ERRATUM

We provide a small correction to the statement and proof of [1, Proposition 2.1]. The key component is the Ingham inequality [3]. Suppose that λ_n, $n = 1, 2, \cdots$ is a sequence of real numbers satisfying

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad n = 1, 2, \cdots . \tag{1}$$

Then there exists a constant A depending only on γ such that for any $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}; \mathbb{C})$,

$$\int_0^{2\pi} \left| \sum_{n=1}^{\infty} a_n e^{i\lambda_n t} \right|^2 dt \leq A \sum_{n=1}^{\infty} |a_n|^2. \tag{2}$$

If γ in (1) satisfies $\gamma > 1$ then we also have

$$\int_0^{2\pi} \left| \sum_{n\in\mathbb{N}} a_n e^{i\lambda_n t} \right|^2 dt \geq B \sum_{n=1}^{\infty} |a_n|^2, \tag{3}$$

for some constant B depending only on γ. Here we will only need (2) but we note that both inequalities are very important in control theory – see for instance [2].

Going back to [1] we now correct the proposition and its proof:

Proposition 2.1 *For any $W \in L^2(\mathbb{T}^1)$, there exists $C > 0$ such that for any $k \in [0, 1)$, and $u_0 \in L^2(\mathbb{T}^1)$ the solution to the Schrödinger equation*

$$(i\partial_t + (\partial_x + ik)^2 - W)u = 0, \qquad v\,|_{t=0} = u_0 \tag{4}$$

satisfies

$$\|u\|_{L^\infty(\mathbb{T}^1_x; L^2(0,T))} \leq (C_0 + T)(C_1 + \|W\|^2_{L^2(\mathbb{T}^1)})\|u_0\|_{L^2(\mathbb{T}^1)}, \quad T > 0, \tag{5}$$

where the constants C_0, C_1 are independent of k.

Proof. For $W \equiv 0$ we put $T = 2\pi$ so that , with $c_n = \hat{u}_0(n)$, we have

$$\begin{aligned} \|e^{it(\partial_x+ik)^2}u_0\|^2_{L^\infty_x L^2_t} &= \sup_x \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} c_n e^{-it|n+k|^2 + inx} \right|^2 dt \\ &\leq 3 \sup_x \Big(2\pi|c_0|^2 + \sum_{\pm} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} c_{\pm n} e^{-it|n\pm k|^2 \pm inx} \right|^2 dt \Big). \end{aligned} \tag{6}$$

Since for $k \in [0, 1)$,

$$|n + 1 \pm k|^2 - |n \pm k|^2 = 2n + 1 \pm 2k \geq 1, \quad n = 1, 2, \cdots ,$$

we can apply (2) to get

$$\|e^{it(\partial_x+ik)^2}u_0\|^2_{L^\infty_x L^2_t([0,T])} \leq C\|u_0\|^2_{L^2}, \quad), \quad 0 \leq T \leq 2\pi.$$

with the constant C is independent of k.

For a non-zero potential $W \in L^2(\mathbb{T}^1)$ we use Duhamel's formula and write

$$u(t) = e^{it(\partial_x+ik)^2}u_0 + \frac{1}{i}\int_0^T \mathbb{1}_{s<t} e^{i(t-s)(\partial_x+ik)^2}\left(Wu(s)\right)\right) ds.$$

Applying (6) (now with $T > 0$ to be chosen later) and the Minkowski inequality we obtain

$$\begin{aligned}
\|u\|_{L^\infty_x L^2_t([0,T])} &\leq C\|u_0\|_{L^2_x} + \int_0^T \|\mathbb{1}_{s<t} e^{i(t-s)(\partial_x+ik)^2}(Wu(s))\|_{L^\infty_x L^2_t([0,T])} ds \\
&\leq C\|u_0\|_{L^2_x} + \int_0^T \|e^{i(t-s)(\partial_x+ik)^2}(Wu(s))\|_{L^\infty_x L^2_t([0,T])} ds \\
&\leq C\|u_0\|_{L^2_x} + C\int_0^T \|Wu(s)\|_{L^2_x} ds \\
&\leq C\|u_0\|_{L^2_x} + C\sqrt{T}\|W\|_{L^2_x}\|u\|_{L^\infty_x L^2_t([0,T])}.
\end{aligned} \tag{7}$$

Hence,

$$\|u\|_{L^\infty_x L^2_t([0,T])} \leq 2C\|u_0\|_{L^2_x}, \quad \text{if} \quad C\sqrt{T}\|W\|_{L^2} \leq \tfrac{1}{2}. \tag{8}$$

To obtain the estimate for multiples of $T = KT_0$, $T_0 = 1/(1 + 4\|W\|^2C^2)$, we note that, by the invariance of the L^2_x norm of $u(t)$, $\int_{(k-1)T_0}^{kT_0} \|u(t)\|^2_{L^\infty_x} dt \leq 2C\|u((k-1)T_0)\|_{L^2_x} = 2C\|u_0\|_{L^2_x}$. Iterating this inequality gives (5). □

References

[1] J. Bourgain, N. Burq and M. Zworski, *Control for Schrdinger operators on 2-tori: rough potentials.* J. Eur. Math. Soc. **15**(2013), 1597–1628.

[2] Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of the multi-dimensional wave and Schrdinger equations in quantum ergodic domains.* J. Eur. Math. Soc. **18**(2016), 1043–1111.

[3] A.E. Ingham, *Some trigonometrical inequalities with applications to the theory of series.* Math. Z. **41**(1936), 367–379.