Linear vs Chaotic

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Plan of the talk

- Dynamics and statistics
- Zeta functions
- Integrability, chaos and beyond
Dynamical systems
Dynamical systems: a statistical approach
Dynamical systems: a statistical approach

Linear

Non-linear
Dynamical systems: a statistical approach

Completely integrable

Chaotic
For general aspects see

**Brin–Stuck Introduction to Dynamical Systems**

The billiard on the right is a form of **Sinai** billiard.

For the discussion of chaotic flows on billiard tables see **Wojtkowski**


and references given there.

We should stress that the results discussed below are really for
smooth dynamical systems. There are technical difficulties in
extending them (e.g. meromorphy of the Ruelle zeta functions) to
billiards because of singularities and the presence of glancing
trajectories.
In the chaotic case positions and directions get uniformly distributed:

**Question:** How long do we have to wait to have uniform distribution?
The length of time needed to have uniform distribution can be estimated when we know \emph{exponential decay of correlations}

\[ \int f(\varphi_t(x))g(x)\,dm(x) = \int f(x)\,dx \int g(x)\,dx + O(e^{-\gamma t}) \]

This has a long tradition but general results are recent:

\textbf{Dolgopyat} \emph{On decay of correlations in Anosov flows}, Ann. of Math. \textbf{147}(1998)

\textbf{Liverani} \emph{On contact Anosov flows}, Ann. of Math. \textbf{159}(2004),

\textbf{Tsujii} \emph{Contact Anosov flows and the FBI transform},

\textbf{Nonnenmacher–Zworski} \emph{Decay of correlations for normally hyperbolic trapping}, Inv. Math., \textbf{200}(2015)
The most famous function of mathematics: Riemann zeta function

\[ \zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \quad p = \text{a prime number} \]

\( \zeta(s) \) has a pole (singularity) at \( s = 1 \).

It has a lot of important zeros

\[ -\log |\zeta(s)| \]
A dynamical analogue: Ruelle zeta function

Replace primes with prime closed orbits
A dynamical analogue: Ruelle zeta function

Replace primes with prime closed orbits

Replace $p$ by $\log T_\gamma$ where $T_\gamma$ is the length of a prime closed orbit.
A dynamical analogue: Ruelle zeta function

Replace primes with prime closed orbits

\[ \zeta_D(s) = \prod_{\gamma} (1 - e^{-sT_\gamma})^{-1} \]

Replace \( p \) by \( e^{T_\gamma} \) where \( T_\gamma \) is the length of a prime closed orbit.

It turns out that the zeros and poles of \( \zeta_D \) contain information about statistical properties of the chaotic dynamical system.

That includes the time at which we achieve uniform distribution.
For an introduction to dynamical zeta functions and to the literature see

http://homepages.warwick.ac.uk/~masdbl/grenoble-16july.pdf

Recent papers about meromorphic continuation of dynamical zeta functions:


Microlocal approach to Anosov systems started with

Computational methods for zeta functions were developed by Cvitanovic, Eckhardt, Gaspard...

A recent mathematical account and references:


A slightly different zeta function is needed to obtain the rate of decay to equilibrium: in addition to closed orbits it also includes the instability factors:

\[
\zeta_1(s) := \exp \left( - \sum_{\gamma} \frac{T_{\gamma}^\# \exp^{-sT_{\gamma}}}{T_{\gamma}|\det(I - P_{\gamma})|} \right).
\]
Trouble with all this: Few real systems are purely completely integrable or purely chaotic.

The simplest (?) flow exhibiting chaotic behaviour:

\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + x_2 x_3, \\
\dot{x}_3 &= 1 - x_2^2.
\end{align*}

Technical asides:

simple: it is the contact flow for $e^{-|x|^2/2}(x_2 dx_1 + dx_3)$

chaotic behaviour: positive Lyapounov exponents
These ordinary differential equations are called the Nosé–Hoover system and have origins in molecular dynamics.

They were rediscovered by Sprott in a computer search for simple systems with positive Lyapunov exponents. The system is simpler than the famed Lorenz equations and easier remember for mathematicians because of the simple contact form.

For a recent account and references see

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A useful visualization: Poincaré section
Question 1: Do we have equidistribution in the chaotic sea?

chaotic sea: the blue region
Although the movies and the picture suggest existence of invariant sets of positive measure, failure of ergodicity for the Nosé–Hoover was established only recently in Legoll–Luskin–Moeckel *Non-Ergodicity of the Nosé–Hoover thermostatted harmonic oscillator*, Arch. Ration. Mech. Anal. 184 (2007).
Question 1: Do we have equidistribution in the chaotic sea?

Question 2: What happens in many degrees of freedom/infinite dimensions?

A relevant example: Power spectrum of El Niño
The El Niño example comes from

Question 1: Do we have equidistribution in the chaotic sea?

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Figure credit: Proc. Nat. Acad. Sci. 2014
To conclude:

- Can we bring methods which have been successful in the study of chaotic systems to the study of mixed systems?
- What happens on the quantum level? (Mathematically, what are possible results about eigenvalues and eigenfunctions.)
- Can we visualize/understand complicated multidimensional systems using ideas from chaotic dynamics?
- Can we find complete integrability behind some interesting mixed systems? (Just as random matrix theory models quantum systems with underlying chaotic dynamics.)
For a recent result for mixed systems close to completely integrable systems see


For a survey of results on partially hyperbolic systems see
Hasselblatt–Pesin


For surveys of problems in *quantum chaos* see Nonnenmacher

http://arxiv.org/abs/1005.5598
http://arxiv.org/abs/1105.2457

For results on eigenfunction statistics for (special) mixed systems:

Galkowski http://arxiv.org/abs/1209.2968
Riviere http://arxiv.org/abs/1209.3576