MATHEMATICS OF INTERNAL WAVES IN A 2D AQUARIUM

SEMYON DYATLOV, JIAN WANG, AND MACIEJ ZWORSKI

ABSTRACT. Following theoretical and experimental work of Maas et al [21] we consider a linearized model for internal waves in effectively two dimensional aquaria. We provide a precise description of singular profiles appearing in long time wave evolution and associate them to classical attractors. That is done by microlocal analysis of the spectral Poincaré problem, leading in particular to a limiting absorption principle. Some aspects of the paper (for instance §6) can be considered as a natural microlocal continuation of the work of John [18] on the Dirichlet problem for hyperbolic equations in two dimensions.

1. Introduction

Internal waves are a central topic in oceanography and the theory of rotating fluids – see [20] and [27] for reviews and references. They can be described by linear perturbations of the initial state of rest of a stable-stratified fluid (dense fluid lies everywhere below less-dense fluid and the isodensity surfaces are all horizontal). Forcing

Figure 1. Contour plots of a numerical solution to (1.1) for $\Omega$ given by a unit square tilted by $\pi/10$ (see §2.5), with $f(x) = e^{-5\pi^2(x-x^0)^2}$, where $x^0$ is the center and $\lambda = 0.8$. In that case the rotation number of the billiard ball map is $\frac{1}{2}$ (see Figure 8) and the classical attractor is given by a parallelogram on which $u$ develops a singularity – see Theorem 1.
can take place at linear level by pushing fluid away from this equilibrium state either mechanically, by wind, a piston, a moving boundary, or thermodynamically, by spatially differential heating or evaporation/rain.

The mechanism behind formation of internal waves comes from dynamics of the classical system which underlies wave equations – see §1.1 for the case of nonlinear dynamics relevant to the case we consider. When parameters of the system produce hyperbolic dynamics, attractors are observed in wave evolution – see Figure 1. This phenomenon is both physically and theoretically more accessible in dimension two. The analysis in the physics literature, see [20], [30], has focused on constructions of standing and propagating waves and did not address the evolution problem analytically. (See however [1] for an analysis of a numerical approach to the evolution problem.) In this paper we prove the emergence of singular profiles in the long time evolution of linear waves for two dimensional domains.

The model we consider is described as follows. Let $\Omega \subset \mathbb{R}^2 = \{ x = (x_1, x_2) : x_j \in \mathbb{R} \}$ be a bounded simply connected open set with a smooth boundary. Following the fluid mechanics literature we consider the following evolution problem, sometimes referred to as the Poincaré problem:

$$
(\partial_t^2 \Delta + \partial_{x_2}^2) u = f(x) \cos \lambda t, \quad u|_{t=0} = \partial_t u|_{t=0} = 0, \quad u|_{\partial \Omega} = 0, \quad (1.1)
$$

where $\lambda > 0$ and $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2$, see Ralston [25], Maas et al [21], Brouzet [3, §§1.1.2–3], Dauxois et al [7], Colin de Verdière–Saint-Raymond [5] and references given there. It models internal waves in a stratified fluid in an effectively two-dimensional aquarium $\Omega$ with an oscillatory forcing term (here we follow [5] rather than change the boundary condition). The geometry of $\Omega$ and the forcing frequency $\lambda$ can produce concentration of the fluid velocity $v = (\partial_{x_2} u, -\partial_{x_1} u)$ on attractors. This phenomenon was predicted by Maas–Lam [22] and was then observed experimentally by Maas–Benielli–Sommeria–Lam [21]. In this paper we provide a mathematical explanation: as mentioned above the physics papers concentrated on the analysis of modes and classical dynamics rather than on the long time behaviour of solutions to (1.1).

1.1. Assumptions on $\Omega$ and $\lambda$. The assumptions on $\Omega$ and $\lambda$ which guarantee existence of singular profiles (internal waves) in long time evolution of (1.1) are formulated using a “chess billiard” – see [24], [19] for recent studies and references. It was first considered in similar context by John [18] and was the basis of the analysis in [22]. It is defined as the reflected bicharacteristic flow for $(1 - \lambda^2)\xi_2^2 - \lambda^2 \xi_1^2$, which is the Hamiltonian for the $1 + 1$ wave equation with $x_2$ corresponding to time and the speed given by $c = \lambda/\sqrt{1 - \lambda^2}$ – see Figure 2 and §2.1.
This flow has a simple reduction to the boundary which we describe using a factorization of the quadratic form dual to \((1 - \lambda^2)\xi_2^2 - \lambda^2\xi_1^2:\)

\[
- \frac{x_1^2}{\lambda^2} + \frac{x_2^2}{1 - \lambda^2} = \ell^+(x, \lambda)\ell^-(x, \lambda), \quad \ell^\pm(x, \lambda) := \pm \frac{x_1}{\lambda} + \frac{x_2}{\sqrt{1 - \lambda^2}}.
\]  

We often suppress the dependence on \(\lambda\), writing simply \(\ell^\pm(x)\). Same applies to other \(\lambda\)-dependent objects introduced below.

**Definition 1.** Let \(0 < \lambda < 1\). We say that \(\Omega\) is \(\lambda\)-simple if each of the functions \(\partial \Omega \ni x \mapsto \ell^\pm(x, \lambda)\) has only two critical points, which are both nondegenerate. We denote these minimum/maximum points by \(x_{\min}^\pm(\lambda), x_{\max}^\pm(\lambda)\).

Under the assumption of \(\lambda\)-simplicity we define the following two smooth orientation reversing involutions on the boundary (see §2.1 for more details):

\[
\gamma^\pm(\bullet, \lambda) : \partial \Omega \to \partial \Omega, \quad \ell^\pm(x) = \ell^\pm(\gamma^\pm(x)).
\]  

These maps correspond to interchanging intersections of the boundary with lines with slopes \(\mp 1/c\), respectively – see Figure 2. The *chess billiard map* \(b(\bullet, \lambda)\) is defined as the composition

\[
b := \gamma^+ \circ \gamma^-
\]  

and is a \(C^\infty\) orientation preserving diffeomorphism of \(\partial \Omega\).

Denoting by \(b^n\) the \(n\)-th iterate of \(b\), we consider the set of periodic points

\[
\Sigma_\lambda := \{x \in \partial \Omega \mid b^n(x, \lambda) = x \text{ for some } n \geq 1\}.
\]  

If \(\Sigma_\lambda \neq \emptyset\), then all the periodic points in \(\Sigma_\lambda\) have the same minimal period, see §2.1.
We are now ready to state the dynamical assumptions on the chess billiard:

**Definition 2.** Let $0 < \lambda < 1$. We say that $\lambda$ satisfies the Morse–Smale condition if:

1. $\Omega$ is $\lambda$-simple;
2. the map $b$ has periodic points, that is $\Sigma_\lambda \neq \emptyset$;
3. the periodic points are hyperbolic, that is $\partial_x b^n(x, \lambda) \neq 1$ for all $x \in \Sigma_\lambda$ where $n$ is the minimal period.

Under the Morse–Smale condition we have $\Sigma_\lambda = \Sigma_\lambda^+ \cup \Sigma_\lambda^−$ where $\Sigma_\lambda^+, \Sigma_\lambda^−$ are the sets of attractive, respectively repulsive, periodic points of $b$:

$$\Sigma_\lambda^+ := \{x \in \Sigma_\lambda \mid \partial_x b^n(x, \lambda) < 1\}, \quad \Sigma_\lambda^− := \{x \in \Sigma_\lambda \mid \partial_x b^n(x, \lambda) > 1\}. \quad (1.6)$$

For $y \in \partial\Omega$, let

$$\Gamma_\lambda^\pm(y) := \{x \in \Omega \mid \ell^\pm(x, \lambda) = \ell^\pm(y, \lambda)\} \quad (1.7)$$

be the open line segment connecting $y$ with $\gamma^\pm(y, \lambda)$. Denote $\Gamma_\lambda(y) := \Gamma_\lambda^+(y) \cup \Gamma_\lambda^−(y)$. Then $\Gamma_\lambda(\Sigma_\lambda)$ gives the closed trajectories of the chess billiard inside $\Omega$.

For $y \in \partial\Omega$ which is not a critical point of $\ell^+$, we split the conormal bundle $N^*\Gamma_\lambda^+(y)$ into the positive/negative directions:

$$N^*\Gamma_\lambda^+(y) \setminus 0 = N^*_+\Gamma_\lambda^+(y) \cup N^*_\Gamma_\lambda^+(y),$$

$$N^*_\pm\Gamma_\lambda^+(y) := \{(x, \tau d\ell^+(x)) \mid x \in \Gamma_\lambda^+(y), \pm(\partial_\theta \ell^+(y))\tau > 0\} \quad (1.8)$$

and similarly for $N^*\Gamma_\lambda^−(y)$. Note that the orientation depends on the choice of $y$ and not just on $\Gamma_\lambda^+(y)$: we have $N^*_+\Gamma_\lambda^+(\gamma^+(y)) = N^*_\Gamma_\lambda^+(y)$.

We now define Lagrangian submanifolds $\Lambda^\pm(\lambda) \subset T^*\Omega \setminus 0$ by

$$\Lambda^\pm(\lambda) := N^*_+\Gamma^+(\Sigma_\lambda^+) \cup N^*_+\Gamma^−(\Sigma_\lambda^+), \quad (1.9)$$

see Figure 3. We note that $\pi(\Lambda^\pm(\lambda)) = \Gamma_\lambda(\Sigma_\lambda)$ and $N^*_+\Gamma^\pm(\Sigma_\lambda^+) = N^*_+\Gamma^\pm(\Sigma_\lambda^−)$.

**1.2. Statement of results.** The main result of this paper is formulated using the concept of wave front set, see [17, Theorem 18.1.27] but the proof provides a much finer information about the singular profile of the solution to (1.1) as a Lagrangian distribution, see [17, §18.1.2] and §3.2. The wave front set of a distribution, $WF(u)$, is a closed subspace of the cotangent bundle of $T^*\Omega \setminus 0$ and it provides phase space information about singularities. Its projection to the base, $\pi(WF(u))$, is the set where the distribution is singular, the singular support, $\text{singsupp } u$.

**Theorem 1.** Suppose that $\Omega$ and $\lambda \in (0, 1)$ satisfy the Morse–Smale conditions of Definition 2. Then the solution to (1.1) with $f \in C^\infty_c(\Omega; \mathbb{R})$ is given by

$$u(t) = \text{Re } e^{i\lambda t} u^+ + b(t) + \epsilon(t), \quad u^+ \in H^{1/2−}(\Omega), \quad b(t) \in H^1_0(\Omega),$$

$$WF(u^+) \subset \Lambda^+(\lambda), \quad \|b(t)\|_{H^1(\Omega)} \leq C, \quad \|\epsilon(t)\|_{H^{1/2−}(\Omega)} \to 0, \quad t \to \infty. \quad (1.10)$$
Figure 3. A visualization of the Lagrangian submanifolds (1.9) corresponding to attractive and repulsive cycles of $b$ given in (1.4). The rectangle represents the projection the attractive (+) and repulsive (−) Lagrangians $\Lambda^\pm(\lambda)$ and the arrows perpendicular to the sides represent the conormal directions distinguishing the two Lagrangians. We also indicate the corresponding sets on the boundary: $\Sigma^\pm_\lambda$ are the attractive (+) and repulsive (−) periodic points of $b$ given by (1.4) and the directions indicate the sign of the conormal directions.

where $\Lambda^+(\lambda) \subset T^*\Omega \setminus 0$ is the attracting Lagrangian – see (1.9) and Figure 3. In particular, $\text{singsupp} u^+$ is contained in the union of closed orbits of the chess billiard flow. In addition, $u^+$ is a Lagrangian distribution, $u^+ \in \mathcal{I}^{-1}(\Omega, \Lambda^+(\lambda))$ (see §3.2) and $u^+|_{\partial \Omega} = 0$ (well defined because of the wave front set condition).

For a numerical illustration of (1.10) see Figure 1. We remark that numerically it is easier to consider polygonal domains – see §2.4 for a discussion of the stability of our assumptions for smoothed out polygonal domains.

Theorem 1 is proved using spectral properties of a self-adjoint operator associated to the evolution equation (1.1). To define it, let $\Delta_\Omega$ be the (negative definite) Dirichlet Laplacian of $\Omega$ with the inverse denoted by $\Delta^{-1}_\Omega : H^{-1}(\Omega) \to H^1_0(\Omega)$. Then,

$$P := \partial_{xx}^{-2} \Delta^{-1}_\Omega : H^{-1}(\Omega) \to H^{-1}(\Omega), \quad \langle u, w \rangle_{H^{-1}(\Omega)} := \langle \nabla \Delta^{-1}_\Omega u, \nabla \Delta^{-1}_\Omega w \rangle_{L^2(\Omega)},$$

is a bounded non-negative (hence self-adjoint) operator studied by Ralston [25] – see §7.1. Studying the spectrum of $P$ is referred to as a Poincaré problem.

The evolution equation (1.1) is equivalent to

$$(\partial_t^2 + P)w = f \cos \lambda t, \quad w|_{t=0} = \partial_t w|_{t=0} = 0, \quad f \in C^\infty_c(\Omega; \mathbb{R}), \quad u = \Delta^{-1}_\Omega w.$$

(1.12)
Figure 4. Numerical illustration of Theorem 2: contour plots of $|u(x)|$ for $u$ solving $(\partial^2_{x^2} - (\lambda^2 + i\varepsilon)\Delta)^{-1}f$ where $\varepsilon = 0.005$ and $f(x) = e^{-10(x-(\frac{1}{2},\frac{1}{2}))^2}$ and $\Omega = \mathcal{T}_{0,5}$ (see §2.5). On the left, the rotation number is given by $\frac{1}{3}$ and we see concentration on an attractor; on the right, the rotation number is (nearly) irrational and, as $\varepsilon \to 0^+$, $u$ is expected be uniformly distributed [20]. Morse–Smale assumptions do not hold, at least not on scales relevant to numerical calculations and trajectories are uniformly distributed in the trapezium. In the contour plots of $|u(x)|$ black corresponds to 0.

This equation is easily solved using the functional calculus of $P$:

$$w(t) = \text{Re} \int_0^t \sqrt{P^{-1}} \sin \left( (t - s)\sqrt{P} \right) f e^{i\lambda s} ds$$

$$= \text{Re} e^{i\lambda t} \frac{1}{2\sqrt{P}} \sum \pm (\lambda \pm \sqrt{P})^{-1} (1 - e^{-it(\lambda \pm \sqrt{P})}) f.$$  \hfill (1.13)

The expression on the right hand side is well defined for finite values $t$ but as $t \to \infty$ we formally have

$$\frac{1}{2\sqrt{P}} \sum \pm (\lambda \pm \sqrt{P})^{-1} (1 - e^{-it(\lambda \pm \sqrt{P})}) f \rightarrow (P - \lambda^2 - i0)^{-1} f.$$  \hfill (1.14)
(This holds if we consider the limit in the sense of distributions in the variable \( \lambda > 0 \) with \( \sqrt{P} \) considered as a positive parameter.) This suggests that the *limiting absorption principle* for the operator \( P \) near \( \lambda^2 \) plays the crucial role in the proof of Theorem 1 and that is the content of

**Theorem 2.** Suppose that for \( \omega \in J \subset (0, 1) \), \( \Omega \) satisfies the Morse–Smale conditions of Definition 2. Then for each \( f \in C_c^\infty(\Omega) \) the limit
\[
(P - \omega^2 - i0)^{-1} f = \lim_{\varepsilon \to 0^+} (P - (\omega + i\varepsilon)^2)^{-1} f \quad \text{in } D'(\Omega)
\] (1.15)
exists and the spectrum of \( P \) is absolutely continuous in \( J \):
\[
\sigma(P) \cap J = \sigma_{ac}(P) \cap J.
\] (1.16)

Moreover,
\[
(P - \omega^2 - i0)^{-1} f \in I^1(\overline{\Omega}, \Lambda^+(\omega)),
\] (1.17)
where \( \Lambda^+(\omega) \) is given in (1.9) and the definition of the Lagrangian spaces \( I^1(\overline{\Omega}, \Lambda^+(\omega)) \) is reviewed in §3.2.

**Remarks.** 1. The proof provides a more precise statement based on a reduction to the boundary – see §7. We also have a smooth dependence on \( \lambda \) which plays a role in proving Theorem 2 as in [12, §5] – see §8. Rather that working with \( P \), we consider the closely related stationary *Poincaré problem*
\[
(\partial_{x_2}^2 - \lambda^2 \Delta) u_\lambda = f \in C_c^\infty, \quad u_\lambda|_{\partial\Omega} = 0, \quad \text{Re } \lambda \in (0, 1), \quad \text{Im } \lambda > 0.
\]
Then \( u_\lambda \in C^\infty(\overline{\Omega}) \) has a limit in \( H_{-1}^{1/2}(\Omega) \) which satisfies \( u_{\lambda+0} \in I^{-1}(\overline{\Omega}, \Lambda^+(\Lambda)) \). Then
\[
(P - \lambda^2 - i0)^{-1} f = \Delta u_{\lambda+0}.
\]
The singular profile in Theorem 1 satisfies
\[
u^+ = u_{\lambda+0},
\]
which agrees with the heuristic argument in (1.14).

2. As noted in [25], \( \sigma(P) = [0, 1] \) but as emphasized there and in numerous physics papers the structure of the spectrum of \( P \) is far from clear. Here we only characterize the spectrum (1.16) under the Morse–Smale assumption of Definition 2.

1.3. **Related mathematical work.** Motivated by the study of internal waves results similar to Theorems 1 and 2 were obtained for self-adjoint 0th order pseudodifferential operators on 2D tori with dynamical conditions in Definitions 1 and 2 replaced by demanding that a naturally defined flow is Morse–Smale. That was done first by Colin de Verdière–Saint-Raymond [5],[4], with different proofs provided by Dyatlov–Zworski [12]. The question of modes of viscosity limits in such models (addressing physics questions formulated for domains with boundary – see Rieutord–Valdettaro [26] and references given there) were investigated by Galkowski–Zworski [14] and Wang [33]. Finer questions related to spectral theory were also answered in [33]. Unlike in the
situation considered in this paper, embedded eigenvalues are possible in the case of 0th order pseudodifferential operators [29].

The dynamical system (1.4) was recently studied by Nogueira–Troubetzkoy [24] and by Lenci et al [19]. We refer to those papers for additional references and dynamical results.

1.4. Organization of the paper. In §2 we provide a self-contained analysis of the dynamical system given by the diffeomorphism (1.4). We emphasize properties needed in the analysis of the operator (1.11): properties of pushforwards by \( \ell^\pm \) and existence of suitable escape/Lyapounov functions. The next section is devoted to a review of microlocal analysis used in this paper and in particular to definitions and properties of conormal/Lagrangian spaces used in the formulations of Theorems 1 and 2. In §4 we describe reduction to the boundary using 1+1 Feynman propagators which arise naturally in the limiting absorption principles. Despite the presence of characteristic points, the restricted operator enjoys good microlocal properties – see Proposition 4.15. Microlocal analysis of that operator is given in §5 with the key estimate (5.19) motivated by Lasota–Yorke inequalities and radial estimates. The self-contained §6 analyses wave front set properties of distributions invariant under the diffeomorphisms (1.4). These results are combined in §7 to give the proof of the limiting absorption principle of Theorem 2. Finally, in §8 we follow the strategy of [12] to describe long time properties of solutions to (1.1) – see Theorem 2.

2. Geometry and dynamics

In this section we assume that \( \Omega \subset \mathbb{R}^2 \) is an open bounded simply connected set with \( C^\infty \) boundary \( \partial \Omega \) and review the basic properties of the involutions \( \gamma^\pm \) and the chess billiard \( b \) defined in (1.3), (1.4). We orient \( \partial \Omega \) in the positive direction as the boundary of \( \Omega \) (i.e., counterclockwise if \( \Omega \) is convex).

2.1. Basic properties. Fix \( \lambda \in (0,1) \) such that \( \Omega \) is \( \lambda \)-simple in the sense of Definition 1. We first show that the involutions \( \gamma^\pm \) defined in (1.3) are smooth. Away from the critical set \( \{ x^\pm_{\min}, x^\pm_{\max} \} \) this is immediate. Next, we write

\[
\ell^\pm(x) = \ell^\pm(x^\pm_{\min}) + \theta^\pm_{\min}(x)^2 \quad \text{for } x \text{ near } x^\pm_{\min},
\]

\[
\ell^\pm(x) = \ell^\pm(x^\pm_{\max}) - \theta^\pm_{\max}(x)^2 \quad \text{for } x \text{ near } x^\pm_{\max},
\]

where \( \theta^\pm_{\min}, \theta^\pm_{\max} \) are local coordinate functions on \( \partial \Omega \) which map \( x^\pm_{\min}, x^\pm_{\max} \) to 0. Then for \( x \) near \( x^\pm_{\min} \) the point \( \gamma^\pm(x) \) satisfies the equation

\[
\theta^\pm_{\min}(\gamma^\pm(x)) = -\theta^\pm_{\min}(x)
\]

and similarly near \( x^\pm_{\max} \). This shows the smoothness of \( \partial \Omega \ni x \mapsto \gamma^\pm(x) \) near the critical points.
Next, note that since $\gamma^\pm$ are involutions, $b$ is conjugate to its inverse:

$$b^{-1} = \gamma^\pm \circ b \circ \gamma^\pm.$$ 

Therefore $\Sigma^+_\lambda = \gamma^\pm(\Sigma^-_\lambda)$ where $\Sigma^\pm_\lambda$ are defined in (1.6). Since $x^\pm_{\text{min}}, x^\pm_{\text{max}}$ are fixed points of $\gamma^\pm$, the Morse–Smale conditions (see Definition 2) implies that there are no characteristic periodic points:

$$\Sigma_\lambda \cap C_\lambda = \emptyset \quad \text{where} \quad C_\lambda := C^+_\lambda \cup C^-_\lambda, \quad C^\pm_\lambda := \{x^\pm_{\text{min}}(\lambda), x^\pm_{\text{max}}(\lambda)\}. \quad (2.2)$$

2.1.1. **Useful identities.** For $x \in \partial \Omega$ and $\lambda \in (0, 1)$ we define the signs

$$\nu^\pm(x, \lambda) := \text{sgn} \partial_\theta \ell^\pm(x, \lambda) \quad (2.3)$$

where $\partial_\theta$ is the derivative along $\partial \Omega$ with respect to a positively oriented parametrization.

**Lemma 2.1.** Assume that $\Omega$ is $\lambda$-simple. Then for all $x \in \partial \Omega$

$$\text{sgn} \ell^\pm(\gamma^\pm(x) - x) = \pm \nu^\pm(x), \quad (2.4)$$

$$\nu^\pm(\gamma^\pm(x)) = -\nu^\pm(x), \quad (2.5)$$

$$\partial_\lambda \ell^\pm(x, \lambda) = \frac{2\lambda^2 - 1}{2\lambda(1 - \lambda^2)} \ell^\pm(x, \lambda) + \frac{1}{2\lambda(1 - \lambda^2)} \ell^\mp(x, \lambda). \quad (2.6)$$

**Proof.** To see (2.4), we first notice that it holds when $x \in \{x^\pm_{\text{min}}, x^\pm_{\text{max}}\}$, as then both sides are equal to 0. Now, assume that $\gamma^\pm(x) \neq x$ (that is, $x \notin \{x^\pm_{\text{min}}, x^\pm_{\text{max}}\}$). Denote by $v(x) \in \mathbb{R}^2$ the velocity vector of the parametrization at the point $x \in \partial \Omega$. The vector $\gamma^\pm(x) - x \in \mathbb{R}^2$ is pointing into $\Omega$ at the point $x \in \partial \Omega$. Since we use a positively oriented parametrization, the vectors $v(x), \gamma^\pm(x) - x$ form a positively oriented basis. We now note that $\ell^+, \ell^-$ form a positively oriented basis of the dual space to $\mathbb{R}^2$, and hence

$$\det \begin{pmatrix} \ell^+(v(x)) & \ell^+(\gamma^\pm(x) - x) \\ \ell^-(v(x)) & \ell^-(\gamma^\pm(x) - x) \end{pmatrix} > 0.$$ 

Since $\partial_\theta \ell^\pm(x) = \ell^\mp(v(x))$, this gives (2.4). The identity (2.5) follows from (2.4), and (2.6) is verified by a direct computation. \qed

**Lemma 2.2.** Assume that $\Omega$ is $\lambda$-simple. Then for all $y \in \partial \Omega$ and $x \in \Omega$

$$\nu^+(y)\ell^-(x - y) > 0 \quad \text{or} \quad \nu^-(y)\ell^+(x - y) < 0 \quad (\text{or both}). \quad (2.7)$$

**Proof.** Let $\Gamma^\pm_\lambda(y)$ be the sets defined in (1.7) and recall that they are open line segments with endpoints $y, \gamma^\pm(y)$. Then by (2.4),

$$\Omega \cap R^\pm(y) = \emptyset \quad \text{where} \quad R^\pm(y) := \{x \in \mathbb{R}^2 \mid \ell^\pm(x - y) = 0, \pm \nu^\pm(y)\ell^\mp(x - y) \leq 0\}.$$ 

The sets $R^\pm(y)$ are closed rays starting at $y$ when $\nu^\pm(y) \neq 0$ and lines passing through $y$ when $\nu^\pm(y) = 0$. Any continuous curve starting at the set of $x \in \mathbb{R}^2$ satisfying (2.7)
and ending in the complement of this set has to intersect $R^+(y) \cup R^-(y)$, as can be seen (in the case $\nu^\pm(y) \neq 0$) by applying the Intermediate Value Theorem to the pullback to that curve of the function $x \mapsto \max(\nu^+(y)\ell^- (x - y), -\nu^-(y)\ell^+(x - y))$. Thus, since $\Omega$ is connected and contains at least one point $x$ satisfying (2.7) (for instance, take any point in $\Gamma^\pm_\lambda(y)$), all points $x \in \Omega$ satisfy (2.7). \hfill \Box

2.1.2. Properties of pushforwards. We next show basic properties of pushforwards of smooth functions by the maps $\partial \Omega \ni x \mapsto \ell^\pm(x, \lambda)$, which are used in the proof of Lemma 4.8. Fix $\lambda \in (0, 1)$ such that $\Omega$ is $\lambda$-simple and define

$$
\ell^\pm_{\min} := \ell^\pm(x^\pm_{\min}), \quad \ell^\pm_{\max} := \ell^\pm(x^\pm_{\max}),
$$

(2.8)

so that $\ell^\pm$ maps $\partial \Omega$ onto the interval $[\ell^\pm_{\min}, \ell^\pm_{\max}]$. We again fix a positively oriented coordinate $\theta$ on $\partial \Omega$.

**Lemma 2.3.** 1. Assume that $f \in C^\infty(\partial \Omega)$ and define $\Pi^\pm_\lambda f \in \mathcal{E}'(\mathbb{R})$ by the formula

$$
\int_\mathbb{R} \Pi^\pm_\lambda f(s) \varphi(s) \, ds = \int_{\partial \Omega} f(x) \varphi(\ell^\pm(x)) \, d\theta(x) \quad \text{for all} \quad \varphi \in C^\infty(\mathbb{R}).
$$

(2.9)

Then $\text{supp} \Pi^\pm_\lambda f \subset [\ell^\pm_{\min}, \ell^\pm_{\max}]$ and

$$
\sqrt{(s - \ell^\pm_{\min})(\ell^\pm_{\max} - s)} \Pi^\pm_\lambda f(s) \in C^\infty([\ell^\pm_{\min}, \ell^\pm_{\max}]).
$$

(2.10)

2. Assume that $f \in C^\infty(\partial \Omega)$ and define the functions $\Upsilon^\pm_\lambda f$ on $(\ell^\pm_{\min}, \ell^\pm_{\max})$ by

$$
\Upsilon^\pm_\lambda f(s) := \sum_{x \in \partial \Omega, \ell^\pm(x) = s} f(x), \quad s \in (\ell^\pm_{\min}, \ell^\pm_{\max}).
$$

Then $\Upsilon^\pm_\lambda f \in C^\infty([\ell^\pm_{\min}, \ell^\pm_{\max}])$.

**Proof.** 1. The support property follows immediately from the definition: if $\text{supp} \varphi \cap [\ell^\pm_{\min}, \ell^\pm_{\max}] = \emptyset$, then $\varphi \circ \ell^\pm = 0$ on $\partial \Omega$ and thus $\int (\Pi^\pm_\lambda f) \varphi = 0$.

