ROUGH CONTROLS FOR SCHRÖDINGER OPERATORS ON 2-TORI

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Abstract. The purpose of this note is to use the results and methods of [BBZ13] and [BZ12] to obtain control and observability by rough functions and sets on 2-tori, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z} \oplus \gamma\mathbb{Z}$. We show that for a non-trivial $W \in L^\infty(\mathbb{T}^2)$, solutions to the Schrödinger equation, $(i\partial_t + \Delta)u = 0$, satisfy $\|u|_{t=0}\|_{L^2(\mathbb{T}^2)} \leq K_T \|Wu\|_{L^2([0,T]\times\mathbb{T}^2)}$. In particular, any set of positive Lebesgue measure can be used for observability. This leads to controllability with localization functions in $L^2(\mathbb{T}^2)$ (or L^4) and controls in $L^4([0,T]\times\mathbb{T}^2)$ (or L^2). For continuous W this follows from the results of Haraux [Ha89] and Jaffard [Ja90], while for $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and $T > \pi$ this can be deduced from the results of Jakobson [Ja97].

1. Introduction

The purpose of this paper is to investigate the general question of control theory with localized control functions. When the localization is performed by a continuous function, the question is completely settled for wave equations [BLR92, BG96] and well understood for Schrödinger equations on tori [Ha89, Ja90, Ko92, BZ12, AM14].

In this paper we localize only to sets of *positive measure* or more generally use control functions in L^4. The understanding is then much poorer and only partial results are available even for the simpler case of wave equations [BG17, Bu17]. Using the work with Bourgain [BBZ13] and [BZ12] we completely settle the question for Schrödinger equation on the two dimensional torus taking advantage, as in previous papers, of the particular simplicity of the dynamical structure.

To state the control result consider

$$\begin{aligned} &\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z} \times \gamma\mathbb{Z}\,, \quad \gamma \in \mathbb{R} \setminus \{0\}\,, \quad a \in L^2(\mathbb{T}^2), \\ &(i\partial_t + \Delta)u(t,z) = a(z)1_{(0,T)} f\,, \quad u(0,z) = u_0(z), \end{aligned} \tag{1.1}$$

where a is a localisation function and f a control. From [BBZ13, Proposition 2.2] (see Theorem 4 below) we know that for $f \in L^4(\mathbb{T}^2; L^2(0,T))$ (so that $af \in L^{4/3}(\mathbb{T}^2; L^2(0,T))$), and any $u_0 \in L^2(\mathbb{T}^2)$, there exists a unique solution

$$u \in C^0([0,T]; L^2(\mathbb{T}^2)) \cap L^4(\mathbb{T}^2; L^2(0,T)).$$

A classical question of control is to fix a and ask for which $u_0 \in L^2$ does there exist a control f such that the solution of (1.1) satisfies $u|_{t>T} = 0$? We show that on $\mathbb{T}^2$ it is always the case as soon as $a \in L^2$ is non-trivial:

Theorem 1. *Let $a \in L^2(\mathbb{T}^2), \|a\|_{L^2} > 0$ and $T > 0$. Then for any $u_0 \in L^2(\mathbb{T}^2)$ there exists $f \in L^4(\mathbb{T}^2; L^2(0,T))$ such that the solution u of (1.1) satisfies $u|_{t=T} = 0$.*

If in addition $a \in L^4(\mathbb{T}^2)$ then the same statement holds with $f \in L^2((0,T) \times \mathbb{T}^2)$.

The next result shows that adding an L^2 damping term results in exponential decay:

Theorem 2. *For $a \in L^2(\mathbb{T}^2)$, $a \geq 0$, $\|a\|_{L^2} > 0$, there exist $C, c > 0$ such that for any $u_0 \in L^2(\mathbb{T}^2)$, the equation*

$$(i\partial_t + \Delta + ia)u = 0, \qquad u|_{t=0} = u_0, \tag{1.2}$$

has a unique global solution $u \in L^\infty(\mathbb{R}; L^2(\mathbb{T}^2)) \cap L^4(\mathbb{T}^2; L^2_{\rm loc}(\mathbb{R}))$ and

$$\|u\|_{L^2(\mathbb{T}^2)}(t) \leq Ce^{-ct}\|u_0\|_{L^2(\mathbb{T}^2)}. \tag{1.3}$$

As shown in §4 both results follow from an the observability estimate. We should think of a in Theorem 1 as W^2 where W appears in the following statement:

Theorem 3. *Suppose that $W \in L^4(\mathbb{T}^2)$, $\|W\|_{L^4} > 0$. Then for any $T > 0$ there exists K such that for $u \in L^2(\mathbb{T}^2)$,*

$$\|u\|_{L^2(\mathbb{T}^2)} \leq K\|We^{it\Delta}u\|_{L^2((0,T)_t \times \mathbb{T}^2)}. \tag{1.4}$$

To keep the paper easily accessible we present proofs in the case when $\gamma \in \mathbb{Q}$ in (1.1). Irrational tori require a more complicated reduction to rectangular coordinates – see [BZ12, Lemma 2.7 and Fig.1] but the modification can be done as in that paper. The crucial [BBZ13, Proposition 2.2] is valid for all tori. Another approach to treating (higher dimensional) irrational tori can be found in the work of Anantharaman–Fermanian-Kammerer–Macià, see [AFM15, Corollary 1.19, Theorem 1.20].

Since, as is already clear, [BBZ13, Proposition 2.2] plays a central role in many proofs we recall it in a version used here:

Theorem 4. *Let $T > 0$. There exists $C = C_T$ such that for*

$$u_0 \in L^2(\mathbb{T}^2), \quad f \in L^{\frac{4}{3}}(\mathbb{T}^2; L^2(0,T)),$$

the solution to $(i\partial_t + \Delta)u = f$, $u|_{t=0} = u_0$, satisfies

$$\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^2)\cap L^4(\mathbb{T}^2;L^2((0,T)))} \leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)} + \|f\|_{L^1((0,T);L^2(\mathbb{T}^2))+L^{\frac{4}{3}}(\mathbb{T}^2;L^2(0,T))}\right).$$

Remarks. 1. Theorem 3 is equivalent to the same statement with $W \in L^\infty(\mathbb{T}^2)$ (by replacing $W \in L^4$ by $\mathbb{1}_{|W|\leq N} W \in L^\infty$ with N sufficiently large). Both the proof and derivations of Theorems 1 and 2 are easier with the L^4 formulation.

2. For rational tori and for $T > \pi$, Theorem 3, and by Proposition 4.1 below, Theorems 1 and 2, follow from the results of Jakobson [Ja97]. That is done by using the complete description of microlocal defect measures for eigenfuctions of $\mathbb{R}^2/2\pi\mathbb{Z}^2$. We explain this in detail in the appendix.

