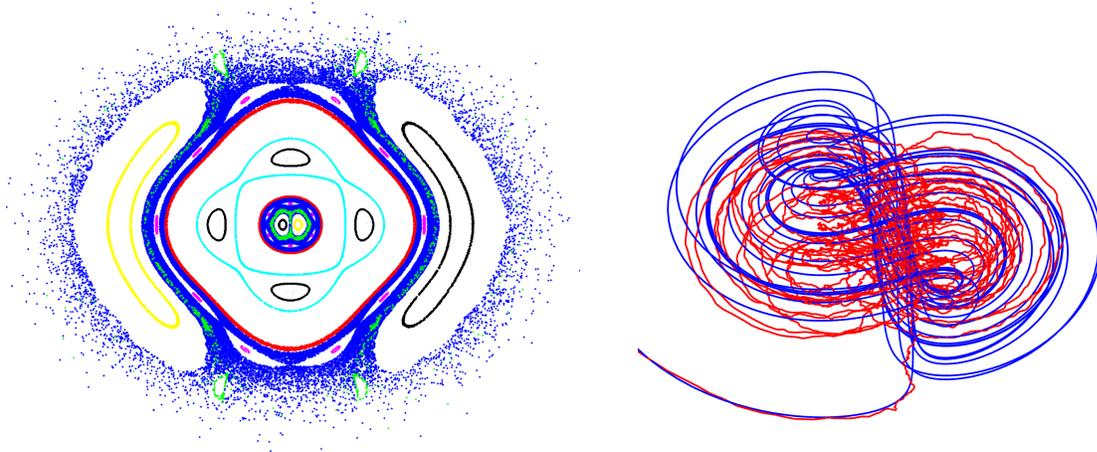


FIGURE 2. An illustration of chaotic and stochastic trajectories: we consider the Nosé–Hoover oscillator [PHV] which is possibly the simplest chaotic system: $W = x_2\partial_{x_1} + (-x_1 + x_2x_3)\partial_{x_2} + (1 - x_2^2)\partial_{x_3}$, $x \in \mathbb{R}^3$. The vector field $V = e^{|x|^2/2}W$ is the Reeb vector field for the contact form $\alpha = e^{-|x|^2/2}(x_2dx_1 + dx_3)$. On the left the Poincaré section $\{x_3 = 0\}$ showing the chaotic sea and islands of quasi-periodicity (each colour corresponds to a *numerical* iteration of a single point). On the right a chaotic trajectory and the stochastic trajectory, $\varepsilon = 0.01$, with the same initial data. We stress that the results of our paper do not apply to mixed systems and this example is meant as an illustration of chaotic and stochastic trajectories. However, as in [CNKMG], Pollicott–Ruelle resonances are expected to be relevant for mixed systems as well.

parts above $-A$ is finite though the number will grow with ε . In fact, [JiZw] shows that the number of Pollicott–Ruelle resonances, λ_j , with $\text{Im } \lambda_j > -A$ is always infinite if A is sufficiently large.

The validity of a modification of (1.3) for $\varepsilon = 0$ is only known for *contact Anosov flows* (see §6) and for $A > -\gamma_0/2$, where γ_0 is an averaged Lyapounov exponent (see (1.5)). That is due to Tsujii [Ts1, Ts2] who followed earlier advances by Dolgopyat [Do] and Liverani [Li1]. It is also a consequence of more general results obtained in [NoZw].

The modification in (1.3) is needed since the corresponding u_j 's are now distributions and the expansion provides fine aspects of the decay of correlations. Let $d\mu(x)$ be the volume form obtained from the contact form on X , $\mu(X) = 1$. For $f, g \in C^\infty(X)$ and

any $\delta > 0$,

$$\begin{aligned} \int_X [e^{-itP_0} f](x)g(x)d\mu(x) &= \int_X f(x)d\mu(x) \int_X g(x)d\mu(x) \\ &+ \sum_{-\frac{1}{2}(\gamma_0-\delta) < \text{Im } \lambda_j < 0} e^{-it\lambda_j} v_j(f)u_j(g) + \mathcal{O}_{f,g}(e^{-\frac{1}{2}t(\gamma_0-\delta)}), \end{aligned} \quad (1.4)$$

where γ_0 is the minimal asymptotic growth rate of the unstable Jacobian, that is the largest constant such that for each $\delta > 0$

$$|\det(d\varphi_{-t}|_{E_u(x)})| \leq C_\delta e^{-(\gamma_0-\delta)t}, \quad t \geq 0; \quad \varphi_{-t}^* = e^{-itP_0} : C^\infty(X) \rightarrow C^\infty(X), \quad (1.5)$$

with $E_u(x) \subset T_x X$ the unstable subspace of the flow at x – see §2. Now u_j and v_j are distributional eigenfunctions of P_0 and P_0^t , $\text{WF}(u_j) \subset E_u^*$ and $\text{WF}(\bar{v}_j) \subset E_s^*$. Here again we make the simplifying assumption that the spectrum is semisimple; that is always the case for geodesic flows in constant negative curvature as shown by Dyatlov–Faure–Guillarmou [DFG, Theorem 3].

Hence it is natural to ask the question if the gap $\gamma_0/2$ is uniform with respect to ε , that is, if the expansion (1.3) with $A > -\frac{1}{2}(\gamma_0 - \varepsilon)$ uniformly approaches the expansion (1.4). That is indeed a consequence of the next theorem:

Theorem 2. *Suppose that X is an odd dimensional compact manifold and that $V \in C^\infty(X; TX)$ generates a contact Anosov flow. There exists a constant s_0 such that for any $\delta > 0$ there exist $N_0, R > 0$ such that for all $\varepsilon > 0$,*

$$(P_\varepsilon - \lambda)^{-1} = \mathcal{O}(\lambda^{N_0}) : H^{s_0}(X) \rightarrow H^{-s_0}(X), \quad (1.6)$$

for γ_0 defined in (1.5) and $\lambda \in [R, \infty) - i[0, \frac{1}{2}(\gamma_0 - \delta)]$.

The same estimate is true for $\lambda \in (-\infty, -R] - i[0, \frac{1}{2}(\gamma_0 - \delta)]$ by recalling (1.2). Since on the compact set $[-R, R] - i[0, \frac{1}{2}(\gamma_0 - \delta)]$, $\lambda_j(\varepsilon)$ converge uniformly to λ_j 's, we see that for ε small enough the number of eigenvalues of P_ε in that set is independent of ε . We should remark that the estimate (1.6) can be made more precise by using microlocally weighted spaces reviewed in §4 – see (6.10).

The proof of Theorem 2 combines the approach of Faure–Sjöstrand [FaSj] and [DyZw1] with the work on resonance gaps for general differential operators [NoZw]. As in that paper we also use the resolvent gluing method of Datchev–Vasy [DaVa].

For a class of maps on \mathbb{T}^2 a similar result has been obtained by Nakano–Wittsten [NaWi].

Negative examples. It is important to point out that the existence of a discrete limit set for the eigenvalues of the operator P_ε is very special to chaotic flows and for mixed flows could only hold under some special domain restrictions. The simplest “counterexample” is given by considering $X = \mathbb{S}^1 \times \mathbb{S}^1$ with $V = \partial_{x_1} + \alpha \partial_{x_2}$, $x_j \in$

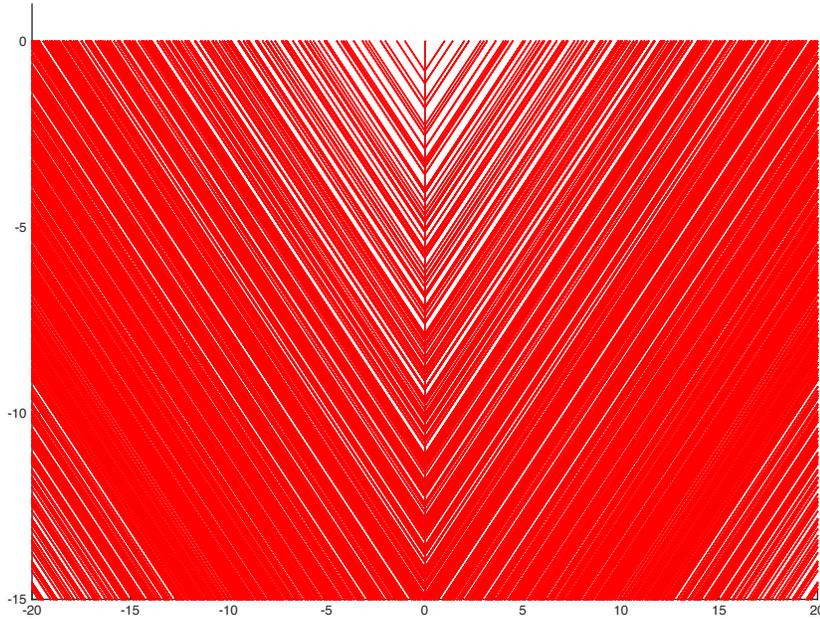


FIGURE 3. The case of geodesic flow on a torus: the unit tangent bundle is given by $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. If V is the generator of the geodesic flow and Δ is the (flat) Laplacian on \mathbb{T}^3 then accumulation points of spectra of $V/i + i\varepsilon\Delta$ as $\varepsilon \rightarrow 0+$ form a discrete set of lines. That is dramatically different from the Anosov case shown in Fig. 1.

$\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$. When α is irrational then accumulation points of the spectrum of P_ε as $\varepsilon \rightarrow 0+$ form the lower half plane. When $\alpha = p/q$ with p and q coprime the limit set is equal to $\mathbb{Z}/q - i[0, \infty)$.

A more interesting example is given by the geodesic flow on the torus, $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with the flat metric. That is a contact flow on the unit cotangent bundle $S^*\mathbb{T}^2 = \mathbb{T}^2_{x_1, x_2} \times \mathbb{S}^1_\theta$, generated by and it is generated by

$$V = \cos \theta \partial_{x_1} + \sin \theta \partial_{x_2}.$$

Defining P_ε using the flat Laplacian Δ on \mathbb{T}^2 , and by expanding in Fourier modes in x we see that

$$\text{Spec}(P_\varepsilon) = \bigcup_{n \in \mathbb{Z}^2} \text{Spec}(P_\varepsilon(n)), \quad P_\varepsilon(n) := n_1 \cos \theta + n_2 \sin \theta - i\varepsilon(n_1^2 + n_2^2 + D_\theta^2),$$

$D_\theta = \frac{1}{i}\partial_\theta$. We rewrite the operator $P_\varepsilon(n)$ as follows:

$$P_\varepsilon(n) = -i\varepsilon D_\theta^2 + |n| \cos(\theta - \delta_n) - i|n|^2\varepsilon, \quad \delta_n = \tan^{-1}(n_1/n_2).$$

For $n = 0$ the spectrum is simply $-i\varepsilon m^2$, $m \in \mathbb{Z}$ and it accumulates on the negative imaginary axis. For $n \neq 0$ the asymptotic behaviour of the spectrum is determined by the asymptotic behaviour of the spectrum of the semiclassical operator

$$Q(h) := (hD_\theta)^2 + i \cos \theta, \quad h^2 := \varepsilon/|n|.$$

That has been determined by Galtsev–Shafarevitch [GaSh] who showed that as $h \rightarrow 0$ the spectrum concentrates on a rotated “Y” shape with the vertices at $\pm i$ and the junction at a special value $E^* \approx 0.85$.