To show (2.10), we compute

$$
\Pi^\pm_\lambda f(s) = \sum_{x \in \partial \Omega, \ell^\pm(x) = s} \frac{f(x)}{|\partial_\theta \ell^\pm(x)|} \quad \text{for all} \quad s \in (\ell^\pm_{\min}, \ell^\pm_{\max}).
$$

(2.11)

It follows that $\Pi^\pm_\lambda f$ is smooth on the open interval $(\ell^\pm_{\min}, \ell^\pm_{\max})$. Next, note that (2.10) does not depend on the choice of the parametrization $\theta$ since changing the parametrization amounts to multiplying $f$ by a smooth positive function. Thus we can use the local coordinate $\theta = \theta^\pm_{\min}$ near $x^\pm_{\min}$ introduced in (2.1). With this choice we have $\ell^\pm(x) = \ell^\pm_{\min} + \theta^2$ and the formula (2.11) gives for $s$ near $\ell^\pm_{\min}$

$$
\Pi^\pm_\lambda f(s) = \frac{f(\sqrt{s - \ell^\pm_{\min}}) + f(-\sqrt{s - \ell^\pm_{\min}})}{2\sqrt{s - \ell^\pm_{\min}}}.
$$
where we view $f$ as a function of $\theta$. It follows that $\sqrt{s - \ell_{\text{min}}^\pm \Pi^\pm_\lambda f(s)}$ is smooth at the left endpoint of the interval $(\ell_{\text{min}}^\pm, \ell_{\text{max}}^\pm)$. Similar analysis shows that $\sqrt{\ell_{\text{max}}^\pm - s \Pi^\pm_\lambda f(s)}$ is smooth at the right endpoint of this interval.

2. This is proved similarly to part 1, where we no longer have $|\partial_\theta \ell^\pm(x)|$ in the denominator in (2.11).

2.1.3. Dynamics of the chess billiard. We now give a description of the dynamics of the orientation preserving diffeomorphism $b = \gamma^+ \circ \gamma^-$ in the presence of periodic points.

**Lemma 2.4.** Assume that $\Sigma^\lambda \neq \emptyset$ (see (1.5)). Then:

1. all periodic points of $b$ have the same minimal period;
2. for each $x \in \partial \Omega$, the trajectory $b^k(x)$ converges to $\Sigma^\lambda$ as $k \to \pm \infty$;
3. if $\partial_x b^n \neq 1$ on $\Sigma^\lambda$ where $n$ denotes the minimal period, then the set $\Sigma^\lambda$ is finite.

**Proof.** See for example [9, §1.1] or [32] for the proof of the first two claims. The last claim follows from the fact that $\Sigma^\lambda$ is the set of solutions to $b^n(x) = x$ and thus $\partial_x b^n(x) \neq 0$ implies that it consists of isolated points. □

We finally discuss the rotation number of $b$. Fix a positively oriented parametrization on $\partial \Omega$ which identifies it with the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\pi : \mathbb{R} \to \partial \Omega$ the covering map. Consider a lift of $b(\theta, \lambda)$ to $\mathbb{R}$, i.e. an orientation preserving diffeomorphism $b(\theta, \lambda) : \mathbb{R} \to \mathbb{R}$ such that

$$\pi(b(\theta, \lambda)) = b(\pi(\theta), \lambda) \quad \text{for all} \quad \theta \in \mathbb{R}.$$  

Denote by $b^k(\theta, \lambda)$ the $k$-th iterate of $b(\theta, \lambda)$. Define the rotation number of $b(\theta, \lambda)$ as

$$r(\lambda) := \lim_{k \to \infty} \frac{b^k(\theta, \lambda) - \theta}{k} \mod \mathbb{Z} \in \mathbb{R}/\mathbb{Z}. \quad (2.12)$$

The limit exists and is independent of the choice of $\theta \in \mathbb{R}$ and of the lift $b$. We refer to [32] for a proof of this fact as well of the following

**Lemma 2.5.** The rotation number $r(\lambda)$ is rational if and only if $\Sigma^\lambda \neq \emptyset$. In this case $r(\lambda) = \frac{q}{n} \mod \mathbb{Z}$ where $n > 0$ is the minimal period of the periodic points and $q \in \mathbb{Z}$ is coprime with $n$.

We remark that $b(\theta, \lambda)$ cannot have fixed points: indeed, if $x \in \partial \Omega$ and $b(x) = x$, then $\gamma^+(x) = \gamma^-(x)$ which is impossible. We then fix the lift $b$ for which

$$0 < b(0, \lambda) < 1. \quad (2.13)$$

With this choice we have $0 < b^k(0, \lambda) < k$ for all $k \geq 0$ and thus (2.12) defines the rotation number $r(\lambda)$ which satisfies $0 < r(\lambda) < 1$. 

2.2. Dependence on $\lambda$. We now discuss the dependence of the dynamics of the chess billiard map $b(\bullet, \lambda)$ on $\lambda$. We first give a stability result:

**Lemma 2.6.** The set of $\lambda \in (0, 1)$ satisfying the Morse–Smale conditions (see Definition 2) is open. Moreover, the maps $\gamma^\pm(x, \lambda)$ and $b(x, \lambda)$, as well as the sets $\Sigma_\lambda$, depend smoothly on $\lambda$ as long as $\lambda$ satisfies the Morse–Smale conditions.

**Proof.** Assume that $\lambda_0$ satisfies the Morse–Smale conditions. We need to show that all $\lambda$ close enough to $\lambda_0$ satisfy this condition as well. From (1.2) we see that the functions $\ell^\pm(x, \lambda)$ depend smoothly on $x \in \partial \Omega, \lambda \in (0, 1)$. Therefore, $\Omega$ is $\lambda$-simple for $\lambda$ close to $\lambda_0$. Moreover, $\gamma^\pm(x, \lambda)$ and $b(x, \lambda)$ depend smoothly on $\lambda$ as long as $\Omega$ is $\lambda$-simple.

Next, let $m > 0$ be the number of points in $\Sigma_{\lambda_0}$ and let $n$ be their minimal period under $b(\bullet, \lambda_0)$. Since $\partial_x b^n(x, \lambda_0) \neq 1$ on $\Sigma_{\lambda_0}$, by the Implicit Function Theorem for $\lambda$ close to $\lambda_0$ the equation $b^n(x, \lambda) = x$ has exactly $m$ solutions, which depend smoothly on $\lambda$. It follows that $\lambda$ satisfies the Morse–Smale conditions. $\Box$

Lemmas 2.5 and 2.6 imply in particular that when $\lambda_0$ satisfies the Morse–Smale conditions, the rotation number $r$ is constant in a neighborhood of $\lambda_0$. A partial converse to this fact is given by the second part of the following

**Lemma 2.7.** Assume that $J \subset (0, 1)$ is an open interval such that $\Omega$ is $\lambda$-simple for each $\lambda \in J$. Then:

1. $r(\lambda)$ is a continuous increasing function of $\lambda \in J$;
2. if $r$ is constant on $J$, then this constant is rational and the Morse–Smale conditions hold for Lebesgue almost every $\lambda \in J$.

**Proof.** 1. Fix a positively oriented coordinate $\theta$ on $\partial \Omega$. Using (1.3), (2.6) we compute

$$\partial_\lambda \gamma^\pm(x, \lambda) = \frac{\partial_x \ell^\pm(x - \gamma^\pm(x, \lambda), \lambda)}{\partial_\theta \ell^\pm(\gamma^\pm(x, \lambda), \lambda)} = \frac{\ell^\pm(x - \gamma^\pm(x, \lambda), \lambda)}{2\lambda(1 - \lambda^2) \partial_\theta \ell^\pm(\gamma^\pm(x, \lambda), \lambda)}.$$ 

By (2.4) and (2.5) we have

$$\partial_\lambda \gamma^+ > 0, \quad \partial_\lambda \gamma^- < 0.$$ 

We then compute

$$\partial_\lambda b(x, \lambda) = \partial_\lambda \gamma^+(-\gamma^-(x, \lambda), \lambda) + \partial_\theta \gamma^+(\gamma^-(x, \lambda), \lambda) \partial_\lambda \gamma^-(x, \lambda).$$

Since $\partial \Omega \ni x \mapsto \gamma^+(x, \lambda) \in \partial \Omega$ is orientation reversing this gives

$$\partial_\lambda b(x, \lambda) > 0 \quad \text{for all} \quad x \in \partial \Omega, \lambda \in J. \quad (2.14)$$

Fix the lift $b(\theta, \lambda)$ satisfying (2.13). Then (2.14) gives $\partial_\lambda b(\theta, \lambda) > 0$. This implies that for each two points $\lambda_1 < \lambda_2$ in $J$ and every $k \geq 1$

$$b^k(\theta, \lambda_1) < b^k(\theta, \lambda_2).$$
Recalling the definition (2.12) of \( r(\lambda) \), we see that \( r(\lambda_1) \leq r(\lambda_2) \), that is \( r(\lambda) \) is an increasing function of \( \lambda \in J \).

2. We now show that \( r(\lambda) \) is a continuous function of \( \lambda \in J \). Fix arbitrary \( \lambda_0 \in J \) and \( \varepsilon > 0 \); since \( r \) is an increasing function it suffices to show that there exists \( \delta > 0 \) such that

\[
\lambda \in J \quad \text{if necessary) we may assume that } \lambda \in J \quad \text{and } \varepsilon > 0; \quad \text{since } r \text{ is an increasing function it suffices to show that there exists } \delta > 0 \text{ such that }
\]

\[
r(\lambda_0 + \delta) < r(\lambda_0) + \varepsilon, \quad r(\lambda_0 - \delta) > r(\lambda_0) - \varepsilon.
\]

We show the first statement, with the second one proved similarly. Choose a rational number \( \frac{q}{n} \in (r(\lambda_0), r(\lambda_0) + \varepsilon) \) where \( n \in \mathbb{N} \) and \( q \in \mathbb{Z} \) are coprime. Since \( r(\lambda_0) < \frac{q}{n} \), the definition (2.12) implies that there exists \( k_0 > 0 \) such that

\[
\frac{b^{k_0}(0, \lambda_0)}{nk_0} < \frac{q}{n},
\]

that is \( b^{k_0n}(0, \lambda_0) < k_0q \). Since \( b^{k_0n}(0, \lambda) \) is continuous in \( \lambda \), we can choose \( \delta > 0 \) small enough so that

\[
b^{k_0n}(0, \lambda_0 + \delta) < k_0q.
\]

By induction on \( j \) we see that

\[
b^{jk_0n}(0, \lambda_0 + \delta) < jk_0q \quad \text{for all } j \geq 1.
\]

(2.15)

3. Assume now that \( r \) is constant on \( J \). We first show that this constant is a rational number. Assume the contrary and take arbitrary \( \lambda_0 \in J \). By (2.14) (shrinking \( J \) slightly if necessary) we may assume that \( \partial_\lambda b(x, \lambda) \geq c > 0 \) for some \( c > 0 \) and all \( x \in \partial \Omega, \lambda \in J \). Then \( \partial_\lambda b^p(x, \lambda) \geq c \) for all \( n \geq 0 \) as well. Fix \( \varepsilon > 0 \) such that \( \lambda_1 := \lambda_0 + \varepsilon/c \) lies in \( J \). Then \( b^n(\theta, \lambda_1) \geq b^n(\theta, \lambda_0) + \varepsilon \) for all \( \theta \in \mathbb{R} \).

Fix arbitrary \( x_0 = \pi(\theta_0) \in \partial \Omega, \theta_0 \in \mathbb{R} \). Since \( r(\lambda_0) \) is irrational and \( b(\bullet, \lambda_0) \) is smooth, by Denjoy’s Theorem [9, \S I.2] every orbit of \( b(\bullet, \lambda_0) \) is dense, in particular the orbit \( \{b^n(x_0, \lambda_0)\}_{n \geq 1} \) intersects the \( \varepsilon \)-sized interval on \( \partial \Omega \) whose right endpoint is \( x_0 \). That is, there exist \( n \in \mathbb{N}, m \in \mathbb{Z} \) such that

\[
\theta_0 + m - \varepsilon \leq b^n(\theta_0, \lambda_0) \leq \theta_0 + m.
\]

It follows that

\[
b^n(\theta_0, \lambda_0) \leq \theta_0 + m \leq b^n(\theta_0, \lambda_0) + \varepsilon \leq b^n(\theta_0, \lambda_1).
\]

By the Intermediate Value Theorem, there exists \( \lambda \in [\lambda_0, \lambda_1] \subset J \) such that \( b^n(\theta_0, \lambda) = \theta_0 + m \). Then \( x_0 = \pi(\theta_0) \) is a periodic orbit of \( b(\bullet, \lambda) \), which contradicts our assumption that \( r(\lambda) \) is irrational for all \( r \in J \).
4. Under the assumption of Step 3, we now have $r(\lambda) = \frac{q}{n} \mod \mathbb{Z}$ for some coprime $q \in \mathbb{Z}$, $n \in \mathbb{N}$ and all $\lambda \in J$. By Lemma 2.5, for each $\lambda \in J$ the set of periodic points $\Sigma_\lambda$ is nonempty and each such point has minimal period $n$. Define

$$\Sigma_J := \{ (x, \lambda) \mid \lambda \in J, \ x \in \Sigma_\lambda \} = \{ (x, \lambda) \in \partial \Omega \times J \mid b^n(x, \lambda) = x \}.$$ 

From (2.14) we see that $\partial \lambda b^n(x, \lambda) > 0$ for all $x \in \partial \Omega$, $\lambda \in J$. Shrinking $J$ if needed, we may assume that $\Sigma_J$ is a one-dimensional submanifold of $J \times \partial \Omega$ projecting diffeomorphically onto the $x$ variable, that is $\Sigma_J = \{ (x, \psi(x)) \mid x \in U \}$ for some open set $U \subset \partial \Omega$ and smooth function $\psi : U \to J$, $\partial_x \psi(x) = (1 - \partial_x b^n(x, \lambda))/\partial_\lambda b^n(x, \lambda)$, $\lambda = \psi(x)$. Then $\lambda \in J$ satisfies the Morse–Smale conditions if and only if $\lambda$ is a regular value of $\psi$, which by the Morse–Sard theorem happens for Lebesgue almost every $\lambda \in J$. □

2.3. Escape functions. We now construct an adapted parametrization of $\partial \Omega$ and a family of escape functions, which are used § 5 below. Throughout this section we assume that $\lambda \in (0, 1)$ satisfies the Morse–Smale conditions of Definition 2. Recall the sets $\Sigma^\pm_\lambda$ of attractive/repulsive periodic points of the map $b(\bullet, \lambda)$ defined in (1.6). Let $n \in \mathbb{N}$ be the minimal period of the corresponding trajectories of $b$.

We first construct a parametrization of $\partial \Omega$ with a bound on $\partial_x b^n|_{\Sigma^\pm_\lambda}$ rather than on the derivative of the $n$-th iterate $\partial_x b^n|_{\Sigma^\pm_\lambda}$:

**Lemma 2.8.** Let $\Sigma^\pm_\lambda$ be given by (1.6). There exists a positively oriented coordinate $\theta : \partial \Omega \to \mathbb{S}^1$ such that, taking derivatives on $\partial \Omega$ with respect to $\theta$,

$$\partial_x b(x, \lambda) < 1 \quad \text{for all} \quad x \in \Sigma^+_\lambda,$$

$$\partial_x b(x, \lambda) > 1 \quad \text{for all} \quad x \in \Sigma^-_\lambda.$$  

(2.17)

**Proof.** Fix any Riemannian metric $g_0$ on $\partial \Omega$ and consider the metric $g$ on $\partial \Omega$ given by

$$|v|_g(x) := \sum_{j=0}^{n-1} |\partial_x b^j(x)v|_{g_0(b^j(x))} \quad \text{for all} \quad (x, v) \in T(\partial \Omega).$$

We have for all $(x, v) \in T(\partial \Omega)$

$$|\partial_x b(x)v|_{g(b(x))} - |v|_g(x) = |\partial_x b^n(x)v|_{g_0(b^n(x))} - |v|_{g_0(x)}.$$ 

Thus by (1.6) we have for $v \neq 0$

$$|\partial_x b(x)v|_{g(b(x))} < |v|_g(x) \quad \text{when} \quad x \in \Sigma^+_\lambda;$$

$$|\partial_x b(x)v|_{g(b(x))} > |v|_g(x) \quad \text{when} \quad x \in \Sigma^-_\lambda.$$ 

It remains to choose the coordinate $\theta$ so that $|\partial \theta|_g$ is constant. □
We next use the global dynamics of \( b(\bullet, \lambda) \) described in Lemma 2.4 to construct an *escape function* in Lemma 2.9 below. Fix a parametrization on \( \partial \Omega \) which satisfies (2.17) and denote by
\[
\Sigma^\pm_\lambda(\delta) \subset \partial \Omega
\]
the open \( \delta \)-neighborhoods of the sets \( \Sigma^\pm_\lambda \) with respect to this parametrization. Here \( \delta > 0 \) is a constant small enough so that the closures \( \Sigma^+_\lambda(\delta) \) and \( \Sigma^-_\lambda(\delta) \) do not intersect each other. We also choose \( \delta \) small enough so that
\[
b(\Sigma^+_\lambda(\delta)) \subset \Sigma^+_\lambda(\delta), \quad b^{-1}(\Sigma^-_\lambda(\delta)) \subset \Sigma^-_\lambda(\delta);
\]
this is possible by (2.17) and since \( \Sigma^\pm_\lambda \) are \( b \)-invariant.

**Lemma 2.9.** Let \( \alpha_+ < \alpha_- \) be two real numbers. Then there exists a function \( g \in C^\infty(\partial \Omega; \mathbb{R}) \) such that:

1. \( g(b(x)) \leq g(x) \) for all \( x \in \partial \Omega \);
2. \( g(b(x)) < g(x) \) for all \( x \in \partial \Omega \setminus (\Sigma^+_\lambda(\delta) \cup \Sigma^-_\lambda(\delta)) \);
3. \( g(x) \geq \alpha_+ \) for all \( x \in \partial \Omega \);
4. \( g(x) \geq \alpha_- \) for all \( x \in \partial \Omega \setminus \Sigma^+_\lambda(\delta) \);
5. \( g = \alpha_+ \) on some neighbourhood of \( \Sigma^+_\lambda \);
6. for \( M \gg 1 \), \( M(g(b(x)) - g(x)) + g(x) \leq \alpha_+ \) for all \( x \in \partial \Omega \setminus \Sigma^-_\lambda(\delta) \).

See Figure 5.

**Remark.** We note that the same construction works for \( b^{-1} \) with the roles of \( \Sigma^\pm_\lambda \) reversed. Hence for any real numbers \( \alpha_- < \alpha_+ \) we can find \( g \in C^\infty(\partial \Omega; \mathbb{R}) \) such that

1. \( g(x) \leq g(b(x)) \) for all \( x \in \partial \Omega \);
(2) \( g(x) < g(b(x)) \) for all \( x \in \partial \Omega \setminus (\Sigma^+_\lambda(\delta) \cup \Sigma^-_\lambda(\delta)) \);
(3) \( g(x) \geq \alpha_- \) for all \( x \in \partial \Omega \);
(4) \( g(x) \geq \alpha_+ \) for all \( x \in \partial \Omega \setminus \Sigma^-_\lambda(\delta) \);
(5) \( g = \alpha_- \) on some neighbourhood of \( \Sigma^-_\lambda \);
(6) for \( M \gg 1 \), \( M(g(x) - g(b(x))) + g(b(x)) \leq \alpha_- \) for all \( x \in \partial \Omega \setminus \Sigma^+_\lambda(\delta) \).

**Proof.** In view of (2.18) there exists \( 0 < \delta_1 < \delta \) such that
\[
b(\Sigma^+_\lambda(\delta)) \subset \Sigma^+_\lambda(\delta_1). \tag{2.19}
\]

1. We first show that there exists \( N \geq 0 \) such that
\[
b^N(\partial \Omega \setminus \Sigma^-_\lambda(\delta)) \subset \Sigma^+_\lambda(\delta_1). \tag{2.20}
\]
We argue by contradiction. Assume that (2.20) does not hold for any \( N \). Then there exist sequences
\[
x_j \in \partial \Omega \setminus \Sigma^-_\lambda(\delta), \quad m_j \to \infty, \quad b^{m_j}(x_j) \not\in \Sigma^+_\lambda(\delta_1). \tag{2.21}
\]
Passing to subsequences, we may assume that \( x_j \to x_\infty \) for some \( x_\infty \in \partial \Omega \). Since \( x_j \not\in \Sigma^-_\lambda(\delta) \), we have \( x_\infty \not\in \Sigma^-_\lambda(\delta) \) as well. Then by (2.18) the trajectory \( b^k(x_\infty) \), \( k \geq 0 \), does not intersect \( \Sigma^-_\lambda(\delta) \). On the other hand, by Lemma 2.4 this trajectory converges to \( \Sigma_\lambda = \Sigma^-_\lambda \cup \Sigma^+_\lambda \) as \( k \to \infty \). Thus this trajectory converges to \( \Sigma^+_\lambda \), in particular there exists \( k \geq 0 \) such that \( b^k(x_\infty) \in \Sigma^+_\lambda(\delta_1) \).

Since \( x_j \to x_\infty \), we have \( b^k(x_j) \to b^k(x_\infty) \) as \( j \to \infty \). Since \( \Sigma^+_\lambda(\delta_1) \) is an open set, there exists \( j \geq 0 \) such that \( m_j \geq k \) and \( b^k(x_j) \in \Sigma^+_\lambda(\delta_1) \). But then by (2.19) we have \( b^{m_j}(x_j) \in \Sigma^+_\lambda(\delta_1) \) which contradicts (2.21).

2. Choose \( N \) such that (2.20) holds and fix a cutoff function
\[
\chi_+ \in C^\infty_c(\Sigma^+_\lambda(\delta); [0, 1]), \quad \chi_+ = 1 \quad \text{on} \quad \Sigma^+_\lambda(\delta_1).
\]
Define the function \( \tilde{g} \in C^\infty(\partial \Omega; \mathbb{R}) \) as an ergodic average of \( \chi_+ \):
\[
\tilde{g}(x) := \frac{1}{N} \sum_{j=0}^{N-1} \chi_+(b^j(x)) \quad \text{for all} \quad x \in \partial \Omega.
\]
It follows from the definition and (2.19) that
\[
0 \leq \tilde{g}(x) \leq 1 \quad \text{for all} \quad x \in \partial \Omega,
\]
\[
\tilde{g}(x) = 1 \quad \text{for all} \quad x \in \Sigma^+_\lambda(\delta_1),
\]
\[
\tilde{g}(x) \leq 1 - \frac{1}{N} \quad \text{for all} \quad x \in \partial \Omega \setminus \Sigma^+_\lambda(\delta).
\]

Next, we compute
\[
\tilde{g}(b(x)) - \tilde{g}(x) = \frac{1}{N}(\chi_+(b^N(x)) - \chi_+(x)).
\]
It follows that
\[ \tilde{g}(b(x)) \geq \tilde{g}(x) \quad \text{for all} \quad x \in \partial \Omega, \]
\[ \tilde{g}(b(x)) = \tilde{g}(x) + \frac{1}{N} \quad \text{for all} \quad x \in \partial \Omega \setminus (\Sigma_+^\delta(\delta) \cup \Sigma_-^\delta(\delta)). \]  
(2.23)

Indeed, take arbitrary \( x \in \partial \Omega \). We have \( \chi^+(x) = 0 \) unless \( x \in \Sigma_+^\delta(\delta) \). By (2.20), we have \( \chi^+(b^N(x)) = 1 \) unless \( x \in \Sigma_-^\delta(\delta) \). Recalling that \( 0 \leq \chi^+ \leq 1 \) and \( \Sigma_+^\delta(\delta) \cap \Sigma_-^\delta(\delta) = \emptyset \), we get (2.23).

3. Now put
\[ g(x) := N\alpha_- - (N - 1)\alpha_+ - N(\alpha_- - \alpha_+)\tilde{g}(x). \]  
(2.24)

Using (2.22) and (2.23), we see that the function \( g \) satisfies the first five properties, with the following quantitative versions of parts (2) and (5):
\[ g(b(x)) - g(x) = \alpha_+ - \alpha_- < 0 \quad \text{for all} \quad x \in \partial \Omega \setminus (\Sigma_+^\delta(\delta) \cup \Sigma_-^\delta(\delta)), \]
\[ g(x) = \alpha_+ \quad \text{for all} \quad x \in \Sigma_+^\delta(\delta_1). \]  
(2.25)

To prove part (6) we first use (2.24) and (2.25) to see that for all \( M \geq N \) and \( x \in \partial \Omega \setminus (\Sigma_+^\delta(\delta) \cup \Sigma_-^\delta(\delta)) \),
\[ M(g(b(x)) - g(x)) + g(x) \leq \alpha_+. \]  
(2.26)

To establish (2.26) for \( x \in \Sigma_+^\delta(\delta) \) we use (2.19) and the fact that \( g|_{\Sigma_+^\delta(\delta_1)} = \alpha_+ \) by (2.25). Then, for \( M \geq 1 \) and \( x \in \Sigma_+^\delta(\delta) \), property (1) gives
\[ M(g(b(x)) - g(x)) + g(x) \leq g(b(x)) = \alpha_+, \]
which completes the proof of the lemma. \( \Box \)

Remark. We discuss here the dependence of the objects in this section on the parameter \( \lambda \). The parametrization \( \theta \) constructed in Lemma 2.8 depends smoothly on \( \lambda \) as follows immediately from its construction (recalling from the proof of Lemma 2.6 that the period \( n \) is locally constant in \( \lambda \)). Next, for each \( \lambda_0 \in (0, 1) \) satisfying the Morse–Smale conditions there exists a neighborhood \( U(\lambda_0) \) such that we can construct a function \( g(x, \lambda) \) for each \( \lambda \in U(\lambda_0) \) satisfying the conclusions of Lemma 2.9 in such a way that it is smooth in \( \lambda \). Indeed, the sets \( \Sigma_\lambda^\pm \) depend smoothly on \( \lambda \) by Lemma 2.6, so the cutoff function \( \chi^+ \) can be chosen \( \lambda \)-independent. The function \( g(x, \lambda) \) is constructed explicitly using this cutoff, the map \( b(\bullet, \lambda) \), and the number \( N \). The latter can be chosen \( \lambda \)-independent as well: if (2.20) holds for some \( \lambda \), then it holds with the same \( N \) and all nearby \( \lambda \).

2.4. Domains with corners. We now discuss the case when the boundary of \( \partial \Omega \) has corners. This includes the situation when \( \partial \Omega \) is a convex polygon, which is the setting of the experiments. Our results do not apply to such domains, however they apply to appropriate ‘roundings’ of these domains described below.
We first define domains with corners. Let $\Omega \subset \mathbb{R}^2$ be an open set of the form

$$\Omega = \{ x \in \mathbb{R}^2 \mid F_1(x) > 0, \ldots, F_k(x) > 0 \}$$

where $F_1, \ldots, F_k : \mathbb{R}^2 \to \mathbb{R}$ are $C^\infty$ functions such that:

1. the set $\overline{\Omega} := \{ F_1 \geq 0, \ldots, F_k \geq 0 \}$ is compact and simply connected, and
2. for each $x \in \overline{\Omega}$, at most 2 of the functions $F_1, \ldots, F_k$ vanish at $x$.

If only one of the functions $F_1, \ldots, F_k$ vanishes at $x \in \overline{\Omega}$, then we call $x$ a regular point of the boundary $\partial \Omega := \overline{\Omega} \setminus \Omega$. If two of the functions $F_1, \ldots, F_k$ vanish at $x \in \overline{\Omega}$, then we call $x$ a corner of $\Omega$. We make the following natural nondegeneracy assumptions:

1. if $x \in \partial \Omega$ is a regular point and $F_j(x) = 0$, then $dF_j(x) \neq 0$;
2. if $x \in \partial \Omega$ is a corner and $F_j(x) = F_j'(x) = 0$ where $j \neq j'$, then $dF_j(x), dF_{j'}(x)$ are linearly independent.

We call $\Omega$ a domain with corners if it satisfies the assumptions (1)–(4) above.

Since $\Omega$ is simply connected, the boundary $\partial \Omega$ is a Lipschitz continuous piecewise smooth curve. We parametrize $\partial \Omega$ in the positively oriented direction by a Lipschitz continuous map

$$\theta \in S^1 := \mathbb{R}/\mathbb{Z} \mapsto x(\theta) \in \partial \Omega \subset \mathbb{R}^2$$

where the corners are given by $x(\theta_j)$ for some $\theta_1 < \cdots < \theta_m$ and the map (2.27) is smooth on each interval $[\theta_j, \theta_{j+1}]$. See Figure 6.