3. The starting point of [Ja97] and [BBZ13] was the classical inequality of Zygmund:

$$\forall\, \lambda \in \mathbb{N}, \quad \| \sum_{|n|^2=\lambda} c_n e^{in\cdot z} \|^2_{L^4(\mathbb{T}^2_z)} \leq \frac{\sqrt{5}}{2\pi} \sum_{|n|^2=\lambda} |c_n|^2, \quad z \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2, \quad n \in \mathbb{Z}^2. \tag{1.5}$$

In particular for $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, we easily see how the homogeneous part ($f = 0$) in Theorem 4 follows from (1.5). For that put $u = \sum_\lambda u_\lambda$, $u_\lambda = \sum_{|n|^2=\lambda} c_n e^{in\cdot x}$. Then, using (1.5) in the third line,

$$\begin{aligned}
\|e^{it\Delta}u\|^4_{L^4(\mathbb{T}^2, L^2((0,2\pi)))} &= \int_{\mathbb{T}^2} \left(\int_0^{2\pi} \left| \sum_\lambda e^{it\lambda} u_\lambda(z) \right|^2 dt \right)^2 dz = (2\pi)^2 \int_{\mathbb{T}^2} \left(\sum_\lambda |u_\lambda(z)|^2 \right)^2 dz \\
&= (2\pi)^2 \int_{\mathbb{T}^2} \sum_{\lambda,\mu} |u_\lambda(z)|^2 |u_\mu(z)|^2 dz \leq (2\pi)^2 \sum_{\lambda,\mu} \|u_\lambda\|^2_{L^4} \|u_\mu\|^2_{L^4} \\
&\leq 5 \sum_{\lambda,\mu} \|u_\lambda\|^2_{L^2} \|u_\mu\|^2_{L^2} = 5 \left(\sum_\lambda \|u_\lambda\|^2_{L^2} \right)^2 = 5\|u\|^4_{L^2}.
\end{aligned}$$

Generalizations for the time dependent Schrödinger equation in higher dimensions were obtained by Aïssiou–Jakobson–Macià [AJM12].

4. Other than tori, the only other manifolds for which (1.4) is known for *any* non-trivial continuous W are compact hyperbolic surfaces. That was proved by Jin [Ji17] using results of Bourgain–Dyatlov [BD16] and Dyatlov–Jin [DJ17].

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2. Semiclassical observability

We follow the strategy of [BZ12] and [BBZ13] and first prove a semiclassical observability result. For that we define

$$\Pi_{h,\rho}(u_0) := \chi\left(\frac{-h^2\Delta - 1}{\rho}\right) u_0\,, \quad \rho > 0\,, \tag{2.1}$$

where $\chi \in \mathcal{C}_c^\infty((-1,1))$ is equal to 1 near 0. With this notation the main result of this section is

Proposition 2.1. *Suppose that $a \in L^2(\mathbb{T}^2)$, $a \geq 0$, $\|a\|_{L^2} > 0$. For any $T > 0$ there exist K, $\rho_0 > 0$ and $h_0 > 0$ such that for any $u_0 \in L^2(\mathbb{T}^2)$,*

$$\|\Pi_{h,\rho} u_0\|_{L^2}^2 \leq K \int_0^T \int_{\mathbb{T}^2} a(z) |e^{it\Delta} \Pi_{h,\rho} u_0(z)|^2 dz dt, \tag{2.2}$$

for $0 < \rho < \rho_0$ and $0 < h < h_0$.

The proof of the Proposition proceeds by contradiction: if (2.2) does not hold then there exists $T > 0$ such that for any $n \in \mathbb{N}$ there exist $0 < h_n < 1/n$, $0 < \rho_n < 1/n$ and $u_n \in L^2$ for which

$$1 = \|u_n\|_{L^2}^2 > n \int_0^T \int_{\mathbb{T}^2} a(z) |e^{it\Delta} u_n(z)|^2 dz dt, \quad u_n = \Pi_{h_n,\rho_n} u_n. \tag{2.3}$$

We will use semiclassical limit measures associated to subsequences of u_n's.

2.1. Semiclassical limit measures.

Each sequence $u_n(t) := e^{it\Delta} u_n$, is bounded in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^2)$. After possibly choosing a subsequence, u_n's define a semiclassical defect measure μ on $\mathbb{R}_t \times T^*(\mathbb{T}^2_z)$ such that for any function $\varphi \in \mathcal{C}_c^0(\mathbb{R}_t)$ and any $A \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2_z)$, we have

$$\langle \mu, \varphi(t) A(z,\zeta) \rangle = \lim_{n\to\infty} \int_{\mathbb{R}_t \times \mathbb{T}^2} \varphi(t) \langle A(z, h_n D_z) u_n(t), u_n(t) \rangle_{L^2(\mathbb{T}^2)} dt\,. \tag{2.4}$$

The measure μ enjoys the following properties:

$$\begin{gathered} \mu((t_0,t_1) \times T^*\mathbb{T}^2_z) = t_1 - t_0, \quad \operatorname{supp}\mu \subset \Sigma := \{(t,z,\zeta) \in \mathbb{R}_t \times \mathbb{T}^2_z \times \mathbb{R}^2_\zeta \ : \ |\zeta| = 1\}, \\ \int_{\mathbb{R}} \int_{T^*\mathbb{T}^2} \varphi(t) A(z + s\zeta, \zeta) d\mu = 0, \quad \varphi \in \mathcal{C}_c^0(\mathbb{R}_t),\ A \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2_z), \end{gathered} \tag{2.5}$$

see [Ma09] for the derivation and references.

We have an additional property which follows from an easy part of Theorem 4 (in the rational case related to the Zygmund inequality (1.5)): for any $\tau \geq 0$ there exists $m_\tau \in L^2(\mathbb{T}^2)$ such that for all $f \in \mathcal{C}(\mathbb{T}^2)$

$$\int_0^\tau \int_{T^*\mathbb{T}^2} f(z) d\mu(t,z,\zeta) = \int_{\mathbb{T}^2} m_\tau(z) f(z) dz. \tag{2.6}$$

In fact, Theorem 4 shows that

$$U_n^\tau(z) := \int_0^\tau |u_n(t,z)|^2 dt \tag{2.7}$$

satisfies

$$\|U_n^\tau\|_{L^2(\mathbb{T}^2_z)} = \|u_n(t,z)\|^2_{L^4(\mathbb{T}^2_z, L^2((0,\tau))} \leq C\|u_n\|^2_{L^2(\mathbb{T}^2)} = C. \tag{2.8}$$

But then, after passing to a subsequence, U_n^τ converges weakly to $m_\tau \in L^2(\mathbb{T}^2)$. Since

$$\int_0^\tau \int_{T^*\mathbb{T}^2} f(z) d\mu(t,z,\zeta) = \lim_{n\to\infty} \int_0^\tau \int_{\mathbb{T}^2} f(z) U_n^\tau(z) dz,$$

this proves (2.6).