This shows that the accumulation points in the case of the generator of the geodesic flow on the two torus regularized using the flat Laplacian are given by

$$-i[0, \infty) \cup \bigcup_{n \in \mathbb{Z}^2 \setminus \{0,0\}} \{z : |\operatorname{Re} z| \leq n, \operatorname{Im} z = -E^*|n| + E^*|\operatorname{Re} z|\},$$

and part of this set is shown in Fig. 3.

Acknowledgements. We would like express thanks to Carlangelo Liverani for suggesting this problem, to Viviane Baladi and Stéphane Nonnenmacher for helpful comments on an earlier version of the paper, and to Michael Hitrik for informing us of the reference [GaSh]. We are also grateful for the support by the Clay Research Fellowship (SD) and by the National Science Foundation grant DMS-1201417 (MZ).

2. PRELIMINARIES

We review some definitions and basic facts mostly to fix notation and to provide references. The needed results from microlocal/semiclassical analysis are presented in detail in [DyZw1, §2.3, Appendix C] and we will rely on the presentation given there.

Notation. We use the following notation: $f = \mathcal{O}_\ell(g)_H$ means that $\|f\|_H \leq C_\ell g$ where the norm (or any seminorm) is in the space H , and the constant C_ℓ depends on ℓ . When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_\ell(g)_{H_1 \rightarrow H_2}$ then the operator $G : H_1 \rightarrow H_2$ has its norm bounded by $C_\ell g$.

2.1. Dynamical systems. In this paper X is a compact manifold and $\varphi_t : X \rightarrow X$ a C^∞ flow, $\varphi_t = \exp tV$, $V \in C^\infty(X; TX)$. The flow is *Anosov* if the tangent space to X has a continuous decomposition $T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x)$ which is invariant, $d\varphi_t(x)E_\bullet(x) = E_\bullet(\varphi_t(x))$, $E_0(x) = \mathbb{R}V(x)$, and for some C and $\theta > 0$ fixed

$$\begin{aligned} |d\varphi_t(x)v|_{\varphi_t(x)} &\leq C e^{-\theta|t|} |v|_x, \quad v \in E_u(x), \quad t < 0, \\ |d\varphi_t(x)v|_{\varphi_t(x)} &\leq C e^{-\theta|t|} |v|_x, \quad v \in E_s(x), \quad t > 0. \end{aligned} \tag{2.1}$$

where $|\bullet|_y$ is given by a smooth Riemannian metric on X .

Following Faure–Sjöstrand [FaSj] we exploit the analogy between dynamical systems and quantum scattering, with the fiber (ξ) infinity playing the role of x -infinity in scattering theory. The pull-back map can be written analogously to the Schrödinger propagator

$$\varphi_{-t} = e^{-itP_0}, \quad P_0 := \frac{1}{i}V.$$

The symbol of P_0 and its Hamiltonian flow are

$$p(x, \xi) = \xi(V_x), \quad e^{tH_p}(x, \xi) = (\varphi_t(x), ({}^T d\varphi_t(x))^{-1}\xi).$$

Here H_p denotes the Hamilton vector field of p : $\omega(\bullet, H_p) = dp$, where $\omega = d(\xi dx)$ is the symplectic form on T^*X .

In the study of P_0 we need the dual decomposition of the cotangent space:

$$T_x^*X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x), \quad (2.2)$$

where $E_0^*(x), E_s^*(x), E_u^*(x)$ are symplectic annihilators of $E_s(x) \oplus E_u(x)$, $E_0(x) \oplus E_s(x)$, and $E_0(x) \oplus E_u(x)$. Hence they are dual to $E_0(x), E_u(x), E_s(x)$.

A special class of Anosov flows is given by contact Anosov flows. In that case X is a contact manifold, that is a manifold equipped with a contact 1-form α : that means that if the dimension of X is $2k - 1$ then $(d\alpha)^{\wedge k} \wedge \alpha$ is non-degenerate. A contact flow is the flow generated by the Reeb vector field V :

$$\alpha(V) = 1, \quad d\alpha(V, \bullet) = 0. \quad (2.3)$$

For an example of a non-Anosov contact flow see Fig. 2. An important class of examples of Anosov contact flows is obtained from negatively curved Riemannian manifolds (M, g) : $X = S^*M := \{(z, \zeta) \in T^*M; |\zeta|_g = 1\}$, $\alpha = \zeta dz|_{S^*M}$.

2.2. Wave front set of distributions and operators. Semiclassical quantization on a compact manifold [DyZw1, Appendix C], [Zw1, Chapter 14] is central to our analysis.

Let X be a compact manifold and $h \in (0, 1)$ a parameter (the asymptotic parameter in the semiclassical analysis). A family of h -dependent distributions $u \in \mathcal{D}'(X)$ is called h -tempered if for some N , $\|u\|_{H^{-N}} \leq Ch^{-N}$. A phase space description of singularities of u is given by the wave front set:

$$\text{WF}_h(u) \subset \overline{T^*}X,$$

where $\overline{T^*}X$ is the fiber-radially compactified cotangent bundle, a manifold with interior T^*X and boundary,

$$\partial\overline{T^*}X = S^*X = (T^*X \setminus 0)/\mathbb{R}^+, \quad \kappa : T^*X \setminus 0 \longrightarrow S^*X = \partial\overline{T^*}X. \quad (2.4)$$

In addition to singularities, WF_h measures oscillations on the h -scale. We also refer to it as the *microsupport* of u or as having u *microlocalized* to some region containing $\text{WF}_h(u)$ – see §C.2 for the definitions.

For families of (h -tempered) operators we define the wave front set $\text{WF}'_h(B)$ using the Schwartz kernel of B , K_B :

$$\text{WF}'_h(B) = \{(x, \xi, y, -\eta) : (x, y, \xi, \eta) \in \text{WF}_h(K_B)\}.$$

This convention guarantees that $\text{WF}_h(I) = \Delta_{T^*X}$ is the diagonal, $\{(x, \xi, x, \xi)\}$, in $T^*X \times T^*X$.

2.3. Pseudodifferential operators. We only use the standard class of semiclassical pseudodifferential operators, $\Psi_h^m(X)$ with the symbol map σ_h , for which

$$0 \longrightarrow h\Psi_h^{m-1}(X) \hookrightarrow \Psi_h^m(X) \xrightarrow{\sigma_h} S^m(X)/hS^{m-1}(X) \longrightarrow 0,$$

is a short exact sequence of algebra homomorphisms and

$$S^m(X) := \{a \in C^\infty(T^*X) : \partial_x^\alpha \partial_\xi^\beta a(x, \xi) = \mathcal{O}_{\alpha\beta}(\langle \xi \rangle^{m-|\beta|})\}$$

(where we were informal about coordinates on X).

One of our uses of the pseudodifferential calculus is that for $\chi \in C_0^\infty(\mathbb{R})$, the operator $\chi(-h^2\Delta_g)$, defined via spectral theory on L^2 , is pseudodifferential in the class Ψ_h^{-N} for each N , and $\sigma_h(\chi(-h^2\Delta_g)) = \chi(|\xi|_g^2)$ – see [Zw1, Theorem 14.9]. Moreover, we implicitly use in the analysis of the operator $\tilde{P}_\varepsilon(\lambda)$ in §4 that the S^0 -seminorms of the full symbol of $\chi(-h^2\Delta_g)$ are controlled by the $S^0(\mathbb{R})$ -seminorms of χ . To see that, we use the proof of [Zw1, Theorem 14.9] to write the full symbol of $\chi(-h^2\Delta_g)$ in the form (see [Dy, Propositions 2.2 and 2.4] for details)

$$\sum_{j=0}^{\infty} h^j \sum_{k=0}^{2j} \chi^{(k)}(|\xi|_g^2) a_{j,k}(x, \xi), \quad a_{j,k} \in S^{2k-j}(T^*X). \quad (2.5)$$

If we control $\sup_{\lambda \in \mathbb{R}} \langle \lambda \rangle^k \chi^{(k)}(\lambda)$ for all $k \geq 0$, then we control $\chi^{(k)}(|\xi|_g^2)$ in S^{-2k} and thus we control (2.5) in S^0 .

The semiclassical Sobolev spaces on X are defined as

$$H_h^s(X) = (I - h^2\Delta_g)^{-s/2} L^2(X) \subset \mathcal{D}'(X), \quad (2.6)$$

for a choice of a Laplacian $\Delta_g \leq 0$ on X and with the inner product inherited from L^2 .

For $A \in \Psi_h^m(X)$ the elliptic set $\text{ell}_h(A) \subset \overline{T^*X}$ is defined as the set of $(x, \xi) \in \overline{T^*X}$ such $\langle \xi' \rangle^{-k} |\sigma_h(A)(x', \xi'; h)| \geq c > 0$ for h small enough and all $(x', \xi') \in T^*X$ in a neighbourhood of (x, ξ) . We recall [DyZw1, Proposition 2.4]:

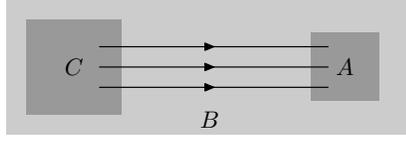


FIGURE 4. The assumptions on the microsupports of the operators in the propagation estimate (2.10) and the flow lines of the Hamilton vector field H_p . This is the simplest of the cases illustrated in Fig. 6 with A_3 and C_3 (in that case B_3 is the identity).

Proposition 2.1. *Suppose that $P \in \Psi^k(X)$ and that $u(h) \in \mathcal{D}'(X; \mathcal{E})$ be h -tempered. Then*

$$\mathrm{WF}_h(u) \cap \mathrm{ell}_h(P) \subset \mathrm{WF}_h(Pu). \quad (2.7)$$

If $A \in \Psi_h^0(X)$ and $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(P)$, then for each m ,

$$\|Au\|_{H_h^m(X)} \leq C\|Pu\|_{H_h^{m-k}(X)} + \mathcal{O}(h^\infty). \quad (2.8)$$

2.4. Propagation estimates. The crucial components of the proofs of Theorems 1 and 2 are propagation results presented in [DyZw1, §2.3] and proved in [DyZw1, §C.3].