We next extend the concept of $\lambda$-simplicity to domains with corners. Let $\ell \in C^\infty(\mathbb{R}^2; \mathbb{R})$ and $x = x(\theta_j)$ be a corner of $\Omega$. Consider the one-sided derivatives $\partial_\theta (\ell \circ x)(\theta_j \pm 0)$. There are three possible cases:

1. Both derivatives are nonzero and have the same sign – then we call $x$ not a critical point of $\ell$;
(2) Both derivatives are nonzero and have opposite signs – then we call $x$ a nondegenerate critical point of $\ell$.

(3) At least one of the derivatives is zero – then we call $x$ a degenerate critical point of $\ell$.

If $x = x(\theta)$ is instead a regular point of the boundary, then we use the standard definition of critical points: $x$ is a critical point of $\ell$ if $\partial_\theta (\ell \circ x)(\theta) = 0$, and a critical point is nondegenerate if $\partial^2_\theta (\ell \circ x)(\theta) \neq 0$. With the above convention for critical points, we follow Definition 1: we say that a domain with corners $\Omega$ is $\lambda$-simple if each of the functions $\ell^\pm(\cdot, \lambda)$ defined in (1.2) has exactly 2 critical points on $\partial \Omega$, which are both nondegenerate.

If $\Omega$ is $\lambda$-simple, then the involutions $\gamma^\pm(\cdot, \lambda) : \partial \Omega \to \partial \Omega$ from (1.3) are well-defined and Lipschitz continuous. Thus $b = \gamma^+ \circ \gamma^-$ is an orientation preserving bi-Lipschitz homeomorphism of $\partial \Omega$. We now revise the Morse–Smale conditions of Definition 2 as follows:

**Definition 3.** Let $\Omega$ be a domain with corners. We say that $\lambda \in (0, 1)$ satisfies the Morse–Smale conditions if:

1. $\Omega$ is $\lambda$-simple;
2. the set $\Sigma_\lambda$ of periodic points of the map $b(\cdot, \lambda)$ is nonempty;
3. the set $\Sigma_\lambda$ does not contain any corners of $\Omega$;
4. for each $x \in \Sigma_\lambda$, $\partial_x b^n(x, \lambda) \neq 1$ where $n$ is the minimal period.

The new condition (3) in Definition 3 ensures that $b$ is smooth near the $\gamma^\pm$-invariant set $\Sigma_\lambda$, so condition (4) makes sense. Without this condition we could have trajectories of $b$ converging to a corner, see Figure 6.

We finally show that if $\Omega$ is a domain with corners satisfying the Morse–Smale conditions of Definition 3 then an appropriate ‘rounding’ of $\Omega$ satisfies the Morse–Smale conditions of Definition 2:

**Proposition 2.10.** Let $\Omega$ be a domain with corners and $\lambda \in (0, 1)$ satisfy the Morse–Smale conditions for $\Omega$. Then there exists $\varepsilon > 0$ such that for any open simply connected $\hat{\Omega} \subset \mathbb{R}^2$ with $C^\infty$ boundary and such that:

- $\hat{\Omega}$ is an $\varepsilon$-rounding of $\Omega$ in the sense that for each $x \in \mathbb{R}^2$ which lies distance $\geq \varepsilon$ from all the corners of $\Omega$, we have $x \in \Omega \iff x \in \hat{\Omega}$; and
- the domain $\hat{\Omega}$ is $\lambda$-simple in the sense of Definition 1,

the Morse–Smale conditions is satisfied for $\lambda$ and $\hat{\Omega}$.

**Proof.** Fix a parametrization $x(\theta)$ of $\partial \Omega$ as in (2.27). Take a parametrization

$$\theta \in S^1 \mapsto x(\theta) \in \partial \hat{\Omega}$$
which coincides with $x(\theta)$ except $\varepsilon$-close to the corners:

$$\hat{x}(\theta) = x(\theta) \text{ for all } \theta \notin \bigcup_{j=1}^{m} I_j(\varepsilon), \quad I_j(\varepsilon) := [\theta_j - C\varepsilon, \theta_j + C\varepsilon].$$

(2.28)

Here $C$ denotes a constant depending on $\Omega$ and the parametrization $x(\theta)$, but not on $\hat{\Omega}$ or $\varepsilon$, whose precise value might change from place to place in the proof.

Denote by $\gamma^\pm, \hat{\gamma}^\pm$ the involutions (1.3) corresponding to $\Omega, \hat{\Omega}$, and consider them as homeomorphisms of $S^1$ using the parametrizations $x, \hat{x}$. Then by (2.28)

$$\gamma^\pm(\theta) = \hat{\gamma}^\pm(\theta) \text{ if } \theta, \gamma^\pm(\theta) \notin \bigcup_{j=1}^{m} I_j(\varepsilon).$$

(2.29)

Let $b = \gamma^+ \circ \gamma^-, \hat{b} = \hat{\gamma}^+ \circ \hat{\gamma}^-$ be the chess billiard maps of $\Omega, \hat{\Omega}$ and $\Sigma_\lambda, \hat{\Sigma}_\lambda$ be the corresponding sets of periodic trajectories. Choose $\varepsilon > 0$ such that the intervals $I_j(\varepsilon)$ do not intersect $\Sigma_\lambda$; this is possible since $\Sigma_\lambda$ does not contain any corners of $\Omega$. Since $\Sigma_\lambda$ is invariant under $\gamma^\pm$, we see from (2.29) that $b = \hat{b}$ in a neighborhood of $\Sigma_\lambda$. That is, the periodic points for the original domain $\Omega$ are also periodic points for the rounded domain $\hat{\Omega}$, with the same period $n$. It also follows that $\partial_x \hat{b}^n(x, \lambda) = \partial_x b^n(x, \lambda) \neq 1$ for all $x \in \Sigma_\lambda$.

It remains to show that $\hat{\Sigma}_\lambda \subset \Sigma_\lambda$, that is the rounding does not create any new periodic points for $\hat{b}$. Note that all periodic points have the same period $n$, and it is enough to show that

$$\hat{b}^n(\theta) \neq \theta \text{ for all } \theta \in \bigcup_{j=1}^{m} I_j(\varepsilon).$$

(2.30)

From (2.29), the monotonicity of $\gamma^\pm, \hat{\gamma}^\pm$, and the Lipschitz continuity of $\gamma^\pm$ we have

$$|\gamma^\pm(\theta) - \hat{\gamma}^\pm(\theta)| \leq C\varepsilon \text{ for all } \theta \in S^1.$$  

Iterating this and using the Lipschitz continuity of $\hat{\gamma}^\pm$ again, we get

$$|b^n(\theta) - \hat{b}^n(\theta)| \leq C\varepsilon \text{ for all } \theta \in S^1.$$  

Since $b^n(\theta_j) \neq \theta_j$ for all $j = 1, \ldots, m$, taking $\varepsilon$ small enough we get (2.30), finishing the proof. \[
\]

2.5. Examples of Morse–Smale chess billiards. Here we present two examples of Morse–Smale chess billiards. See Figures 7 and 8.

**Example 1.** For $\alpha \in (0, \frac{\pi}{2})$, let $\Omega_\alpha \subset \mathbb{R}^2$ be the open square with vertices $(0, 0), \ (\cos \alpha, \sin \alpha), \ \sqrt{2}(\cos(\alpha + \frac{\pi}{4}), \sin(\alpha + \frac{\pi}{4})), \ (\cos(\alpha + \frac{\pi}{2}), \sin(\alpha + \frac{\pi}{2}))$. We parametrize $\partial \Omega_\alpha$ by $\theta \in \mathbb{R}/4\mathbb{Z}$ so that the parametrization $x(\theta)$ is affine on each side of the square
and the vertices of the square listed above correspond to $\theta = 0, 1, 2, 3$ respectively. For $\lambda \in (0,1)$, we define

$$\beta \in (0, \pi/2), \quad \tan \beta = \sqrt{1-\lambda^2}/\lambda, \quad t_1 := \tan(\beta - \alpha), \quad t_2 := \tan(\beta + \alpha).$$

We will show that if

$$0 < \alpha < \pi/8, \quad \pi/4 - \alpha < \beta < \pi/4 + \alpha,$$

or equivalently

$$0 < \alpha < \pi/8, \quad \cos(\pi/4 + \alpha) < \lambda < \cos(\pi/4 - \alpha),$$

then $\lambda$ and $\Omega_\alpha$ satisfy the Morse–Smale conditions (Definition 3). Moreover, for $\alpha, \lambda$ satisfying (2.31), we have

$$\Sigma_\lambda = \left\{ x(\theta) \middle| \theta = \frac{1-t_1}{t_2-t_1}, \quad 2 + \frac{1-t_1}{t_2-t_1}, \quad 1 + \frac{t_1(t_2-1)}{t_2-t_1}, \quad 3 + \frac{t_1(t_2-1)}{t_2-t_1} \right\},$$

and the rotation number $r(\lambda) = \frac{1}{2}$.

In fact, assume $\alpha, \lambda$ satisfy (2.31), then $\ell^+(\bullet, \lambda)$ has exactly two nondegenerate critical points $x(0), x(2)$ on $\partial \Omega_\alpha$; $\ell^-(\bullet, \lambda)$ also has two nondegenerate critical points $x(1), x(3)$ on $\partial \Omega$. This shows that $\Omega_\alpha$ is $\lambda$-simple. We also have

$$b^2(\theta) = (t_1/t_2)^2 \theta + (t_1 + t_2)(1-t_1)/t_2^2, \quad \theta \in [0,1].$$

By solving $b^2(\theta_0) = \theta_0, \theta_0 \in [0,1]$, we find $\theta_0 = (1-t_1)/(t_2-t_1)$ and

$$\{ x(\theta) \mid \theta = \theta_0, \gamma^-(\theta_0), b(\theta_0), \gamma^+(\theta_0) \} \subset \Sigma_\lambda.$$

On the other hand, suppose $\theta_1 \in \mathbb{R}/4\mathbb{Z}$ and $x(\theta_1) \in \Sigma_\lambda$. By Lemma 2.4, we have $b^2(\theta_1) = \theta_1$.

Figure 7. Chess billiards in $\Omega_\alpha$ in Example 1, on the left, and for $T_d$ in Example 2, on the right.
Notice that $[0, 1] \cup \gamma^+[0, 1] \cup b^{-1}[0, 1] \cup \gamma^+b^{-1}[0, 1] = \mathbb{R}/4\mathbb{Z} \simeq \partial \Omega_\alpha$. If $\theta_1 \in [0, 1]$, by the definition of $\theta_0$, we must have $\theta_1 = \theta_0$. If $\theta_1 \in \gamma^+[0, 1]$, then there exists $\theta_2 \in [0, 1]$ such that $\theta_1 = \gamma^+(\theta_2)$. Use the assumption $b^2(\theta_1) = \theta_1$ and we find
\[ b^2\gamma^+(\theta_2) = \gamma^+(\theta_2) \Rightarrow \gamma^+b^{-2}(\theta_2) = \gamma^+(\theta_2) \Rightarrow b^2(\theta_2) = \theta_2 \Rightarrow \theta_2 = \theta_0. \]
Therefore we must have $\theta_1 = \gamma^+(\theta_0)$. Similarly, if $\theta_1 \in b^{-1}[0, 1]$ or $\gamma^+b^{-1}[0, 1]$, then there exists $\theta_2 \in [0, 1]$ such that $\theta_1 = \gamma^+(\theta_2)$ and we find $b^2(\theta_2) = \theta_2$, $\theta_2 = \gamma^+(\theta_0)$. As a result, we have
\[ \Sigma_\lambda \subset \{ x(\theta) | \theta = \theta_0, \gamma^-(\theta_0), b(\theta_0), \gamma^+(\theta_0) \}. \]
Thus we have (2.32).

Near $\Sigma_\lambda$, the chess billiard map $b(\theta)$ is given by
\[ b(\theta) = \begin{cases} (t_1/t_2)\theta + (1 - t_1)/t_2 + 2, & 0 \leq \theta < 1, \\ (t_2/t_1)\theta - t_2/t_1 - t_2 + 4, & 1 + t_1(1 - t_2^{-1}) \leq \theta < 1 + t_1, \\ (t_1/t_2)\theta + (1 - 3t_1)/t_2, & 2 \leq \theta < 3, \\ (t_2/t_1)\theta - 3t_2/t_1 - t_2 + 2, & 3 + t_1(1 - t_2^{-1}) \leq \theta < 3 + t_1. \end{cases} \]
Thus
\[ \partial_b b^2(\theta) = t_1^2/t_2^2 < 1, \quad \theta = \theta_0, b(\theta_0); \quad \partial_b b^2(\theta) = t_2^2/t_1^2 > 1, \quad \theta = \gamma^-(\theta_0), \gamma^+(\theta_0). \]

We have now checked that under the condition (2.31), $\Omega_\alpha$ and $\lambda$ satisfy all conditions in Definition 3.

**Example 2.** Let $\mathcal{T}_d \subset \mathbb{R}^2$ be the open trapezium with vertices $(0, 0)$, $(1 + d, 0)$, $(1, 1)$, $(0, 1)$, $d > 0$. We parametrize $\partial \mathcal{T}_d$ by $x : \mathbb{R}/4\mathbb{Z} \to \partial \mathcal{T}_d$, such that the parametrization $x(\theta)$ is affine on each side $\mathcal{T}_d$ and the vertices of $\mathcal{T}_d$ listed above correspond to $\theta = 0, 1, 2, 3$ respectively.
For $\lambda \in (0, 1)$, we put $c = \lambda / \sqrt{1 - \lambda^2}$. We assume that

$$\max(1, d) < c < d + 1. \quad (2.33)$$

Under the condition (2.33) we know $\ell^+(\bullet, \lambda)$ has exactly two nondegenerate critical points $x(0), x(2)$; $\ell^-(\bullet, \lambda)$ also has two nondegenerate critical points $x(1), x(3)$. Hence $\mathcal{T}_d$ is $\lambda$-simple. We also have

$$b^2(s) = \frac{c - d}{c + d} \theta + \frac{(c - 1)(c - d)}{(1 + d)(c + d)}, \quad \theta \in [0, 1].$$

Suppose $\theta_0 \in [0, 1], b^2(\theta_0) = \theta_0$, then we find

$$\theta_0 = \frac{(c - 1)(c - d)}{2d(1 + d)}.$$

Thus by the same argument as in Example 1,

$$\Sigma_\lambda = \{ x(\theta)| \theta = \theta_0, \gamma^-(\theta_0), b(\theta_0), \gamma^+(\theta_0) \}.$$

Near $\Sigma_\lambda$, we have

$$b(s) = \begin{cases} 
\frac{(c-d)(1+d)}{c+d} \theta + \frac{2(1-d)c+4d}{c+d}, & 0 \leq \theta < 1, \\
\frac{c+d}{c} \theta - \frac{1+2d}{c} + 3, & \frac{c+2d}{c+d} \leq \theta < 1 + \frac{1+d}{c+d}, \\
\frac{1}{1+d} \theta + \frac{c-3}{1+d}, & 2 \leq \theta < 3, \\
\frac{c}{c-d} \theta - \frac{c+2d+1}{c-d}, & 2 + \frac{1+d}{c} \leq \theta < 3 + \frac{1}{c}.
\end{cases}$$

Thus we have

$$\partial_\theta b^2(\theta) = \frac{c - d}{c + d} < 1, \quad \theta = \theta_0, b(\theta_0); \quad \partial_\theta b^2(\theta) = \frac{c + d}{c - d} > 1, \quad \theta = \gamma^-(\theta_0), \gamma^+(\theta_0).$$

We can now conclude that $\mathcal{T}_d$ and $\lambda$ satisfy the Morse-Smale conditions if (2.33) holds.

3. Microlocal preliminaries

In this section we present some general results needed in the proof. Most of the microlocal analysis in this paper takes place on the one dimensional boundary $\partial \Omega$; we review the basic notions in §3.1. In §3.2 we review definitions and basic properties of conormal distributions (needed in dimensions one and two). These are used to prove and formulate Theorem 1: the singularities of $(P - \lambda \mp i0)^{-1}f$ using conormal distributions. In our approach, this structure of $(P - \lambda \mp i0)^{-1}f$ is essential for describing the long time evolution profile in Theorem 2. Finally, §§3.3–3.4 contain technical results needed in §4.
3.1. Microlocal analysis on $\partial \Omega$. We first briefly discuss pseudodifferential operators on the circle $S^1 = \mathbb{R}/\mathbb{Z}$, referring to [17, §18.1] for a detailed introduction to the theory of pseudodifferential operators. Pseudodifferential operators on $S^1$ are given by quantizations of 1-periodic symbols. More precisely, if $0 \leq \delta < \frac{1}{2}$ and $m \in \mathbb{R}$, then we say that $a \in C^\infty(\mathbb{R}^2)$ lies in $S^m_\delta(T^*S^1)$ if

$$a(x + 1, \xi) = a(x, \xi), \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m + \delta - (1 - \delta)\beta}. \quad (3.1)$$

For brevity we just write $S^m_\delta := S^m_\delta(T^*S^1)$. Each $a \in S^m_\delta$ is quantized by the operator $\text{Op}(a) : C^\infty(S^1) \to C^\infty(S^1)$, $\mathcal{D}'(S^1) \to \mathcal{D}'(S^1)$ defined by

$$\text{Op}(a)u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi}a(x, \xi)u(y) \, dy \, d\xi,$$

where $u \in C^\infty(\mathbb{R})$ is 1-periodic and the integral is understood in the sense of oscillatory integrals [16, §7.8]. We introduce the following spaces of pseudodifferential operators:

$$\Psi^m_\delta := \{\text{Op}(a) : a \in S^m_\delta\}, \quad \Psi^m_\delta^+ = \bigcap_{m' > m} \Psi^{m'}_\delta, \quad \Psi^m_\delta^+ = \bigcap_{\delta' > \delta} \Psi^m_{\delta'}. \quad (3.2)$$

We remark that $S^m_\delta^+ \subset S^m_\delta$; moreover, $a \in S^m_\delta^+$ lies in $S^m_\delta$ if and only if $a(x, \xi) = \mathcal{O}(\langle \xi \rangle^m)$. We henceforth denote $\Psi^m := \Psi^m_0$. The space $\Psi^{-\infty} := \bigcap_m \Psi^m$ consists of smoothing operators.

In terms of Fourier series on $S^1$, we have

$$\text{Op}(a)u(x) = \sum_{k, n \in \mathbb{Z}} e^{2\pi i nx}a_{n-k}(k)u_k,$$

$$a_\ell(k) := \int_0^1 a(x, 2\pi k) e^{-2\pi i \ell x} \, dx, \quad u_k := \int_0^1 u(x) e^{-2\pi i k x} \, dx.$$  \quad (3.3)

This shows that $\text{Op}(a)$ does not determine $a$ uniquely. This representation also shows boundedness on Sobolev spaces $\text{Op}(a) : H^s(S^1) \to H^{s-m}(S^1)$, $s \in \mathbb{R}$, $a \in S^m_\delta$. Indeed, smoothness of $a$ in $x$ shows that $a_\ell(k) = \mathcal{O}(\langle \ell \rangle^{-\infty} \langle k \rangle^m)$ and the bound on the norm follows from the Schur criterion [11, (A.5.3)]. Despite the fact that $A := \text{Op}(a)$ does not determine $a$ uniquely, it does determine its essential support, which is the right hand side in the definition of the wave front set of a pseudodifferential operator:

$$\text{WF}(A) := \mathbb{C}\{(x, \xi) : \xi \neq 0, \exists \rho > 0 : a(y, \eta) = \mathcal{O}(\langle \eta \rangle^{-\infty}) \text{ when } |x - y| < \rho, |\frac{\eta}{\xi} > 0\},$$

see [11, §E.2]. We refer to that section and [17, §18.1] for a discussion of wave front sets. We also recall a definition of the wave front set of a distribution,

$$\text{WF}(u) := \bigcap_{A \in \mathcal{C}^\infty, A \in \Psi^0} \text{Char}(A),$$

$$\text{Char}(A) := \mathbb{C}\{(x, \xi) : \xi \neq 0, \exists \rho, c > 0 : |a(y, \eta)| > c, \quad |x - y| < \rho, |\frac{\eta}{\xi} > 0, \quad |\eta > 1/\rho\}$$
The symbol calculus on $S^1$ translates directly from the symbol calculus of pseudodifferential operators on $\mathbb{R}$. We record in particular the composition formula [17, Theorem 18.1.8]: for $b_1 \in S^m_\delta$, $b_2 \in S^{m_2}_\delta$,

$$\text{Op}(b_1) \text{Op}(b_2) = \text{Op}(b), \quad b \in S^{m_1+m_2}_\delta,$$

$$b(x, \xi) = \exp(-i\partial_x \partial_\eta) \left[ b_1(x, \eta)b_2(y, \xi) \right] |_{(y, \eta)=(x, \xi)},$$

$$b(x, \xi) = \sum_{0 \leq k < N} \frac{(-i)^k}{k!} \partial^k_x b_1(x, \xi) \partial^k_\xi b_2(x, \xi) + b_N(x, \xi), \quad b_N \in S^{m-N(1-2\delta)}_\delta.$$  \hspace{1cm} (3.3)

where expanding the exponential gives an asymptotic expansion of $b$.

We record here a norm bound for pseudodifferential operators at high frequency:

**Lemma 3.1.** Assume that $a \in S^0_\delta$, $r \in S^{-1+}$, and $\sup |a| \leq R$. Then for all $N$, $\nu > 0$, and $u \in L^2(S^1)$ we have

$$\| \text{Op}(a + r)u \|_{L^2} \leq (R + \nu)\|u\|_{L^2} + C\|u\|_{H^{-N}}$$  \hspace{1cm} (3.4)

where the constant $C$ depends on $R$, $\nu$, $N$, and some seminorms of $a$ and $r$ but not on $u$.

**Proof.** By [15, Lemma 4.6] we can write

$$(R + \nu)^2 I = \text{Op}(a + r)^* \text{Op}(a + r) + \text{Op}(b)^* \text{Op}(b) + \text{Op}(q)$$

for some $b \in S^0_\delta$ and $q \in S^{-\infty}$. The bound (3.4) follows. \hfill \Box

Although $a$ in (3.2) is not unique, the principal symbol of $\text{Op}(a)$ defined as

$$\sigma(\text{Op}(a)) = [a] \in S^m_\delta/S^{m-1+2\delta}_\delta$$  \hspace{1cm} (3.5)

is, and we have a short exact sequence $0 \to \Psi^{m-1+2\delta}_\delta \to \Psi^m_\delta \to S^m_\delta/S^{m-1+2\delta}_\delta \to 0$. Somewhat informally, we write $\sigma(\text{Op}(a)) = b$ for any $b$ satisfying $a - b \in S^{m-1+2\delta}_\delta$.

In our analysis, we also consider families $\varepsilon \mapsto a_\varepsilon$, $\varepsilon \geq 0$, such that $a_\varepsilon \in S^{-\infty}$ for $\varepsilon > 0$ and $a_0 \in S^m_\delta$. In that case, for $A_\varepsilon = \text{Op}(a_\varepsilon)$,

$$\sigma(A_\varepsilon) = [b_\varepsilon], \quad b_\varepsilon - a_\varepsilon \in S^{m-1+2\delta}_\delta \text{ uniformly for } \varepsilon \geq 0.$$  \hspace{1cm} (3.6)

Again, we drop $[\bullet]$ when writing $\sigma(A)$ for a specific operator.

We will crucially use mild exponential weights which result in pseudodifferential operators of varying order – see [31], and in a related context, [13].

**Lemma 3.2.** Suppose that (in the sense of (3.1)) $m_j \in S^0$, $m_0$ is real-valued, and

$$G(x, \xi) := m_0(x, \xi) \log(\xi) + m_1(x, \xi), \quad m_0(x, t\xi) = m_0(x, \xi), \quad t, |\xi| \geq 1.$$  \hspace{1cm} (3.7)

Then

$$e^G \in S^M_{0+}, \quad e^{-G} \in S^{-m}_{0+}, \quad M := \max_{|\xi| = 1} m_0(x, \xi), \quad m := \min_{|\xi| = 1} m_0(x, \xi).$$  \hspace{1cm} (3.8)
and there exists \( r_G \in S^{-1+} \) such that
\[
\text{Op}(e^G) \text{Op}(e^{-G}(1 + r_G)) - I, \text{ Op}(e^{-G}(1 + r_G)) \text{Op}(e^G) - I \in \Psi^{-\infty}.
\] (3.9)

Also, if \( G_j(x, \xi) \) are given by (3.7) with \( m_0 \) and \( m_1 \) replaced by \( m_{0j}, m_{1j} \), respectively, then for \( a_j \in S^0, r_j \in S^{-1+}, j = 1, 2 \), there exists \( r_3 \in S^{-1+} \) such that
\[
\text{Op}(e^{G_1}(a_1 + r_1)) \text{Op}(e^{G_2}(a_2 + r_2)) = \text{Op}(e^{G_1+G_2}(a_1a_2 + r_3)).
\] (3.10)

**Proof.** Since \( \log(\xi) = \mathcal{O}_\varepsilon(\langle \xi \rangle^\varepsilon) \) for all \( \varepsilon > 0 \), (3.8) follows from (3.1). In fact, we have the stronger bound
\[
|\partial_\xi x_{1/2} \partial_\xi \xi^\beta (e^{\pm G(x, \xi)})| \leq C_{\alpha\varepsilon} e^{\pm G(x, \xi)} \langle \xi \rangle^{\varepsilon - |\beta|}, \quad \varepsilon > 0.
\] (3.11)

This gives (3.10). Indeed, the remainder in the expansion (3.3) is in \( S^{M_1+M_2-N^+} \) and the \( k \)-th term is in \( e^{G_1+G_2}S^{-k^+} \) by (3.11); it suffices to take \( N \geq M + M_2 - 1 \).

To obtain (3.9) we note that (3.10) gives \( \text{Op}(e^{\pm G}) \text{Op}(e^{\mp G}) = I - \text{Op}(r_\pm), r_\pm \in S^{-1+} \). We then have parametrices for the operators \( I - \text{Op}(r_\pm) \) [17, Theorem 18.1.9], \( I + \text{Op}(b_\pm) \), which give left and right approximate inverses (in the sense of (3.9)) \( (I + \text{Op}(b_-)) \text{Op}(e^{-G}), \text{Op}(e^{-G})(I + \text{Op}(b_+)) \). Those have the required form by (3.10) (where one of \( G_1, G_2 \) is equal to \(-G\) and the other one is equal to 0).

We also record a change of variables formula. Suppose \( f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) is a diffeomorphism with a lift \( \tilde{f} : \mathbb{R} \to \mathbb{R}, \tilde{f}(x + 1) = f(x) \pm 1 \) (with the + sign for orientation preserving \( f \) and the − sign otherwise). For symbols 1-periodic in \( x \) we can use the standard formula given in [17, Theorem 18.1.17] and an argument similar to (3.10). That gives, for \( G \) given by (3.7), and \( r \in S^{-1+} \),
\[
f^* \circ \text{Op}(e^G(1 + r)) = \text{Op}(e^{G^f}(1 + r_f)) \circ f^*,
\]
\[
G_f(x, \xi) := G(f(x), f'(x)^{-1} \xi), \quad r_f \in S^{-1+}.
\] (3.12)

In §4.6 below we will use pseudodifferential operators acting on 1-forms on \( S^1 \). Using the canonical 1-form \( dx, x \in S^1 = \mathbb{R}/\mathbb{Z} \), we identify 1-forms with functions, and this gives an identification of the class \( \Psi^m_\delta(S^1; T^*S^1) \) (operators acting on 1-forms) with \( \Psi^m_\delta(S^1) \) (operators acting on functions). This defines the principal symbol map, which we still denote by \( \sigma \).

Fixing a positively oriented coordinate \( \theta : \partial \Omega \to S^1 \), we can identify functions / distributions on \( \partial \Omega \) with functions / distributions on \( S^1 \). The change of variables formula used for (3.12) also shows the invariance of \( \sigma(A) \) under changes of variables and allows pseudodifferential operators acting on section of bundles – see [17, Definition 18.1.32]. In particular, we can define the class of pseudodifferential operators \( \Psi^m_\delta(\partial \Omega; T^*\partial \Omega) \) acting on 1-forms on \( \partial \Omega \) and the symbol map
\[
\sigma : \Psi^m_\delta(\partial \Omega; T^*\partial \Omega) \to S^m_\delta(T^*\partial \Omega)/S^{m-1+2\delta}_\delta(T^*\partial \Omega),
\] (3.13)
3.2. Conormal distributions. We now review conormal distributions associated to hypersurfaces, referring the reader to [17, §18.2] for details. Although we consider the case of manifolds with boundaries, the hypersurfaces are assumed to be transversal to the boundaries and conormal distributions are defined as restrictions of conormal distributions in the no-boundary case.