The assumption (2.3) gives the following

Lemma 2.2. *Let m_τ be defined by (2.6) with the measure μ obtained from $e^{it\Delta}u_n$ satisfying (2.3). Then*

$$\int_{\mathbb{T}^2} a(z) m_T(z) dz = 0. \tag{2.9}$$

Proof. We choose

$$a_j \in \mathcal{C}^\infty(\mathbb{T}^2), \quad a_j \geq 0, \quad \lim_{j\to\infty} \|a - a_j\|_{L^2} = 0. \tag{2.10}$$

Using (2.3) and then (2.8) (with the notation introduced in (2.7)),

$$\begin{aligned} \int_{\mathbb{T}^2} m_T(z) a_j(z) dz &= \lim_{n\to\infty} \int_{\mathbb{T}^2} U_n^T(z)(a_j(z) - a(z)) dz \\ &= \mathcal{O}(\|U_n^T\|_{L^2(\mathbb{T}^2)}) \|a - a_j\|_{L^2(\mathbb{T}^2)} = \mathcal{O}(1)\|a - a_j\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

Since $m_T \in L^2(\mathbb{T}^2)$, letting $j \to \infty$ shows (2.9). $\square$

The next lemma shows that our measure has most of its mass on the set of rational directions:

Lemma 2.3. *Suppose that μ is defined by u_n satisfying (2.3). For $m \in \mathbb{N}$ define,*

$$W^m := \left\{ (z,\zeta) \in T^*\mathbb{T}^2 \ : \ \zeta = \frac{(p,q)}{\sqrt{p^2+q^2}}, \ \max(|p|,|q|) \leq m, (p,q) \in \mathbb{Z}^2, \ \gcd(p,q) = 1 \right\},$$

its complement, $W_m := \complement W^m$, and a measure $\widetilde{\mu}_T$ on $T^*\mathbb{T}^2$:

$$\int_{T^*\mathbb{T}^2} A(z,\zeta) d\widetilde{\mu}_T := \int_0^T \int_{T^*\mathbb{T}^2} A(z,\zeta) d\mu(t,z,\zeta), \quad A \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2). \tag{2.11}$$

Then,

$$\forall\, \epsilon > 0 \ \exists\, m \ \ \textit{such that} \ \ \widetilde{\mu}_T(W_m) < \epsilon. \tag{2.12}$$

Proof. With a_j's from (2.10) we then have

$$\int_{T^*\mathbb{T}^2} a_j(z) d\widetilde{\mu}_T(z,\zeta) = \int_{\mathbb{T}^2} (a_j(z) - a(z)) m_T(z) dz = \mathcal{O}(\|a - a_j\|_{L^2}). \tag{2.13}$$

With the notation $\langle b \rangle_S(z,\zeta) := \frac{1}{S}\int_0^S b(z+s\zeta)ds$, $b \in \mathcal{C}^\infty(\mathbb{T}^2)$, the last property in (2.5) and the fact that W_∞ is invariant under the flow shows that for any $S > 0$,

$$\int_{W_\infty} a_j(z) d\widetilde{\mu}_T(z,\zeta) = \int_{W_\infty} \langle a_j \rangle_S(z,\zeta) d\widetilde{\mu}_T(z,\zeta).$$

We note that

$$W_{m+1} \subset W_m, \quad W_\infty := \bigcap_{m=1}^{\infty} W_m = \{(z,\zeta) : |\zeta| = 1, \zeta \in \mathbb{R}^2 \setminus \mathbb{Q}^2\}. \tag{2.14}$$

For $(z,\zeta) \in W_\infty$, unique ergodicity of the flow $z \mapsto z + s\zeta$ shows that $\langle a_j \rangle_S \to \langle a_j \rangle := \int_{\mathbb{T}^2} a_j(z) dz/(2\pi)^2$. Fatou's Lemma then shows that

$$\begin{aligned}\int_{W_\infty} a_j(z) d\widetilde{\mu}_T(z,\zeta) &= \liminf_{S\to\infty} \int_{W_\infty} \langle a_j \rangle_S(z,\zeta) d\widetilde{\mu}_T(z,\zeta) \\ &\geq \int_{W_\infty} \liminf_{S\to\infty} \langle a_j \rangle_S(z,\zeta) d\widetilde{\mu}_T(z,\zeta) = \widetilde{\mu}_T(W_\infty)\langle a_j \rangle.\end{aligned}$$

Combining this with (2.13) shows that

$$\widetilde{\mu}_T(W_\infty) \leq \frac{C\|a - a_j\|_{L^2}}{\langle a_j \rangle} \to 0, \quad j \to \infty,$$

(since $\|a\|_{L^2} > 0$ and $a \geq 0$, $\langle a_j \rangle \to \langle a \rangle > 0$) which gives $\widetilde{\mu}_T(W_\infty) = 0$. But then (2.14) implies that $\lim_{m\to\infty} \widetilde{\mu}_T(W_m) = \widetilde{\mu}_T(W_\infty) = 0$, concluding the proof. □

2.2. **Reduction to one dimension.** We start with the following

Lemma 2.4. *Suppose that in (2.12) m is large enough so that $\widetilde{\mu}_T(W_m) < T$ and that $(z,\zeta_0) \in \operatorname{supp}(\widetilde{\mu}_T|_{\complement W_m})$. Then there exists $F \in L^2(\mathbb{T}^2)$ such that*

$$\widetilde{\mu}_T|_{\mathbb{T}^2\times\{\zeta_0\}} = F \otimes \delta_{\zeta=\zeta_0}, \quad \|F\|_{L^2(\mathbb{T}^2)} \neq 0, \quad F \geq 0. \tag{2.15}$$

Proof. Let $\pi : T^*\mathbb{T}^2 \to \mathbb{T}^2$ be the natural projection map, $\pi(x,\xi) = x$. Then, using (2.6) and (2.11), for any Lebesgue measurable set $E \subset \mathbb{T}^2$,

$$\pi_*(\widetilde{\mu}_T|_{\mathbb{T}^2\times\{\zeta_0\}})(E) \leq \pi_*(\widetilde{\mu}_T)(E) = \int_E m_T(z)dz, \quad m_T \in L^2. \tag{2.16}$$

The Radon–Nikodym theorem then shows that $\pi_*(\widetilde{\mu}_T|_{\mathbb{T}^2\times\{\zeta_0\}}) = g m_T$ where g is measurable, m_T-a.e. finite. The inequality (2.16) gives $F := g m_T \leq m_T$ almost everywhere which shows that $F \in L^2$. □

