ADDENDUM TO "MAGNETIC OSCILLATIONS IN A MODEL OF GRAPHENE"

SIMON BECKER AND MACIEJ ZWORSKI

In the proof of [BZ19, Proposition 5.1] we misquoted [HR84] when claiming that $F_1 \equiv 0$. Results of [HR84] do however produce this conclusion and to see it we will use an elegant presentation of higher order Bohr–Sommerfeld rules from [CdV05]. The more recent paper [ILR18] can also be consulted for a different approach and for references on this old subject. The statement that $F_1 \equiv 0$ is sometimes interpreted in the physics literature as the cancellation of the Maslov index by the Berry phase – see for instance [CU08]. Here we will confine ourselves to Bohr–Sommerfeld analysis as we will also discuss the next term.

We consider (compared to [BZ19] we remove $\frac{1}{3}$ as it does not change the calculations)

$$P := \lambda_+^{\rm w} \lambda_-^{\rm w}, \quad \lambda_\pm(x,\xi) := 1 + e^{\pm ix} + e^{\pm i\xi}, \quad (\lambda_\pm^{\rm w})^* = \lambda_\mp^{\rm w}, \quad \lambda_\pm^{\rm w} = \lambda_\pm^{\rm w}(x, hD),$$

and work microlocally near

$$(x_0, \xi_0) = (\tfrac{2\pi}{3}, -\tfrac{2\pi}{3}), \quad \lambda_\pm(x_0,\xi_0) = 0, \quad \{\lambda_+, \lambda_-\}(x_0,\xi_0) = \sqrt{3} i.$$

We start by computing the full symbol of P:

$$P = P^{\rm w}(x, hD, h), \quad P(x,\xi,h) = p(x,\xi) + hp_1(x,\xi) + h^2 p_2(x,\xi) + \cdots . \tag{1}$$

From the product formula [Zw, (4.3.10),(4.4.18)] we immediately have

$$P = \lambda_+^{\rm w}\lambda_-^{\rm w} = (\lambda_+\lambda_-)^{\rm w} + \frac{h}{2i}\{\lambda_+, \lambda_-\}^{\rm w} + \mathcal{O}(h^2) \tag{2}$$

To see an exact formula we recall (see for instance [Zw, Theorem 4.7]) that $(e^{i(ax+b\xi)})^{\rm w} = e^{i(ax+bhD)}$ (where the right hand side is defined as an exponential of an anti-self-adjoint operator) and that $e^{i(ax+bhD)}e^{i(cx+dhD)} = e^{\frac{i}{2}h(cb-ad)}e^{i(a+c)x+(d+b)hD}$. Hence,

$$\begin{aligned}
\lambda_+^{\rm w}\lambda_-^{\rm w} &= 3 + e^{ix} + e^{-ix} + e^{ihD} + e^{-ihD} + e^{ix}e^{-ihD} + e^{ihD}e^{-ix} \\
&= 3 + 2\cos x + 2(\cos\xi)^{\rm w} + e^{\frac{i}{2}h}\left(e^{i(x-\xi)}\right)^{\rm w} + e^{-\frac{i}{2}h}\left(e^{-i(x-\xi)}\right)^{\rm w} \\
&= 3 + 2\cos x + 2(\cos\xi)^{\rm w} + 2\cos(h/2)(\cos(x-\xi))^{\rm w} - 2\sin(h/2)(\sin(x-\xi))^{\rm w}.
\end{aligned}$$

Returning to (1) we see that for $j \geq 0$,

$$p_{2j+1}(x,\xi) = \frac{(-1)^{j+1}}{(2j+1)!4^j}\sin(x-\xi), \quad p_{2j+2}(x,\xi) = \frac{1}{2}\frac{(-1)^{j+1}}{(2j+2)!4^j}\cos(x-\xi).$$

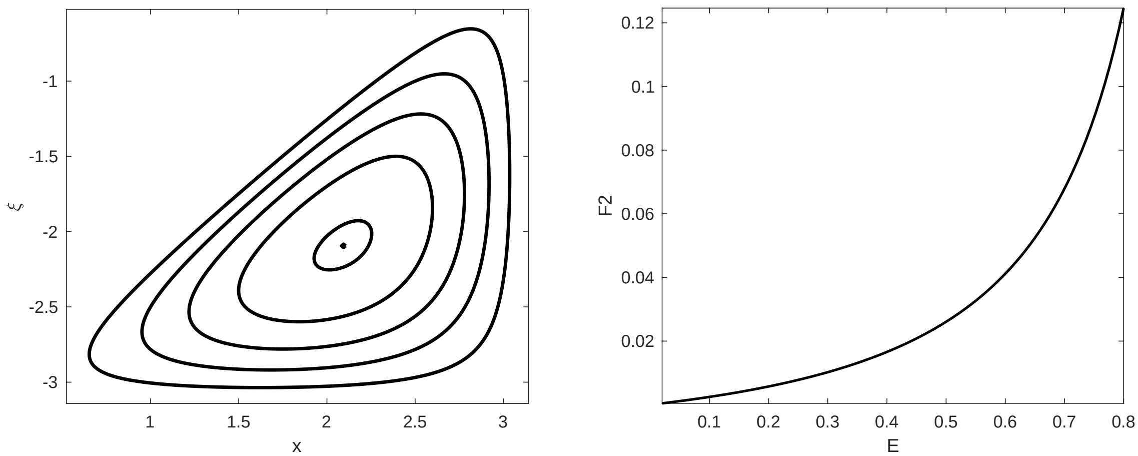

FIGURE 1. On the left: level sets $p(x,\xi)=E$ for $0.02\leq E\leq 0.8$ with the point (x_0,ξ_0), $p(x_0,\xi_0)=0$, indicated. On the right: the plot of $F_2(E)$, $0.02\leq E\leq 0.8$.