Let \( M \) be a compact manifold with boundary and \( \Sigma \subset M \) be a compact hypersurface transversal to the boundary (i.e. \( \Sigma \) is a compact codimension 1 submanifold of \( M \) with \( \partial \Sigma \subset \partial M \) and \( T_x \Sigma \neq T_x \partial M \) for all \( x \in \partial \Sigma \)).

The conormal bundle to \( \Sigma \) is given by \( N^* \Sigma := \{(x, \xi) \in T^*M : x \in \Sigma, \xi|_{T_xM} = 0\} \), which is a Lagrangian submanifold of \( T^*M \) and a one-dimensional vector bundle over \( \Sigma \). For \( s \in \mathbb{R} \), define the symbol class \( S^k(N^*\Sigma) \) consisting of functions \( a \in C^\infty(N^*\Sigma) \) satisfying the derivative bounds

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \theta)| \leq C_{\alpha \beta} \langle \theta \rangle^{k-|\beta|}
\]

where we use coordinates \((x, \theta) \in \mathbb{R}^{m-1} \times \mathbb{R} \) in \( N^*\Sigma \) and put \( \langle \xi \rangle := \sqrt{1 + |\xi|^2} \). Here the estimates are supposed to be valid uniformly up to the boundary of \( \Sigma \). In other words we can consider \( a \) as a restriction of a symbol on an extension of \( \Sigma \).

Denote by \( I^s(M, N^*\Sigma) \subset \mathcal{D}'(M^\circ) \) the space of extendible distributions on the interior \( M^\circ \) (see [17, §B.2]) which are conormal to \( \Sigma \) of order \( s \) smoothly up to the boundary of \( M \). To each \( u \in I^s(M, N^*\Sigma) \) is associated its principal symbol

\[
\sigma(u) \in S^m_{\frac{m}{2} - \frac{1}{2} + s}(N^*\Sigma)/S^m_{\frac{m}{2} - \frac{3}{2} + s}(N^*\Sigma) \quad \text{where} \quad m := \dim M.
\]

To describe the class \( I^s \) and the principal symbol we first consider two model cases:

- if \( M = \mathbb{R}^m \), we write points in \( \mathbb{R}^m \) as \((x_1, x') \in \mathbb{R} \times \mathbb{R}^{m-1} \), and \( \Sigma = \{x_1 = 0\} \), then a compactly supported distribution \( u \in \mathcal{E}'(\mathbb{R}^m) \) lies in \( I^s(M, N^*\Sigma) \) if and only its Fourier transform in the \( x_1 \) variable, \( \hat{u}(\xi_1, x') \), lies in \( S^m_{\frac{m}{2} - \frac{1}{2} + s}(N^*\Sigma) \) where \( N^*\Sigma = \{(0, x', \xi_1, 0) \mid x' \in \mathbb{R}^{m-1}, \xi_1 \in \mathbb{R}\} \).
- if \( M = \mathbb{R}_{x_1} \times [0, \infty)_{x_2} \times \mathbb{R}_{x'}^{m-2} \) and \( \Sigma = \{x_1 = 0, \ x_2 \geq 0\} \), then a distribution \( u \in \mathcal{D}'(M^\circ) \) with bounded support lies in \( I^s(M, N^*\Sigma) \) if and only if \( u = \hat{u}|_{M^\circ} \) for some \( \hat{u} \in \mathcal{E}'(\mathbb{R}^m) \) which lies in \( I^s(\mathbb{R}^m; N^*\Sigma) \), with \( \Sigma := \{x_1 = 0\} \subset \mathbb{R}^m \). Alternatively, \( \hat{u}(\xi_1, x') \) lies in \( S^m_{\frac{m}{2} - \frac{1}{2} + s}(N^*\Sigma) \) where the derivative bounds are uniform to the boundary.

In those model cases, elements of \( I^s(M, N^*\Sigma) \) are given by (in the notation of (3.14))

\[
u(x) = (2\pi)^{-\frac{m}{2} - \frac{1}{2}} \int_{\mathbb{R}} e^{ix_1 \xi_1} a(x', \xi_1) d\xi_1, \quad a \in S^m_{\frac{m}{2} - \frac{1}{2} + s}(N^*\{x_1 = 0\}) \quad (3.15)
\]
We note that in both of the above cases the distribution $u$ is in $C^\infty(M)$ (up to the boundary in the second case) outside of any neighborhood of $\Sigma$. The symbol $\sigma(u)$ is the equivalence class of $a$ in $S^\infty_{2-\frac{1}{2}+s}/S^\infty_{2-\frac{1}{2}+s}$.

For the case of general compact manifold $M$ and hypersurface $\Sigma$ transversal to the boundary of $M$, we say that $u \in I^s(M, N^*\Sigma)$ if

1. $u$ is in $C^\infty(M)$ (up to the boundary) outside of any neighborhood of $\Sigma$; and
2. the localizations of $u$ to the model cases using coordinates lie in $I^s$ as defined above.

Note that the wavefront set of any $u \in I^s(M, N^*\Sigma)$ is contained in $N^*\Sigma$. The principal symbol is the defined (in the second case, as in the boundaryless case) using local representations (3.15) – see [17, Theorem 18.2.11]. We should emphasize that in our case, the hypersurfaces $\Sigma$ take a particularly simple form: we either have $M = \Omega$ and $\Sigma$ given by straight lines transversal to $\partial\Omega$ (see Theorems 1, 2) or $M = \partial\Omega$ and $\Sigma$ is given by points (see Proposition 5.6).

In addition we define the space

$$I^{s+}(M, N^*\Sigma) := \bigcap_{s' > s} I^{s'}(M, N^*\Sigma).$$

Such spaces are characterized in terms of the Sobolev spaces (on manifolds with boundary, hence extendible [17, §B.2]), $H^{s-}(M) := \bigcap_{s' < s} H^{s'}(M)$, as follows:

$$u \in I^{s+}(M, N^*\Sigma) \iff X_1 \ldots X_\ell u \in H^{-\frac{m}{2}-s-}(M),$$

for any collection of vector fields $X_1, \ldots, X_\ell$ on $M$ which are tangent to $\Sigma$, see [17, Lemma 18.4].

Assume now that the conormal bundle $N^*\Sigma$ is oriented; for $(x, \xi) \in N^*\Sigma \setminus 0$ we say that $\xi > 0$ if $\xi$ is positively oriented and $\xi < 0$ if $\xi$ is negatively oriented. This gives the splitting

$$N^*\Sigma \setminus 0 = N^*_+\Sigma \sqcup N^*_\Sigma, \quad N^*_\pm\Sigma := \{(x, \xi) \in N^*\Sigma \mid \pm\xi > 0\}.$$  

(3.17)

Denote by $I^*(M, N^*_\pm\Sigma)$ the space of distributions $u \in I^*(M, N^*\Sigma)$ such that $\text{WF}(u) \subset N^*\Sigma$, up to the boundary. Since $\Sigma$ is transversal to the boundary this means that an extension of $u$ satisfies this condition. In the model case (and effectively in the cases considered in this paper) $M = \mathbb{R}^m$, $\Sigma = \{x_1 = 0\}$ they can be characterized as follows: $u(\xi_1, x')$ lies in $S^\infty_{2-\frac{1}{2}+s}(N^*\Sigma)$ and $u(\xi_1, x') = \mathcal{O}(\langle \xi_1 \rangle^{-\infty})$ as $\xi_1 \to \pm\infty$.

We will also need the notion of conormal distributions depending smoothly on a parameter – see [12, Lemma 4.4] for a more general Lagrangian version. Here we restrict ourselves to the specific conormal distributions appearing in this paper and define relevant smooth families of conormal distributions in Lemma 5.6.
3.3. Convolution with logarithm. In §4 we need information about mapping properties between spaces of conormal distributions on the boundary and conormal distributions in the interior. In preparation for Lemma 4.9 below we now prove the following

Lemma 3.3. Let \( f \in C^\infty_c(\mathbb{R}) \) and define
\[
g(x) := \frac{1}{2} \int_0^\infty \log |x - y| \frac{f(y)}{\sqrt{y}} \, dy, \quad x > 0. \tag{3.18}
\]
Then \( g \in C^\infty([0, \infty)) \).

Remark. In general \( g \) is not smooth on \((-\infty, 0]\). In fact, changing variables \( y = s^2|x| \), we obtain (see (3.19) below)
\[
g'(x) = -|x|^{-\frac{1}{2}} \int_0^\infty f(s^2|x|) \frac{1}{1 + s^2} \, ds, \quad x < 0,
\]
which blows up as \( x \to 0^- \) for \( f \equiv 1 \) near 0. These, and the conclusion of Lemma 3.3, can also be seen from analysis on the Fourier transform side.

Proof. Denote \( x^{-1/2} := H(x)x^{-1/2} \) where \( H(x) = [x > 0] \) is the Heaviside function. Since \( g \) is the convolution of \( \log |x| \) and \( x^{-1/2}f(x) \), which are both smooth except at \( x = 0 \), the function \( g \) is smooth on \((0, \infty)\). Thus it suffices to prove that \( g \) is smooth on \([0, 1]\) up to the boundary.

1. Assume first that \( f \) is real valued and extends holomorphically to the disk \( \{|z| < 4\} \). Making the change of variables \( y = s^2|x| \), we write
\[
g(x) = \int_\mathbb{R} \log |t^2 - x| f(t^2) \, dt. \tag{3.19}
\]
Assume that \( x \in (0, 1] \) and consider the holomorphic function
\[
\psi_z(z) := \log(z - \sqrt{x}) + \log(z + \sqrt{x}), \quad z \in \mathbb{C} \setminus ((\sqrt{x} - i[0, \infty)) \cup (-\sqrt{x} - i[0, \infty))),
\]
where we use the branch of the logarithm on \( \mathbb{C} \setminus -i[0, \infty) \) which takes real values on \((0, \infty)\). Then \( \text{Re} \psi_z(t) = \log |t^2 - x| \) for all \( t \in \mathbb{R} \setminus \{\sqrt{x}, -\sqrt{x}\} \).

Fix an \( x \)-independent contour \( \Gamma = \{t + iw(t) \mid t \in \mathbb{R}\} \subset \mathbb{C} \) such that \( w(t) \geq 0 \) everywhere, \( w(t) = 0 \) for \( |t| \geq \frac{3}{2} \), \( |t + iw(t)| < 2 \) for \( |t| \leq \frac{3}{2} \), and \( w(t) > 0 \) for \( |t| \leq 1 \). (See Figure 9.) Deforming the contour in (3.19), we get
\[
g(x) = \text{Re} \int_\Gamma \psi_z(z) f(z^2) \, dz \quad \text{for all} \quad x \in (0, 1].
\]
Since \( \partial_z \psi_z(z) = (x - z^2)^{-1} \), the function \( \psi_z(z) \) and all its \( x \)-derivatives are bounded uniformly in \( x \in (0, 1] \) and locally uniformly in \( z \in \Gamma \). It follows that \( g \) is smooth on the interval \([0, 1]\).
For the general case, fix a cutoff $\chi \in C^{\infty}_c(\mathbb{R})$ such that $\chi = 1$ near $[-4, 4]$. Take arbitrary $N \in \mathbb{N}$. Using the Taylor expansion of $f$ at $0$, we write

$$f(x) = f_1(x) + f_2(x), \quad f_1(x) = p(x)\chi(x)$$

where $p$ is a polynomial of degree at most $N$ and $f_2 \in C^{\infty}_c(\mathbb{R})$ satisfies $f_2(x) = O(|x|^{N+1})$ as $x \to 0$. We write $g = g_1 + g_2$ where $g_j$ are constructed from $f_j$ using (3.18). By Step 1 of the present proof, we see that $g_1$ is smooth on $[0, 1]$. On the other hand, $x^{-1/2}f_2(x) \in C^N(\mathbb{R})$; since $\log|x|$ is locally integrable we get $g_2 \in C^N([0, 1])$. Since $N$ can be chosen arbitrary, this gives $g \in C^{\infty}([0, 1])$ and finishes the proof. \hfill $\square$

We also give the following general mapping property of convolution with logarithm on conormal spaces:

**Lemma 3.4.** Let $\Sigma \subset \mathbb{R}$ be a finite set and put $\log_\pm x := \log(x \pm i0)$. Then

$$I^s(\mathbb{R}, N^+_\Sigma) \cap C^\infty_\Sigma(\mathbb{R}) \ni f \mapsto \log_\bullet * f \in \begin{cases} I^{s-1}(\mathbb{R}, N^+_\Sigma), \quad \bullet = +, \\ C^\infty, \quad \bullet = -, \end{cases}$$

with a similar statement for $N^-_\Sigma$.

**Proof.** Since $\partial_x$ is an elliptic operator, the local definition (3.15) (with no $x'$ variable) shows that it is enough to show that $\partial_x \log_\bullet * f$ is in $I^s(\mathbb{R}, N^+_\Sigma)$ for $\bullet = +$ and in $C^\infty$ for $\bullet = -$. Since $\partial_\nu \log_\pm = (x \pm i0)^{-1}$ and, since the Fourier transform of $x \mapsto (x \pm i0)^{-1}$ is given by $\pm 2\pi iH(\pm \xi)$ (see [16, Example 7.1.17]; here $H$ is the Heaviside function), the result follows. \hfill $\square$

### 3.4. Microlocal structure of $(x \pm i\varepsilon \psi(x))^{-1}$

In this section we study the behaviour as $\varepsilon \to 0+$ of functions of the form

$$\chi(x, \varepsilon)(x \pm i\varepsilon \psi(x, \varepsilon))^{-1} \in C^\infty(J), \quad 0 < \varepsilon < \varepsilon_0$$

(3.20)
where $J \subset \mathbb{R}$ is an open interval containing 0 and
\[ \chi, \psi \in C^\infty(J \times [0, \varepsilon_0); \mathbb{C}), \quad \text{Re} \psi > 0 \quad \text{on} \quad J \times [0, \varepsilon_0). \] (3.21)
We first decompose (3.20) into the sum of
\[ J \]
where $q$ but not on $x$, and a function which is smooth uniformly in $\varepsilon$:

**Lemma 3.5.** Under the conditions (3.21) we have for all $\varepsilon \in (0, \varepsilon_0)$
\[ \chi(x, \varepsilon)(x \pm i\varepsilon \psi(x, \varepsilon))^{-1} = r^\pm(\varepsilon)(x \pm i\varepsilon z^\pm(\varepsilon))^{-1} + q^\pm(x, \varepsilon) \] (3.22)
where $r^\pm, z^\pm \in C^\infty([0, \varepsilon_0))$ and $q^\pm \in C^\infty(J \times [0, \varepsilon_0))$ are complex valued, $\text{Re} z^\pm > 0$ on $[0, \varepsilon_0)$, $z^\pm(0) = \psi(0, 0)$, and $\chi(x, 0) = x q^\pm(x, 0) + r^\pm(0)$.

**Proof.** Since $(x \pm i\varepsilon \psi(x, \varepsilon))^{-1}$ is a smooth function of $(x, \varepsilon) \in J \times [0, \varepsilon_0)$ outside of $(0, 0)$, it is enough to show that (3.22) holds for $|x|, \varepsilon$ small enough.

The complex valued function $F^\pm(x, \varepsilon) := x \pm i\varepsilon \psi(x, \varepsilon)$ is smooth in $(x, \varepsilon) \in J \times [0, \varepsilon_0)$ and satisfies $F^\pm(0, 0) = 0$ and $\partial_x F^\pm(0, 0) = 1$. Thus by the Malgrange Preparation Theorem [16, Theorem 7.5.6], we have for $(x, \varepsilon)$ in some neighbourhood of $(0, 0)$ in $J \times [0, \varepsilon_0)$
\[ x = q^\pm(x, \varepsilon)(x \pm i\varepsilon \psi(x, \varepsilon)) + r^\pm(\varepsilon) \]
where $q^\pm_1, r^\pm_1$ are smooth. Taking $\varepsilon = 0$ we get $r^\pm_1(0) = 0$ and $q^\pm_1(x, 0) = 1$; differentiating in $\varepsilon$ and then putting $x = \varepsilon = 0$ we get $\partial_\varepsilon r^\pm_1(0) = \mp i\psi(0, 0)$. We put $z^\pm(\varepsilon) := \pm i\varepsilon^{-1}r^\pm_1(\varepsilon)$, so that when $\varepsilon > 0$
\[ (x \pm i\varepsilon \psi(x, \varepsilon))^{-1} = q^\pm_1(x, \varepsilon)(x \pm i\varepsilon z^\pm(\varepsilon))^{-1}. \] (3.23)
Note that $z^\pm(0) = \psi(0, 0)$ and thus $\text{Re} z^\pm(\varepsilon) > 0$ for small $\varepsilon$.

Now, we use the Malgrange Preparation Theorem again, this time for the function $F^\pm(x, \varepsilon) := x \pm i\varepsilon z^\pm(\varepsilon)$, to get for $(x, \varepsilon)$ in some neighbourhood of $(0, 0)$ in $J \times [0, \varepsilon_0)$
\[ \chi(x, \varepsilon) q^\pm_1(x, \varepsilon) = q^\pm(x, \varepsilon)(x \pm i\varepsilon z^\pm(\varepsilon)) + r^\pm(\varepsilon) \]
where $q^\pm, r^\pm$ are again smooth. Taking $\varepsilon = 0$ we get $\chi(x, 0) = x q^\pm(x, 0) + r^\pm(0)$. Together with (3.23) this gives the decomposition (3.22). \hfill \Box

As an application of Lemma 3.5, we give

**Lemma 3.6.** Assume that $\psi$ satisfies (3.21). Then we have for all $s < -\frac{1}{2}$
\[ (x \pm i\psi(x, \varepsilon))^{-1} \rightarrow (x \pm i0)^{-1} \quad \text{as} \quad \varepsilon \rightarrow 0 + \quad \text{in} \quad H^s_{\text{loc}}(J). \] (3.24)

**Proof.** Put $\chi \equiv 1$ and let $z^\pm(\varepsilon), r^\pm(\varepsilon), q^\pm(x, \varepsilon)$ be given by Lemma 3.5; note that $1 = q^\pm(x, 0) + r^\pm(0)$, thus $r^\pm(0) = 1$ and $q^\pm(x, 0) = 0$. We have
\[ (x \pm i\varepsilon z^\pm(\varepsilon))^{-1} \rightarrow (x \pm i0)^{-1} \quad \text{in} \quad H^s(\mathbb{R}). \]
Indeed, the Fourier transform of the left-hand side is equal to $\mp 2\pi i H(\pm \xi) e^{\mp \varepsilon z^\pm(\varepsilon) \xi}$ and the Fourier transform of the right-hand side is equal to $\mp 2\pi i H(\pm \xi)$, where $H$ is the
Heaviside function; we have convergence of these in $L^2(\mathbb{R}; \langle \xi \rangle^{2s} d\xi)$ by the Dominated Convergence Theorem.

By (3.22) this implies that the left-hand side of (3.24) converges in $H^s_{\text{loc}}(J)$ to

$$r^\pm(0)(x \pm i0)^{-1} + q^\pm(x, 0) = (x \pm i0)^{-1}$$

which finishes the proof. \hfill \square

The functions $r^\pm, z^\pm, q^\pm$ in Lemma 3.5 are not uniquely determined by $\chi, \psi$, however they are unique up to $O(\varepsilon^{\infty})$:

**Lemma 3.7.** Assume that $r_j^\pm, z_j^\pm \in C^\infty([0, \varepsilon_0]), j = 1, 2$, are complex valued functions such that $\text{Re } z_j^\pm > 0$ on $[0, \varepsilon_0)$, $r_j^\pm(0) \neq 0$, and

$$\tilde{q}^\pm(x, \varepsilon) := r_1^\pm(\varepsilon)(x \pm i\varepsilon z_1^\pm(\varepsilon))^{-1} - r_2^\pm(\varepsilon)(x \pm i\varepsilon z_2^\pm(\varepsilon))^{-1} \in C^\infty(J \times [0, \varepsilon_0)).$$

Then $r_1^\pm(\varepsilon) - r_2^\pm(\varepsilon), z_1^\pm(\varepsilon) - z_2^\pm(\varepsilon)$, and $\tilde{q}^\pm(x, \varepsilon)$ are $O(\varepsilon^{\infty})$, that is all their derivatives in $\varepsilon$ vanish at $\varepsilon = 0$.

**Proof.** Differentiating $k - 1$ times in $x$ and then putting $x = 0$ we see that for all $k \geq 1$

$$\frac{r_1^\pm(\varepsilon)}{\varepsilon^k z_1^\pm(\varepsilon)^k} - \frac{r_2^\pm(\varepsilon)}{\varepsilon^k z_2^\pm(\varepsilon)^k} \in C^\infty([0, \varepsilon_0)).$$

Therefore

$$\partial_{\varepsilon}^0 (r_1^\pm(\varepsilon) \frac{z_1^\pm(\varepsilon)}{z_2^\pm(\varepsilon)} - r_2^\pm(\varepsilon)) = 0 \quad \text{for all } 0 \leq \ell < k.$$  \hfill (3.25)

Taking $\ell = 0$ and $k = 1, 2$, we see that $r_1^\pm(0) = r_2^\pm(0)$ and $z_1^\pm(0) = z_2^\pm(0)$. Arguing by induction on $\ell$ and using $k = \ell + 1, \ell + 2$ in (3.25) we see that $\partial_{\varepsilon}^\ell r_1^\pm(0) = \partial_{\varepsilon}^\ell r_2^\pm(0)$ and $\partial_{\varepsilon}^\ell z_1^\pm(0) = \partial_{\varepsilon}^\ell z_2^\pm(0)$. Thus $r_1^\pm(\varepsilon) - r_2^\pm(\varepsilon)$ and $z_1^\pm(\varepsilon) - z_2^\pm(\varepsilon)$ are $O(\varepsilon^{\infty})$, which implies that $\tilde{q}^\pm(x, \varepsilon)$ is $O(\varepsilon^{\infty})$ as well. \hfill \square

**Remark.** If $\chi$ and $\psi$ depend smoothly on some additional parameter $y$, then the proof of Lemma 3.5 shows that $r^\pm, z^\pm, q^\pm$ can be chosen to depend smoothly on $y$ as well. Lemma 3.6 also holds, with convergence locally uniform in $y$, as does Lemma 3.7. In §4.6 below we use this to study expressions of the form

$$\chi(\theta, \theta', \varepsilon)(\theta - \theta' \pm i\varepsilon\psi(\theta, \theta', \varepsilon))^{-1},$$  \hfill (3.26)

where $(\theta, \theta')$ is in some neighbourhood of 0 and we put $x := \theta - \theta'$, $y := \theta$.

For the use in §4 we record the fact that operators with Schwartz kernels of the form (3.26) are pseudodifferential:

**Lemma 3.8.** Assume that $c^\pm_{\varepsilon}(\theta')$ and $z^\pm_{\varepsilon}(\theta')$ are complex valued functions smooth in $\theta' \in S^1 := \mathbb{R}/\mathbb{Z}$ and $\varepsilon \in [0, \varepsilon_0)$ and such that $\text{Re } z^\pm_{\varepsilon} > 0$. Let $\chi \in C^\infty(S^1 \times S^1)$ be
supported in a neighbourhood of the diagonal and equal to 1 on a smaller neighbourhood of the diagonal. Consider the operator $A^\pm_\varepsilon$ on $C^\infty(S^1)$ given by

$$A^\pm_\varepsilon f(\theta) = \int_{S^1} K^\pm_\varepsilon(\theta, \theta') f(\theta') d\theta',$$

$$K^\pm_\varepsilon(\theta, \theta') = \begin{cases} 
\chi(\theta, \theta')c^\pm_\varepsilon(\theta')(\theta - \theta' \pm i\varepsilon z^\pm_\varepsilon(\theta'))^{-1}, & \varepsilon > 0; \\
\chi(\theta, \theta')c^\pm_\varepsilon(\theta')(\theta - \theta' \pm i0)^{-1}, & \varepsilon = 0.
\end{cases} \tag{3.27}$$

Then $A^\pm_\varepsilon \in \Psi^0(S^1)$ uniformly in $\varepsilon$ and we have, uniformly in $\varepsilon$,

$$\WF(A^\pm_\varepsilon) \subset \{ \pm \xi > 0 \}, \quad \sigma(A^\pm_\varepsilon)(\theta, \xi) = \mp 2\pi i c^\pm_\varepsilon(\theta)e^{-\varepsilon z^\pm_\varepsilon(\theta)|\xi|} H(\pm \xi), \tag{3.28}$$

where for $\varepsilon > 0$ the principal symbol is understood as in (3.6) and $H$ is the Heaviside function (with the symbol considered for $|\xi| > 1$).

**Remark.** We note that the definition (3.6) of the symbol of a family of operators and Lemma 3.7 show that the principal symbol is independent of the (not unique) $c^\pm_\varepsilon, z^\pm_\varepsilon$.

**Proof.** We recall first that $A \in \Psi^{m+}(\partial \Omega)$ if and only if the Schwartz kernel of $A$, $K_A$, belongs to $I^{m+}(\partial \Omega \times \partial \Omega, N^*\Delta)$ (in the notation of §3.2), $\Delta := \{(x, x) : x \in \partial \Omega\}$ – see [17, §18.2] (incidentally, this explains the convention used for the order of the space of conormal distributions). We also recall the Fourier transforms:

$$\mathcal{F}_{x \rightarrow \xi}(x \pm iq)^{-1} = e^{-q|x|}H(\pm \xi), \quad q > 0, \quad \mathcal{F}_{y \rightarrow \eta}(x \pm i0)^{-1} = H(\pm \xi). \tag{3.29}$$

This and (3.27) show that $K_\varepsilon \in H^{-\frac{1}{2}+}(\partial \Omega \times \partial \Omega)$ and then (3.16) gives $A^\pm_\varepsilon \in \Psi^{0+}(\partial \Omega)$ (we need to use locally defined vector fields $(\theta - \theta')\partial_\theta$ and $\partial_\theta + \partial_\theta$. Writing, $K^\pm_\varepsilon(\theta, \theta') = \tilde{K}^\pm_\varepsilon(\theta', \theta - \theta')$ and taking the Fourier transform in the second variable using (3.29) give (3.28). That in particular shows that the order of $A^\pm_\varepsilon$ is 0. \hfill \Box

4. **Boundary layer potentials**

In this section we describe microlocal properties of boundary layer potentials for the operator $P - \lambda^2 = \partial_x^2 \Delta_\Omega^{-1} - \lambda^2$, or rather for the related partial differential operator $P(\lambda)$ defined in (4.1). The key issue is the transition from elliptic to hyperbolic behaviour as $\text{Im} \lambda \to 0$. To motivate the results we explain the analogy with the standard boundary layer potentials in §4.2. In §4.3 we compute fundamental solutions for $P(\lambda)$ on $\mathbb{R}^2$ and in §4.4 we use these to study the Dirichlet problem for $P(\lambda)$ on $\Omega$. This will lead us to single layer potentials: in §4.5 we study their mapping properties (in particular relating Lagrangian distributions on the boundary to Lagrangian distributions in the interior) and in §4.6 we give a microlocal description of their restriction to $\partial \Omega$ uniformly as $\text{Im} \lambda \to 0$, which is crucially used in §5.
4.1. Basic properties. Consider the second order constant coefficient differential operator on $\mathbb{R}^2_{x_1,x_2}$

\[ P(\lambda) := (1 - \lambda^2)\partial^2_{x_2} - \lambda^2\partial^2_{x_1} \quad \text{where} \quad \lambda \in \mathbb{C}, \quad 0 < \Re \lambda < 1. \quad (4.1) \]

Formally,

\[ P(\lambda) = (P - \lambda^2)\Delta_\Omega, \quad P(\lambda)^{-1} = \Delta_\Omega^{-1}(P - \lambda^2)^{-1}. \quad (4.2) \]

We note that $P(\lambda)$ is hyperbolic when $\lambda \in (0,1)$ and elliptic otherwise. We factorize $P(\lambda)$ as follows:

\[ P(\lambda) = 4L^+_\lambda L^-_\lambda, \quad L^\pm_\lambda := \frac{1}{2}(\pm \lambda \partial_{x_1} + \sqrt{1 - \lambda^2} \partial_{x_2}). \quad (4.3) \]

Here $\sqrt{1 - \lambda^2}$ is defined by taking the branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ which takes positive values on $(0, \infty)$. We note that for $\lambda \in (0,1)$ the operators $L^\pm_\lambda$ are two linearly independent constant vector fields on $\mathbb{R}^2$. For $\Im \lambda \neq 0$, $L^\pm_\lambda$ are Cauchy–Riemann type operators.