Using [BZ12, Lemma 2.7] (see also [BZ12, Fig.1]) we can assume (by changing the torus but not Δ_z) that $\zeta_0 = (0,1)$, $z = (x,y)$, $x \in \mathbb{R}/A_1\mathbb{Z}$, $y \in \mathbb{R}/B_1\mathbb{Z}$, $A_1/B_1 \in \mathbb{Q}$. Abusing the notation we will keep the notation u_n and μ for the transformed functions. The invariance property in (2.5) and the proof of Lemma 2.2 show now that

$$
\begin{gathered}
(\widetilde{\mu}_T|_{\mathbb{T}^2\times\{(0,1)\}}) = g(x)dxdy \otimes \delta_0(\xi) \otimes \delta_1(\eta), \quad g \in L^2(\mathbb{T}^1), \quad \|g\|_{L^2(\mathbb{T}^2)} \neq 0 \\
\int_{\mathbb{T}^2} g(x)a(x,y)dxdy = 0.
\end{gathered}
\tag{2.17}
$$

Let us choose $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^2)$ supported near $|\zeta| = 1$ and such that

$$
\operatorname{supp}\chi \cap \complement W_m = \{(0,1)\}, \quad \chi(0,1) = 1. \tag{2.18}
$$

We then define $v_n := \chi(hD_z)u_n$ and $\nu := |\chi(\zeta)|^2\mu \neq 0$. Definition (2.4) shows that ν is the semiclassical defect measure associated to $v_n(t) := e^{it\Delta}v_n = \chi(hD_z)e^{it\Delta}u_n$ which in particular shows that

$$
\|v_n\|^2_{L^2(\mathbb{T}^2)} = T^{-1}\nu([0,T]\times T^*\mathbb{T}^2) \geq T^{-1}\int_{\mathbb{T}^2} g(x)dxdy =: \beta > 0. \tag{2.19}
$$

The reduction to a one dimensional problem is based, as in [BZ12], on a Fourier expansion in y (assuming $B_1 = 2\pi$ for notational simplicity):

$$
v_n(t)(x,y) := [e^{it\Delta}v_n](x,y) = \sum_{k\in\mathbb{Z}} [e^{it\partial_x^2}v_{n,k}](x)e^{-itk^2+iky} \tag{2.20}
$$

We will now use a one dimensional result proved in §2.3 below:

Lemma 2.5. *Suppose that $b \in L^1(\mathbb{T}^1)$, $b \geq 0$, $\|b\|_{L^1} > 0$ and that $T > 0$. Then there exists C such that for $w \in L^2(\mathbb{T}^1)$,*

$$
\|w\|^2_{L^2(\mathbb{T}^1)} \leq C\int_0^T b(x)[e^{it\partial_x^2}w](x)|^2 dxdt. \tag{2.21}
$$

Let a_j be again given by (2.10). We apply (2.21) to (2.20) with $b = \langle a\rangle_y := \frac{1}{2\pi}\int_{\mathbb{T}^1} a(x,y)dy$. That gives,

$$
\begin{aligned}
0 < \beta = \|v_n\|^2_{L^2(\mathbb{T}^2)} &= 2\pi\sum_{k\in\mathbb{Z}}\|v_{n,k}\|^2_{L^2(\mathbb{T}^1)} \leq C'\int_0^T \langle a\rangle_y(x)\sum_{k\in\mathbb{Z}}|[e^{it\partial_x^2}v_{n,k}](x)|^2 dxdt \\
&= C\int_0^T\int_{\mathbb{T}^2}\langle a\rangle_y(x)|[e^{it\Delta}v_n](x,y)|^2 dxdydt \\
&= C\int_0^T\int_{\mathbb{T}^2}\langle a_j\rangle_y(x)|[e^{it\Delta}v_n](x,y)|^2 dxdydt + \mathcal{O}(\|a-a_j\|_{L^2(\mathbb{T}^2)}) \\
&\longrightarrow C\int_{T^*\mathbb{T}^2}\langle a_j\rangle_y(x)d\widetilde{\nu}_T(x,y,\xi,\eta) + \mathcal{O}(\|a-a_j\|_{L^2(\mathbb{T}^2)}), \quad n \longrightarrow \infty,
\end{aligned}
$$

where $\widetilde{\nu}_T = |\chi(\xi,\eta)|^2 \widetilde{\mu}_T$ (see (2.11)). In particular for every j,

$$0 < \alpha \leq \int_{T^*\mathbb{T}^2} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x,y,\xi,\eta) + \mathcal{O}(\|a - a_j\|_{L^2(\mathbb{T}^2)}), \quad \alpha := \frac{\beta}{C}. \tag{2.22}$$

We now decompose the integral in (2.22) as $I_1 + I_2$ and use (2.17):

$$\begin{aligned} I_1 := \int_{\mathbb{T}^2 \times \{(0,1)\}} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x,y,\xi,\eta) &= \int_{\mathbb{T}^2} g(x) a_j(x,y) dx dy \\ &= \int_{\mathbb{T}^2} g(x)(a_j(x,y) - a(x,y)) dx dy \\ &\leq \sqrt{2\pi} \|g\|_{L^2(\mathbb{T}^1)} \|a_j - a\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{2.23}$$

We then use (2.12) and (2.18) to estimate the remainder:

$$\begin{aligned} I_2 &:= \int_{(\xi,\eta)\neq(0,1)} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x,y,\xi,\eta) \leq \int_{W_m} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x,y,\xi,\eta) \\ &\leq \|a_j\|_{L^\infty}\, \widetilde{\nu}_T(W_m) \leq \epsilon \|a_j\|_{L^\infty}, . \end{aligned}$$

We now combine these two estimates with (2.22) to obtain:

$$0 < \alpha \leq K \|a_j - a\|_{L^2(\mathbb{T}^2)} + \epsilon \|a_j\|_{L^\infty(\mathbb{T}^2)},$$

where the constant K depends on a, u_n and $\zeta_0 = (0,1)$ but not on χ and m. Hence, we first choose j large enough so that $K\|a_j - a\|_{L^2(\mathbb{T}^2)} < \alpha/2$ and then m large enough and χ satisfying (2.18) so that $\epsilon \|a_j\|_{L^\infty} < \alpha/2$. This provides a contradiction and proves Proposition 2.1.

2.3. **One dimensional estimate.** We now prove Lemma 2.5. The semiclassical part proceeds along the lines of the proof of Proposition 2.1. The derivation of (2.21) from the semiclassical estimate follows the same arguments needed in §3 and we will refer to that section for details.