We start by recalling a modification of the result of Duistermaat–Hörmander:

Proposition 2.2. *Assume that $\tilde{P} \in \Psi_h^1(X)$ and the semiclassical principal symbol, $\sigma_h(\tilde{P}) \in S_h^1(X)/hS_h^0(X)$, has a representative $\tilde{p} - iq$, where for some $\delta > 0$,*

$$\tilde{p} = p + \mathcal{O}(h^\delta)_{S^{1/2}(T^*X)}, \quad p(x, t\xi) = tp(x, \xi) \in \mathbb{R}, \quad \|\xi\|_g \geq 1, \quad t \geq 1, \quad q \geq 0. \quad (2.9)$$

*Let e^{tH_p} be the Hamiltonian flow of p on \bar{T}^*X and $u(h) \in \mathcal{D}'(X; \mathcal{E})$ be an h -tempered family of distributions. Then (see Figure 4):*

1. *Assume that $A, B, C \in \Psi_h^0(X)$ and for each $(x, \xi) \in \mathrm{WF}_h(A)$, there exists $T \geq 0$ with $e^{-TH_p}(x, \xi) \in \mathrm{ell}_h(C)$ and $e^{tH_p}(x, \xi) \in \mathrm{ell}_h(B)$ for $t \in [-T, 0]$. Then for each m ,*

$$\|Au\|_{H_h^m(X; \mathcal{E})} \leq K\|Cu\|_{H_h^m(X; \mathcal{E})} + Kh^{-1}\|BPu\|_{H_h^m(X; \mathcal{E})} + \mathcal{O}(h^\infty). \quad (2.10)$$

2. *If $\gamma(t)$ is a flow line of H_p , then for each $T > 0$,*

$$\gamma(-T) \notin \mathrm{WF}_h(u), \quad \gamma([-T, 0]) \cap \mathrm{WF}_h(Pu) = \emptyset \implies \gamma(0) \notin \mathrm{WF}_h(u). \quad (2.11)$$

Proof. We explain the modifications needed in the proof of [DyZw1, Proposition 2.5] where $\tilde{p} = p$. We again construct the escape function f using the homogeneous part of the symbol given by p . The difference $\tilde{p} - p$ produces an additional $\mathcal{O}(h^\delta)_{\Psi^{2m-1/2}}$ term in the operator \mathbf{T}_ε of [DyZw1], which is uniform in the parameter ε of [DyZw1] (note that in [DyZw1] the letter ε has a different meaning than in the current paper). The $H_h^{m-1/2}$ norm should be replaced by the $H_h^{m-1/4}$ norm on the right-hand side of [DyZw1, (C.12)], which leads to the same modification on the right-hand side of [DyZw1, (C.5)]; the rest of the proof is carried out the same way as in [DyZw1]. \square

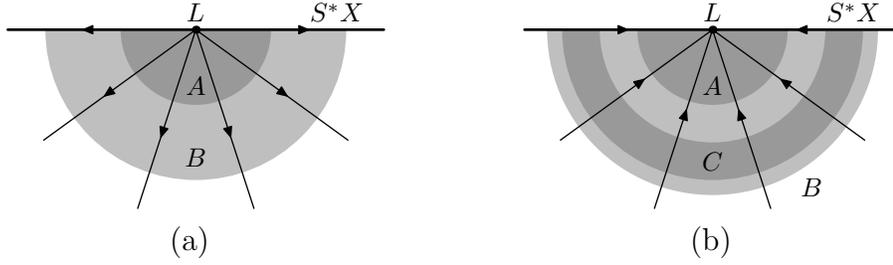


FIGURE 5. (a) The assumptions of Proposition 2.3. (b) The assumptions of Proposition 2.4. Here S^*X is the boundary of \bar{T}^*X and the flow lines of H_p are pictured.

This propagation result is applied away from the radial sinks and sources given by $\kappa(E_s^*)$ and $\kappa(E_u^*)$ where κ is the projection in (2.4) and E_\bullet^* are from (2.2). Near $\kappa(E_\bullet^*)$ we use radial estimates obtained in the context of scattering theory by Melrose [Me, Propositions 9,10] (see also Vasy [Va, Propositions 2.3,2.4]). These less standard estimates guarantee regularity of u near sources/sinks, provided that u lies in a sufficiently high/low Sobolev space.

Let p satisfy the assumptions in (2.9). Assume that $L \subset T^*X \setminus 0$ is a closed conic set invariant under the flow e^{tH_p} . It is called a *radial source* if there exists an open conic neighbourhood U of L with the following properties valid for some constant $\theta > 0$:

$$\begin{aligned} d(\kappa(e^{-tH_p}(U)), \kappa(L)) &\rightarrow 0 \quad \text{as } t \rightarrow +\infty; \\ (x, \xi) \in U &\implies |e^{-tH_p}(x, \xi)| \geq C^{-1}e^{\theta t}|\xi|, \quad \text{for any norm on the fibers.} \end{aligned} \quad (2.12)$$

A *radial sink* is defined analogously, reversing the direction of the flow.

We now have a propagation estimate near radial sources. It shows that Pu controls u there for sufficiently regular solutions:

Proposition 2.3. *Let $P \in \Psi_h^1(X)$ and assume that $\sigma_h(P)$ has a representative of the form $p - iq$ and p and Q satisfy (2.9). Assume that $L \subset T^*X \setminus 0$ is a radial source for the flow of H_p . Then there exists $m_0 > 0$ such that (see Figure 5(a))*

1. *For each $B \in \Psi_h^0(X)$ elliptic on $\kappa(L) \subset S^*X = \partial\bar{T}^*X$, there exists $A \in \Psi_h^0(X)$ elliptic on $\kappa(L)$ such that if $u(h) \in \mathcal{D}'(X; \mathcal{E})$ is h -tempered, then for each $m \geq m_0$,*

$$Au \in H_h^{m_0} \implies \|Au\|_{H_h^m} \leq Kh^{-1}\|BPu\|_{H_h^m} + \mathcal{O}(h^\infty). \quad (2.13)$$

2. *If $u(h) \in \mathcal{D}'(X; \mathcal{E})$ is h -tempered and $B \in \Psi_h^0(X)$ is elliptic on $\kappa(L)$, then*

$$Bu \in H_h^{m_0}, \quad \text{WF}_h(Pu) \cap \kappa(L) = \emptyset \implies \text{WF}_h(u) \cap \kappa(L) = \emptyset. \quad (2.14)$$

The second result shows that for sufficiently low regularity we have a propagation result at radial sinks analogous to (2.10).

Proposition 2.4. *Assume that $P \in \Psi_h^1(X)$ is as in Proposition 2.3 and $L \subset T^*X \setminus 0$ is a radial sink. Then there exists $m_0 > 0$ such that for each $B \in \Psi_h^0(X)$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0(X)$ elliptic on $\kappa(L)$ and $C \in \Psi_h^0(X)$ with $\text{WF}_h(C) \subset \text{ell}_h(B) \setminus \kappa(L)$, such that if $u(h) \in \mathcal{D}'(X)$ is h -tempered, then for each $m \leq -m_0$ (see Figure 5(b))*

$$\|Au\|_{H_h^m} \leq K\|Cu\|_{H_h^m} + Kh^{-1}\|BPu\|_{H_h^m} + \mathcal{O}(h^\infty). \quad (2.15)$$

The proofs of Propositions 2.3 and 2.4 can be found in [DyZw1, §C.3].

3. DEFINITION OF POLLICOTT–RUELLE RESONANCES

The resonances for Anosov flows are defined as spectra of the generator of the flow acting on suitably modified spaces – see Baladi–Tsuji [BaTs], Faure–Sjöstrand [FaSj], Gouëzel–Liverani [GoLi1], Liverani [Li2], and references given there.

Here we follow [DyZw1, §3.1–3.2] where the spaces are defined using microlocal weights with simple properties:

$$\begin{aligned} H_{sG(h)}(X) &:= \exp(-sG(x, hD))L^2(X), \quad G \in \Psi_h^{0+}(X), \\ \sigma_h(G) &= (1 - \psi_0(x, \xi))m_G(x, \xi) \log |\xi|_g, \end{aligned} \quad (3.1)$$

where $\psi_0 \in C_c^\infty(T^*X, [0, 1])$ is 1 near $\{\xi = 0\}$, $m_G(x, \xi) \in C^\infty(T^*X \setminus 0, [-1, 1])$ is homogeneous of degree 0 and satisfies

$$m_G(x, \xi) = \begin{cases} 1 & \text{near } E_s^* \\ -1 & \text{near } E_u^* \end{cases} \quad H_p m_G(x, \xi) \leq 0, \quad (x, \xi) \in T^*X \setminus 0. \quad (3.2)$$

The existence of such m_G is shown in [DyZw1, Lemma C.1]. For convenience we choose $|\xi|_g^2$ to be the same metric as in the definition of the Laplacian $-\Delta_g$. We can also assume that for some $\chi_0 \in C_c^\infty(\mathbb{R})$, $\chi_0 \equiv 1$ near 0,

$$G(x, hD) = (1 - \chi_0(-h^2\Delta_g))G(x, hD). \quad (3.3)$$

(Simply multiply $G(x, hD)$ by $(1 - \tilde{\chi}_0(-h^2\Delta_g))$ for $\tilde{\chi}_0 \in C_c^\infty$ such that if $|\xi|_g \in \text{supp } \tilde{\chi}_0$ then $\psi_0(x, \xi) = 1$ and then choose χ_0 so that $\text{supp } \chi_0 \subset \tilde{\chi}_0^{-1}(1)$.)

We note that as a set $H_{sG(h)}$ is independent of h and that for some N and C ,

$$h^N \|u\|_{H_{sG(1)}}/C \leq \|u\|_{H_{sG(h)}} \leq Ch^{-N} \|u\|_{H_{sG(1)}}. \quad (3.4)$$

We also need a version of weighted Sobolev spaces associated to $H_{sG(h)}$:

$$\begin{aligned} H_{sG(h)}^r &:= \exp(-G_{r,s}(x, hD))L^2(X), \quad G_{r,s} \in \Psi_h^{0+}(X), \\ \sigma_h(G_{r,s}) &= (1 - \psi_0(x, \xi))(sm_G(x, \xi) + r) \log |\xi|_g. \end{aligned} \quad (3.5)$$

We can also assume that (3.3) holds for $G_{s,r}$ as well.