The definition $(1.2)$ of the functions $\ell^\pm(x,\lambda)$ extends to complex values of $\lambda$:

\[ \ell^\pm(x,\lambda) := \pm \frac{x_1}{\lambda} + \frac{x_2}{\sqrt{1 - \lambda^2}}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad \lambda \in \mathbb{C}, \quad 0 < \Re \lambda < 1. \]

Then the linear functions $\ell^\pm(\bullet, \lambda)$ are dual to the operators $L^\pm_\lambda$:

\[ L^\pm_\lambda \ell^\pm(x, \lambda) = 1, \quad L^\mp_\lambda \ell^\pm(x, \lambda) = 0. \quad (4.4) \]

We record here the following statement:

**Lemma 4.1.** Assume that $\Im \lambda > 0$. Then the map $x \in \mathbb{R}^2 \mapsto \ell^\pm(x, \lambda) \in \mathbb{C}$ is orientation preserving in the case of $\ell^+$ and orientation reversing in the case of $\ell^-$. If $\Im \lambda < 0$ then a similar statement holds with the roles of $\ell^\pm$ switched.

**Proof.** This follows immediately from the sign identity

\[ \sgn \Im \frac{\lambda}{\sqrt{1 - \lambda^2}} = \sgn \Im \lambda, \quad \lambda \in \mathbb{C}, \quad 0 < \Re \lambda < 1 \quad (4.5) \]

which can be verified by noting that $\Re \sqrt{1 - \lambda^2} > 0$ and $\sgn \Im \sqrt{1 - \lambda^2} = -\sgn \Im \lambda$. \hfill $\Box$

4.2. Motivational discussion. When $\Im \lambda > 0$ the decomposition $(4.3)$ is similar to the factorization of the Laplacian,

\[ \Delta = 4\partial_z \partial_{\bar{z}}, \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \]

The functions $z = x + iy$ and $\bar{z}$ play the role of $\ell^\pm(x, \lambda)$ for $\pm$, respectively (which matches the orientation in Lemma 4.1) and $\partial_z, \partial_{\bar{z}}$ play the role of $L^+_\lambda, L^-_\lambda$. Hence to explain the structure of the fundamental solution of $P(\lambda)$ and to motivate the restricted
boundary layer potential in §4.6 we review the basic case when $\Omega = \{ y > 0 \}$ and $P(\lambda)$ is replaced by $\Delta$. The fundamental solution is given by (see e.g. [16, Theorem 3.3.2])

$$\Delta E = \delta_0, \quad E := c \log(z\bar{z}), \quad \partial_x E = c/z, \quad \partial_z E = c/\bar{z}, \quad c = 1/4\pi.$$  

We consider the single layer potential $S : C^\infty_c(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{ y = 0 \}),$

$$Sv(x, y) := \int_{\mathbb{R}} E(x - x', y)v(x') \, dx', \quad v \in C^\infty_c(\mathbb{R}).$$

We then have limits as $y \to 0 \pm$, 

$$C_\pm v(x) = C v(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \log |x - x'| v(x') \, dx',$$

and we consider

$$\partial_x C v(x) = \lim_{y \to 0 \pm} \partial_x Sv(x, y) = \lim_{y \to 0 \pm} (\partial_x + \partial_z) Sv(x, y).$$

Then (where we recall $c = 1/4\pi$)

$$\lim_{y \to 0 \pm} \partial_x Sv(x, y) = \lim_{y \to 0 \pm} \int_{\mathbb{R}} \frac{c}{x - x' + iy} v(x') \, dx' = \int_{\mathbb{R}} \frac{c}{x - x' \pm i0} v(x') \, dx,$$

and similarly,

$$\lim_{y \to 0 \pm} \partial_z Sv(x, y) = \lim_{y \to 0 \pm} \int_{\mathbb{R}} \frac{c}{x - x' - iy} v(x') \, dx' = \int_{\mathbb{R}} \frac{c}{x - x' \mp i0} v(x') \, dx,$$

where the right hand sides are understood as distributional pairings. This gives

$$\partial_x C v(x) = \frac{1}{4\pi} \int_{\mathbb{R}} \sum_\pm (x - x' \pm i0)^{-1} v(x') \, dx' \tag{4.6}$$

which is $1/2$ times the Hilbert transform, that is, the Fourier multiplier with symbol $-i \text{sgn}(\xi)$. (We note that $\sum_\pm (x - x' \pm i0)^{-1} = 2 \partial_x \log |x|$ which is the principal value of $2/x$.)

In §4.6 we describe the analogue of $\partial_x C$ in our case. It is similar to (4.6) when $\text{Im} \lambda > 0$ but when $\text{Im} \lambda \to 0^+$ it has additional singularities described using the chess billiard map $b(x, \lambda)$, or rather its building components $\gamma^\pm$. The operator becomes an elliptic operator of order 0 (just as is the case in (4.6) if we restrict our attention to compact sets) plus a Fourier integral operator – see Proposition 4.15.

4.3. Fundamental solutions. We now construct a fundamental solution of the operator $P(\lambda)$ defined in (4.1), that is a distribution $E_\lambda \in \mathcal{D}'(\mathbb{R}^2)$ such that

$$P(\lambda)E_\lambda = \delta_0. \tag{4.7}$$

For that we use the complex valued quadratic form

$$A(x, \lambda) := \ell^+(x, \lambda)\ell^-(x, \lambda) = -\frac{x_1^2}{\lambda^2} + \frac{x_2^2}{1 - \lambda^2}.$$
Since $0 < \text{Re} \lambda < 1$ we have $\text{sgn} \text{Im}(-\lambda^{-2}) = \text{sgn} \text{Im}((1 - \lambda^2)^{-1}) = \text{sgn} \text{Im} \lambda$, thus
\begin{equation}
\text{sgn} \text{Im} A(x, \lambda) = \text{sgn} \text{Im} \lambda \quad \text{for all} \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{4.8}
\end{equation}

4.3.1. The non-real case. We first consider the case $\text{Im} \lambda \neq 0$. In this case our fundamental solution is the locally integrable function
\begin{equation}
E_\lambda(x) := c_\lambda \log A(x, \lambda), \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad c_\lambda := \frac{i \text{sgn} \text{Im} \lambda}{4\pi \lambda \sqrt{1 - \lambda^2}}. \tag{4.9}
\end{equation}
Here we use the branch of logarithm on $\mathbb{C} \setminus (-\infty, 0]$ which takes real values on $(0, \infty)$. Note that the function $E_\lambda$ is smooth on $\mathbb{R}^2 \setminus \{0\}$.

\textbf{Lemma 4.2.} The function $E_\lambda$ defined in (4.9) solves (4.7).

\textit{Proof.} We first check that $P(\lambda)E_\lambda = 0$ on $\mathbb{R}^2 \setminus \{0\}$: this follows from (4.3), (4.4), and the identities
\begin{equation}
L^\pm_\lambda \log A(x, \lambda) = \frac{1}{\ell^\pm(x, \lambda)} \quad \text{for all} \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{4.10}
\end{equation}
Next, denote by $B_\varepsilon$ the ball of radius $\varepsilon > 0$ centered at 0 and orient $\partial B_\varepsilon$ in the counterclockwise direction. Using the Divergence Theorem twice, we compute for each $\varphi \in C^\infty_c(\mathbb{R}^2)$
\begin{align*}
\int_{\mathbb{R}^2} E_\lambda(x)(P(\lambda)\varphi(x)) \, dx &= 4 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 \setminus B_\varepsilon} E_\lambda(x)(L^+_\lambda L^-_\lambda \varphi(x)) \, dx \\
&= -4c_\lambda \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 \setminus B_\varepsilon} \frac{L^-_\lambda \varphi(x)}{\ell^+(x, \lambda)} \, dx \\
&= -2c_\lambda \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon} \frac{\varphi(x)(\sqrt{1 - \lambda^2} \, dx_1 + \lambda \, dx_2)}{\ell^+(x, \lambda)} \\
&= -2c_\lambda \sqrt{1 - \lambda^2} \varphi(0) \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon} \frac{d\ell^+(x, \lambda)}{\ell^+(x, \lambda)} = \varphi(0)
\end{align*}
which gives (4.7). Here in the last equality we make the change of variables $z = \ell^+(x, \lambda)$ and use Lemma 4.1.
\hfill \Box

4.3.2. The real case. We now discuss the situation when $\lambda$ is real, that is $\lambda \in (0, 1)$, taking the limits of the fundamental solution (4.9) as $\text{Im} \lambda \to 0 \pm$:

\textbf{Lemma 4.3.} Let $\lambda \in (0, 1)$. We have the following limit locally in $L^1$ and thus also in the space of distributions $D'(\mathbb{R}^2)$:
\begin{equation}
\lim_{\varepsilon \to 0^+} E_{\lambda \pm i \varepsilon} = E_{\lambda \pm i 0} \tag{4.11}
\end{equation}
where the locally integrable functions $E_{\lambda \pm i0}$ are given by

$$E_{\lambda \pm i0}(x) = \pm c_\lambda \log(A(x, \lambda) \pm i0), \quad c_\lambda := \frac{i}{4\pi\lambda\sqrt{1 - \lambda^2}},$$

$$\log(A(x, \lambda) \pm i0) = \begin{cases} \log A(x, \lambda), & A(x, \lambda) > 0; \\ \log(-A(x, \lambda)) \pm i\pi, & A(x, \lambda) < 0. \end{cases} \quad (4.12)$$

**Proof.** Using (4.8) we see that $\lim_{\varepsilon \to 0^+} E_{\lambda \pm i\varepsilon}(x) = E_{\lambda \pm i0}(x)$ for almost every $x \in \mathbb{R}^2$, more specifically for $A(x, \lambda) \neq 0$. We now use the Dominated Convergence Theorem. To apply it one needs to find a function $F \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $\varepsilon_0 > 0$ such that we have the bound $| \log A(x, \lambda \pm i\varepsilon)| \leq F(x)$ for all $\varepsilon \in (0, \varepsilon_0)$ and almost every $x \in \mathbb{R}^2$. Note that $|\Im \log A(x, \lambda \pm i\varepsilon)| \leq \pi$, so it is enough to estimate the real part, which is

$$\log|A(x, \lambda \pm i\varepsilon)| = \log|\ell^+(x, \lambda \pm i\varepsilon)| + \log|\ell^-(x, \lambda \pm i\varepsilon)|.$$

We bound $\log|\ell^+(x, \lambda + i\varepsilon)|$, with the other cases handled similarly. Denote by $C$ a constant depending only on $\lambda$, whose precise value might change from place to place. We have $\log|\ell^+(x, \lambda + i\varepsilon)| \leq C + \log|x|$, so it remains to give a lower bound. Taking the Taylor expansion of $\ell^+(x, \lambda + i\varepsilon)$ in $\varepsilon$, we get

$$\ell^+(x, \lambda + i\varepsilon) = \ell^+(x, \lambda) + i\varepsilon \partial_\lambda \ell^+(x, \lambda) + O(\varepsilon^2|x|).$$

Since $\partial_\lambda \ell^+(x, \lambda)$ is real, we bound

$$|\ell^+(x, \lambda + i\varepsilon)| \geq \frac{1}{2}(|\ell^+(x, \lambda)| + \varepsilon|\partial_\lambda \ell^+(x, \lambda)|) - C\varepsilon^2|x|. \quad (4.13)$$

Since $\ell^+(x, \lambda), \partial_\lambda \ell^+(x, \lambda)$ are linearly independent linear forms in $x$ (see (2.6)), we have

$$|x| \leq C(|\ell^+(x, \lambda)| + |\partial_\lambda \ell^+(x, \lambda)|). \quad (4.14)$$

Together (4.13) and (4.14) show that for $\varepsilon$ small enough

$$|\ell^+(x, \lambda + i\varepsilon)| \geq \frac{1}{3}|\ell^+(x, \lambda)|$$

which gives the lower bound $\log|\ell^+(x, \lambda + i\varepsilon)| \geq \log|\ell^+(x, \lambda)| - \log 3$. Since $\log|\ell^+(x, \lambda)|$ is locally integrable, this finishes the proof. \qed

Passing to the limit in (4.7) we see that

$$P(\lambda)E_{\lambda \pm i0} = \delta_0 \quad \text{for all} \quad \lambda \in (0, 1). \quad (4.15)$$

Note that $E_{\lambda \pm i0}(x)$ is smooth except on the union of the two lines $\{\ell^+(x, \lambda) = 0\}$ and $\{\ell^-(x, \lambda) = 0\}$. We remark that $E_{\lambda \pm i0}$ are the Feynman propagators in dimension one.
4.4. Reduction to the boundary. We now let $\Omega \subset \mathbb{R}^2$ be a bounded open set with $C^\infty$ boundary and consider the elliptic boundary value problem

$$
P(\lambda)u = f, \quad u|_{\partial\Omega} = 0, \quad \text{Re} \lambda \in (0, 1), \quad \text{Im} \lambda \neq 0. \quad (4.16)
$$

**Lemma 4.4.** For each $f \in C_c^\infty(\Omega)$, the problem (4.16) has a unique solution $u \in C^\infty(\bar{\Omega})$.

**Proof.** 1. We first show that for each $\omega \in \mathbb{C} \setminus [1, \infty)$ and $s \geq 2$, the map

$$
\mathcal{H}^s(\Omega) \ni u \mapsto ((\Delta - \omega\partial^2_{x_2})u, u|_{\partial\Omega}) \in \mathcal{H}^{s-2}(\Omega) \oplus H^{s-\frac{1}{2}}(\partial\Omega)
$$

is a Fredholm operator. (Here $\mathcal{H}^s(\Omega)$ denotes the space of distributions on $\Omega$ which extend to $H^s$ distributions on $\mathbb{R}^2$.) We apply [17, Theorem 20.1.2]. The operator $\Delta - \omega\partial^2_{x_2}$ is elliptic, so it remains to verify that the Shapiro–Lopatinski condition [17, Definition 20.1.1(ii)] holds for any domain $\Omega$. (An example of an operator for which this condition fails is $(\partial_{x_1} + i\partial_{x_2})^2$.) In our specific case the Shapiro–Lopatinski condition can be reformulated as follows: for each basis $(\xi, \eta)$ of $\mathbb{R}^2$, if we denote by $\mathcal{M}$ the space of all bounded solutions on $[0, \infty)$ to the ODE

$$
p(\xi - i\eta\partial_t)u(t) = 0, \quad p(\xi) := \xi_1^2 + (1 - \omega)\xi_2^2
$$

then the map $u \in \mathcal{M} \mapsto u(0)$ is an isomorphism. This is equivalent to the requirement that the quadratic equation $p(\xi + z\eta) = 0$ have two roots, one with $\text{Im} z > 0$ and one with $\text{Im} z < 0$. To see that the latter condition holds, we argue by continuity: since $\Delta - \omega\partial^2_{x_2}$ is elliptic, the equation $p(\xi + z\eta) = 0$ cannot have any real roots $z$, so the condition either holds for all $\omega, \xi, \eta$ or fails for all $\omega, \xi, \eta$. However, it is straightforward to check that the condition holds when $\omega = 0, \xi = (1, 0), \eta = (0, 1)$, as the roots are $\pm i$.

2. We next claim that the Fredholm operator (4.17) is invertible. We first show that it has index 0, arguing by continuity: since the operator (4.17) is continuous in $\omega$ in the operator norm topology, its index should be independent of $\omega$. However, for $\omega = 0$ we get the Dirichlet problem for the Laplacian, where (4.17) is invertible.

To show that (4.17) is invertible it remains to prove injectivity, namely

$$
u \in H^2(\Omega), \quad (\Delta - \omega\partial^2_{x_2})u = 0, \quad u|_{\partial\Omega} = 0 \implies u = 0. \quad (4.18)
$$

Multiplying the equation $(\Delta - \omega\partial^2_{x_2})u = 0$ by $\overline{u}$ and integrating by parts over $\Omega$, we get $\|\nabla u\|_{L^2(\Omega)}^2 = \omega\|\partial_{x_2} u\|_{L^2(\Omega)}^2$. Since $0 \leq \|\partial_{x_2} u\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^2(\Omega)}^2$ and $\omega \not\in [1, \infty)$, we see that $\|\nabla u\|_{L^2(\Omega)} = 0$, which implies that $u = 0$, giving (4.18).

3. Writing

$$
P(\lambda) = \partial^2_{x_2} - \lambda^2 \Delta = -\lambda^2(\Delta - \omega\partial^2_{x_2}), \quad \omega := \lambda^{-2} \in \mathbb{C} \setminus [1, \infty)
$$

and using the invertibility of (4.17), we see that for each $s \geq 2$ and $f \in \mathcal{H}^{s-2}(\Omega)$, the problem (4.16) has a unique solution $u \in \mathcal{H}^{s}(\Omega)$. When $f \in C_c^\infty(\Omega)$, we may take arbitrary $s$ which gives that $u \in C^\infty(\bar{\Omega})$. \qed
We will next express the solution to (4.16) in terms of boundary data and single layer potentials. Let us first define the operators used below. Let $T^*\partial\Omega$ be the cotangent bundle of the boundary $\partial\Omega$. Sections of this bundle are differential 1-forms on $\partial\Omega$ (where we use the positive orientation on $\partial\Omega$); they can be identified with functions on $\partial\Omega$ by fixing a coordinate $\theta$. Define the operator $I: D'(\partial\Omega; T^*\partial\Omega) \to E'(\mathbb{R}^2)$ as follows: for $v \in D'(\partial\Omega; T^*\partial\Omega)$ and $\varphi \in C^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \mathcal{I}v(x)\varphi(x) \, dx := \int_{\partial\Omega} \varphi v. \quad (4.19)$$

Note that supp$(\mathcal{I}v) \subset \partial\Omega$ and we can think of $\mathcal{I}v$ as multiplying $v$ by the delta function on $\partial\Omega$.

Next, let $E_\lambda$ be the fundamental solution constructed in (4.9) and define the convolution operator

$$R_\lambda: \mathcal{E}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2), \quad R_\lambda g(x) := \int_{\mathbb{R}^2} E_\lambda(x - y)g(y) \, dy. \quad (4.20)$$

We can now reduce the problem (4.16) to the boundary:

**Lemma 4.5.** Assume that $u \in C^\infty(\overline{\Omega})$ is the solution to (4.16) for some $f \in C^\infty_c(\Omega)$. Denote by $j: \partial\Omega \to \overline{\Omega}$ the embedding map and let

$$v := -2\lambda \sqrt{1 - \lambda^2} j^*(L_\lambda^+ u \, dl^+) \in C^\infty(\partial\Omega; T^*\partial\Omega). \quad (4.21)$$

Put $U := \mathbb{1}_\Omega u \in \mathcal{E}'(\mathbb{R}^2)$. Then

$$P(\lambda)U = f - \mathcal{I}v, \quad (4.22)$$

$$U = R_\lambda f - R_\lambda \mathcal{I}v. \quad (4.23)$$

**Remark.** Note that we also have

$$v = 2\lambda \sqrt{1 - \lambda^2} j^*(L_\lambda^- u \, dl^-).$$

Indeed, $0 = j^* du = j^*(L_\lambda^+ u \, dl^+ + L_\lambda^- u \, dl^-)$ since $u|_{\partial\Omega} = 0$ and by (4.4).

**Proof.** Let $\varphi \in C^\infty_c(\mathbb{R}^2)$, then by (4.3)

$$\int_{\mathbb{R}^2} (P(\lambda)U)\varphi \, dx = 4 \int_{\Omega} u L_\lambda^+ L_\lambda^- \varphi \, dx = -4 \int_{\Omega} (L_\lambda^+ u)(L_\lambda^- \varphi) \, dx$$

$$= \int_{\Omega} f \varphi \, dx - 4 \int_{\Omega} L_\lambda^- (\varphi L_\lambda^+ u) \, dx$$

$$= \int_{\Omega} f \varphi \, dx + 2\lambda \sqrt{1 - \lambda^2} \int_{\partial\Omega} \varphi L_\lambda^+ u \, dl^+$$

$$= \int_{\Omega} (f - \mathcal{I}v)\varphi \, dx$$
which gives (4.22). The identity (4.23) follows from (4.22), the fundamental solution equation (4.7), and the fact that $U$ is a compactly supported distribution: $E_\lambda * P(\lambda)U = (P(\lambda)E_\lambda) * U = U$. □

In the notation of Lemma 4.5, define $S_\lambda v := (R_\lambda \mathcal{I}v)|_{\Omega} \in \mathcal{D}'(\Omega)$. Then (4.23) implies that

$$u = (R_\lambda f)|_{\Omega} - S_\lambda v. \tag{4.24}$$

Since $R_\lambda f \in C^\infty(\mathbb{R}^2)$, we have $S_\lambda v \in C^\infty(\overline{\Omega})$. Denote by $C_\lambda v := (S_\lambda v)|_{\partial\Omega}$ its boundary trace, then the boundary condition $u|_{\partial\Omega} = 0$ gives the following equation on $v$:

$$C_\lambda v = (R_\lambda f)|_{\partial\Omega}. \tag{4.25}$$

This motivates the study of the operator $S_\lambda$ in §4.5 and of the operator $C_\lambda$ in §4.6.

4.5. **Single layer potentials.** We now introduce single layer potentials. For $\lambda \in \mathbb{C}$ with $0 < \text{Re} \lambda < 1$ and $\text{Im} \lambda \neq 0$ the single layer potential is the operator $S_\lambda : \mathcal{D}'(\partial\Omega; T^*\partial\Omega) \to \mathcal{D}'(\Omega)$ given by

$$S_\lambda v := (E_\lambda * \mathcal{I}v)|_{\Omega}, \quad v \in \mathcal{D}'(\partial\Omega; T^*\partial\Omega). \tag{4.26}$$

Here $E_\lambda \in \mathcal{D}'(\mathbb{R}^2)$ is the fundamental solution defined in (4.9) and the operator $\mathcal{I} : \mathcal{D}'(\partial\Omega; T^*\partial\Omega) \to \mathcal{E}'(\mathbb{R}^2)$ is defined in (4.19). Similarly, if $\lambda \in (0, 1)$ then we can define operators

$$S_{\lambda \pm i0} : \mathcal{D}'(\partial\Omega; T^*\partial\Omega) \to \mathcal{D}'(\Omega) \tag{4.27}$$

by the formula (4.26), using the limiting distributions $E_{\lambda \pm i0}$ defined in (4.12).

If we fix a positively oriented coordinate $\theta$ on $\partial\Omega$ and use it to identify $\mathcal{D}'(\partial\Omega; T^*\partial\Omega)$ with $\mathcal{D}'(\partial\Omega)$, then the action of $S_\lambda$ on smooth functions is given by

$$S_\lambda (f \, d\theta)(x) = \int_{\partial\Omega} E_\lambda(x - y)f(y) \, d\theta(y), \quad f \in C^\infty(\partial\Omega), \quad x \in \Omega \tag{4.28}$$

and similarly for $S_{\lambda \pm i0}$.

We now discuss the mapping properties of $S_\lambda$, in particular showing that $S_\lambda v \in C^\infty(\overline{\Omega})$ when $v \in C^\infty(\partial\Omega; T^*\partial\Omega)$. We break the latter into two cases:

4.5.1. **Non-real $\lambda$.** We first consider the case $\text{Im} \lambda \neq 0$. We use the following standard result, which is a version of the Sochocki–Plemelj theorem:

**Lemma 4.6.** Assume that $\Omega_0 \subset \mathbb{C}$ is a bounded open set with $C^\infty$ boundary (oriented in the positive direction). For $f \in C^\infty(\partial\Omega_0)$, define $u \in C^\infty(\Omega_0)$ by

$$u(z) = \int_{\partial\Omega_0} \frac{f(w) \, dw}{w - z}, \quad z \in \Omega_0.$$

Then $u$ extends smoothly to the boundary and the operator $f \mapsto u$ is continuous $C^\infty(\partial\Omega_0) \to C^\infty(\overline{\Omega}_0)$. 
Remark. In the (unbounded) model case $\Omega_0 = \{ \text{Im } z > 0 \}$, we have for each $f \in C^\infty_c(\mathbb{R})$
\[ u(x + iy) = \int_{\mathbb{R}} \frac{f(t)}{t - x - iy}, \quad y > 0. \]
We see in particular that the function $x \mapsto \lim_{y \to 0^+} \partial_y^k u(x + iy)$ is given by the convolution of $f$ with $(-1)^{k+1} \frac{k!}{(x + i0)^{k+1}}$.

Proof. Let $\tilde{f} \in C^\infty_c(\mathbb{C})$ be an almost analytic extension of $f$: that is, $\tilde{f}|_{\partial \Omega_0} = f$ and $\partial \bar{z} \tilde{f}$ vanishes to infinite order on $\partial \Omega_0$. (See for example [11, Lemma 4.30] for the existence of such an extension.) Denote by $dm$ the Lebesgue measure on $\mathbb{C}$. By the Cauchy–Green formula (see for instance [16, (3.1.11)]), we have
\[ u(z) = 2\pi i \tilde{f}(z) + 2i \int_{\Omega_0} \frac{\partial \bar{w} \tilde{f}(w)}{w - z} \, dm(w), \quad z \in \Omega_0 \]
and this extends smoothly to $z \in \mathbb{C}$: indeed, the second term on the right-hand side is the convolution of the distribution $-2i z^{-1} \in L^1_{\text{loc}}(\mathbb{C})$ with $\mathbb{1}_{\Omega_0} \partial \bar{z} \tilde{f} \in C^\infty_c(\mathbb{C})$. □

We now come back to the mapping properties of single layer potentials:

Lemma 4.7. Assume that $0 < \text{Re } \lambda < 1$ and $\text{Im } \lambda \neq 0$. Then $S_\lambda$ is a continuous operator from $C^\infty(\partial \Omega; T^* \partial \Omega)$ to $C^\infty(\Omega)$.

Remark. With more work, it is possible to show that $S_\lambda$ is actually continuous $C^\infty(\partial \Omega; T^* \partial \Omega) \to C^\infty(\Omega)$ uniformly as $\text{Im } \lambda \to 0$, with limits being the operators $S_{\lambda \pm i0}$. However, our proof of Lemma 4.7 only shows the mapping property for any fixed non-real $\lambda$. This is enough for our purposes since we have weak convergence of $S_{\lambda \pm i\epsilon}$ to $S_{\lambda \pm i0}$ (Lemma 4.3, see also Lemma 4.10 below) and in § 4.6 we analyze the behavior of the restricted single layer potentials uniformly as $\text{Im } \lambda \to 0$.

Proof. Let $v \in C^\infty(\partial \Omega; T^* \partial \Omega)$. Since $E_\lambda$ is smooth on $\mathbb{R}^2 \setminus \{0\}$ and $\mathcal{I}v$ is supported on $\partial \Omega$, we have $S_\lambda v \in C^\infty(\Omega)$. It remains to show that $S_\lambda v$ is smooth up to the boundary, and for this it is enough to verify the smoothness of the derivatives $L^{\pm}_\lambda S_\lambda v$ where $L^{\pm}_\lambda$ are defined in (4.3). By (4.10) we have (suppressing the dependence of $\ell^{\pm}$ on $\lambda$ in the notation)
\[ L^{\pm}_\lambda S_\lambda v(x) = c_\lambda \int_{\partial \Omega} \frac{v(y)}{\ell^{\pm}(x - y)}, \quad x \in \Omega. \]
Since $\text{Im } \lambda \neq 0$, the maps $x \mapsto \ell^{\pm}(x)$ are linear isomorphisms from $\mathbb{R}^2$ onto $\mathbb{C}$ (considered as a real vector space). Using this we write
\[ L^{\pm}_\lambda S_\lambda v(x) = \pm \text{sgn}(\text{Im } \lambda) c_\lambda \int_{\partial \Omega^{\pm}} \frac{f^{\pm}(w)}{z - w} \, dw, \quad z := \ell^{\pm}(x) \in \Omega^{\pm} \] (4.29)
where we put \( \Omega_\pm := \ell^\pm(\Omega) \subset \mathbb{C} \) and define the functions \( f_\pm \in C^\infty(\partial\Omega_\pm) \) by the equality of differential forms \( v(y) = f_\pm(\ell^\pm(y)) d\ell^\pm(y) \) on \( \partial \Omega \). Here \( \partial \Omega_\pm \) are positively oriented and the sign factor \( \pm \text{sgn}(\text{Im } \lambda) \) accounts for the orientation of the map \( \ell^\pm \), see Lemma 4.1.