Proof of Lemma 2.5. We start with a semiclassical statement: for every T there exist K, ρ_0 and h_0 such that for $0 < h < h_0$ and $0 < \rho < \rho_0$ we have the analogue of (2.2):

$$\|\pi_{h,\rho} u_0\|^2_{L^2(\mathbb{T}^1)} \leq K \int_0^T \int_{\mathbb{T}^1} b(z) |e^{it\Delta} \pi_{h,\rho} u_0(z)|^2 dz dt, \quad \pi_{h,\rho}(u_0) := \chi\left(\frac{h^2 D_x^2 - 1}{\rho}\right) u_0\,. \tag{2.24}$$

We proceed by contradiction which leads to an analogue of (2.3) and then to a measure ω_T analogous to $\widetilde{\mu}_T$ (see (2.11)) on $T^*\mathbb{T}^1$ and satisfying: $\operatorname{supp} \omega_T \subset \{\xi = \pm 1\}$, $\partial_x \omega_T = 0$, where the derivative is taken in the distributional sense.

From [BBZ13, Proposition 2.1][†] and the argument in Lemma 2.4 (with weak convergence in L^2 replaced by the weak$*$ convergence in $L^\infty = (L^1)^*$) we obtain

$$d\omega_T = \sum_{\pm} f_\pm(x) dx \otimes \delta_{\pm 1}(\xi) d\xi, \quad f_\pm \in L^\infty(\mathbb{T}^1), \quad f_\pm \geq 0.$$

But the fact that $\partial_x \omega_T = 0$ and the analogue of Lemma 2.2 show that $f_\pm(x) = c_\pm \geq 0$, $c_+ + c_- > 0$, $(c_+ + c_-) \int_{\mathbb{T}^1} b(x) dx = 0$, which is a contradiction proving (2.24).

From the semiclassical estimate we obtain

$$\|u_0\|_{L^2(\mathbb{T}^1)} \leq C \int_0^T \int_{\mathbb{T}^1} b(z) |e^{it\Delta} u_0(z)|^2 dz dt + C \|u_0\|_{H^{-2}(\mathbb{T}^1)}.$$

That is done by the same argument recalled in §3.1 below. Finally the error term $\|u_0\|_{H^{-1}(\mathbb{T}^1)}$ is removed – see §3.2 for review of the procedure for doing (applying [BBZ13, Proposition 2.1] again). $\square$

3. Observability estimate

To prove Theorem 3 we first prove a weaker statement involving an error term:

Proposition 3.1. *Suppose that $W \in L^4(\mathbb{T}^2)$, $a \geq 0$ and $\|W\|_{L^4} \neq 0$. Then for any $T > 0$, there exists K such that for $u \in L^2$,*

$$\|u\|^2_{L^2(\mathbb{T})} \leq C \int_0^T \int_{\mathbb{T}^2} |W(z) e^{it\Delta} u(z)|^2 dz dt + C \|u\|_{H^{-2}(\mathbb{T}^2)}. \tag{3.1}$$

3.1. **Dyadic decomposition.** The proof of (3.1) uses a dyadic decomposition as in [BBZ13, §5.1] and [BZ12, §4] and we recall the argument adapted to the setting of this paper. For that let $1 = \varphi_0(r)^2 + \sum_{k=1}^\infty \varphi_k(r)^2$, where

$$\varphi_k(r) := \varphi(R^{-k}|r|), \;\; R > 1, \quad \varphi \in \mathcal{C}^\infty_{\rm c}((R^{-1}, R); [0,1]), \quad (R^{-1}, R) \subset \{r \;:\; \chi(r/\rho) \geq \tfrac{1}{2}\},$$

with χ and ρ same as in (2.1) and (2.2). Then, we decompose u_0 dyadically: $\|u_0\|^2_{L^2} = \sum_{k=0}^\infty \|\varphi_k(-\Delta) u_0\|^2_{L^2}$, which will allow an application of Proposition 2.1.

Proof of Proposition 3.1. Let $\psi \in \mathcal{C}^\infty_{\rm c}((0,T); [0,1])$ satisfy $\psi(t) > 1/2$, on $T/3 < t < 2T/3$. Proposition 2.1 applied with $a = W^2$ shows that

$$\|\Pi_{h,\rho} u_0\|^2_{L^2} \leq K \int_{\mathbb{R}} \psi(t)^2 \|W e^{it\Delta} \Pi_{h,\rho} u_0\|^2_{L^2(\mathbb{T}^2)} dt, \quad 0 < h < h_0. \tag{3.2}$$

[†]See https://math.berkeley.edu/~zworski/corr_bbz.pdf for a corrected version.

Taking K large enough so that $R^{-K} \leq h_0$ we apply (3.2) to the dyadic pieces:

$$\begin{aligned}\|u_0\|_{L^2}^2 &= \sum_{k\in\mathbb{Z}} \|\varphi_k(-\Delta)u_0\|_{L^2}^2 \\ &\leq \sum_{k=0}^{K} \|\varphi_k(-\Delta)u_0\|_{L^2}^2 + C\sum_{k=K+1}^{\infty}\int_0^T \psi(t)^2\|W\varphi_k(-\Delta)\,\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^2(\mathbb{T}^2)}^2 dt \\ &= \sum_{k=0}^{K} \|\varphi_k(-\Delta)u_0\|_{L^2}^2 + C\sum_{k=K+1}^{\infty}\int_{\mathbb{R}} \|\psi(t)W\varphi_k(D_t)\,\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^2(\mathbb{T}^2)}^2 dt.\end{aligned}$$

In the last equality we used the equation and replaced $\varphi(-\Delta)$ by $\varphi(D_t)$.

We need to consider the commutator of $\psi \in \mathcal{C}_{\mathrm{c}}^\infty((0,T))$ and $\varphi_k(D_t) = \varphi(R^{-k}D_t)$. If $\widetilde{\psi} \in \mathcal{C}_{\mathrm{c}}^\infty((0,T))$ is equal to 1 on $\operatorname{supp}\psi$ then the semiclassical pseudo-differential calculus with $h = R^{-k}$ (see for instance [Zw12, Chapter 4]) gives

$$\psi(t)\varphi_k(D_t) = \psi(t)\varphi_k(D_t)\widetilde{\psi}(t) + E_k(t, D_t), \quad \partial^\alpha E_k = \mathcal{O}(\langle t\rangle^{-N}\langle\tau\rangle^{-N}R^{-Nk}), \tag{3.3}$$

for all N and uniformly in k.