The spaces with $r \neq 0$ will be used to control applications of differential operators:

$$\Psi_h^m(X) \ni A : H_{sG(h)}^r(X) \longrightarrow H_{sG(h)}^{r-m}(X). \quad (3.6)$$

Since (see [DyZw1, (3.9)])

$$H_p \sigma_h(G_{r,s}) = s \log |\xi|_g H_p m_G + \mathcal{O}(1)_{S_h^0},$$

we can use the estimates reviewed in §2.4 as in the proof of [DyZw1, Proposition 3.4]. That shows that for any $r \in \mathbb{R}$, $\lambda \in D(0, R)$, $s > s_0 = s_0(R, r)$ and $0 < h < h_0$,

$$(hP_0 - iQ - h\lambda)^{-1} = \mathcal{O}(1/h) : H_{sG(h)}^r \longrightarrow H_{sG(h)}^r. \quad (3.7)$$

Here Q is a *complex absorbing operator*

$$Q = \chi(-h^2 \Delta_g), \quad \chi \in C_0^\infty((-2, 2); [0, 1]), \quad \chi(t) = 1, \quad t \in [-1, 1]. \quad (3.8)$$

It is introduced to damp the trapped set which, on $p^{-1}(0)$, is equal to the zero section. Writing

$$P_0 - \lambda = h^{-1}(I + iQ(hP_0 - iQ - h\lambda)^{-1})(hP_0 - iQ - h\lambda),$$

and noting that

$$Q(hP_0 - iQ - h\lambda)^{-1} : H_{sG(1)} \rightarrow C^\infty(X), \quad (3.9)$$

is compact as an operator $H_{sG(1)} \rightarrow H_{sG(1)}$, analytic Fredholm theory (see for instance [Zw1, Theorem D.4]) shows that $(P_0 - \lambda)^{-1}$ is a meromorphic family:

Proposition 3.1. *For $\lambda \in D(0, R)$ and $s > s_0 = s_0(R)$,*

$$(P_0 - \lambda)^{-1} : H_{sG(1)} \rightarrow H_{sG(1)},$$

is a meromorphic family of operators with poles of finite rank. These poles are independent of s and are called Pollicott–Ruelle resonances.

The mapping property (3.9) also shows that the operator there is of trace class. Combined with Gohberg–Sigal theory (see for instance [DyZw2, (C.4.6)]) this gives the following characterization of Pollicott–Ruelle resonances:

Proposition 3.2. *Let $R > 0$ and assume that $s > s_0(R)$. For $0 < h < h_0(R, s)$ define*

$$D_R(\lambda) := \det_{H_{sG(1)}}(I + iQ(hP_0 - iQ - h\lambda)^{-1}), \quad \lambda \in D(0, R).$$

Then Pollicott–Ruelle resonances in $D(0, R)$ are given, with multiplicities, by the zeros of D_R .

4. MICROLOCAL BOUNDS ON THE MODIFIED OPERATOR

Let P_ε be given by (1.1) and let Q be the complex absorbing operator (3.8). The goal of this section is to prove that for $0 < h < h_0$ and $0 \leq \varepsilon < h/C$ the operator $hP_\varepsilon - iQ - h\lambda$ is invertible on the same weighted spaces on which $hP_0 - iQ - h\lambda$ is invertible. Note that for $\varepsilon > 0$, $hP_\varepsilon - iQ - h\lambda$ is a Fredholm operator $H_{sG(h)}^2 \rightarrow H_{sG(h)}$ of index 0 by the standard elliptic theory applied to the conjugation of this operator by $e^{sG(x, hD)}$ (see (4.6) below and [HöIII, Theorem 19.2.1]).

We first prove an elliptic estimate, which does not involve the parameter h :

Lemma 4.1. *Suppose that $\chi_1 \in C_0^\infty((-2, 2), [0, 1])$ satisfies $\chi_1 = 1$ on $[-1, 1]$, and put $\chi_2(t) := \chi_1(3t)$. Then for $\lambda \in D(0, R)$,*

$$\begin{aligned} & \| (1 - \chi_1(-\varepsilon^2 \Delta_g))u \|_{H_{sG(1)}} + \| \varepsilon^2 \Delta_g (1 - \chi_1(-\varepsilon^2 \Delta_g))u \|_{H_{sG(1)}} \\ & \leq C\varepsilon \| (1 - \chi_2(-\varepsilon^2 \Delta_g))(P_\varepsilon - \lambda)u \|_{H_{sG(1)}} + \mathcal{O}_R(\varepsilon^\infty) \| u \|_{H_{sG(1)}}. \end{aligned} \quad (4.1)$$

Proof. In (3.3) we can assume that $\text{supp } \chi_0 \subset \chi_2^{-1}(1)$: changing χ_0 corresponds to changing ψ_0 in the definition of $H_{sG(1)}$ that produces an equivalent norm (see [Zw1, Theorem 8.8]).

The weight of the space $H_{sG(1)}$ is not smooth at the zero section when one considers the ε -quantization. To counteract this problem, we introduce a new, ε -dependent, norm on $H_{sG(1)}$ using a modified weight:

$$\| u \|_{s,\varepsilon} := \| e^{sG_\varepsilon(x,\varepsilon D)} u \|_{L^2}, \quad G_\varepsilon(x, \varepsilon D) := (1 - \chi_0(-\varepsilon^2 \Delta_g))G(x, D), \quad (4.2)$$

where $G_\varepsilon(x, \varepsilon D) \in \log(1/\varepsilon)\Psi_\varepsilon^{0+}(X)$ and

$$\sigma_\varepsilon(G_\varepsilon(x, \varepsilon D)) := (1 - \chi_0(|\xi|_g^2)) \log(|\xi|_g/\varepsilon) m_G(x, \xi) \pmod{\varepsilon \log(1/\varepsilon) S^{-1+}(T^*X)}. \quad (4.3)$$

(We used here the homogeneity of m_G : $m_G(x, \xi/\varepsilon) = m_G(x, \xi)$.)

We now claim that for $j = 1, 2$,

$$(e^{sG(x,D)} - e^{sG_\varepsilon(x,\varepsilon D)})(1 - \chi_j(-\varepsilon^2 \Delta_g)) = \mathcal{O}(\varepsilon^\infty)_{\mathcal{D}'(X) \rightarrow C^\infty(X)}. \quad (4.4)$$

This can be rewritten as the following identity for $t = s$:

$$(e^{tG(x,D)} e^{-tG_\varepsilon(x,\varepsilon D)} - I) e^{sG_\varepsilon(x,\varepsilon D)} (1 - \chi_j(-\varepsilon^2 \Delta_g)) = \mathcal{O}(\varepsilon^\infty)_{\mathcal{D}'(X) \rightarrow C^\infty(X)}.$$

Differentiating the left-hand side in t , we obtain

$$e^{tG(x,D)} C(t, s), \quad C(t, s) = \chi_0(-\varepsilon^2 \Delta_g) G(x, D) e^{(s-t)G_\varepsilon(x,\varepsilon D)} (1 - \chi_j(-\varepsilon^2 \Delta_g)).$$

We now consider $C(t, s)$ as an operator in $\Psi^{s-t+}(X)$. Since $\text{supp } \chi_0 \cap \text{supp}(1 - \chi_j) = \emptyset$, we see that all the terms in the symbolic composition formula for the four factors in $C(t, s)$ vanish. The remainder (estimated, for instance, as in [Zw1, (9.3.7)]) is of size ε^N for any N . Hence $C(t, s) \in \varepsilon^\infty \Psi^{-\infty}(X)$ and consequently

$$e^{tG(x,D)} C(t, s) \in \varepsilon^\infty \Psi^{-\infty}(X),$$

uniformly for bounded t, s . Integration then gives (4.4).

By (4.4), we may replace the $H_{sG(1)}$ norms in (4.1) by the $\| \bullet \|_{s,\varepsilon}$ norms. We now consider our operator in the ε -pseudodifferential calculus:

$$\varepsilon(P_\varepsilon - \lambda) \in \Psi_\varepsilon^2, \quad p_\varepsilon(x, \xi) := \sigma_\varepsilon(\varepsilon P_\varepsilon) = -i|\xi|_g^2 + \xi(V_x).$$

This operator is elliptic in the class Ψ_ε^2 for $\xi \neq 0$. By the choice of χ_j 's, we see that both $\varepsilon(P_\varepsilon - \lambda) \in \Psi_\varepsilon^2$ and $(1 - \chi_2(-\varepsilon^2 \Delta_g)) \in \Psi_\varepsilon^0$ are elliptic on $\text{WF}_\varepsilon(1 - \chi_1(-\varepsilon^2 \Delta_g))$. Hence the estimate (4.1) holds for $s = 0$ – see Proposition 2.1 above.

To prove (4.1) for the $\|\bullet\|_{s,\varepsilon}$ -norms, we consider conjugated operators:

$$P_{\varepsilon,s} := e^{sG_\varepsilon(x,\varepsilon D)} P_\varepsilon e^{-sG_\varepsilon(x,\varepsilon D)}, \quad A_{j,s}(x, hD) := e^{sG_\varepsilon(x,\varepsilon D)} (1 - \chi_j(-\varepsilon^2 \Delta_g)) e^{-sG_\varepsilon(x,\varepsilon D)},$$

and need to prove that

$$\|A_{1,s}u\|_{H_g^2} \leq C \|A_{2,s}(\varepsilon P_{\varepsilon,s} - \varepsilon \lambda)\|_{L^2} + \mathcal{O}(\varepsilon^\infty) \|u\|_{L^2}. \quad (4.5)$$

(The conjugation of $\varepsilon^2 \Delta_g$ appearing in (4.1) is handled in the same way as $\varepsilon P_{\varepsilon,s}$ below.)

We have, as in [DyZw1, §3.3], $\varepsilon P_{\varepsilon,s} \in \Psi_\varepsilon^2$, $A_{j,s} \in \Psi_\varepsilon^0$, and

$$\varepsilon P_{\varepsilon,s} = \varepsilon P_\varepsilon - i\varepsilon s \frac{i}{\varepsilon} [G_\varepsilon, \varepsilon P_\varepsilon] + \mathcal{O}(\varepsilon^2 \log(1/\varepsilon))_{\Psi_\varepsilon^{0+}}, \quad (4.6)$$

so that

$$\sigma_\varepsilon(\varepsilon P_{\varepsilon,s}) = -i|\xi|_g^2 + \xi(V) + \varepsilon s H_{|\xi|_g^2} \sigma_\varepsilon(G_\varepsilon) + i\varepsilon H_{\xi(V)} \sigma_\varepsilon(G_\varepsilon) \pmod{\varepsilon S^1(T^*X)}.$$

Recalling (4.3) we see that

$$|H_{|\xi|_g^2} \sigma_\varepsilon(G_\varepsilon)| + |H_{\xi(V)} \sigma_\varepsilon(G_\varepsilon)| \leq C \log(1/\varepsilon) \langle \xi \rangle^{1+}.$$

Hence, $\varepsilon P_{\varepsilon,s} - \varepsilon \lambda$ is elliptic in Ψ_ε^2 on the set $|\xi| > \delta$ for any $\delta > 0$. Composition of pseudodifferential operators in Ψ_ε^* shows that

$$\text{WF}_\varepsilon(A_{1,s}) \subset \{|\xi| > 1\} \subset \text{ell}_\varepsilon(\varepsilon P_{\varepsilon,s} - \varepsilon \lambda), \quad \text{WF}_\varepsilon(I - A_{2,s}) \cap \text{WF}_\varepsilon(A_{1,s}) = \emptyset.$$

We can apply Proposition 2.1 again to obtain (4.5) and hence (4.1). \square

We turn to the question of invertibility of $hP_\varepsilon - iQ - \lambda h$ and suppose that

$$(hP_\varepsilon - iQ - \lambda h)u = f.$$

For $\varepsilon < h/C$ we have $(1 - \chi_2(-\varepsilon^2 \Delta_g))Q = 0$. Hence in view of (3.4) and (4.1),

$$\begin{aligned} & \| (1 - \chi_1(-\varepsilon^2 \Delta_g))u \|_{H_{sG(h)}} + \| \varepsilon^2 \Delta_g (1 - \chi_1(-\varepsilon^2 \Delta_g))u \|_{H_{sG(h)}} \\ & \leq Ch^{-N} \varepsilon \| (1 - \chi_2(-\varepsilon^2 \Delta_g))f \|_{H_{sG(h)}} + \mathcal{O}_R(h^{-N} \varepsilon^\infty) \| u \|_{H_{sG(h)}}, \end{aligned} \quad (4.7)$$

for $\lambda \in D(0, R)$, $\varepsilon < h/C$, and some N depending on s .