Now \( L_\lambda^\pm S_\lambda v \) extends smoothly to the boundary by Lemma 4.6. \( \square \)

4.5.2. Real \( \lambda \). We now consider the case \( \lambda \in (0, 1) \):

**Lemma 4.8.** Assume that \( \lambda \in (0, 1) \) and \( \Omega \) is \( \lambda \)-simple (see Definition 1). Then \( S_{\lambda \pm i0} \) are continuous operators from \( C^\infty(\partial\Omega; T^*\partial\Omega) \) to \( C^\infty(\Omega) \).

**Proof.**

1. We focus on the operator \( S_{\lambda+i0} \), noting that \( S_{\lambda-i0} \) is related to it by the identity

\[
S_{\lambda-i0} v = S_{\lambda+i0} v
\]

for all \( v \in \mathcal{D}'(\partial\Omega; T^*\partial\Omega) \).

We again suppress the dependence on \( \lambda \) in the notation, writing simply \( \ell^\pm(x) \) and \( A(x) \). Denoting by \( H(x) = \mathbb{I}_{(0,\infty)}(x) \) the Heaviside function, we can rewrite (4.12) as

\[
\log(A(x) + i0) = \log|\ell^+(x)| + \log|\ell^-(x)| + i\pi H(-A(x)).
\]

We then decompose

\[
S_{\lambda+i0} = c_\lambda(S_\lambda^+ + S_\lambda^- + i\pi S_\lambda^0)
\]

where for all \( x \in \Omega \) and \( v \in C^\infty(\partial\Omega; T^*\partial\Omega) \)

\[
S_\lambda^\pm v(x) = \int_{\partial\Omega} \log|\ell^\pm(x-y)|v(y),
\]

\[
S_\lambda^0 v(x) = \int_{\partial\Omega} H(-A(x-y))v(y).
\]

2. Let \( v \in C^\infty(\partial\Omega; T^*\partial\Omega) \). Fix a positively oriented coordinate \( \theta \) on \( \partial\Omega \) and write \( v = f d\theta \) for some \( f \in C^\infty(\partial\Omega) \). We first analyze \( S_\lambda^\pm v \), writing it as

\[
S_\lambda^\pm v(x) = g_\pm(\ell^\pm(x)), \quad g_\pm(t) := \int_R (\Pi_\lambda^\pm f)(s) \log|t - s| ds
\]

where \( \Pi_\lambda^\pm f \in \mathcal{E}'(\mathbb{R}) \) are the pushforwards of \( f \) by the maps \( \ell^\pm \) defined in (2.9). Let \( \ell^\pm_{\min} < \ell^\pm_{\max} \) be defined in (2.8). By part 1 of Lemma 2.3, \( \Pi_\lambda^\pm f \) is supported in \([\ell^\pm_{\min}, \ell^\pm_{\max}]\) and

\[
\sqrt{(s - \ell^\pm_{\min})(\ell^\pm_{\max} - s)} \Pi_\lambda^\pm f(s) \in C^\infty([\ell^\pm_{\min}, \ell^\pm_{\max}]).
\]

Using Lemma 3.3, we then get

\[
g_\pm \in C^\infty([\ell^\pm_{\min}, \ell^\pm_{\max}])
\]

which implies that \( S_\lambda^\pm v \in C^\infty(\Omega) \).

3. It remains to show that \( S_\lambda^0 v \in C^\infty(\Omega) \). We may assume that \( v = dF \) for some \( F \in C^\infty(\partial\Omega) \), that is \( \int_{\partial\Omega} v = 0 \). Indeed, if we are studying \( S_\lambda^0 v \) near some point
Lemma 4.9. Assume that \( \partial \Omega \) which are transverse to the boundary \( \Sigma \). Let \( \Gamma^{\pm} \) for \( x \in \Omega \) such that \( A(x - y) < 0 \).

\( \Omega \) then we may take \( y_0 \in \partial \Omega \) such that \( A(x_0 - y_0) > 0 \) and change \( v \) in a small neighborhood of \( y_0 \) so that \( S_\Lambda^0 v(x) \) does not change for \( x \) near \( x_0 \) and \( v \) integrates to 0.

For \( s \in (\ell^{\pm}_{\min}, \ell^{\pm}_{\max}) \), define \( x^{\pm}(s), x^{\pm}(s) \in \partial \Omega \) by

\[
\ell^{\pm}(x^{\pm}(s)) = \ell^{\pm}(x^{\pm}(s)) = s, \quad \ell^{\mp}(x^{\pm}(s)) < \ell^{\pm}(x^{\pm}(s)).
\]

Then for any \( x \in \Omega \), the set of \( y \in \partial \Omega \) such that \( A(x - y) < 0 \) consists of two intervals of the circle \( \partial \Omega \), from \( x^{\pm}(1)(\ell^{\pm}(x)) \) to \( x^{\pm}(2)(\ell^{\pm}(x)) \) (with respect to the positive orientation on \( \partial \Omega \)) and from \( x^{\pm}(2)(\ell^{\pm}(x)) \) to \( x^{\pm}(1)(\ell^{\pm}(x)) \) – see Figure 10. Since \( v = dF \), we compute for \( x \in \Omega \)

\[
S_\Lambda^0 v(x) = F_-(\ell^{\pm}(x)) - F_+(\ell^{\pm}(x)), \quad F_{\pm}(s) := F(x^{\pm}(s)) + F(x^{\pm}(s)).
\]

By part 2 of Lemma 2.3, we have \( F_{\pm} = T_{\Lambda}^\pm F \in C^\infty([\ell^{\pm}_{\min}, \ell^{\pm}_{\max}]) \). Thus \( S_\Lambda^0 v \in C^\infty(\overline{\Omega}) \) as needed.

4.5.3. Conormal singularities. We now study the action of \( S_{\Lambda + i0} \) on conormal distributions. Assume that \( \lambda \in (0, 1) \) and \( \Omega \) is \( \lambda \)-simple. Let \( \Sigma \subset \partial \Omega \) be a finite set such that \( \Sigma \cap \mathcal{C}_\Lambda = \emptyset \), where the characteristic set \( \mathcal{C}_\Lambda \) was defined in (2.2). We define the positive/negative halves of the conormal bundle \( N^+_\Sigma \subset T^* \partial \Omega \) using the positive orientation. Let \( \Gamma_{\Lambda}^+(\Sigma) \subset \Omega \) be defined using (1.7); these are one-dimensional submanifolds of \( \Omega \) which are transverse to the boundary \( \partial \Omega \).

**Lemma 4.9.** Assume that \( f \in I^*(\partial \Omega, N^+_\Sigma) \). Then \( S_{\Lambda + i0} f \in \mathcal{I}^{s - \frac{2}{3}}(\overline{\Omega}, N^+_\Sigma \Gamma_{\Lambda}^+(\Sigma)) \), where \( N^+_\Sigma \Gamma_{\Lambda}^+(\Sigma) \) is defined in (1.8).

**Proof.** We may assume that \( \Sigma = \{y_0\} \) where \( y_0 \in \partial \Omega \setminus \mathcal{C}_\Lambda \). By Lemma 4.8 and since \( f \) is smooth away from \( y_0 \), we may further assume that

\[
\text{supp} f \subset U := \{y \in \partial \Omega \mid \nu^+(y) = \nu^+(y_0), \nu^-(y) = \nu^-(y_0)\}
\]
where \( \nu^\pm(y) = \text{sgn} \partial_\theta \ell^\pm(y) \), see (2.3). We denote \( \nu^+ := \nu^+(y_0), \nu^- := \nu^-(y_0) \). By Lemma 2.2, for each \( y \in U \) and \( x \in \Omega \) we have

\[
\nu^+ \ell^-(x-y) > 0 \quad \text{or} \quad \nu^- \ell^+(x-y) < 0 \quad \text{(or both)}.
\]

Suppose it is the latter at some point \( x = x_0 \in \Omega, y = y_0 \in U \). Then for some \( M \gg 1 \), and with the argument \( x-y, x, y \) near \( x_0, y_0 \), respectively, we have

\[
A + i\varepsilon = \ell^+ \ell^- + i\varepsilon = \ell^+ \ell^- + i\varepsilon (\nu^+ \ell^- - i\nu^- M \ell^+)
\]

\[
= (\ell^+ + i\nu^+ \varepsilon)(\ell^- - i\nu^- \varepsilon M) + O(\varepsilon^2), \quad \varepsilon \approx \varepsilon > 0,
\]

with the same argument when \( \nu^+ \ell^-(x-y) > 0 \). Hence (as always with the branch of \( \log \) real on the positive real axis),

\[
\log(A(x-y) + i0) = \log(\ell^+(x-y) + i\nu^+0) + \log(\ell^-(x-y) - i\nu^-0) + c_0,
\]

\[
c_0 = \begin{cases} 2\pi i, & \text{if } \nu^+ = -1 \quad \text{and} \quad \nu^- = 1; \\ 0, & \text{otherwise.} \end{cases}
\]

Similarly to Step 2 in the proof of Lemma 4.8 we then have

\[
S_{\lambda+0} f(x) = c_\lambda \left( g_+(\ell^+(x)) + g_-(\ell^-(x)) + c_0 \int_{\partial \Omega} f(y) \, d\theta(y) \right)
\]

where, denoting \( \log^+ x := \log(x+i0) \), \( \log^- x := \log(x-i0) \) and using (2.9),

\[
g_+ := (\Pi^+_\lambda f) * \log^+ \nu, \quad g_- := (\Pi^-_\lambda f) * \log^- \nu. \tag{4.30}
\]

(Here, \( \pm \nu^* \) is meant at \( \pm \) if \( \nu^* = 1 \) and \( \mp \) when \( \nu^* = -1 \).)

Since \( f \in I^s(\partial \Omega, N^*_{\pm \nu}(y_0)) \) is supported away from \( \mathcal{C}_\lambda, \ell^\pm: \partial \Omega \to \mathbb{R} \) are diffeomorphisms and hence

\[
\Pi^+_\lambda f \in I^s(\mathbb{R}, N^*_{\pm \nu}(\ell^+(y_0))), \quad \Pi^-_\lambda f \in I^s(\mathbb{R}, N^*_{\pm \nu}(\ell^-(y_0))).
\]

By Lemma 3.4, we see that

\[
g_\pm \in I^{s-1}(\mathbb{R}, N^*_{\pm \nu}(\ell^\pm(y_0))), \quad g_\mp \in C^\infty.
\]

This implies that \( S_{\lambda+0} f \in I^{s-\frac{1}{2}}(\overline{\Omega}, N^*_{\pm \nu}(\ell^\pm(y_0))) \) as needed. \( \square \)

**Remark.** In §7 we will apply this result to elements of

\[
I^s(\partial \Omega, N^*_{\pm \nu}(\Sigma^\pm_\lambda)) = I^s(\partial \Omega, N^*_{\pm \nu}(\Sigma^\pm_\lambda)) + I^s(\partial \Omega, N^*_{\pm \nu}(\Sigma^\pm_\lambda)).
\]

(The equality from the disjointness of the sets \( \Sigma^\pm_\lambda \) defined in (1.6).) Lemma 4.9 then gives

\[
S_{\lambda+0} : I^s(\partial \Omega, N^*_{\pm \nu}(\Sigma^\pm_\lambda)) \to I^{s-\frac{1}{2}}(\overline{\Omega}, \Lambda^\pm(\lambda)),
\]

\[
\Lambda^\pm(\lambda) := N^*_{\pm \nu}(\Sigma^\pm_\lambda) \cup N^*_{\pm \nu}(\Sigma^\pm_\lambda). \tag{4.31}
\]
4.6. The restricted single layer potentials. We now study the restricted operators
\[ C_\lambda : C^\infty(\partial\Omega; T^*\partial\Omega) \to C^\infty(\partial\Omega), \quad C_\lambda v := (S_\lambda v)|_{\partial\Omega}, \tag{4.32} \]
given by the boundary trace of \( S_\lambda v \in C^\infty(\bar{\Omega}) \), see Lemma 4.7. When \( \omega \) is real and \( \Omega \) is \( \omega \)-simple (see Definition 1) we have two operators \( C_{\omega \pm i0} \) obtained by restricting \( S_{\omega \pm i0} \), see Lemma 4.8. From (4.28) we have for \( v \in C^\infty(\partial\Omega; T^*\partial\Omega) \)
\[ C_\lambda v(x) = \int_{\partial\Omega} E_\lambda(x-y) v(y), \quad x \in \partial\Omega, \tag{4.33} \]
with the integration in \( y \), and same is true for \( \lambda \) replaced with \( \omega \pm i0 \). Later in (4.66) we show that \( C_\lambda \) and \( C_{\omega \pm i0} \) extend to continuous operators \( D'(\partial\Omega; T^*\partial\Omega) \to D'(\partial\Omega) \).

In this section we assume that \( \lambda = \omega + i\varepsilon, \quad \varepsilon > 0, \tag{4.34} \)
where \( \omega \in (0,1) \) is fixed and \( \Omega \) is \( \omega \)-simple. Our main result here is a microlocal description of \( dC_\lambda \) uniformly as \( \varepsilon \to 0+ \), see Proposition 4.15 below. We note that \( C_{\omega+i\varepsilon} \to C_{\omega+i0} \) as \( \varepsilon \to 0+ \) in the following weak sense:

**Lemma 4.10.** Assume that \( v, w \in C^\infty(\partial\Omega; T^*\partial\Omega) \). Then
\[ \int_{\partial\Omega} (C_{\omega+i\varepsilon} v) w \to \int_{\partial\Omega} (C_{\omega+i0} v) w \quad \text{as} \quad \varepsilon \to 0+. \]

**Proof.** Given (4.33), it suffices to show that
\[ C_{\omega+i\varepsilon}(x,y) \to C_{\omega+i0}(x,y) \quad \text{in} \quad L^1(\partial\Omega \times \partial\Omega) \]
where we define \( C_{\omega+i\varepsilon}(x,y) := E_{\omega+i\varepsilon}(x-y), \quad x, y \in \partial\Omega, \) and similarly for \( C_{\omega+i0} \). This is done similarly to the proof of Lemma 4.3, where we use that the functions \( \text{log}|\ell^\pm(x-y,\omega)| \) are integrable in \( (x,y) \in \partial\Omega \times \partial\Omega \), which follows from \( \omega \)-simplicity of \( \Omega \). \( \square \)

For convenience, we fix a positively oriented coordinate \( \theta \in S^1 \) on \( \partial\Omega \) and identify 1-forms on \( \partial\Omega \) with functions on \( S^1 \) by writing \( v = f(\theta) d\theta \). Let \( x : S^1 \to \partial\Omega \) be the corresponding parametrization map. Let
\[ \gamma^\pm_\omega : S^1 \to S^1, \quad \gamma^\pm(x(\theta),\omega) = x(\gamma^\pm_\omega(\theta)) \tag{4.35} \]
be the orientation reversing involutions on \( S^1 \) induced by the maps \( \gamma^\pm(\cdot,\omega) \) defined in (1.3).
4.6.1. **Decomposition into** $T^±_λ$. Since the linear functions $ℓ^±(x, λ)$ are dual to the vector fields $L^±_λ$ (see (4.4)), we have

$$dC_λ = T^+_λ + T^-_λ$$

(4.36)

where the operators $T^±_λ : C^∞(∂Ω; T^*∂Ω) → C^∞(∂Ω; T^*∂Ω)$ are given by (with $j$ the embedding map)

$$T^±_λ v = j^∗((L^±_λ S_λ v)dℓ^±), \quad j : ∂Ω → Ω.$$  

(4.37)

Let $K^±_λ(θ, θ') ∈ D'(S^1 × S^1)$ be the Schwartz kernel of $T^±_λ$, that is

$$T^±_λ v(θ) = (∂_θ ℓ^±(x(θ), λ))L^±_λ S_λ v(x(θ)) dθ = \left( ∫_{S^1} K^±_λ(θ, θ') f(θ') dθ' \right) dθ,$$

(4.38)

where we put $v = f(θ) dθ$. Recalling the integral definition (4.28) of $S_λ$, the formula (4.9) for $E_λ$ (which in particular shows that $E_λ$ is smooth on $R^2 \setminus \{0\}$), and the identity (4.10), we see that $K^±_λ$ is smooth on $(S^1 × S^1) \setminus \{θ ≠ θ'\}$ and

$$K^±_λ(θ, θ') = c_λ \frac{∂_θ ℓ^±(x(θ), λ)}{ℓ^±(x(θ) − x(θ'), λ)}, \quad θ ≠ θ'.$$

(4.39)

4.6.2. **Away from the singularities.** Define the sets

$$\text{Diag} := \{((θ, θ) | θ ∈ S^1\},$$

$$\text{Ref}_ω^± := \{((θ, γ^±_ω(θ)) | θ ∈ S^1\}. \quad (4.40)$$

Note that the intersection

$$\text{Diag} ∩ \text{Ref}_ω^± = \{((θ, θ) | θ ∈ S^1, ∂_θ ℓ^±(x(θ), ω) = 0\}$$

(4.41)

corresponds to the critical points $x^±_\text{min}(ω), x^±_\text{max}(ω)$ of $ℓ^±(•, ω)$ on $∂Ω$ (see Definition 1). At these points the operator $P(ω)$ is characteristic with respect to $∂Ω$.

We start the analysis of the uniform behavior of $K^±_λ$ as $ε = \text{Im} λ → 0$ by showing that the singularities are contained in $\text{Diag} ∪ \text{Ref}_ω^±$:

**Lemma 4.11.** We have

$$K^±_λ|_{(S^1 × S^1) \setminus (\text{Diag} ∪ \text{Ref}_ω^±)} ∈ C^∞((S^1 × S^1) \setminus (\text{Diag} ∪ \text{Ref}_ω^±))$$

smoothly in $ε$ up to $ε = 0$.

**Proof.** This follows immediately from (4.39). Indeed, for $(θ, θ') ∉ \text{Diag} ∪ \text{Ref}_ω^±$, we have $ℓ^±(x(θ), ω) ≠ ℓ^±(x(θ'), ω)$ and thus the denominator in (4.39) is nonvanishing when $ε = 0$. □
4.6.3. Noncharacteristic diagonal. We next consider the singularities of $K^\pm_\lambda(\theta, \theta')$ near
the diagonal but away from the characteristic set $\text{Diag} \cap \text{Ref}^\pm_\omega$. In that case the structure of the kernel is similar to the model case (4.6):

**Lemma 4.12.** Take $\theta_0 \in \mathbb{S}^1$ such that $\gamma^\pm_\omega(\theta_0) \neq \theta_0$. Then for $\theta, \theta'$ in some neighborhood $U$ of $\theta_0$ and $\varepsilon = \text{Im} \lambda > 0$ small enough, we have

$$K^\pm_\lambda(\theta, \theta') = c_\lambda(\theta - \theta' \pm i0)^{-1} + \mathcal{K}^\pm_\lambda(\theta, \theta'),$$

(4.42)

where $\mathcal{K}^\pm_\lambda \in C^\infty(U \times U)$ is smooth in $\theta, \theta', \varepsilon$ up to $\varepsilon = 0$.

**Proof.** 1. Fix some smooth vector field $v(\theta)$ on $\partial \Omega$ which points inwards. We have for all $v = f(\theta) d\theta \in C^\infty(\partial \Omega; T^* \Omega)$,

$$\int_{\mathbb{S}^1} K^\pm_\lambda(\theta, \theta') f(\theta') d\theta' = \left( \partial_{\theta} \ell^\pm(\theta(\theta), \lambda) \right) \lim_{\delta \to 0^+} L^\pm_\lambda S^\pm_\lambda v(\theta(\theta) + \delta v(\theta), \lambda)$$

$$= c_\lambda \left( \partial_{\theta} \ell^\pm(\theta(\theta), \lambda) \right) \lim_{\delta \to 0^+} \int_{\mathbb{S}^1} \ell^\pm(\theta(\theta) - \theta(\theta') + \delta v(\theta, \lambda)) d\theta'$$

where the limit is in $C^\infty(\mathbb{S}^1)$. Here in the first equality we use the definition (4.38) of $K^\pm_\lambda$ (recalling that $S^\pm v \in C^\infty(\Omega)$ by Lemma 4.7). In the second equality we use the definition (4.28) of $S_\lambda$, the formula (4.9) for $E_\lambda$, and the identity (4.10).

Since $\partial_{\theta} \ell^\pm(\theta(\theta), \omega) \neq 0$ at $\theta = \theta_0$, we factorize for $\theta, \theta'$ in some neighborhood $U$ of $\theta_0$ and $\varepsilon = \text{Im} \lambda$ small enough

$$\ell^\pm(\theta(\theta) - \theta(\theta'), \lambda) = G^\pm_\lambda(\theta, \theta')(\theta - \theta')$$

where $G^\pm_\lambda(\theta, \theta')$ is a nonvanishing smooth function of $\theta, \theta', \varepsilon$ up to $\varepsilon = 0$ and

$$G^\pm_\lambda(\theta, \theta) = \partial_{\theta} \ell^\pm(\theta(\theta), \lambda), \quad \theta \in U.$$ (4.43)

Therefore, for $(\theta, \theta') \in U \times U$ we have

$$K^\pm_\lambda(\theta, \theta') = \frac{c_\lambda \partial_{\theta} \ell^\pm(\theta(\theta), \lambda)}{G^\pm_\lambda(\theta, \theta')} \lim_{\delta \to 0^+} \left( \theta - \theta' + \delta \ell^\pm(\theta(\theta), \lambda) \right)^{-1}$$ (4.44)

with the limit in $\mathcal{D}'(U \times U)$.

2. We next claim that if $U$ is a small enough neighborhood of $\theta_0$, then for all $(\theta, \theta') \in U \times U$ and $\text{Im} \lambda = \varepsilon > 0$ small enough

$$\pm \text{Im} \frac{\ell^\pm(\theta(\theta), \lambda)}{G^\pm_\lambda(\theta, \theta')} > 0.$$ (4.45)

When $\lambda = \omega$ is real, the expression (4.45) is equal to 0. Thus it suffices to check that for all $(\theta, \theta') \in U \times U$

$$\pm \partial_\varepsilon |_{\varepsilon=0} \text{Im} \frac{\ell^\pm(\theta(\theta), \omega + i\varepsilon)}{C^\pm_\omega + i\varepsilon(\theta, \theta')} > 0.$$ (4.46)
It is enough to consider the case $\theta = \theta' = \theta_0$, in which case the left-hand side of (4.46) equals
\[
\pm \partial_\varepsilon|_{\varepsilon=0} \mathrm{Im} \left[ \frac{\ell^\pm(\v(\theta_0), \omega + i\v)}{\ell^\pm(\partial_\v \v(\theta_0), \omega + i\v)} \right].
\]
By (2.6) and since $\ell^\pm$ is holomorphic in $\lambda$ it then suffices to check that
\[
\pm \left( \ell^+(\v(\theta_0), \omega)\ell^\pm(\partial_\v \v(\theta_0), \omega) - \ell^+(\v(\theta_0), \omega)\ell^+(\partial_\v \v(\theta_0), \omega) \right) > 0. \tag{4.47}
\]
The inequality (4.47) follows from the fact that $x \mapsto (\ell^+(x, \omega), \ell^-(x, \omega))$ is an orientation preserving linear map on $\mathbb{R}^2$ and $\partial_\v \v(\theta_0), \v(\theta_0)$ form a positively oriented basis of $\mathbb{R}^2$ since the parametrization $\v(x)$ is positively oriented and $\v(\theta)$ points inside $\Omega$. This finishes the proof of (4.45).

3. By Lemma 3.6 (see also (3.26)), with $\delta$ taking the role of $\v$, the distributional limit on the right-hand side of (4.44) is equal to $(\theta - \theta' \pm i0)^{-1}$. Therefore
\[
K_\theta^\pm(\theta, \theta') = \frac{c_\v \partial_\v \ell^\pm(\v(x), \lambda)}{G_\lambda^\pm(\theta, \theta')} (\theta - \theta' \pm i0)^{-1}. \tag{4.48}
\]
By (4.43) we can write for some $H_\theta^\pm(\theta, \theta')$ which is smooth in $\theta, \theta', \v$ up to $\v = 0$,
\[
c_\v \partial_\v \ell^\pm(\v(x), \lambda) \bigg/ G_\lambda^\pm(\theta, \theta') = c_\v + H_\theta^\pm(\theta, \theta')(\theta - \theta'),
\]
which gives (4.42) since $(\theta - \theta')(\theta - \theta' \pm i0)^{-1} = 1$.

4.6.4. Noncharacteristic reflection. We now move to the singularities on the reflection sets $\text{Ref}_\omega^\pm$, again staying away from the characteristic set $\text{Diag} \cap \text{Ref}_\omega^\pm$:

**Lemma 4.13.** Take $\theta_0 \in S^1$ such that $\gamma_\omega^\pm(\theta_0) \neq \theta_0$. Then there exists neighborhoods $U, U' = \gamma_\omega^\pm(U)$ of $\gamma_\omega^\pm(\theta_0), \theta_0$ such that for $(\theta, \theta') \in U \times U'$ and $\v = \text{Im} \lambda > 0$ small enough, we have
\[
K_\theta^\pm(\theta, \theta') = \bar{c}_\lambda^\pm(\theta')(\gamma_\omega^\pm(\theta) - \theta' \pm i\v z_\lambda^\pm(\theta'))^{-1} + H_\theta^\pm(\theta, \theta'), \tag{4.49}
\]
where $H_\theta^\pm \in C^\infty(U \times U')$ is smooth in $\theta, \theta', \v$ up to $\v = 0$, the functions $\bar{c}_\lambda^\pm(\theta')$ and $z_\lambda^\pm(\theta')$ are smooth in $\theta', \v$ up to $\v = 0$, and
\[
\bar{c}_\lambda^\pm(\theta') = \frac{c_\v}{\partial_\theta \gamma_\omega^\pm(\theta')} + O(\v), \quad \text{Re} z_\lambda^\pm(\theta') \geq c > 0 \tag{4.50}
\]
where $c$ is independent of $\v, \theta'$.

**Proof.** 1. Recall that $\lambda = \omega + i\v$. We take Taylor expansions of $\ell^\pm(x, \lambda)$ at $\v = 0$, using its holomorphy in $\lambda$:
\[
\ell^\pm(x, \lambda) = \ell^\pm(x, \omega) + i\v \ell_1^\pm(x, \omega) + \v^2 \ell_2^\pm(x, \omega, \v), \quad \ell_1^\pm(x, \omega) := \partial_\v \ell^\pm(x, \omega) \tag{4.51}
\]
where the coefficients of the linear maps $x \mapsto \ell_\pm^\pm(x,\omega,\varepsilon)$ are smooth in $\varepsilon$ up to $\varepsilon = 0$. Since $\partial_\theta \ell_\pm^\pm(x(\theta),\omega) \neq 0$ at $\theta = \theta_0$, we factorize for $\theta, \theta'$ in some neighborhoods $U, U' = \gamma_\omega^\pm(U)$ of $\gamma_\omega^\pm(\theta_0), \theta_0$

$$
\ell_\pm^\pm(x(\theta) - x(\theta'), \omega) = \ell_\pm^\pm(x(\gamma_\omega^\pm(\theta)) - x(\theta'), \omega) = G_\omega^\pm(\theta, \theta')(\gamma_\omega^\pm(\theta) - \theta')
$$

where $G_\omega^\pm \in C^\infty(U \times U'; \mathbb{R})$ is nonvanishing and

$$
G_\omega^\pm(\gamma_\omega^\pm(\theta'), \theta') = \partial_\theta \ell_\pm^\pm(x(\theta'), \omega). \tag{4.52}
$$

Hence for $(\theta, \theta') \in U \times U'$

$$
K_\omega^\pm(\theta, \theta') = F_\omega^\pm(\theta, \theta')(\gamma_\omega^\pm(\theta) - \theta' \pm i\varepsilon \psi_\omega^\pm(\theta, \theta'))^{-1}, \quad F_\omega^\pm(\theta, \theta') := c_\lambda \frac{\partial_\theta \ell_\pm^\pm(x(\theta), \lambda)}{G_\omega^\pm(\theta, \theta')}. \quad \tag{4.53}
$$

Note that $\psi_\omega^\pm(\theta, \theta')$ and $F_\omega^\pm(\theta, \theta')$ are smooth in $\theta, \theta', \varepsilon$ up to $\varepsilon = 0$.