The errors obtained from E_k can be absorbed into the $\|u_0\|_{H^{-2}(\mathbb{T}^2)}$ term on the right-hand side. Hence we obtain,

$$\begin{aligned}\|u_0\|_{L^2}^2 &\leq C\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + C\sum_{k=0}^{\infty}\int_0^T \|\psi(t)\varphi_k(D_t)\,W\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^2(\mathbb{T}^2)}^2 dt \\ &\leq \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\sum_{k=0}^{\infty}\langle\varphi_k(D_t)^2\widetilde{\psi}(t)\,W\mathrm{e}^{\mathrm{i}t\Delta}u_0, \widetilde{\psi}(t)\,W\mathrm{e}^{\mathrm{i}t\Delta}u_0,\rangle_{L^2(\mathbb{R}_t\times\mathbb{T}^2)} \\ &\leq \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\int_{\mathbb{R}}\|\widetilde{\psi}(t)\,W\mathrm{e}^{\mathrm{i}t\Delta}u_0\|_{L^2(\mathbb{T}^2)}^2 dt \\ &\leq \widetilde{C}\|u_0\|_{H^{-2}(\mathbb{T}^2)}^2 + K\int_0^T \|\,W\mathrm{e}^{\mathrm{i}t\Delta}u_0\,\|_{L^2(\mathbb{T}^2)}^2 dt,\end{aligned}$$

where the last inequality is (3.1) in the statement of the proposition. □

3.2. **Elimination of the error term.** We now eliminate the error term on the right hand side of (3.1). For that we adapt the now standard method of Bardos–Lebeau–Rauch [BLR92] just we did at the end of [BZ12, §4]. The argument recalled there shows that if

$$N := \{u \in L^2(\mathbb{T}^2) \ : \ We^{it\Delta}u \equiv 0 \ \text{ on } (0,T)\times\mathbb{T}^2\} \tag{3.4}$$

is non-trivial then since $iWe^{it\Delta}\Delta u = \partial_t We^{it\Delta}u \equiv 0$ on $(0,T)\times\mathbb{T}^2$, then N is invariant by the action of Δ, and hence it contains a nontrivial $w \in L^2(\mathbb{T}^2)$ such that for some λ,

$$(-\Delta - \lambda)w = 0, \quad Ww \equiv 0.$$

But then w is a trigonometric polynomial vanishing on a set of positive measure which implies that $w \equiv 0$. Hence

$$N = \{0\}. \tag{3.5}$$

Proof of Theorem 3. Suppose the conclusion (1.4) were not to valid. Then there exists a sequence $u_n \in L^2(\mathbb{T}^2)$ such that

$$\|u_n\|_{L^2(\mathbb{T}^2)} = 1, \quad \|We^{it\Delta}u_n\|_{L^2((0,T)\times\mathbb{T}^2)} \to 0, \quad n \to \infty. \tag{3.6}$$

By passing to a subsequence we can then assume that u_n converging weakly in $L^2(\mathbb{T}^2)$ and strongly in $H^{-2}(\mathbb{T}^2)$ to some $u \in L^2$. From Proposition 3.1 we would also have

$$1 = \|u_n\|^2_{L^2(\mathbb{T}^2)} \le C \int_0^T \|We^{it\Delta}u_n\|^2_{L^2(\mathbb{T}^2)} dt + C\|u_n\|^2_{H^{-2}(\mathbb{T}^2)}.$$

Hence,

$$1 \le C \lim_{n\to\infty} \|u_n\|_{H^{-2}(\mathbb{T}^2)} = C\|u\|_{H^{-2}(\mathbb{T}^2)} \implies u \not\equiv 0. \tag{3.7}$$

Let $W_j \in \mathcal{C}^\infty(\mathbb{T}^2)$ satisfy $\|W - W_j\|_{L^4(\mathbb{T}^2)} \to 0$. For $\varphi \in \mathcal{C}^\infty_{\rm c}((0,T)\times\mathbb{T}^2)$, due to distributional convergence, Theorem 4 and (3.6),

$$\begin{aligned} |\langle W_j e^{it\Delta}u, \varphi\rangle| &= \lim_{n\to\infty} |\langle e^{it\Delta}u_n, W_j\varphi\rangle| \le \lim_{n\to\infty} \big(|\langle (W_j - W)e^{it\Delta}u_n, \varphi\rangle| + \langle We^{it\Delta}u_n, \varphi\rangle\big) \\ &\le \|\varphi\|_{L^2}\|(W_j - W)e^{it\Delta}u_n\|_{L^2((0,T)\times\mathbb{T}^2)} \le C\|\varphi\|_{L^2}\|W_j - W\|_{L^4(\mathbb{T}^2)}. \end{aligned}$$

On the other hand the same argument shows that

$$|\langle We^{it\Delta}u, \varphi\rangle| \le |\langle W_j e^{it\Delta}u, \varphi\rangle| + C\|\varphi\|_{L^2}\|W_j - W\|_{L^4(\mathbb{T}^2)}.$$

Combining the two inequalities we see that $|\langle We^{it\Delta}u, \varphi\rangle| \le C \lim_{j\to\infty} \|W_j - W\|_{L^4(\mathbb{T}^2)} = 0$. which means that $We^{it\Delta}u \equiv 0$. Thus $u \in N$ given by (3.4) and by (3.5), $u = 0$. This contradicts (3.7) completing the proof.

□

4. The HUM method: proofs of Theorems 1 and 2

We now show the equivalence of the stabilization, control and observability properties in our context. The proof is a variation on the classical HUM method [Li88], but since our damping and localization functions are *not* in L^∞ it requires additional care.

Proposition 4.1. *The following are equivalent (for fixed $T > 0$).*

(1) *Let $a \in L^2(\mathbb{T}^2;\mathbb{R})$, $\|a\|_{L^2} > 0$. Then for any $u_0 \in L^2(\mathbb{T}^2)$ there exists $f \in L^4(\mathbb{T}^2; L^2(0,T))$ such that the solution u of (1.1) satisfies $u|_{t=T} = 0$*
(2) *Let $a \in L^4(\mathbb{T}^2;\mathbb{R})$, $\|a\|_{L^4} > 0$. Then for any $u_0 \in L^2(\mathbb{T}^2)$ there exists $f \in L^2((0,T)\times\mathbb{T}^2)$ such that the solution u of (1.1) satisfies $u|_{t=T} = 0$*

(3) *Let* $a \in L^4(\mathbb{T}^2;\mathbb{R}), \|a\|_{L^4} > 0$. *Then there exists* $C > 0$ *such that for any* $v_0 \in L^2(\mathbb{T}^2)$,

$$\|v_0\|^2_{L^2(\mathbb{T}^2)} \leq C\|ae^{it\Delta}v_0\|_{L^2((0,T)\times\mathbb{T}^2)}. \tag{4.1}$$

Proof. Let us prove that (1) implies (2). Indeed, for $a \in L^4$, we can apply (1) to $a^2 \in L^2$ and get a function $g \in L^4(\mathbb{T}^2; L^2(0,T))$ such that $a^2 g$ drives the system to rest, and (2) follows by defining $f = ag \in L^2(\mathbb{T}^2; L^2(0,T))$.