Put

$$\tilde{P}_\varepsilon(\lambda) := \frac{h}{i} V + i\varepsilon h \Delta_g \chi_1(-\varepsilon^2 \Delta_g) - iQ - \lambda h, \quad (4.8)$$

Then

$$\tilde{P}_\varepsilon(\lambda)u = -i\varepsilon h \Delta_g (1 - \chi_1(-\varepsilon^2 \Delta_g))u + f =: F. \quad (4.9)$$

From (4.7) we see immediately that

$$\|F\|_{H_{sG(h)}} \leq Ch^{-N} \|f\|_{H_{sG(h)}} + \mathcal{O}(h^{-N} \varepsilon^\infty) \|u\|_{H_{sG(h)}}, \quad (4.10)$$

where N depends on s .

The operator $\tilde{P}_\varepsilon(\lambda)$ on the left-hand side of (4.9) is an h -pseudodifferential operator in Ψ_h^1 and

$$\sigma_h(\tilde{P}_\varepsilon(\lambda)) = \xi(V_x) - i|\xi|_g \chi_1 \left(\frac{\varepsilon^2}{h^2} |\xi|_g^2 \right) \frac{\varepsilon}{h} |\xi|_g - i\chi(|\xi|_g^2) \in S^1(T^*X),$$

uniformly in $\varepsilon \in (0, Ch)$, $\lambda \in D(0, R)$. The domain of this operator is given by the domain of V acting on $H_{sG(h)}$:

$$\begin{aligned} D_{sG(h)} &= \{u \in H_{sG(h)} \mid Vu \in H_{sG(h)} \subset \mathcal{D}'(X)\}, \\ \|u\|_{D_{sG(h)}} &= \|u\|_{H_{sG(h)}} + h\|Vu\|_{H_{sG(h)}}. \end{aligned}$$

We now verify that the main estimate of [DyZw1, §3.3] is valid for the operator $\tilde{P}_\varepsilon(\lambda)$. The key fact is that the operator is now of order 1 in ξ as, using Lemma 4.1, we can control F by f .

Lemma 4.2. *Suppose that $\lambda \in D(0, R)$ and that $0 \leq \varepsilon \leq h/C_0$. Then there exist $h_0 = h_0(R)$, $s_0 = s_0(R)$, $C = C(R)$ (independent of ε) such that for $u \in D_{sG(h)}$ and the operator $\tilde{P}_\varepsilon(\lambda)$ defined in (4.8)*

$$\|u\|_{H_{sG(h)}} \leq Ch^{-1} \|\tilde{P}_\varepsilon(\lambda)u\|_{H_{sG(h)}}, \quad s_0 < s, \quad 0 < h < h_0. \quad (4.11)$$

Proof. We refer to the proof of [DyZw1, Proposition 3.4] for details and explain the differences between the operator $\tilde{P}_\varepsilon(\lambda)$ and the operator $\tilde{P}_0(\lambda) = \frac{h}{i}V - iQ - h\lambda$ considered there. We recall that the proof is based on propagation results recalled in §2.4.

First of all, near $\kappa(E_s^*)$, where $\kappa : T^*X \setminus 0 \rightarrow S^*X = \partial\bar{T}^*X$ is the projection to fiber infinity, we use the *radial source* estimate (Proposition 2.3). The operator $\tilde{P}_\varepsilon(\lambda)$ satisfies the assumptions of Proposition 2.3 and we get for each N

$$\|A_1u\|_{H_h^s} \leq Ch^{-1} \|B_1\tilde{P}_\varepsilon(\lambda)u\|_{H_h^s} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}}, \quad s > s_0, \quad (4.12)$$

where both $A_1, B_1 \in \Psi_h^0$ are microlocalized in a small neighborhood of $\kappa(E_s^*)$ and A_1 is elliptic near $\kappa(E_s^*)$ – see Fig. 6. From the properties of the weight G – see (3.2) – we see that

$$\|A_1u\|_{H_{sG(h)}} = \|A_1u\|_{H_h^s} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}}, \quad \|B_1f\|_{H_{sG(h)}} = \|B_1f\|_{H_h^s} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}},$$

and hence we can replace H_h^s by $H_{sG(h)}$ in (4.12).

Similarly if $A_2 \in \Psi_h^0$ is microlocalized near $\kappa(E_u^*)$ there exist $B_2, C_2 \in \Psi_h^0$ microlocalized near $\kappa(E_u^*)$ with $\text{WF}_h(C_2) \cap \kappa(E_u^*) = \emptyset$ (see Fig. 6) such that

$$\|A_2u\|_{H_h^{-s}} \leq C \|C_2u\|_{H_h^{-s}} + Ch^{-1} \|B_2\tilde{P}_\varepsilon(\lambda)u\|_{H_h^{-s}} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}}, \quad s > s_0. \quad (4.13)$$

This follows from Proposition 2.4. Recalling (3.2) again we see that

$$\|A_2u\|_{H_{sG(h)}} = \|A_2u\|_{H_h^{-s}} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}}, \quad \|B_2f\|_{H_{sG(h)}} = \|B_2u\|_{H_h^{-s}} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}},$$

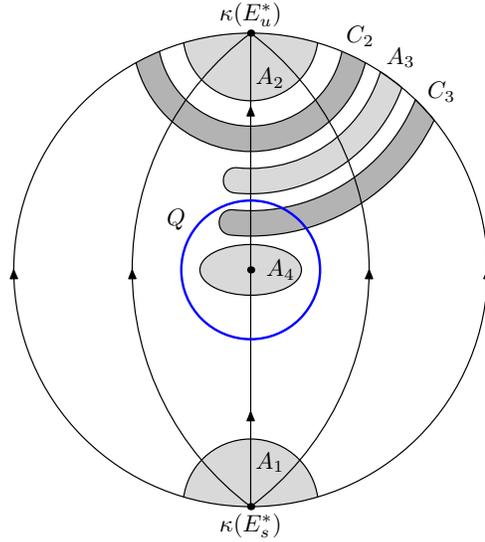


FIGURE 6. A schematic representation of the flow on $\overline{T^*X}$. Different regions (we denote by \bullet_j the region of microlocalization of \bullet_j ; control in C_j is needed for the estimate in A_j) in which different propagation results are applied: for A_1 we use the radial source estimates (Proposition 2.3); for A_2 the radial sink estimates (Proposition 2.4); for A_3 the standard propagation result (Proposition 2.2) applied to the conjugated operator; for A_4 we use elliptic estimates (Proposition 2.1). Since for A_1 and A_4 we do not need any initial control (given by C_j), C_3 can be dynamically controlled by regions of the type A_1 and A_4 , and C_2 is a region of the type A_2 , a partition of unity provides a global estimate (4.15).

and similarly for C_2 , so that again the estimate (4.13) is valid with H_h^{-s} replaced by $H_{sG(h)}$.

We now have to consider the case of $A_3 \in \Psi_h^0$ microlocalized away from $\kappa(E_u^*) \cup \kappa(E_s^*)$. For that we need to see that the conjugated operator satisfies the assumptions of the Duistermaat–Hörmander propagation theorem (Proposition 2.2). As in (4.6) we have

$$\tilde{P}_{\varepsilon,s}(\lambda) := e^{sG(h)} \tilde{P}_\varepsilon(\lambda) e^{-sG(h)} = \tilde{P}_\varepsilon(\lambda) - ihs \frac{i}{h} [G(h), \tilde{P}_\varepsilon(\lambda)] + \mathcal{O}(h^2)_{\Psi_h^{-1+}},$$

where now, as the operators $\tilde{P}_\varepsilon(\lambda)$ and $G(h)$ are uniformly bounded in Ψ_h^1 and Ψ_h^{0+} , respectively, the error is in Ψ_h^{-1+} . Hence we have

$$\sigma_h(P_{\varepsilon,s}(\lambda)) = p_{\varepsilon,s}(x, \xi) - iq_{\varepsilon,s}(x, \xi) \quad \text{mod } (h\Psi_h^0)$$

where, with $p(x, \xi) := \xi(V_x)$, away from $\xi = 0$ we can take

$$\begin{aligned} p_{\varepsilon,s}(x, \xi) &= p(x, \xi) - hs \log |\xi|_g H_{m_G} \left(|\xi|_g \chi_1 \left(\frac{\varepsilon^2}{h^2} |\xi|_g^2 \right) \frac{\varepsilon}{h} |\xi|_g \right), \\ q_{\varepsilon,s}(x, \xi) &= \chi(|\xi|_g^2) + |\xi|_g \chi_1 \left(\frac{\varepsilon^2}{h^2} |\xi|_g^2 \right) \frac{\varepsilon}{h} |\xi|_g - hs \log |\xi|_g H_p m_G(x, \xi) \geq 0. \end{aligned} \quad (4.14)$$

We note that $\tilde{p} := p_{\varepsilon,s} = p + \mathcal{O}(h)_{s_0+}$ satisfies the assumptions of Proposition 2.2 with $\delta = 1$ and $q_{\varepsilon,s} \leq 0$. Hence the propagation estimate (2.10) applies.