2. We next claim that $\text{Re} \psi_\omega^\pm(\theta, \theta') \geq \epsilon > 0$ for $\varepsilon$ small enough and $(\theta, \theta') \in U \times U'$, if $U, U'$ are sufficiently small neighborhoods of $\gamma_\omega^\pm(\theta_0), \theta_0$. For that it suffices to show that

$$
\pm \frac{\ell_1^\pm(x(\gamma_\omega^\pm(\theta_0)) - x(\theta_0), \omega)}{G_\omega^\pm(\gamma_\omega^\pm(\theta_0), \theta_0)} > 0. \tag{4.53}
$$

By (2.6) and (4.52), and since $\ell_\pm^\pm(x(\gamma_\omega^\pm(\theta_0)) - x(\theta_0), \omega) = 0$, the left-hand side of (4.53) has the same sign as

$$
\pm \frac{\ell_\pm^\pm(x(\gamma_\omega^\pm(\theta_0)) - x(\theta_0), \omega)}{\partial_\theta \ell_\pm^\pm(x(\theta), \omega)}|_{\theta = \theta_0}
$$

which is positive by (2.4) with $x := x(\theta_0)$.

3. Now (4.49) and the second part of (4.50) follow from Lemma 3.5, see also (3.26) where we replace $\theta$ with $\gamma_\omega^\pm(\theta)$. Finally, by (4.52) and differentiating the identity $\ell_\pm^\pm(x(\gamma_\omega^\pm(\theta')), \omega) = \ell_\pm^\pm(x(\theta'), \omega)$ in $\theta'$ we compute

$$
F_\omega^\pm(\gamma_\omega^\pm(\theta'), \theta') = c_\lambda \frac{\partial_\theta \ell_\pm^\pm(x(\theta), \omega)}{\partial_\theta \ell_\pm^\pm(x(\theta'), \omega)}|_{\theta = \gamma_\omega^\pm(\theta')} + \mathcal{O}(\varepsilon) = \frac{c_\lambda}{\partial_\theta \gamma_\omega^\pm(\theta')} + \mathcal{O}(\varepsilon)
$$

which gives the first part of (4.50). \hfill \Box

4.6.5. Characteristic points. We finally study the singularities of $K_\omega^\pm$ near the characteristic set $\text{Diag} \cap \text{Ref}_\omega^\pm$. Recalling (4.41), we see that this set consists of two points $(\theta_{\min}^\pm, \theta_{\min}^\pm)$ and $(\theta_{\max}^\pm, \theta_{\max}^\pm)$ where $x(\theta_{\min}^\pm) = x_{\min}^\pm(\omega), x(\theta_{\max}^\pm) = x_{\max}^\pm(\omega)$ are the critical points of $\ell_\pm^\pm(\cdot, \omega)$ (see Definition 1).
Lemma 4.14. Assume that \( \theta_0 \in \{ \theta^\pm_{\min}, \theta^\pm_{\max} \} \). Then there exists a neighborhood \( U = \gamma_\omega^\pm(U) \) of \( \theta_0 \) such that for \( (\theta, \theta') \in U \times U \) and \( \varepsilon = \text{Im} \lambda > 0 \) small enough, we have
\[
K_\omega^\pm(\theta, \theta') = c_\lambda(\theta - \theta' \pm 10)^{-1} + c_\lambda^\pm(\theta')^{(\gamma_\omega^\pm(\theta) - \theta' \pm i\varepsilon z_\omega^\pm(\theta'))^{-1} + \mathcal{K}_\omega^\pm(\theta, \theta') \quad (4.54)
\]
where \( \mathcal{K}_\omega^\pm \in C^\infty(U \times U) \) is smooth in \( \theta, \theta', \varepsilon \) up to \( \varepsilon = 0 \), and \( \mathcal{K}_\omega^\pm \) are smooth in \( \theta', \varepsilon \) up to \( \varepsilon = 0 \), and (4.50) holds.

Remarks. 1. Note that Lemma 4.14 implies Lemmas 4.12 and 4.13 in a neighborhood of the characteristic set, since the first term on the right-hand side of (4.54) is smooth away from the diagonal \text{Diag} and the second term is smooth (uniformly in \( \varepsilon \)) away from the reflection set \( \text{Ref}_\omega^\pm \).

2. Since keeping track of the signs is frustrating we present a model situation: \( \ell^+(x) = x_1 + i\varepsilon x_2, \ell^-(x) = x_2 + i\varepsilon x_1 \) (which is compatible with Lemma 4.1) and \( \partial \Omega \) which near \((0, 0)\) is given by
\[
x_1 = q(x_2), \quad q(0) = q'(0) = 0, \quad q''(0) < 0.
\]
This corresponds to the point \( \theta^\pm_{\max} \), since when \( \varepsilon = 0 \) the function \( \ell^+(x) = x_1 \) has a nondegenerate maximum on \( \partial \Omega \).

We can use \( \theta = x_2 \) as a positively oriented parametrization of \( \partial \Omega \) near \((0, 0)\). In that case the involution \( \gamma^+(\theta) \) is given by
\[
q(\gamma^+(\theta)) = q(\theta), \quad \gamma^+(\theta) = -\theta + \mathcal{O}(\theta^2).
\]
This gives
\[
q(\theta) - q(\theta') = Q(\theta, \theta')(\gamma^+(\theta) - \theta'), \quad Q(0, 0) = -\frac{q''(0)}{2} > 0.
\]
The Schwartz kernel of the model restricted single layer potential \( \mathcal{C} \) is given by (with \( Q = Q(\theta, \theta') \) and neglecting the overall constant \( c_\lambda \) in (4.9))
\[
K(\theta, \theta') = \log \left( \ell^+(x(\theta) - x(\theta')) \ell^-(x(\theta) - x(\theta')) \right)
= \log \left( (q(\theta) - q(\theta') + i\varepsilon(\theta - \theta'))(\theta - \theta' + i\varepsilon(q(\theta) - q(\theta'))) \right)
= \log \left( (\theta - \theta')^2(Q(\gamma^+(\theta) - \theta') + i\varepsilon)(1 + i\varepsilon Q(\gamma^+(\theta) - \theta')) \right)
= 2\log |\theta - \theta'| + \log(\gamma^+(\theta) - \theta' + i\varepsilon Q^{-1})
+ \log(1 + i\varepsilon Q(\gamma^+(\theta) - \theta')) + \log Q.
\]
Hence (see §4.2) the Schwartz kernel of \( \partial_\theta \mathcal{C} \) is
\[
\partial_\theta K(\theta, \theta') = \sum_{\pm} (\theta - \theta' \pm i0)^{-1} + \partial_\theta \gamma^+(\theta) + i\varepsilon \partial_\theta Q^{-1}(\theta, \theta') \gamma^+(\theta) - \theta' + i\varepsilon Q^{-1}(\theta, \theta') + \mathcal{K}(\theta, \theta')
\]
where \( Q(0, 0) > 0 \) and \( \mathcal{K} \in C^\infty \) uniformly in \( \varepsilon \). This is consistent with (4.54) and (4.50), where we use Lemma 3.5 and recall that by (4.36) we have \( \partial_\theta K = K^+ + K^- \).
Proof of Lemma 4.14. 1. Recall that $\lambda = \omega + i\varepsilon$, $\varepsilon > 0$ and consider the expansion (4.51):

$$\ell^\pm(x, \lambda) = \ell^\pm(x, \omega) + i\varepsilon \ell_1^\pm(x, \omega) + \varepsilon^2 \ell_2^\pm(x, \omega, \varepsilon).$$

We have for $\theta, \theta'$ in a sufficiently small neighborhood $U$ of $\theta_0 \in \{\theta_{\min}^\pm, \theta_{\max}^\pm\}$

$$\ell^\pm(x(\theta) - x(\theta'), \omega) = G_0(\theta, \theta')(\gamma_\omega^\pm(\theta) - \theta')(\theta - \theta'),$$

$$\ell_1^\pm(x(\theta) - x(\theta'), \omega) = G_1(\theta, \theta')(\theta - \theta'),$$

$$\ell_2^\pm(x(\theta) - x(\theta'), \omega, \varepsilon) = G_2(\theta, \theta', \varepsilon)(\theta - \theta'),$$

where $G_0, G_1, G_2$ are smooth in $\theta, \theta', \varepsilon$ up to $\varepsilon = 0$, and $G_0, G_1$ are real-valued and nonvanishing. Indeed, the first decomposition follows from (2.1) and the second one, from (2.6) and the fact that $\partial_\theta \ell^\pm(x(\theta), \omega) \neq 0$ at $\theta = \theta_0$. We have now (with $G_j = G_j(\theta, \theta')$)

$$\ell^\pm(x(\theta) - x(\theta'), \lambda) = (\theta - \theta')(G_0(\gamma_\omega^\pm(\theta) - \theta') + i\varepsilon G_1 + \varepsilon^2 G_2). \quad (4.55)$$

2. The argument in the proof of Lemma 4.12 (see (4.48)) shows that for any fixed small $\varepsilon > 0$

$$K_\lambda^\pm(\theta, \theta') = \frac{c_x \partial_\theta \ell^\pm(x(\theta), \lambda)}{G_0(\gamma_\omega^\pm(\theta) - \theta') + i\varepsilon G_1 + \varepsilon^2 G_2}(\theta - \theta' \pm i0)^{-1}. \quad (4.56)$$

To apply this argument we need to check the condition (4.45), which we rewrite as

$$\pm \text{Im} \frac{G_0(\gamma_\omega^\pm(\theta) - \theta') + i\varepsilon G_1 + \varepsilon^2 G_2}{\ell^\pm(x(\theta), \lambda)} < 0 \quad (4.57)$$

for $\theta, \theta'$ near $\theta_0$, $\varepsilon = \text{Im} \lambda > 0$ small enough, and $v(\theta)$ an inward pointing vector field on $\partial \Omega$. Here the denominator is separated away from zero since $\ell^\pm(v(\theta_0), \omega) \neq 0$.

For $\varepsilon = 0$, the expression (4.57) is equal to 0. Thus is suffices to check the sign of its derivative in $\varepsilon$ at $\varepsilon = 0$ and $\theta = \theta' = \theta_0$, i.e. show that (where we use (2.6))

$$\pm \ell^\pm(v(\theta_0), \omega)\ell^\mp(\partial_\theta x(\theta_0), \omega) < 0. \quad (4.58)$$

The latter follows from the fact that $\ell^\pm(\partial_\theta x(\theta_0), \omega) = 0$, $x \mapsto (\ell^+(x, \omega), \ell^-(x, \omega))$ is an orientation preserving linear map on $\mathbb{R}^2$, and $\partial_\theta x(\theta_0), v(\theta_0)$ form a positively oriented basis of $\mathbb{R}^2$.

3. Differentiating (4.55) in $\theta$ to get a formula for $\partial_\theta \ell^\pm(x(\theta), \lambda)$ and substituting into (4.56) we get the following identity for $\theta, \theta' \in U$:

$$K_\lambda^\pm(\theta, \theta') = c_x(\theta - \theta' \pm i0)^{-1} + \frac{c_x \partial_\theta (G_0(\gamma_\omega^\pm(\theta) - \theta') + i\varepsilon G_1 + \varepsilon^2 G_2)}{G_0(\gamma_\omega^\pm(\theta) - \theta') + i\varepsilon G_1 + \varepsilon^2 G_2} \quad (4.59)$$

where as before, $G_j = G_j(\theta, \theta')$. Dividing the numerator and denominator of the last term on the right-hand side by $G_0$, we see that the second term on the right-hand side
of (4.59) is equal to \( F_\lambda^\pm(\theta, \theta')(\gamma^\pm_\omega(\theta) - \theta' \pm i\varepsilon \psi^\pm_\lambda(\theta, \theta'))^{-1} \) where the functions

\[
\psi^\pm_\lambda(\theta, \theta') := \pm \frac{G_1(\theta, \theta') - i\varepsilon G_2(\theta, \theta', \varepsilon)}{G_0(\theta, \theta')},
\]

\[
F_\lambda^\pm(\theta, \theta') := \frac{c_\lambda \partial_\theta(G_0(\theta, \theta')(\gamma^\pm_\omega(\theta) - \theta') + i\varepsilon G_1(\theta, \theta') + \varepsilon^2 G_2(\theta, \theta', \varepsilon))}{G_0(\theta, \theta')},
\]

are smooth in \( \theta, \theta', \varepsilon \) up to \( \varepsilon = 0 \) and \( \psi^\pm_\lambda \) is real and nonzero when \( \varepsilon = 0 \).

To get (4.54) we can now use Lemma 3.5 similarly to Step 3 in the proof of Lemma 4.13. Here the sign condition \( \text{Re} \psi^\pm_\lambda \geq c > 0 \) and (4.50) can be verified by comparing (4.54) with (4.49), or alternatively by a direct computation using (2.6) and (4.58).

4.6.6. Summary. We summarize the findings of this section in microlocal terms. Consider the pullback operator by \( \gamma^\pm_\omega \) on 1-forms on \( S^1 \),

\[
(\gamma^\pm_\omega)^*: C^\infty(S^1; T^*S^1) \to C^\infty(S^1; T^*S^1).
\]

In terms of the identification of functions with 1-forms, \( f \mapsto f \, d\theta \), we have

\[
(\gamma^\pm_\omega)^*(f \, d\theta) = ((f \circ \gamma^\pm_\omega) \partial_\theta \gamma^\pm_\omega) \, d\theta.
\]

**Proposition 4.15.** Assume that \( \lambda = \omega + i\varepsilon \) where \( \omega \in (0, 1) \), \( \varepsilon \geq 0 \), and \( \Omega \) is \( \omega \)-simple in the sense of Definition 1. Let \( \mathcal{C}_\lambda \) be the operator defined in (4.32), where for \( \varepsilon = 0 \) we understand it as the operator \( \mathcal{C}_{\omega + 0} \). Using the coordinate \( \theta \), we treat \( d\mathcal{C}_\lambda \) as an operator on \( C^\infty(S^1; T^*S^1) \). Then for all \( \varepsilon \) small enough, we can write

\[
\mathcal{E}_\lambda d\mathcal{C}_\lambda = I + (\gamma^+_\omega)^* A^+_\lambda + (\gamma^-_\omega)^* A^-_\lambda
\]

(4.61)

where \( \mathcal{E}_\lambda, A^\pm_\lambda \) are pseudodifferential operators in \( \Psi^0(S^1; T^*S^1) \) bounded uniformly in \( \varepsilon \) and such that, uniformly in \( \varepsilon \) (see (3.6))

\[
\sigma(\mathcal{E}_\lambda)(\theta, \xi) = \frac{i \text{sgn} \xi}{2\pi c_\lambda}, \quad \text{WF}(A^\pm_\lambda) \subset \{ \pm \xi > 0 \}, \quad \sigma(A^\pm_\lambda)(\theta, \xi) = a^\pm_\lambda(\theta) H(\pm \xi) e^{-\varepsilon z^\pm_\lambda(\theta) |\xi|}
\]

where \( H(\xi) \) denotes the Heaviside function, \( a^\pm_\lambda \) and \( z^\pm_\lambda \) are smooth in \( \theta, \varepsilon \) up to \( \varepsilon = 0 \), \( \text{Re} z^\pm_\lambda(\theta) \geq c > 0 \), and \( a^\pm_\lambda(\theta) = -1 + \mathcal{O}(\varepsilon) \).

Remarks. 1. Proposition 4.15 is stated for a fixed value of \( \omega = \text{Re} \lambda \). However, its proof still works when \( \omega \) varies in some open interval \( J \subset (0, 1) \) such that \( \Omega \) is \( \omega \)-simple for all \( \omega \in J \). The conclusions of Proposition 4.15 hold locally uniformly in \( \omega \in J \) and the functions \( a^\pm_\lambda(\theta) \), \( z^\pm_\lambda(\theta) \) can be chosen depending smoothly on \( \theta \in S^1 \), \( \omega \in J \), and \( \varepsilon = \text{Im} \lambda \geq 0 \).

2. One can formulate a version of (4.61) directly on \( \partial \Omega \) which does not depend on the choice of the (positively oriented) coordinate \( \theta \), using the fact that the principal symbol (3.13) is invariantly defined.
Proof. 1. Recall from (4.36) that \( d\mathcal{C}_\lambda = T^+ \mathcal{C}_\lambda^+ + T^- \mathcal{C}_\lambda^- \), where \( T^\pm \) are defined in (4.37). As with \( d\mathcal{C}_\lambda \), we use the coordinate \( \theta \) to think of \( T^\pm_\lambda \) as operators on \( C^\infty(S^1; T^*S^1) \). We will write \( T^\pm_\lambda \) as a sum of a pseudodifferential operator and a composition of a pseudodifferential operator with \( (\gamma^\pm_\omega)^* \), see (4.64) below. The singular supports of the Schwartz kernels of these two operators will lie in the sets \( \text{Diag} \) and \( \text{Ref}^\pm \) defined in (4.40).

Fix a cutoff \( \chi_{\text{Diag}} \in C^\infty(S^1 \times S^1) \) supported in a small neighborhood of the diagonal \( \text{Diag} \) and equal to 1 on a smaller neighborhood of \( \text{Diag} \). Define the \((\lambda\text{-dependent})\) operator

\[
T^\pm_{\text{Diag}} : C^\infty(S^1; T^*S^1) \to C^\infty(S^1; T^*S^1)
\]

with the Schwartz kernel \( c_\lambda \chi_{\text{Diag}}(\theta, \theta')(\theta - \theta' \pm i0)^{-1} \). Here Schwartz kernels are defined in (4.38). By Lemma 3.8 we have

\[
T^\pm_{\text{Diag}} \in \Psi^0(S^1; T^*S^1), \quad \sigma(T^\pm_{\text{Diag}})(\theta, \xi) = \mp 2\pi ic_\lambda H(\pm \xi).
\]  

(4.62)

2. Next, define the reflected operators

\[
T^\pm_{\text{Ref}} := T^\pm_\lambda - T^\pm_{\text{Diag}}, \quad \tilde{T}^\pm_{\text{Ref}} := (\gamma^\pm_\omega)^* T^\pm_{\text{Ref}}.
\]

Denote by \( K^\pm_{\text{Ref}}, \tilde{K}^\pm_{\text{Ref}} \) the corresponding Schwartz kernels. Combining Lemmas 4.11, 4.12, 4.13, and 4.14 we see that, putting \( \chi^\pm_{\text{Ref}}(\theta, \theta') := \chi_{\text{Diag}}(\gamma^\pm_\omega(\theta), \theta') \),

\[
K^\pm_{\text{Ref}}(\theta, \theta') = \chi^\pm_{\text{Ref}}(\theta, \theta') c^\pm_{\lambda}(\theta') (\gamma^\pm_\omega(\theta) - \theta' \pm i\varepsilon z^\pm_\lambda(\theta'))^{-1} + \mathcal{K}^\pm_{\lambda}(\theta, \theta'), \quad 0 < \varepsilon < \varepsilon_0
\]

where \( \mathcal{K}^\pm_{\lambda} \) is smooth in \( \theta, \theta', \varepsilon \) up to \( \varepsilon = 0 \), \( c^\pm_{\lambda}(\theta') \) and \( z^\pm_\lambda(\theta') \) are smooth in \( \theta', \varepsilon \) up to \( \varepsilon = 0 \), \( \text{Re} z^\pm_\lambda(\theta') \geq c > 0 \) for some constant \( c \), and \( c^\pm_{\lambda}(\theta') = c_\lambda/\partial_{\theta'}(\gamma^\pm_\omega(\theta') + O(\varepsilon)) \). Here we use a partition of unity and Lemma 3.7 to patch together different local representations from Lemmas 4.13 and 4.14 and get globally defined \( c^\pm_{\lambda}, z^\pm_\lambda \). Recalling (4.60), we have

\[
\tilde{K}^\pm_{\text{Ref}}(\theta, \theta') = (\partial_{\theta'}(\gamma^\pm_\omega(\theta))) K^\pm_{\text{Ref}}(\gamma^\pm_\omega(\theta), \theta').
\]

Thus by Lemma 3.8 the operator \( \tilde{T}^\pm_{\text{Ref}} \) is pseudodifferential: we have uniformly in \( \varepsilon > 0 \)

\[
\tilde{T}^\pm_{\text{Ref}} \in \Psi^0(S^1; T^*S^1), \quad \text{WF}(\tilde{T}^\pm_{\text{Ref}}) \subset \{ \pm \xi > 0 \},
\]

(4.63)

\[
\sigma(\tilde{T}^\pm_{\text{Ref}})(\theta, \xi) = \mp 2\pi i c^\pm_{\lambda}(\theta)(\partial_{\theta's}(\gamma^\pm_\omega(\theta))) e^{-\varepsilon z^\pm_\lambda(\theta)\xi} H(\pm \xi).
\]

3. We now have the decomposition for \( \varepsilon > 0 \)

\[
d\mathcal{C}_\lambda = T^+_{\text{Diag}} + T^-_{\text{Diag}} + (\gamma^\pm_\omega)^* \tilde{T}^+_{\text{Ref}} + (\gamma^\mp_\omega)^* \tilde{T}^-_{\text{Ref}}.
\]  

(4.64)

Taking the limit as \( \varepsilon \to 0+ \) and using Lemmas 3.6 (see also (3.26)) and 4.10 we see that the same decomposition holds for \( \varepsilon = 0 \), where we have

\[
K^\pm_{\text{Ref}}(\theta, \theta') = \chi^\pm_{\text{Ref}}(\theta, \theta') c^\pm_{\lambda}(\theta') (\gamma^\pm_\omega(\theta) - \theta' \pm i0)^{-1} + \mathcal{K}^\pm_{\lambda}(\theta, \theta') \quad \text{when} \quad \varepsilon = 0
\]

and by Lemma 3.8 the properties (4.63) hold for \( \varepsilon = 0 \).
The operator $T_{\text{Diag}} := T_{\text{Diag}}^- + T_{\text{Diag}}^+$ lies in $\Psi^0(S^1; T^*S^1)$ and has principal symbol $-2\pi ic_\lambda \text{sgn } \xi$ (away from $\xi = 0$), which is elliptic. Let $E_\lambda$ be the elliptic parametrix of $T_{\text{Diag}}$, so that $E_\lambda T_{\text{Diag}} = I + \Psi^{-\infty}$ (see [17, Theorem 18.1.9]). We have $\sigma(E_\lambda) = 1/\sigma(T_{\text{Diag}}) = i \text{sgn } \xi/(2\pi c_\lambda)$. Multiplying (4.64) on the left by $E_\lambda$ we get (4.61) where the operators $A_\lambda^{\pm}$ have the following form:

$$A_\lambda^{\pm} = (\gamma_\omega^{\pm})^* E_\lambda (\gamma_\omega^{\pm})^* \tilde{T}_{\text{Ref}}^{\pm}.$$ 

By [17, Theorem 18.1.17], $(\gamma_\omega^{\pm})^* E_\lambda (\gamma_\omega^{\pm})^* \in \Psi^0(S^1; T^*S^1)$ has the principal symbol $-i \text{sgn } \xi/(2\pi c_\lambda)$ (as $\gamma_\omega^{\pm}$ is orientation reversing), so from (4.63) we get the needed properties of $A_\lambda^{\pm}$, with

$$a_\lambda^{\pm}(\theta) = -\frac{\tilde{c}_\lambda^{\pm}(\theta)}{c_\lambda} \partial_\theta \gamma_\omega^{\pm}(\theta) = -1 + O(\varepsilon).$$

□

A corollary of Lemma 4.10 and Proposition 4.15 is the following limiting statement:

**Lemma 4.16.** Assume that $\omega \in (0, 1)$, $\Omega$ is $\omega$-simple, and $s + 1 > t$. Then

$$\|C_{\omega+i\varepsilon} - C_{\omega+i0}\|_{H^s(S^1; T^*S^1) \rightarrow H^t(S^1)} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0^+.$$  \hspace{1cm} (4.65)

**Proof.** 1. We first show the following uniform bound: for each $s$ there exists $C_s$ such that for all $0 \leq \varepsilon \ll 1$

$$\|C_{\omega+i\varepsilon}\|_{H^s(S^1; T^*S^1) \rightarrow H^{s+1}(S^1)} \leq C_s \quad \text{for all } s. \hspace{1cm} (4.66)$$

Indeed, Proposition 4.15 (more precisely, (4.64)) implies that

$$\|dC_{\omega+i\varepsilon}\|_{H^s(S^1; T^*S^1) \rightarrow H^{s+1}(S^1; T^*S^1)} \leq C_s \quad \text{for all } s. \hspace{1cm} (4.67)$$

On the other hand the proof of Lemma 4.10 shows that the Schwartz kernel of $C_{\omega+i\varepsilon}$ is bounded in $L^1(S^1 \times S^1)$ uniformly in $\varepsilon$, thus by Sobolev embedding

$$\|C_{\omega+i\varepsilon}\|_{H^s(S^1; T^*S^1) \rightarrow H^{s+t}(S^1)} \leq C_{s,t} \quad \text{for all } s > \frac{1}{2}, \ t < -\frac{1}{2}. \hspace{1cm} (4.68)$$

Together (4.67), (4.68), and the elliptic estimate for $\partial_\theta$ imply that (4.66) holds for all $s > \frac{1}{2}$. The operator $C_{\omega+i\varepsilon}$ is its own transpose under the natural bilinear pairing on $C^\infty(S^1; T^*S^1) \times C^\infty(S^1)$. Since $H^{-s}$ is dual to $H^s$ under this pairing, (4.66) holds for all $s < -\frac{3}{2}$. Then (4.68) holds for all $s, t$ such that $t < \min(s + 1, -\frac{1}{2})$. Together with (4.67) and the elliptic estimate for $\partial_\theta$ this implies that (4.66) holds in general.

2. We now show that, as $\varepsilon \rightarrow 0^+$,

$$C_{\omega+i\varepsilon} v \rightarrow C_{\omega+i0} v \quad \text{in } C^\infty(S^1) \quad \text{for all } v \in C^\infty(S^1; T^*S^1). \hspace{1cm} (4.69)$$

Indeed, by (4.66) the family $\varepsilon \mapsto C_{\omega+i\varepsilon} v$ is precompact in $H^s$ for every $s$, and any convergent subsequence has to converge to $C_{\omega+i0} v$ by since $C_{\omega+i\varepsilon} v \rightarrow C_{\omega+i0} v$ in $\mathcal{D}'$ by Lemma 4.10.
Since $C^\infty$ is dense in $H^s$, we get from (4.66), (4.69), and a standard argument in functional analysis the strong-operator convergence

$$C_{\omega+i\varepsilon} v \to C_{\omega+i0} v \quad \text{in} \quad H^{s+1}(S^1) \quad \text{for all} \quad v \in H^s(S^1;T^*S^1).$$

(4.70)

We are now ready to prove (4.65). Let $s + 1 > t$. Assume that (4.65) fails, then there exists some $c > 0$ and sequences

$$v_j \in H^s(S^1;T^*S^1), \quad \parallel v_j \parallel_{H^s} = 1, \quad \varepsilon_j \to 0+, \quad \parallel (C_{\omega+i\varepsilon_j} - C_{\omega+i0}) v_j \parallel_{H^t} \geq c.$$

Since $H^s$ embeds compactly into $H^{t-1}$, passing to a subsequence we may assume that $v_j \to v_0$ in $H^{t-1}$. But then

$$\parallel (C_{\omega+i\varepsilon_j} - C_{\omega+i0}) v_j \parallel_{H^t} \leq \parallel (C_{\omega+i\varepsilon_j} - C_{\omega+i0})(v_j - v_0) \parallel_{H^t} + \parallel (C_{\omega+i\varepsilon_j} - C_{\omega+i0}) v_0 \parallel_{H^t}.$$

Now the first term on the right-hand side goes to 0 as $j \to \infty$ by (4.66), and the second term goes to 0 by (4.70), giving a contradiction. \qed

5. HIGH FREQUENCY ANALYSIS ON THE BOUNDARY

In this section, we take

$$\lambda = \omega + i\varepsilon, \quad 0 < \varepsilon \ll 1,$$

where $\omega \in (0,1)$ satisfies the Morse–Smale conditions on $\Omega$ (see Definition 2), and consider the elliptic boundary value problem (4.16):

$$P(\lambda) u_\lambda = f, \quad u_\lambda|_{\partial\Omega} = 0.$$

Here $f \in C^\infty_c(\Omega)$ is fixed and the solution $u_\lambda$ lies in $C^\infty(\overline{\Omega})$ (see Lemma 4.4). Our goal is to prove high frequency estimates on $u_\lambda$ which are uniform in the limit $\varepsilon \to 0+$, when the operator $P(\lambda)$ becomes hyperbolic. To do this we combine the detailed analysis of §4.6 with the dynamical properties following from the Morse–Smale conditions.