To prove that (2) and (3) are equivalent, we follow the HUM method. Define the map

$$R : f \in L^2((0,T)\times\mathbb{T}^2) \mapsto Rf = u|_{t=0},$$

where u is the solution of the final value problem

$$(i\partial_t + \Delta)u(z) = a(x)1_{(0,T)}f \in L^{4/3}(\mathbb{T}^2; L^2(0,T)), \quad u|_{T=0} = 0.$$

By Theorem 4 $R : L^{4/3}(\mathbb{T}^2; L^2(0,T)) \to L^2(\mathbb{T}^2)$ and

$$(2) \iff R(L^{4/3}(\mathbb{T}^2; L^2(0,T))) = L^2(\mathbb{T}^2). \tag{4.2}$$

Again by Theorem 4, $e^{it\Delta}v_0 \in L^4(\mathbb{T}^2; L^2(0,T))$ for $v_0 \in L^2(\mathbb{T}^2)$, we define

$$S : v_0 \in L^2(\mathbb{T}^2) \mapsto 1_{(0,T)} \times ae^{it\Delta}v_0 \in L^2((0,T)\times\mathbb{T}^2),$$

and

$$(3) \iff \exists\, K\ \forall\, v_0 \in L^2(\mathbb{T}^2)\ \ \|v_0\|_{L^2(\mathbb{T}^2)} \leq K\|Sv_0\|_{L^2((0,T)\times\mathbb{T}^2)}. \tag{4.3}$$

To relate R and S we integrate by parts:

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} af\overline{v}dxdt &= \int_0^T\int_{\mathbb{T}^2}(i\partial_t+\Delta)u\overline{v}dxdt = i\left[\int_{\mathbb{T}^2}u\overline{v}dx\right]_0^T + \int_0^T\int_{\mathbb{T}^2}u\overline{(i\partial_t+\Delta)v}dxdt\\ &= -i\int_{\mathbb{T}^2}u\overline{v}dx|_{t=0},\end{aligned}$$

which is the same as

$$\big(f, Sv_0\big)_{L^2((0,T)\times\mathbb{T}^2)} = -i\big(Rf, v_0\big)_{L^2(\mathbb{T}^2)}. \tag{4.4}$$

Let us assume (2). By (4.2) and the closed graph theorem there exists $\eta > 0$ such that the image of the unit ball in $L^2((0,T)\times\mathbb{T}^2)$ by R contains the ball $\{v_0 \in L^2(\mathbb{T}^2) : \|v_0\|_{L^2} \leq \eta\}$. Hence for all $v_0 \in L^2(\mathbb{T}^2)$ there exists $f \in L^2((0,T)\times\mathbb{T}^2)$, such that

$$\|f\|_{L^2((0,T)\times\mathbb{T}^2)} \leq \frac{1}{\eta}\|v_0\|_{L^2}, \quad Rf = v_0.$$

Hence, using (4.4),

$$\begin{aligned}\|v_0\|^2_{L^2(\mathbb{T}^2)} &= i\big(f, Sv_0\big)_{L^2((0,T)\times\mathbb{T}^2)} \le \|f\|_{L^2((0,T)\times\mathbb{T}^2}\|Sv_0\|_{L^2((0,T)\times\mathbb{T}^2)} \\ &\le \frac{1}{\eta}\|Sv_0\|_{L^2((0,T)\times\mathbb{T}^2)}\|v_0\|_{L^2(\mathbb{T}^2)},\end{aligned} \tag{4.5}$$

and by (4.3), (3) follows.

On the other, assume that (3) holds. By (4.3), the operator

$$-iR\circ S : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$$

is continuous and, by (4.4), there exists $C > 0$ such that for all $v_0 \in L^2(\mathbb{T}^2)$,

$$\big(-iR\circ Sv_0, v_0\big)_{L^2(\mathbb{T}^2)} = \big(Sv_0, Sv_0\big)_{L^2((0,1)\times\mathbb{T}^2)} \ge \frac{1}{C}\|v_0\|^2_{L^2(\mathbb{T}^2)}.$$

Consequently $-iR\circ S$ is an injective bounded self-adjoint operator, hence bijective. This in turn shows that R is surjective and in view of (4.2) proves (2).

We also deduce that in (2) we can assume that f is of the form $f = Su_0 = ae^{it\Delta}u_0$, which, changing a to a^2 and using that $e^{it\Delta}u_0 \in L^4(\mathbb{T}^2; L^2(0,1))$ implies (1) when $a \ge 0$. By changing f by a phase factor gives the general case of (1). □

In view of Theorem 3 this proves Theorem 1 and provides some additional versions of it. We now turn to the damped Schrödinger equation.

Proof of Theorem 2. For $a \in L^2$ with $a \ge 0$ and $H := (-i\Delta + a)$ we have

$$\big(Hu, u\big)_{L^2} = \int_{\mathbb{T}^2} a|u|^2(x)dx \ge 0, \quad u \in H^2(\mathbb{T}^2).$$

Hence for $\lambda > 0$ the equation $(H+\lambda)u = f \in L^2(\mathbb{T}^2)$ can be solved with $\|f\|_{L^2} \le \lambda^{-1}\|u\|_{L^2}$. Hille–Yosida theorem then shows that H defines a strongly continuous semigroup $[0,\infty) \ni t \mapsto \exp(-tH)$. Furthermore, when $u_0 \in H^2$,

$$u(t) := \exp(-tH)u_0 \in C^1([0,\infty); L^2(\mathbb{T}^2)) \cap C^0([0,\infty); H^2(\mathbb{T}^2)).$$

We then check that

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{T}^2)} &= \|u_0\|^2_{L^2(\mathbb{T}^2)} - \int_0^t\int_{\mathbb{T}^2} a(x)|u|^2(s,x)dxds, \\ u(t) &= e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}(au)(s)ds.\end{aligned} \tag{4.6}$$

Let $a_j \in C^0(\mathbb{T}^2)$ and $\|a_j - a\|_{L^2(\mathbb{T}^2)} \to 0$, $j \to \infty$. Using the second expression in (4.6),

$$\begin{aligned}\|u\|_{L^4(\mathbb{T}^2;L^2(0,T))} &\leq C\|u_0\|_{L^2(\mathbb{T}^2)} + \|a_j u\|_{L^1((0,T);L^2(\mathbb{T}^2))} + \|(a-a_j)u\|_{L^{4/3}(\mathbb{T}^2;L^2(0,T))}\\ &\leq C\|u_0\|_{L^2(\mathbb{T}^2)} + CT\|a_j\|_{L^\infty}\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^2))}\\ &\qquad + C\|a-a_j\|_{L^2(\mathbb{T}^2)}\|u\|_{L^4(\mathbb{T}^2;L^2(0,T))}\end{aligned}$$

Taking j large enough so that $C\|a-a_j\|_{L^2(\mathbb{T}^2)} \leq \frac{1}{2}$, we get

$$\|u\|_{L^4(\mathbb{T}^2;L^2(0,T))} \leq C'\|u_0\|_{L^2(\mathbb{T}^2)}, \quad u_0 \in H^2(\mathbb{T}^2).$$

Since H^2 is dense in L^2, this remains true for initial data $u_0 \in L^2(\mathbb{T}^2)$ and consequently, for $a \in L^2$, we get that

$$\int_0^t \int_{\mathbb{T}^2} a(x)|u|^2(s,x)dxds \leq C\|u_0\|_{L^2(\mathbb{T}^2)}. \tag{4.7}$$

By simple integration by parts (4.6) is true for $u_0 \in H^2$, and consequently from (4.7) it remains true for $u_0 \in L^2$. Now, if for some $T > 0$,

$$\|u_0\|_{L^2(\mathbb{T}^2)} \leq C\int_0^T \int_{\mathbb{T}^2} a(x)|u|^2(t,x)dxdt, \tag{4.8}$$

where u is the solution of (1.3), then (4.6) and semigroup property show that $\|u(kT)\|^2_{L^2(\mathbb{T}^2)} \leq (1-1/C)^N\|u_0\|^2_{L^2(\mathbb{T}^2)}$, and the exponential decay (1.3) follows.

