MATH 54: Complex eigenvalues

Suppose that A is an $n \times n$ matrix and the characteristic polynomial has a complex zero:

$$P(r) = \det(A - rI), \quad P(\alpha \pm i\beta) = 0.$$

If $\alpha + i\beta$ is a root so is $\alpha - i\beta$ so we need to worry about one of them only.

This comes up when we consider systems of equations:

$$\mathbf{x}'(t) = A\mathbf{x}(t). \tag{1}$$

We then need to look for $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^n$ such that

$$\begin{aligned} A\mathbf{a} &= \alpha\mathbf{a} - \beta\mathbf{b} \\ A\mathbf{b} &= \beta\mathbf{a} + \alpha\mathbf{b}. \end{aligned} \tag{2}$$

NO COMPLEX NUMBERS ARE INVOLVED HERE.

The two independent solution of (1) associated to the complex roots are

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}(\cos\beta t\,\mathbf{a} - \sin\beta t\,\mathbf{b}), \\ \mathbf{x}_2(t) &= e^{\alpha t}(\sin\beta t\,\mathbf{a} + \cos\beta t\,\mathbf{b}), \end{aligned}$$

SO, HOW DO WE SOLVE (2)**?**

This is a *homogeneous* system of $2n$ linear equations in with $2n$ unknowns

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

$$\begin{aligned} (A - \alpha I)\mathbf{a} + \beta\mathbf{b} &= 0 \\ -\beta\mathbf{a} + (A - \alpha I)\mathbf{b} &= 0. \end{aligned}$$

You will not be asked to do this for some complicated matrices but 2×2 matrices are fair game.

Those of you who are comfortable with complex numbers can also proceed directly and solve

$$(A - (\alpha + i\beta)I)(\mathbf{a} + i\mathbf{b}) = 0,$$

which is done by row reduction as in the case of real equations but using the complex numbers arithmetic.