As in the proof of [DyZw1, Proposition 3.4], combining (4.12), (4.13), Proposition 2.2, and the elliptic estimate (Proposition 2.1) we obtain uniformly in ε ,

$$\begin{aligned} \|u\|_{H_{sG(h)}} &\leq Ch^{-1} \|\tilde{P}_\varepsilon(\lambda)u\|_{H_{sG(h)}} + \mathcal{O}(h^\infty) \|u\|_{H_h^{-N}}, \quad 0 < h < h_0(R) \\ & \quad s > s_0(R), \quad \lambda \in D(0, R), \quad 0 < \varepsilon \leq h, \end{aligned} \quad (4.15)$$

for any N and that implies (4.11), finishing the proof. \square

We now fix $h < h_0$ and apply Lemma 4.2 to (4.9). That and (4.10) give

$$\|u\|_{H_{sG(h)}} \leq Ch^{-N} \|f\|_{H_{sG(h)}} + \mathcal{O}(h^{-N} \varepsilon^\infty) \|u\|_{H_{sG(h)}}$$

and the $\mathcal{O}(h^{-N} \varepsilon^\infty)$ can be absorbed into the left-hand side for ε/h small enough.

We summarize the result of this section in

Proposition 4.3. *Let P_ε be given by (1.1) and Q by (3.8). Suppose that $\lambda \in D(0, R)$ and that $0 \leq \varepsilon \leq h/C_0$. Then there exist $h_0 = h_0(R)$, $s_0 = s_0(R)$, (independent of ε) such that for $0 < h < h_0$ and $s > s_0(R)$*

$$hP_\varepsilon - iQ - h\lambda : H_{sG(h)}^2 \rightarrow H_{sG(h)},$$

is invertible and for some constants C and N independent of ε ,

$$\|(hP_\varepsilon - iQ - h\lambda)^{-1}\|_{H_{sG(h)} \rightarrow H_{sG(h)}} \leq Ch^{-N}. \quad (4.16)$$

Remark. Same statement is true if we replace the spaces $H_{sG(h)}$ with $H_{sG(h)}^r$ for some fixed r . Indeed, this amounts to replacing sm_G by $sm_G + r$ in the weight G . The proof of Lemma 4.1 remains unchanged. As for Lemma 4.2, its proof uses the inequality $H_p m_G \leq 0$ (which is still true), as well as the fact that $H_{sG(h)}$ is equivalent to H_h^s microlocally near E_s^* and to H_h^{-s} microlocally near E_u^* . The space $H_{sG(h)}^r$ is equivalent to H_h^{r+s} near E_s^* and to H_h^{r-s} near E_u^* ; for s large enough depending on r and R , Lemma 4.2 still holds.

5. STOCHASTIC APPROXIMATION OF POLLICOTT–RUELLE RESONANCES

In this section we prove Theorem 1. Using Proposition 4.3 we see that for $\lambda \in D(0, R)$, we have the following expression for the meromorphic continuation of the resolvent of P_ε :

$$(P_\varepsilon - \lambda)^{-1} = h(hP_\varepsilon - iQ - h\lambda)^{-1}(I + K(\lambda, \varepsilon))^{-1} : H_{sG} \rightarrow H_{sG}, \quad (5.1)$$

where

$$K(\lambda, \varepsilon) := iQ(hP_\varepsilon - iQ - h\lambda)^{-1} : H_{sG} \rightarrow H_{sG}, \quad (5.2)$$

is of trace class and depends holomorphically on λ – see (3.9). Here $0 < h < h_0$, $0 \leq \varepsilon \leq \varepsilon_0 := h/C_0$ and $s > s_0$ with h_0 and s_0 depending on R . We fix h and drop it in the notation for H_{sG} .

As in Proposition 3.2 we see that the spectrum of P_ε in $D(0, R)$ is given (with multiplicities) by the zeros of the following Fredholm determinant:

$$D_R(\lambda, \varepsilon) := \det_{H_{sG}}(I + K(\lambda, \varepsilon)). \quad (5.3)$$

Note that, since Q is compactly microlocalized, $K(\lambda, \varepsilon)$ acts $H_{sG} \rightarrow H^N$ for all N . It follows that $D_R(\lambda, \varepsilon)$ is equal to the H^N determinant of $I + K(\lambda, \varepsilon)$ for each $N \geq s$.

To analyze the determinant $D_R(\lambda, \varepsilon)$, we apply the following two lemmas. We use the notation $f \in C^1([a, b])$ to mean that f and its derivative f' are continuous in $[a, b]$; here $f'(a), f'(b)$ are the left and right derivatives of f at those points. By induction we then define $C^k([a, b])$ and $C^\infty([a, b])$.

Lemma 5.1. *Let R and h be fixed so that (5.1) is valid. Then for every k there exists $s_1 = s_1(k, R)$ such that for $s \geq s_1$,*

$$K(\lambda, \varepsilon) \in C^k([0, \varepsilon_0]_\varepsilon, \text{Hol}(D(0, R)_\lambda, \mathcal{L}^1(H^s, H^s))), \quad (5.4)$$

where $H^s = H^s(X)$ are Sobolev spaces and \mathcal{L}^1 denotes the space of trace class operators.

Proof. We first show that the identity

$$\partial_\varepsilon(hP_\varepsilon - iQ - h\lambda)^{-1} = -ih(hP_\varepsilon - iQ - h\lambda)^{-1}\Delta_g(hP_\varepsilon - iQ - h\lambda)^{-1} \quad (5.5)$$

is true for $\varepsilon \in [0, \varepsilon_0]$ in the space $\text{Hol}(D(0, R), \mathcal{B}(H_{sG}^r, H_{sG}^{r-4}))$, for each r and for s large enough depending on R and r . Here \mathcal{B} stands for the class of bounded operators with operator norm. Indeed, for each $\varepsilon, \varepsilon' \in [0, \varepsilon_0]$,

$$\begin{aligned} & \frac{(hP_\varepsilon - iQ - h\lambda)^{-1} - (hP_{\varepsilon'} - iQ - h\lambda)^{-1}}{\varepsilon - \varepsilon'} \\ &= -ih(hP_\varepsilon - iQ - h\lambda)^{-1}\Delta_g(hP_{\varepsilon'} - iQ - h\lambda)^{-1} \end{aligned} \quad (5.6)$$

where the right-hand side of the equation is uniformly bounded in $\varepsilon, \varepsilon'$ as an operator $H_{sG}^r \rightarrow H_{sG}^{r-2}$. Here we used

$$(hP_{\varepsilon'} - iQ - h\lambda)^{-1} \in \mathcal{B}(H_{sG}^r, H_{sG}^r), \quad (hP_\varepsilon - iQ - h\lambda)^{-1} \in \mathcal{B}(H_{sG}^{r-2}, H_{sG}^{r-2}),$$

(see Proposition 4.3 and the remark following it) and the fact that Δ_g is bounded $H_{sG}^r \rightarrow H_{sG}^{r-2}$. Now, (5.6) implies that $(hP_\varepsilon - iQ - h\lambda)^{-1}$ is Lipschitz (and thus continuous) as an operator $H_{sG}^r \rightarrow H_{sG}^{r-2}$. Passing to the limit $\varepsilon' \rightarrow \varepsilon$ in (5.6), we obtain (5.5) in the class $\mathcal{B}(H_{sG}^r, H_{sG}^{r-4})$. Holomorphy in λ follows automatically from the holomorphy of each of the operators involved.

Iterating (5.5), we see that for each r , each $k > 0$, and for s large enough depending on R, r and k ,

$$(hP_\varepsilon - iQ - h\lambda)^{-1} \in C^k([0, \varepsilon_0]_\varepsilon, \text{Hol}(D(0, R), \mathcal{B}(H_{sG}^r, H_{sG}^{r-4k}))).$$

To obtain (5.4) we recall the definition (5.2) of $K(\lambda, \varepsilon)$, take $r = 0$, note that H^s embeds into H_{sG} and that the operator Q is compactly microlocalized and thus of trace class $H_{sG}^{-4k} \rightarrow H^s$. \square

Lemma 5.2. *Suppose that $\{X_j\}_{j=0}^\infty$ is a nested family of Hilbert spaces, $X_{j+1} \subset X_j$. Let*

$$K(\varepsilon) : X_j \rightarrow \bigcap_{\ell=0}^\infty X_\ell, \quad (5.7)$$

be a family of operators such that $K \in C^k([0, \varepsilon_0], \mathcal{L}^1(X_k, X_k))$. Then

$$F(\varepsilon) := \det_{X_0}(I + K(\varepsilon)) \in C^\infty([0, \varepsilon_0]).$$

Proof. Because of (5.7) we see that $\det_{X_j}(I + K(\varepsilon))$ is independent of j and hence we only need to prove that $\det_{X_j}(I + K(\varepsilon)) \in C^j([0, \varepsilon_0]_\varepsilon)$, for any j . For $j = 1$ we note that $\partial_\varepsilon F(\varepsilon) = F(\varepsilon) \text{tr}_{X_1}((I + K(\varepsilon))^{-1} \partial_\varepsilon K(\varepsilon))$. The operators $\varepsilon \mapsto F(\varepsilon)(I + K(\varepsilon))^{-1}$ form a continuous family of uniformly bounded operators (see for instance [DyZw2, (B.7.4)]). Hence, $|\partial_\varepsilon F(\varepsilon)| \leq C \|\partial_\varepsilon K(\varepsilon)\|_{\mathcal{L}^1(X_1, X_1)}$. Higher order derivatives are handled similarly and smoothness of F follows. \square

Applying this Lemma with $X_j = H^{s_1(j, R)}$ where s_1 comes from Lemma 5.1, we see that $\varepsilon \mapsto D_R(\lambda, \varepsilon)$ is a smooth function of $\varepsilon \in [0, \varepsilon_0]$ with values in $\text{Hol}(D(0, R))$. Rouché's theorem implies that the zeros are continuous in ε up to 0, proving Theorem 1. If μ_0 is a simple zero of $D_R(\lambda, 0)$ then for $0 \leq \varepsilon < \varepsilon_1$, $D_R(\lambda, \varepsilon)$ has a unique zero, $\mu(\varepsilon)$, close to μ_0 . Smoothness of D_R in ε shows that

$$\mu(\varepsilon) \in C^\infty([0, \varepsilon_1]).$$

When the zeros are not simple (in particular, when the eigenvalues of P_0 are not semisimple) the situation is potentially quite complicated. However we have smoothness of spectral projectors:

Proposition 5.3. *Suppose that $\mu_0 \in D(0, R - 1)$ is an eigenvalue of $P_0 : H_{sG}(X) \rightarrow H_{sG}(X)$, $s \geq s_0(R)$, and that multiplicity of μ_0 is m :*

$$m = \operatorname{tr} \Pi_0, \quad \Pi_0 = \frac{1}{2\pi i} \oint_{\gamma_\delta} (\lambda - P_0)^{-1} d\lambda,$$

where $\gamma_\delta : [0, 2\pi) \ni t \rightarrow \mu_0 + \delta e^{it}$, and δ is small enough.