For any fixed $T > 0$ (4.8) is the same as (1.4) with $W = a^{\frac{1}{2}}$, except that here u is the solution of the damped Schrödinger equation, while in (1.4) it is the solution of the free Schrödinger equation.

We now claim that (1.4), $W = a^{\frac{1}{2}}$, implies (4.8). In fact, suppose that (4.8) is not true. Then, there exists a sequence $u_{0,n} \in L^2$,

$$\|u_{0,n}\|_{L^2} = 1, \quad (i\partial_t + \Delta)u_n = au_n, \quad u_n|_{t=0} = u_{0,n}, \quad \|a^{\frac{1}{2}}u_n\|_{L^2((0,T)\times\mathbb{T}^2)} \to 0, \ n \to \infty.$$

Then

$$\|au_n\|_{L^{\frac{4}{3}}(\mathbb{T}^2;L^2((0,T)))} = \|a^{\frac{1}{2}}a^{\frac{1}{2}}u\|_{L^{\frac{4}{3}}(\mathbb{T}^2;L^2((0,T)))} \leq \|a^{\frac{1}{2}}\|_{L^4(\mathbb{T}^2)}\|a^{\frac{1}{2}}u_n\|_{L^2((0,T)\times\mathbb{T}^2)} \to 0,$$

and Theorem 4 shows that $u_n = e^{it\Delta}u_{0,n} + e_n$, $\|e_n\|_{L^4(\mathbb{T}^2;L^2((0,T)))} \to 0$. But then, using (1.4),

$$\begin{aligned}0 &= \limsup_{n\to\infty} \|a^{\frac{1}{2}}u_n\|_{L^2((0,T)\times\mathbb{T}^2)}\\ &\geq \limsup_{n\to\infty}\left(\|a^{\frac{1}{2}}e^{it\Delta}u_{0,n}\|_{L^2((0,T)\times\mathbb{T}^2)} - \|a^{\frac{1}{2}}\|_{L^4}\|e_n\|_{L^4(\mathbb{T}^2;L^2((0,T)))}\right)\\ &\geq \limsup_{n\to\infty}\|a^{\frac{1}{2}}e^{it\Delta}u_{0,n}\|_{L^2((0,T)\times\mathbb{T}^2)} \geq c\limsup_{n\to\infty}\|u_{0,n}\|_{L^2(\mathbb{T})} = c > 0\end{aligned}$$

which gives a contradition. Hence (4.8) holds and that completes the proof. □

Appendix

To see that Theorem 3 for $T > \pi$ and rational tori follows from [Ja97, Theorem 1.2] assume that $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. We then write $u(z) = \sum_\lambda u_\lambda$, where the sum of is over distinct eigenvalues of $-\Delta$ (and u_λ is the projection of u on the corresponding eigenspace). By Ingham's inequality [In36] (this is where $T > \pi$ is used),

$$\int_0^T \|We^{it\Delta}u\|^2_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}^2}\int_0^T \left|\sum_{\lambda\in\mathbb{N}} W(z)u_\lambda(z)e^{it\lambda}\right|^2 dtdz \geq B\int_{\mathbb{T}^2}\sum_{\lambda\in\mathbb{N}}|W(z)u_\lambda(z)|^2dz.$$

Hence, (1.4) follows from the estimate,

$$\sum_\lambda \|u_\lambda\|^2_{L^2(\mathbb{T}^2)} \leq C\int_{\mathbb{T}^2}\sum_{\lambda\in\mathbb{N}}|W(z)u_\lambda(z)|^2dz, \tag{A.1}$$

which it turn follows from a pointwise estimate:

$$\|u_\lambda\|^2_{L^2(\mathbb{T}^2)} \leq C\int_{\mathbb{T}^2}|W(z)u_\lambda(z)|^2dz, \quad -\Delta u_\lambda = \lambda u_\lambda. \tag{A.2}$$

Proof of (A.2). We start with the observation that the zero set of a non-trivial trigonometric polynomial $p(z)$ has measure zero and hence,

$$\int_{\mathbb{T}^2}|W(z)p(z)|^2dz > 0. \tag{A.3}$$

In particular that holds for any fixed eigenfunction of $-\Delta$.

To prove (A.2) we proceed by contradiction, that is we assume that there exists a sequence of e_n's, such that

$$\|e_n\|^2_{L^2} = 1, \quad \|We_n\|^2_{L^2} \to 0, \quad -\Delta e_n = \lambda_n e_n. \tag{A.4}$$

Suppose first that λ_n are bounded. We can then assume that $\lambda_n \to \lambda$. From (A.4) we see that e_n are bounded in H^2 and hence we can assume that $e_n \to e$ in H^1 and, as $H^1 \subset L^4$, also in L^4. Then (A.4) shows that $-\Delta e = \lambda e$, $\|e\|_{L^2} = 1$, $\|We\|_{L^2} = 0$, which contradicts (A.3).

Hence we can assume (by extracting a subsequence) that $\lambda_n \to \infty$ in (A.4). We can then assume that the sequence of probability measures $|e_n|^2dx$ converges weakly to a measure ν. According to [Ja97, Theorem 1.2], $\nu = p(z)dz$ where p is a non-negative trigonometric polynomial, $\int p(z)dz = 1$.

Let $f_k \in C^0$, $f_k \geq 0$, converge to $|W|^2$ in L^2. From Zygmund's bound on the L^4 norm of e_n (1.5), we get

$$\limsup_{n\to+\infty} \left| \int (f_k - |W|^2)|e_n|^2(x)dx \right| \leq C\|f_k - |W|^2\|_{L^2},$$

and from the weak convergence $\lim_{n\to+\infty} \int f_k|e_n|^2(x)dx = \int f_k(x)p(x)dx$. We deduce

$$0 = \lim_{n\to+\infty} \int |We_n|^2(x)dx = \int |W(x)|^2 p(x)dx.$$

This again contradicts (A.3). □

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