As explained in the proof of [BZ19, Proposition 5.1] the quasimodes microlocalized to a neighbourhood of $(x_0,\xi_0)=(2\pi/3,-2\pi/3)$ have energies (approximate eigenvalues) given by the Bohr–Sommerfeld rule $F(\lambda_n(h),h)=nh$, $n=0,1,\cdots$, $F(\omega,h)\sim F_0(\omega)+hF_1(\omega)+h^2F_2(\omega)\cdots$. General arguments there also show that $\lambda_0(h)=\mathcal{O}(h^\infty)$ from which we obtain that $F_j(0)=0$ for all j.

We now analyse F_1 following [CdV05],[ILR18]. In the notation of those papers $F_1(E)=S_1(E)/2\pi$. The Bohr–Sommerfeld rules discussed there apply only to excited states, that is to $E>E_0>0$, for any fixed E_0, but the the formulas apply in our setting. The validity of the Bohr–Sommerfeld rules near 0 energy follows from [HR84], see also [Sj89, §8, Case II, p.292].

Let γ_E be the component of $p^{-1}(E)$ enclosing (x_0,ξ_0) (see the figure). We denote by t the conjugate variable to the energy E so that $\kappa^{-1}:(x,\xi)\mapsto(t,E)$, $0\leq t<T(E)$ (where $T(E)$ is the period of γ_E) is a local symplectomorphism near points on γ_E, $E>0$. For any function $f=f(x,\xi)$ we then have

$$\begin{aligned}\frac{\partial}{\partial E}\int\int_{p(x,\xi)\leq E}f(x,\xi)|dxd\xi| &= \frac{\partial}{\partial E}\int_0^E\int_0^{T(\omega)}\kappa^*f(t,\omega)|dtd\omega|\\ &= \int_0^{T(E)}\kappa^*f(t,E)|dt| = \int_{\gamma_E}f|dt|.\end{aligned}$$

We then quote [CdV05] to obtain

$$S_1(E)=\pi-\int_{\gamma_E}p_1|dt|=\pi-\frac{\partial}{\partial E}\int_{p(x,\xi)\leq E}p_1(x,\xi)|dxd\xi|. \qquad (3)$$

The right hand side is the same as in [HS90a, (6.2.14)] where it is derived from [HR84, Corollary 3.7]. To compute it we use (2) noting that

$$p = |\lambda_+|^2, \;\; p_1 = \frac{1}{2i}\{\lambda_+, \lambda_-\} = -\{\operatorname{Re}\lambda_+, \operatorname{Im}\lambda_+\}, \;\; \{\operatorname{Re}\lambda_+, \operatorname{Im}\lambda_+\}(x_0,\xi_0) = -\sqrt{3}. \quad (4)$$

If we put $y = \operatorname{Re}\lambda_+(x,\xi)$, $\eta = \operatorname{Im}\lambda_+(x,\xi)$ then

$$\left|\frac{\partial(y,\eta)}{\partial(x,\xi)}\right| = |\{\operatorname{Re}\lambda_+, \operatorname{Im}\lambda_+\}| = -\{\operatorname{Re}\lambda_+, \operatorname{Im}\lambda_+\},$$

and we obtain

$$\begin{aligned}\frac{\partial}{\partial E}\int_{p(x,\xi)\leq E} p_1(x,\xi)|dxd\xi| &= -\frac{\partial}{\partial E}\int_{|\lambda_+|^2\leq E}\{\operatorname{Re}\lambda_+, \operatorname{Im}\lambda_-\}|dxd\xi| \\ &= \frac{\partial}{\partial E}\int_{y^2+\eta^2\leq E}|dyd\eta| = \pi.\end{aligned}$$

Returning to (3) we see that $S_1(E) \equiv 0$ and hence the same is true for F_1.

We conclude with comments about $F_2(E) = S_2(E)/2\pi$. Following [CdV05, Theorem 2] we have

$$S_2(E) = \frac{\partial}{\partial E}\left(\int_{\gamma_E}(-\tfrac{1}{24}\Delta + \tfrac{1}{2}p_1^2)|dt|\right) - \int_{\gamma_E} p_2|dt|, \quad \Delta := \det\begin{bmatrix} p_{x\xi} & p_{xx} \\ -p_{\xi\xi} & -p_{x\xi}\end{bmatrix}.$$

This function satisfies $S_2(0) = 0$ but it does not seem to have a simple form. The figure shows its numerical evaluation (one can check, as is indicated by numerics, that, near 0, $S_2(E) \simeq c_0 E$, $c_0 \simeq 0.134$).

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Email address: `simon.becker@damtp.cam.ac.uk`

DAMTP, University of Cambridge, Wilberforce Rd, Cambridge CB3 0WA, UK

Email address: `zworski@math.berkeley.edu`

Department of Mathematics, University of California, Berkeley, CA 94720, USA