Then there exists ε_0 and δ such that for $0 < \varepsilon \leq \varepsilon_0$, P_ε has exactly m eigenvalues in $D(\mu_0, \delta)$:

$$\operatorname{tr} \Pi_\varepsilon = m, \quad \Pi_\varepsilon := \frac{1}{2\pi i} \oint_{\gamma_\delta} (\lambda - P_\varepsilon)^{-1} d\lambda, \quad \Pi_\varepsilon^2 = \Pi_\varepsilon, \quad (5.8)$$

and $\Pi_\varepsilon \in C^\infty([0, \varepsilon_0], \mathcal{L}^1(C^\infty(X), \mathcal{D}'(X)))$. More precisely, the projections Π_ε have rank m and for each j there exists s_j such that

$$\Pi_\varepsilon \in C^j([0, \varepsilon_0], \mathcal{L}(H_{s_jG}, H_{s_jG})) \subset C^j([0, \varepsilon_0], \mathcal{L}(H^{-s_j}, H^{s_j})). \quad (5.9)$$

Proof. From the analysis of the determinants, we already know that there exist ε_0, δ such that for $0 \leq \varepsilon \leq \varepsilon_0$, $\lambda \mapsto D_R(\lambda, \varepsilon)$ has no zeros on $|\lambda - \mu_0| = \delta$ and has exactly m zeros in $D(\mu_0, \delta)$. Hence the spectral projectors are well defined by the formula in (5.8) and their rank is equal to m . To consider regularity, we choose h sufficiently small (depending on R) and write

$$(P_\varepsilon - \lambda)^{-1} = (P_\varepsilon - ih^{-1}Q - \lambda)^{-1} - ih^{-1}(P_\varepsilon - \lambda)^{-1}Q(P_\varepsilon - ih^{-1}Q - \lambda)^{-1}.$$

Since the first term is holomorphic in $\lambda \in D(0, R)$ we have

$$\Pi_\varepsilon := -\frac{1}{2\pi h} \oint_{\gamma_\delta} (\lambda - P_\varepsilon)^{-1} Q(P_\varepsilon - ih^{-1}Q - \lambda)^{-1} d\lambda.$$

Also

$$(\lambda - P_\varepsilon)^{-1} = \mathcal{O}_{R,r,s}(1) : H_{sG}^r \rightarrow H_{sG}^r, \quad s \geq s_0(R, r), \quad \lambda \in \partial D(\mu_0, \delta).$$

Hence the same argument as in the proof of Lemma 5.1 shows j -fold differentiability of Π_ε as bounded operators $H_{s_jG} \rightarrow H_{s_jG}$. \square

6. STOCHASTIC STABILITY IN THE CASE OF CONTACT ANOSOV FLOWS

We now turn to the proof of Theorem 2. The first result concerns values of ε larger than h^2 . Here we do not need to make the contact assumption on the flow.

Lemma 6.1. *Let P_ε be given by (1.1). There exist $K_0 > 0$ and $h_0 > 0$ such that for any $\gamma > 1$, h and ε satisfying*

$$0 < K_0 \gamma h^2 < \varepsilon, \quad 0 < h < h_0,$$

we have

$$(hP_\varepsilon - z)^{-1} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right) : L^2(X) \rightarrow L^2(X), \quad z \in [\tfrac{1}{2}, \tfrac{3}{2}] - i[0, \gamma h]. \quad (6.1)$$

In particular hP_ε does not have any spectrum with $|z - 1| < \frac{1}{2}$ and $\text{Im } z > -\gamma h$.

Remark. The lemma shows that for any fixed ε the number of eigenvalues of P_ε in $\text{Im } \lambda > -C$ is finite. In fact the rescaling from z to λ shows that there are only finitely many eigenvalues of P_ε in $\{\text{Im } \lambda > -\varepsilon|\text{Re } \lambda|^2/C_0\}$, for some fixed C_0 . This leads to an easy justification of the expansion (1.3). We also see that a gap $\text{Im } z > -\gamma h$ for any γ is valid for $\varepsilon > C(\gamma)h^2$. Hence in what follows we will assume that $\varepsilon = \mathcal{O}(h^2)$.

Proof. We fix the volume form on X induced by the metric g , so that the operator Δ_g is symmetric on $L^2(X)$. Take $u \in H^2(X)$ and denote $f := (hP_\varepsilon - z)u$; then

$$\langle f, u \rangle_{L^2} = \langle (\frac{h}{i}V + ih\varepsilon\Delta_g - z)u, u \rangle_{L^2} = \frac{h}{i}\langle Vu, u \rangle_{L^2} - ih\varepsilon\|\nabla_g u\|_{L^2}^2 - z\|u\|_{L^2}^2.$$

Taking the real part, we get

$$\text{Re}\langle f, u \rangle_{L^2} = h \text{Im}\langle Vu, u \rangle_{L^2} - \text{Re } z\|u\|_{L^2}^2.$$

Since $\text{Re } z \geq \frac{1}{2}$ and V is a vector field, we find for some constant C independent of h, z, ε ,

$$\|u\|_{L^2}^2 \leq C\|f\|_{L^2} \cdot \|u\|_{L^2} + Ch\|\nabla_g u\|_{L^2} \cdot \|u\|_{L^2},$$

which implies

$$\|u\|_{L^2} \leq C\|f\|_{L^2} + Ch\|\nabla_g u\|_{L^2}. \quad (6.2)$$

Now, taking the imaginary part, we get for $F := \frac{1}{2} \text{div } V \in C^\infty(X)$,

$$\text{Im}\langle f, u \rangle_{L^2} = h\langle Fu, u \rangle_{L^2} - h\varepsilon\|\nabla_g u\|_{L^2}^2 - (\text{Im } z)\|u\|_{L^2}^2.$$

Since $\text{Im } z \geq -\gamma h$ and F is a bounded function, we get

$$h\varepsilon\|\nabla_g u\|_{L^2}^2 \leq C\|f\|_{L^2} \cdot \|u\|_{L^2} + (C + \gamma)h\|u\|_{L^2}^2,$$

which implies

$$\|\nabla_g u\|_{L^2} \leq Ch^{-1}\varepsilon^{-1/2}\|f\|_{L^2} + (C + \sqrt{\gamma})\varepsilon^{-1/2}\|u\|_{L^2}. \quad (6.3)$$

Combining (6.2) and (6.3), we get

$$\|u\|_{L^2} \leq C\varepsilon^{-1/2}\|f\|_{L^2} + Ch(C + \sqrt{\gamma})\varepsilon^{-1/2}\|u\|_{L^2}.$$

For K_0 large enough and $\varepsilon > K_0\gamma h^2$, $\gamma > 1$, we have $Ch(C + \sqrt{\gamma})\varepsilon^{-1/2} < \frac{1}{2}$, implying (6.1). \square

To prove Theorem 2 we follow [NoZw]. We first prove a result in which damping is introduced near the fiber infinity in T^*X . For that we introduce a complex absorbing operator

$$W_0 := -f(-h^2\Delta_g)h^2\Delta_g, \quad (6.4)$$

where $f \in C^\infty(\mathbb{R})$ satisfies the following conditions:

$$f \geq 0, \quad |f^{(k)}| \leq C_k f^{1-\alpha}, \quad f(t) \equiv 0 \text{ for } t \leq C_0, \quad f(t) \equiv 1 \text{ for } t \geq 2C_0 \quad (6.5)$$

for some $\alpha < \frac{1}{2}$ and some large constant C_0 . The technical condition on $f^{(k)}$ is useful for comparing the propagators of \widehat{P}_ε and P_0 – see [NoZw, Appendix].

With P_ε given by (1.1) we now consider

$$\widehat{P}_\varepsilon := hP_\varepsilon - iW_0. \quad (6.6)$$

Unlike in §§4,5 we will now work near a fixed rescaled energy level $z = h\lambda \sim 1$ rather than near the zero energy.

The next result is an almost immediate application of [NoZw, Theorem 2]:

Lemma 6.2. *Suppose that the flow $\varphi_t : X \rightarrow X$ is a contact Anosov flow (see (2.3)), \widehat{P}_ε is given by (6.6) and that $\varepsilon = \mathcal{O}(h^2)$. Let γ_0 be the averaged Lyapounov exponent defined in (1.5). Then for any $\delta > 0$ and s there exist h_0, c_0, C_1 , such that for $0 < h < h_0$,*

$$\|(\widehat{P}_\varepsilon - z)^{-1}\|_{L^2 \rightarrow L^2} \leq C_1 h^{-1+c_0 \operatorname{Im} z/h} \log(1/h), \quad (6.7)$$

for

$$z \in \left[\frac{1}{2}, \frac{3}{2}\right] - ih[0, \gamma_0/2 - \delta]. \quad (6.8)$$

Remark. The bound (6.7) is more precise than the bound (1.6) which corresponds to $\mathcal{O}(h^{-N})$. It is obtained by interpolation between the bound $1/\operatorname{Im} z$ for $\operatorname{Im} z > 0$ and the polynomial bound $\mathcal{O}(h^{-N})$ – see [Bu, Lemma 4.7], [TaZw, Lemma 2]. Using the fact that H_{sG} are complex interpolation spaces [Ca] the estimate (1.6) can be refined to a form similar to (6.7).

Proof. Put $W = -(\varepsilon/h)h^2\Delta_g + W_0$ (where W_0 appearing in the definition of \widehat{P}_ε is given by (6.4)). Note that $\widehat{P}_\varepsilon = hP_0 - iW$. We have $W \in \Psi_h^2(X)$, $W \geq 0$, and since $\varepsilon = \mathcal{O}(h^2)$,

$$w := \sigma_h(W) = \sigma_h(W_0) = f(|\xi|_g^2)|\xi|_g^2.$$

Hence $P := hP_0$ and W satisfy the assumptions [NoZw, (1.9),(1.10)]. The only difference is that $P \in \Psi_h^1(X)$, so that in the notation of [NoZw], $k = 2$ and $m = 1$. Replacing k with m in the ellipticity condition [NoZw, (1.9)] does not change the proofs in [NoZw] (in particular it does not affect [NoZw, Proposition A.3]): all the arguments are microlocal near the (compact) trapped set

$$\widehat{K} := \{(x, \xi) : |p_0(x, \xi) - 1| < 1/2, \exp(tH_p)(x, \xi) \not\rightarrow \infty, t \rightarrow \pm\infty\}, \quad (6.9)$$

$$p(x, \xi) = \xi(V_x).$$

Since φ_t is a contact Anosov flow, the trapped set is normally hyperbolic in the sense of [NoZw, (1.14)–(1.17)] – see [NoZw, §9]. Hence we can apply [NoZw, Theorem 2] and obtain the bound (6.7). \square

We are now ready for

Proof of Theorem 2. We first note that (1.6) follows by rescaling $z = h\lambda$ from a semiclassical estimate between the weighted spaces (we recall that $H^s \subset H_{sG(1)} \subset H^{-s}$)

$$(hP_\varepsilon - z)^{-1} = \mathcal{O}(h^{-N}) : H_{s_0G(1)}(X) \rightarrow H_{s_0G(1)}(X). \quad (6.10)$$

for the same range of z 's as in (6.8). By Lemma 6.1, we can assume that $\varepsilon = \mathcal{O}(h^2)$.

To prove (6.10) we follow the strategy as in [NoZw, §9] combined with the estimates of §4. For that we choose Q in (3.8) and W_0 in (6.4), so that for the weight G in (3.1) and the trapped set \widehat{K} defined in (6.9) we have

$$\text{WF}_h(Q) \cap \text{WF}_h(G) = \widehat{K} \cap \text{WF}_h(I - Q) = \text{WF}_h(Q) \cap \text{WF}_h(W_0) = \emptyset.$$

Since \widehat{K} is compact that is possible by modifying the conditions on χ in (3.8) and by increasing C_0 in (6.5).

To stay close to the notation of [NoZw, §9] we now put $P_\infty := hP_0 + i\varepsilon h\Delta_g - iQ$. To apply the gluing argument of Datchev–Vasy [DaVa] as in [NoZw, §9] we check that the conclusions of [NoZw, Lemma 9.19] are valid. First,

$$(P_\infty - z)^{-1} = \mathcal{O}(h^{-N_1}) : H_{s_0G(1)} \rightarrow H_{s_0G(1)}, \quad \text{Im } z > -\gamma_0/2, \quad |\text{Re } z - 1| < 1/2, \quad (6.11)$$

is proved similarly as (4.16). Indeed, Lemma 4.1 holds for $\lambda = \mathcal{O}(\varepsilon^{-1/2})$, and this condition is true since $\lambda = \mathcal{O}(h^{-1})$ and $\varepsilon = \mathcal{O}(h^2)$. The proof of Lemma 4.2 goes through as in the case $\lambda = \mathcal{O}(1)$. The proof of (4.16) works as before, using again that $\varepsilon = \mathcal{O}(h^2)$.

Next, we need the propagation statement,

$$u = (P_\infty - z)^{-1}f, \quad \text{WF}_h(f) \cap \partial\overline{T}^*X = \emptyset \implies \\ \text{WF}_h(u) \setminus (\text{WF}_h(f) \cup \partial\overline{T}^*X) \subset \exp([0, \infty)H_p) (\text{WF}_h(f) \cap p^{-1}(\text{Re } z)),$$

where $p(x, \xi) = \xi(V_x)$. This statement follows from the propagation theorems reviewed in the proof of Lemma 4.2; note that $(hP_\varepsilon - \widetilde{P}_\varepsilon(0))u = \mathcal{O}(h^\infty)_{C^\infty}$ since $\varepsilon = \mathcal{O}(h^2)$ and $\text{WF}_h(f)$ does not intersect the fiber infinity.

We can now follow the gluing argument of the resolvent estimates on $(P_\infty - z)^{-1}$ and $(\widehat{P}_\varepsilon - z)^{-1}$ (given in (6.7)) as in [NoZw, §9] to obtain (6.10). In the notation of [NoZw, §9] the parametrix for $(hP_\varepsilon - z)^{-1}$ is given by

$$F(z) := A_1(\widehat{P}_\varepsilon - z)^{-1}A_0 + B_1(P_\infty - z)^{-1}B_0,$$

where $A_j, I - B_j \in \Psi_h^{\text{comp}}(X)$ are suitably chosen, with $\text{WF}_h(A_j) \cap \text{WF}(G) = \emptyset$. Away from the microsupport of G , the spaces $H_{sG(h)}$ are microlocally equivalent to L^2 . Hence the L^2 estimates on $(\widehat{P}_\varepsilon - z)^{-1}$ imply the $H_{sG(h)}$ estimates on $F(z)$. The gluing argument of [DaVa] as recalled in [NoZw, §8] concludes the proof of (6.10). \square

REFERENCES

- [BaTs] Viviane Baladi and Masato Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, Ann. Inst. Fourier **57**(2007), no. 1, 127–154.
- [BKL] Michael Blank, Gerhard Keller and Carlangelo Liverani, *Ruelle–Perron–Frobenius spectrum for Anosov maps*, Nonlinearity **15**(2002), 1905–1973.
- [BLAg] Galya Blum and Oded Agam, *Leading Ruelle resonances of chaotic maps* Phys. Rev. E **62**(2000), 1977–1982.
- [Bu] Nicolas Burq, *Smoothing effect for Schrödinger boundary value problems*, Duke Math. J. **123**(2004), 403–427.
- [BuLi1] Oliver Butterley and Carlangelo Liverani, *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn. **1**(2007), 301–322.
- [BuLi2] Oliver Butterley and Carlangelo Liverani, *Robustly invariant sets in fiber contracting bundle flows*, J. Mod. Dyn. **7**(2013), 255–267.
- [Ca] Alberto P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Mathematica **24**(1964), 113–190.
- [CNKMG] Mickaël David Chekroun, J. David Neelin, Dmitri Kondrashov, James C. McWilliams, and Michael Ghil, *Rough parameter dependence in climate models and the role of Pollicott–Ruelle resonances*, Proc. Nat. Acad. Sci. **111**(2014), no. 5, 1684–1690.
- [CKRZ] Peter Constantin, Alexander Kiselev, Leonid Ryzhik, and Andrej Zlatoš, *Diffusion and mixing in fluid flow*, Ann. of Math. (2) **168**(2008), no. 2, 643–674.
- [DaVa] Kiril Datchev and András Vasy, *Gluing semiclassical resolvent estimates via propagation of singularities*, Int. Math. Res. Not. **23**(2012), 5409–5443.
- [Do] Dmitry Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. **147**(1998), 357–390.
- [Dy] Semyon Dyatlov, *Asymptotic distribution of quasi-normal modes for Kerr–de Sitter black holes*, Ann. Henri Poincaré **13**(2012), 1101–1166.
- [DFG] Semyon Dyatlov, Frédéric Faure, and Colin Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds*, A&PDE **8**(4)(2015), 923–1000.
- [DyZw1] Semyon Dyatlov and Maciej Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, preprint, [arXiv:1306.4203](https://arxiv.org/abs/1306.4203), to appear in Ann. Sci. Éc. Norm. Sup.
- [DyZw2] Semyon Dyatlov and Maciej Zworski, *Mathematical theory of scattering resonances*, book in preparation; <http://math.berkeley.edu/~zworski/res.pdf>
- [El] K. David Elworthy, *Stochastic Differential Equations on Manifolds*, Cambridge University Press, 1982.
- [FNW] Albert Fannjiang, Stéphane Nonnenmacher, and Lech Wołowski, *Dissipation time and decay of correlations*, Nonlinearity **17**(2004), 1481–1508.
- [FRS] Frédéric Faure, Nicolas Roy, and Johannes Sjöstrand, *Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances*, Open Math. Journal **1**(2008), 35–81.
- [FaSj] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308**(2011), no. 2, 325–364.
- [FaTs1] Frédéric Faure and Masato Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows*, C. R. Math. Acad. Sci. Paris **351**(2013), 385–391.
- [FaTs2] Frédéric Faure and Masato Tsujii, *The semiclassical zeta function for geodesic flows on negatively curved manifolds*, preprint, [arXiv:1311.4932](https://arxiv.org/abs/1311.4932).
- [GaSh] S. V. Galtsev, A. I. Shafarevich, *Quantized Riemann surfaces and semiclassical spectral series for a nonselfadjoint Schrödinger operator with periodic coefficients*, Theoret. and Math. Phys., **148**(2006), 206–226.

- [GaRa] Pierre Gaspard and D. Alonzo Ramirez, *Ruelle classical resonances and dynamical chaos: The three- and four-disk scatterers*, Phys. Rev. A **45**(1992), 8383–8397.
- [GoLi1] Sébastien Gouëzel and Carlangelo Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26**(2006), no. 1, 189–217.
- [GoLi2] Sébastien Gouëzel and Carlangelo Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Diff. Geom. **79**(2008), 433–477.
- [HöIII] Lars Hörmander, *The Analysis of Linear Partial Differential Operators III*. Springer, 1994.
- [JiZw] Long Jin and Maciej Zworski *Local trace formula and lower bounds for Pollicott–Ruelle resonances*, with an appendix by Frédéric Naud, [arXiv:1411.6177](https://arxiv.org/abs/1411.6177).
- [KeLi] Gerhard Keller and Carlangelo Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28**(1999), 141–152.
- [Ki] Yuri Kifer, *Random Perturbations of Dynamical Systems*, Birkhäuser, 1988.
- [Li1] Carlangelo Liverani, *On contact Anosov flows*, Ann. of Math. **159**(2004), 1275–1312.
- [Li2] Carlangelo Liverani, *Fredholm determinants, Anosov maps and Ruelle resonances*, Discrete Contin. Dyn. Syst. **13**(2005), no. 5, 1203–1215.
- [Me] Richard B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, in *Spectral and scattering theory* (M. Ikawa, ed.), Marcel Dekker, 1994.
- [NaWi] Yushi Nakano and Jens Wittsten, *On the spectra of a randomly perturbed partially expanding map on the torus*, Nonlinearity **28**(2015), 951–1002.
- [NoZw] Stéphane Nonnenmacher and Maciej Zworski, *Decay of correlations in normally hyperbolic trapping*, Invent. Math. **200**(2015), 345–438.
- [Po] Mark Pollicott, *Meromorphic extensions of generalized zeta functions*, Invent. Math. **85**(1986), 147–164.
- [PHV] Harald A. Posch, William G. Hoover and Franz J. Vesely, *Canonical dynamics of the Nosé oscillator, stability, order, and chaos*, Phys. Rev. A **33**(1986), 4253–4265.
- [Ru] David Ruelle, *Resonances of chaotic dynamical systems*, Phys. Rev. Lett. **56**(1986), 405–407.
- [TaZw] Siu-Hung Tang and Maciej Zworski, *From quasimodes to resonances*, Math. Res. Lett. **5**(1998), 261–272.
- [Ts1] Masato Tsujii, *Quasi-compactness of transfer operators for contact Anosov flows*, Nonlinearity **23** (2010), 1495–1545.
- [Ts2] Masato Tsujii, *Contact Anosov flows and the FBI transform*, Erg. Th. Dyn. Syst. **32**(2012), 2083–2118.
- [Va] András Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces*, with an appendix by Semyon Dyatlov, Invent. Math. **194**(2013), 381–513.
- [Ve] Roberto Venegeroles, *Leading Pollicott–Ruelle resonances and transport in area-preserving maps*, Phys. Rev. Lett. **99**(2007), 014101.
- [Zw1] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics **138**, AMS, 2012.
- [Zw2] Maciej Zworski, *Scattering resonances as viscosity limits*, [arXiv:1505.00721](https://arxiv.org/abs/1505.00721)

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