5.1. Splitting into positive and negative frequencies. Fix a positively oriented coordinate $\theta : \partial\Omega \to S^1$ to identify $\partial\Omega$ with $S^1$. Recall from (4.25) that

$$C_{\lambda} v_\lambda = G_{\lambda} := (R_{\lambda} f)|_{\partial\Omega}. \quad (5.1)$$

Here the 1-form $v_\lambda \in C^\infty(S^1;T^*S^1)$ depends on the Neumann data of $u_\lambda$ and is defined using (4.21); however, we do not have uniform bounds on $v_\lambda$ in $C^\infty$ as $\varepsilon = \text{Im} \lambda \to 0+$. The function $G_{\lambda}$ lies in $C^\infty(S^1)$ uniformly in $\varepsilon$ since $f \in C^\infty_c$ and $R_{\lambda}$ is the convolution operator with the fundamental solution $E_{\lambda}$, which has a distributional limit as $\varepsilon \to 0+$ by Lemma 4.3.

Let $\gamma^\pm_\omega$ be defined in (4.35). By Proposition 4.15 we have

$$E_{\lambda} dG_{\lambda} = E_{\lambda} dC_{\lambda} v_\lambda = v_\lambda + (\gamma^+_\omega)^* A^+_\lambda v_\lambda + (\gamma^-_\omega)^* A^-_\lambda v_\lambda.$$

We rewrite this equation as

$$v_\lambda = -A_\lambda v_\lambda + E_{\lambda} dG_{\lambda}, \quad A_\lambda := (\gamma^+_\omega)^* A^+_\lambda + (\gamma^-_\omega)^* A^-_\lambda. \quad (5.2)$$
The operator $A_\lambda$ exchanges positive and negative frequencies, since $A_\lambda^\pm$ are pseudodifferential and the maps $\gamma_\omega^\pm$ are orientation reversing. We thus study the square of $A_\lambda$, which maps positive and negative frequencies to themselves. It is expressed in terms of the pullback of the chess billiard map $b = \gamma^+ \circ \gamma^-$ to $S^1$:

$$b_\omega := \gamma_\omega^+ \circ \gamma_\omega^-, \quad b_\omega^{-1} = \gamma_\omega^- \circ \gamma_\omega^+,$$

which is an orientation preserving diffeomorphism of $S^1$. Denote the pullback operators by $b_\omega$ and $b_\omega^{-1}$ on 1-forms by

$$b_\omega^*, b_\omega^{-1} : C^\infty(S^1; T^*S^1) \to C^\infty(S^1; T^*S^1).$$

**Lemma 5.1.** We have

$$A_\lambda^2 = B_\lambda^+ b_\omega^* + B_\lambda^- b_\omega^{-*}$$

where $B_\lambda^\pm$ are pseudodifferential, more precisely we have uniformly in $\varepsilon \geq 0$ (see (3.6))

$$B_\lambda^\pm \in \Psi^0(S^1; T^*S^1), \quad \text{WF}(B_\lambda^\pm) \subset \{ \pm \xi > 0 \},$$

$$\sigma(B_\lambda^\pm)(\theta, \xi) = a_{\lambda}^\pm(\theta)H(\pm \xi)e^{-\varepsilon z_{\lambda}^\pm(\theta)\xi}$$

where $H$ denotes the Heaviside function, the functions $\tilde{a}_{\lambda}^\pm(\theta), \tilde{z}_{\lambda}^\pm(\theta)$ are smooth in $\theta \in S^1$ and $\varepsilon \geq 0$, $\text{Re} \tilde{z}_{\lambda}^\pm \geq c > 0$, and $\tilde{a}_{\lambda}^\pm(\theta) = 1 + O(\varepsilon)$.

**Proof.** From Proposition 4.15 and the change of variables formula for pseudodifferential operators [17, Theorem 18.1.17] we see that $(\gamma_\omega^\pm)^*=A_{\lambda}^\pm(\gamma_\omega^\pm)^*$ lies in $\Psi^0(S^1; T^*S^1)$ and has wavefront set inside $\{ \mp \xi > 0 \}$ uniformly in $\varepsilon$. Since products of pseudodifferential operators with nonintersecting wavefront sets are smoothing, we see that

$$((\gamma_\omega^\pm)^* A_{\lambda}^\pm)^2 \in \Psi^{-\infty}(S^1; T^*S^1) \quad \text{uniformly in } \varepsilon \geq 0.$$

Recalling (5.2) we see that (with $\Psi^{-\infty}$ denoting smoothing operators uniformly in $\varepsilon$)

$$A_{\lambda}^2 = ((\gamma_\omega^\pm)^* A_{\lambda}^\pm(\gamma_\omega^\pm)^*) ((b_\omega^{-1})^* A_{\lambda}^\pm(b_\omega^{-1})^*) + \Psi^{-\infty}.$$

This gives the decomposition (5.4) with

$$B_\lambda^\pm = ((\gamma_\omega^\pm)^* A_{\lambda}^\pm(\gamma_\omega^\pm)^*) ((b_\omega^{-1})^* A_{\lambda}^\pm(b_\omega^{-1})^*) + \Psi^{-\infty}.$$

Using the properties of $A_{\lambda}^\pm$ in Proposition 4.15 together with the product formula and the change of variables formula for pseudodifferential operators, we see that $B_\lambda^\pm \in \Psi^0(S^1; T^*S^1)$ and $\text{WF}(B_\lambda^\pm) \subset \{ \pm \xi > 0 \}$ uniformly in $\varepsilon$. This also gives

$$\sigma(B_\lambda^\pm)(\theta, \xi) = \sigma(A_{\lambda}^\pm)\left(\gamma_\omega^\pm(\theta), \frac{\xi}{\partial_\theta \gamma_\omega^\pm(\theta)}\right) \sigma(A_{\lambda}^\pm)\left(b_\omega^{-1}(\theta), \frac{\xi}{\partial_\theta b_\omega^{-1}(\theta)}\right)$$

in the sense of (3.6), which implies the formula for the principal symbol in (5.5) with

$$\tilde{a}_{\lambda}^\pm(\theta) = a_{\lambda}^\pm(\gamma_\omega^\pm(\theta)) a_{\lambda}^\pm(b_\omega^{-1}(\theta)), \quad \tilde{z}_{\lambda}^\pm(\theta) = \frac{z_{\lambda}^\pm(\gamma_\omega^\pm(\theta))}{|\partial_\theta \gamma_\omega^\pm(\theta)|} + \frac{z_{\lambda}^\pm(b_\omega^{-1}(\theta))}{|\partial_\theta b_\omega^{-1}(\theta)|},$$

where $a_{\lambda}^\pm, z_{\lambda}^\pm$ are given in Proposition 4.15. □
Applying (5.2) twice, we get the equation
\[ v_\lambda = B_\lambda^+ b_\omega^* v_\lambda + B_\lambda^- b_\omega^- v_\lambda + g_\lambda \]
(5.6)
where
\[ g_\lambda := (I - A_\lambda)E_\lambda dG_\lambda \]
is in \( C^\infty(S^1; T^*S^1) \) uniformly in \( \varepsilon > 0 \).

We now split \( v_\lambda \) into positive and negative frequencies. Consider a pseudodifferential partition of unity
\[ I = \Pi^+ + \Pi^-, \quad \Pi^\pm \in \Psi^0(S^1, T^*S^1), \]
WF(\( \Pi^\pm \)) \subset \{ \pm \xi > 0 \}, \quad \sigma(\Pi^\pm)(\theta, \xi) = H(\pm \xi). \]
(5.7)
Put
\[ v_\lambda^\pm := \Pi^\pm v_\lambda, \quad g_\lambda^\pm := \Pi^\pm g_\lambda, \]
(5.8)
with \( g_\lambda^\pm \) in \( C^\infty(S^1; T^*S^1) \) uniformly in \( \varepsilon \), and apply \( \Pi^\pm \) to (5.6) to get
\[ v_\lambda^\pm = B_\lambda^\pm (b_\omega^\pm)^* v_\lambda^\pm + g_\lambda^\pm \]
(5.9)
where the operator
\[ R_\lambda^\pm := ([\Pi^\pm, B_\lambda^\pm] + B_\lambda^\pm (\Pi^\pm - (b_\omega^\pm)^* \Pi^\pm (b_\omega^\pm)^*)) (b_\omega^\pm)^* + \Pi^\pm B_\lambda^\pm (b_\omega^\pm)^* \]
is in \( \Psi^{-\infty}(S^1; T^*S^1) \) uniformly in \( \varepsilon \), as follows from (5.7) and the fact that WF(\( B_\lambda^\pm \)) \subset \{ \pm \xi > 0 \}.

5.2. **Microlocal Lasota–Yorke inequalities.** We now show that \( B_\lambda^\pm (b_\omega^\pm)^* \) featured in the equation (5.9) are contractions at high frequencies on appropriately chosen inhomogeneous Sobolev spaces, and use this to prove a high frequency estimate on \( v_\lambda \), see Proposition 5.3 below. This is reminiscent of Lasota–Yorke inequalities (see [2] and references given there) and could be considered a simple version of radial estimates (see [11, §E.4.3] and references given there) for Fourier integral operators. It is also related to microlocal weights used by Faure–Roy–Sjöstrand [13].

Unlike applications to volume preserving Anosov maps in [2, 13], where critical regularity is given by \( L^2 \), for us the critical regularity space is \( H^{-\frac{1}{2}} \). This can be informally explained as follows: if we have \( v_\lambda^\pm = df^\pm \) for some functions \( f^\pm \) then the flux \( \text{Im} \int_{S^1} F^\pm df^\pm \), is invariant under replacing \( f^\pm \) with the pullback \( (b_\omega^{-1})^* f^\pm \) and is well defined for \( f^\pm \in H^\frac{1}{2} \). When WF(\( f^\pm \)) \subset \{ \pm \xi > 0 \}, the flux is related to \( \| f^\pm \|_{H^\frac{1}{2}}^2 \sim \| v_\lambda^\pm \|_{H^{-\frac{1}{2}}}^2 \).

To simplify notation, we only study in detail the case of the ‘+’ sign. The case of the ‘−’ sign is handled similarly by replacing \( b_\omega \) with \( b_\omega^{-1} \), switching \( \Sigma_\omega^+ \) with \( \Sigma_\omega^- \), and using the escape function in Lemma 2.9 (rather than in the remark following it).
We identify \( \partial \Omega \) with \( S^1 \) using the adapted coordinate \( \theta \) constructed in Lemma 2.8, which satisfies for \( \delta > 0 \) small enough
\[
\mp \log \partial_\theta b_\omega > 0 \quad \text{on} \quad \Sigma_\omega^\pm(\delta)\tag{5.10}
\]
where \( \Sigma_\omega^\pm \subset S^1 \) are the attractive (+) and repulsive (−) periodic points of \( b_\omega \) defined in (1.6) and \( \Sigma_\omega^\pm(\delta) \) are their open \( \delta \)-neighborhoods.

Take arbitrary \( \alpha_- < \alpha_+ \) and small \( \delta > 0 \) (in particular, so that (5.10) holds). Let \( g \in C^\infty(S^1; \mathbb{R}) \) be the escape function defined in the remark following Lemma 2.9. We have
\[
\alpha_- \leq g(\theta) \leq N_0 \quad \text{for some} \quad N_0.
\tag{5.11}
\]

Define the symbol
\[
G(\theta, \xi) := g(\theta)(1 - \chi_0(\xi)) \log |\xi|, \quad (\theta, \xi) \in T^*S^1
\tag{5.12}
\]
where \( \chi_0 \in C^{\infty}_c((-1, 1)) \) is equal to 1 near 0. We use Lemma 3.2 to construct
\[
E_G := \text{Op}(e^G) \in \Psi_{0+}^0(S^1; T^*S^1), \quad \tilde{E}_{-G} := \text{Op}(e^{-G}(1 + r_G)) \in \Psi_{0+}^{\alpha_-}(S^1; T^*S^1),
\tag{5.13}
\]
\[
r_G \in S^{-1}, \quad \tilde{E}_{-G}E_G - I, E_G\tilde{E}_{-G} - I \in \Psi^{-\infty}.
\]

By property (4) in the remark following Lemma 2.9 we have \( g \geq \alpha_+ \) on \( S^1 \setminus \Sigma_\omega^0(\delta) \). Therefore by (3.10)
\[
\chi \tilde{E}_{-G} \in \Psi_{0+}^{-\alpha_+}(S^1; T^*S^1) \quad \text{for all} \quad \chi \in C^\infty(S^1), \quad \text{supp} \chi \cap \Sigma_\omega^0(\delta) = \emptyset.
\tag{5.14}
\]

We now apply \( E_G \) to (5.9) (with the ‘+’ sign) to get
\[
v_G = T_Gv_G + \mathcal{R}_Gv_\lambda + g_G \quad \text{where} \quad v_G := E_Gv_\lambda^+, \quad g_G := E_Gg_\lambda^+,
\tag{5.15}
\]
\[
T_G := E_GB_\lambda^+ b_\omega^+ \tilde{E}_{-G}, \quad \mathcal{R}_G := E_GB_\lambda^+ b_\omega^+(I - \tilde{E}_{-G}E_G)\Pi^+ + E_G\mathcal{R}_G^+.
\]

Here \( g_G \in C^\infty(S^1; T^*S^1) \) and \( \mathcal{R}_G \in \Psi^{-\infty}(S^1; T^*S^1) \), both uniformly in \( \varepsilon \). The function \( v_G \) lies in \( C^\infty(S^1; T^*S^1) \) for \( \varepsilon > 0 \), but it is not bounded in this space uniformly in \( \varepsilon \). We also have the following bounds for each \( N \), which follow from (5.13) and (5.14) (writing \( v_\lambda^+ = \tilde{E}_{-G}v_G + (I - \tilde{E}_{-G}E_G)v_\lambda^+ \)):
\[
\|v_\lambda^+\|_{H^{\alpha_-}} \leq C\|v_G\|_{L^2} + C_N\|v_\lambda\|_{H^{-N}},
\tag{5.16}
\]
\[
\|\chi v_\lambda^+\|_{H^{\alpha_-}} \leq C\|v_G\|_{L^2} + C_N\|v_\lambda\|_{H^{-N}} \quad \text{if} \quad \text{supp} \chi \cap \Sigma_\omega^0(\delta) = \emptyset,
\tag{5.17}
\]
\[
\|g_G\|_{L^2} \leq C\|g_\lambda\|_{H^{N_0}}.
\tag{5.18}
\]

The key result in this section is the following lemma. The point is that for \( \alpha_- < -\frac{1}{2} < \alpha_+ \), we can obtain a contraction property of the microlocally conjugated operator \( T_G \):

**Lemma 5.2.** Suppose that \( G \) is given by (5.12) (using a coordinate \( \theta \) in which (5.10) holds) with \( g \) defined with parameters \( \alpha_- < \alpha_+, \delta > 0 \), and that \( T_G \) is defined in (5.15).
Define the norm on $L^2(S^1; T^*S^1)$ using the coordinate $\theta$. Then for any $N$ and $\nu > 0$ there exists $C_N$ such that for all $w \in C^\infty(S^1; T^*S^1)$,

$$\|T_G w\|_{L^2} \leq \left( \max_{\pm} \sup_{\Sigma^\pm_\delta(\theta)} (\partial_\theta b_\omega)^{\frac{1}{2} + \alpha_{\pm}} + \nu \right) \|w\|_{L^2} + C_N \|w\|_{H^{-N}}. \quad (5.19)$$

**Proof.** 1. Recalling the formula (4.60) for pullback operators on 1-forms, we see that the operator

$$(b_\omega^* \partial_\theta b_\omega)^{\frac{1}{2}} : L^2(S^1; T^*S^1) \to L^2(S^1; T^*S^1)$$

is unitary. Multiplying $T_G$ by this operator on the right, we see that it suffices to show that

$$\|\tilde{T}_G w\|_{L^2} \leq \left( \max_{\pm} \sup_{\Sigma^\pm_\delta(\theta)} (\partial_\theta b_\omega)^{\frac{1}{2} + \alpha_{\pm}} + \nu \right) \|w\|_{L^2} + C_N \|w\|_{H^{-N}}$$

where $\tilde{T}_G := E_G B_{\alpha}^+ b_\omega^* \tilde{E}_G (\partial_\theta b_\omega)^{\frac{1}{2}}$,

By (3.12) we have $b_\omega^* \tilde{E}_G b_\omega^* = \text{Op}(e^{-G_b(1 + r)})$ for $G_b(\theta, \xi) := G(b_\omega(\theta), \xi/\partial_\theta b_\omega(\theta))$ and some $r \in S^{1+}$. Recalling the definition (5.12) of $G$, we compute for $|\xi|$ large enough

$$G(\theta, \xi) - G_b(\theta, \xi) = (g(\theta) - g(b_\omega(\theta))) \log |\xi| + g(b_\omega(\theta)) \log \partial_\theta b_\omega(\theta). \quad (5.21)$$

Since $g(\theta) - g(b_\omega(\theta)) \leq 0$ by property (1) in the remark following Lemma 2.9, we see that $G - G_b$ is bounded above by some constant. By (3.10) and Lemma 5.1 we then see that $\tilde{T}_G \in \Psi^0(\mathbb{S}^1; T^*\mathbb{S}^1)$ uniformly in $\varepsilon$ and its principal symbol is (in the sense of (3.6))

$$\sigma(\tilde{T}_G)(\theta, \xi) = \tilde{a}_\lambda^+(\theta) H(\xi) e^{-\varepsilon \tilde{z}_\lambda^+(\theta) \xi} (\partial_\theta b_\omega(\theta))^\frac{1}{2} e^{G(\theta, \xi) - G_b(\theta, \xi)}, \quad |\xi| \geq 1.$$

Thus (5.20) follows from Lemma 3.1 once we show that there exists $C_1 > 0$ such that for all $\xi \geq C_1$

$$|\tilde{a}_\lambda^+(\theta)| e^{-\varepsilon \text{Re} \tilde{z}_\lambda^+(\theta) \xi} (\partial_\theta b_\omega(\theta))^\frac{1}{2} e^{G(\theta, \xi) - G_b(\theta, \xi)} \leq \max_{\pm} \sup_{\Sigma^\pm_\delta(\theta)} (\partial_\theta b_\omega)^{\frac{1}{2} + \alpha_{\pm}}. \quad (5.22)$$

2. Since $\tilde{a}_\lambda^+(\theta) = 1 + \mathcal{O}(\varepsilon)$ and $\text{Re} \tilde{z}_\lambda^+(\theta) \geq c > 0$, for $\xi \geq C_1$ and $C_1$ large enough we have $|\tilde{a}_\lambda^+(\theta)| e^{-\varepsilon \text{Re} \tilde{z}_\lambda^+(\theta) \xi} \leq 1$. Thus (5.22) reduces to showing that for all $\xi \geq C_1$

$$\tilde{G}(\theta, \xi) \leq \max_{\pm} \sup_{\Sigma^\pm_\delta(\theta)} \left( \frac{1}{2} + \alpha_{\pm} \right) \log \partial_\theta b_\omega$$

where $\tilde{G}(\theta, \xi) := (g(\theta) - g(b_\omega(\theta))) \log \xi + \left( \frac{1}{2} + g(b_\omega(\theta)) \right) \log \partial_\theta b_\omega(\theta)$.

This in turn is proved if we show that there exists $c_0 > 0$ such that for $\xi$ large enough

$$\tilde{G}(\theta, \xi) \leq \begin{cases} 
- c_0 \log \xi, & \theta \in \mathbb{S}^1 \setminus (\Sigma^-_\omega(\delta) \cup \Sigma^+_\omega(\delta)), \\
(\frac{1}{2} + \alpha_+) \log \partial_\theta b_\omega(\theta), & \theta \in \Sigma^+_\omega(\delta), \\
(\frac{1}{2} + \alpha_-) \log \partial_\theta b_\omega(\theta), & \theta \in \Sigma^-_\omega(\delta).
\end{cases} \quad (5.23)$$
We now prove (5.24) using properties (1)–(6) in Lemma 2.9 (or rather the remark which follows it). The first inequality follows from property (2), since \( g(\theta) - g(b_\omega(\theta)) \leq -2c_0 \) for some \( c_0 > 0 \). The second inequality follows from properties (1) and (4) together with (5.10). Finally, the third inequality follows from property (6) with \( M := \frac{(\log \xi)}{(\log \partial_\theta b_\omega(\theta))} \gg 1 \), where we again use (5.10).

With Lemma 5.2 in place we give a high frequency estimate on solutions to (5.6) which is uniform as \( \text{Im} \lambda \to 0 \). Together with Proposition 5.4 below it is used in the proof of Limiting Absorption Principle in §7 below.

**Proposition 5.3.** Fix \( \beta > 0, N, \) and some functions \( \chi^\pm \in C^\infty(S^1) \) such that \( \text{supp} \chi^\pm \cap \Sigma^+_\omega = \emptyset \). Then there exist \( N_0 \) and \( C \) such that for all small \( \varepsilon = \text{Im} \lambda > 0 \) and each solution \( v_\lambda \in C^\infty(S^1; T^*S^1) \) to (5.6) we have

\[
\|v_\lambda\|_{H^{-\frac{1}{2}} - \beta} \leq C(\|g_\lambda\|_{H^{N_0}} + \|v_\lambda\|_{H^{-N}}),
\]

(5.25)

\[
\|\chi^\pm \Pi^\pm v_\lambda\|_{H^N} \leq C(\|g_\lambda\|_{H^{N_0}} + \|v_\lambda\|_{H^{-N}}).
\]

(5.26)

**Remarks.**

1. The a priori assumption that \( v_\lambda \) is smooth (without any uniformity as \( \varepsilon \to 0^+ \)) is important in the argument because it ensures that the norm \( \|v_G\|_{L^2} \) is finite.

2. Using the notation (3.17), we see that (5.26) implies that, assuming that the right-hand side of this inequality is bounded uniformly in \( \varepsilon \) for each \( N_0 \) and some \( N, \) we have \( \text{WF}(v_\lambda) \subset N^*_+ \Sigma^-_\omega \cup N^*_- \Sigma^+_\omega \) uniformly in \( \varepsilon \).

**Proof.**

1. Fix \( \alpha_\pm \) satisfying

\[-\frac{1}{2} - \beta \leq \alpha_- < -\frac{1}{2} < \alpha_+ \leq N.\]

Next, fix \( \delta > 0 \) in the construction of the escape function \( g \) small enough so that (5.10) holds and \( \text{supp} \chi^\pm \cap \overline{\Sigma^-_\omega(\delta)} = \emptyset \). By (5.10) and since \( \alpha_- < -\frac{1}{2} < \alpha_+ \), we may choose \( \tau \) such that

\[
\max_{\pm} \sup_{\Sigma^-_\omega(\delta)} (\partial_\theta b_\omega)^{\frac{1}{2} + \alpha_\pm} < \tau < 1.
\]

Take \( N_0 \) so that (5.11) holds. We use the equation (5.15) and (5.18) to get

\[
\|v_G\|_{L^2} \leq \|T_Gv_G\|_{L^2} + C(\|g_\lambda\|_{H^{N_0}} + \|v_\lambda\|_{H^{-N}}).
\]

Applying Lemma 5.2 to \( w := v_G \), we see that

\[
\|T_Gv_G\|_{L^2} \leq \tau\|v_G\|_{L^2} + C\|v_\lambda\|_{H^{-N}}.
\]

Since \( \tau < 1 \), together these two inequalities give

\[
\|v_G\|_{L^2} \leq C(\|g_\lambda\|_{H^{N_0}} + \|v_\lambda\|_{H^{-N}}).
\]

(5.27)
2. From (5.27) and (5.16) we have
\[ \| v_\lambda^+ \|_{H^{-\frac{1}{2}} + \beta} \leq C(\| g_\lambda \|_{H^N_0} + \| v_\lambda \|_{H^{-\infty}}). \] (5.28)

The bound (5.26) for the ‘+’ sign follows from (5.27) and (5.17). Similar analysis (replacing \( b_\omega \) with \( b_\omega^{-1} \), switching the roles of \( \Sigma_\omega^+ \) and \( \Sigma_\omega^- \), and using Lemma 2.9 instead of the remark that follows it) shows that (5.28) holds for \( v_\lambda^- \) and (5.26) holds for the ‘-’ sign. Since \( v_\lambda = v_\lambda^+ + v_\lambda^- \), we obtain (5.25). \( \square \)

5.3. Conormal regularity of \( v_\lambda^\pm \). We now upgrade Proposition 5.3 to obtain iterated conormal regularity uniformly as \( \text{Im} \lambda \to 0+ \). As before we identify 1-forms on \( S^1 \) with functions using the coordinate \( \theta \), which makes it possible to define the operator \( \partial_\theta \) on 1-forms.

**Proposition 5.4.** Let \( \rho \in C^\infty(S^1; \mathbb{R}) \) vanish simply on \( \Sigma_\omega = \Sigma_\omega^+ \sqcup \Sigma_\omega^- \). Fix \( \beta > 0 \), \( K \in \mathbb{N}_0 \), and \( N \). Then there exist \( N_0 = N_0(\beta, K) \) and \( C = C(\beta, K, N) \) such that for all small \( \varepsilon = \text{Im} \lambda > 0 \) and any solution \( v_\lambda \in C^\infty(S^1; T^*S^1) \) to (5.6) we have
\[ \|(\rho \partial_\theta)^K v_\lambda\|_{H^{-\frac{1}{2}} + \beta} \leq C(\| g_\lambda \|_{H^N_0} + \| v_\lambda \|_{H^{-\infty}}). \] (5.29)

**Proof.** 1. It suffices to show (5.29) when \( \rho \) is some defining function of \( \Sigma_\omega \). We choose \( \rho \) such that
\[ \rho^{-1}(0) = \Sigma_\omega, \quad |\partial_\theta \rho| = 1 \quad \text{on} \quad \Sigma_\omega. \] (5.30)

Recalling the formula (4.60) for pullback on 1-forms we have the commutation identity of operators on \( C^\infty(S^1; T^*S^1) \)
\[ \rho \partial_\theta b_\omega^* = \varphi b_\omega^* \rho \partial_\theta + \psi b_\omega^*, \quad \varphi(\theta) = \frac{\rho(\theta) \partial_\theta b_\omega(\theta)}{\rho(b_\omega(\theta))}, \quad \psi(\theta) = \frac{\rho(\theta) \partial_\theta^2 b_\omega(\theta)}{\partial_\theta b_\omega(\theta)}. \] (5.31)

By (5.30) and since \( b_\omega(\Sigma_\omega) = \Sigma_\omega \) we have \( |\varphi| = 1 \) on \( \Sigma_\omega \).

2. Define \( v_k := (\rho \partial_\theta)^k v_\lambda^+ \in C^\infty(S^1; T^*S^1) \). We will show that for each \( k \) there exists \( N_k \) such that
\[ \| v_k \|_{H^{-\frac{1}{2}} + \beta} \leq C(\| g_\lambda \|_{H^N_0} + \| v_\lambda \|_{H^{-\infty}}). \] (5.32)

Similar analysis gives estimates on \( v_\lambda^- \); since \( v_\lambda = v_\lambda^+ + v_\lambda^- \) this implies (5.29).

The base case \( k = 0 \) is covered by Proposition 5.3. To prove (5.32) for \( k = 1 \), we apply \( \rho \partial_\theta \) to (5.9) and use (5.31) to get a similar equation on \( v_1 = \rho \partial_\theta v_\lambda^+ \) which also involves \( v_0 = v_\lambda^+ \):
\[
\begin{align*}
v_1 &= B_\lambda^+ \varphi b_\omega^* v_1 + Q_\lambda^+ b_\omega v_0 + \rho \partial_\theta (\mathcal{R}_\lambda^+ v_\lambda + g_\lambda^+), \\
Q_\lambda^+ &= [\rho \partial_\theta, B_\lambda^+] + B_\lambda^+ \psi \in \Psi^0(S^1; T^*S^1) \quad \text{uniformly in} \quad \varepsilon.
\end{align*}
\] (5.33)

Let \( E_G, \widetilde{E}_G \) be constructed in (5.13) where the escape function \( g \) is constructed using parameters \( \alpha_- < \alpha_+, \delta > 0 \) fixed in Step 3 below. Applying \( E_G \) to (5.33), we get similarly to (5.15)
\[ v_G^1 = T_G^1 v_G^1 + Q_G^0 v_G^0 + \mathcal{R}_G^1 v_\lambda + g_G^1 \] (5.34)
where \( v^k_G := E_G v_k, \mathcal{R}^1_G \in \Psi^{-\infty} \) uniformly in \( \varepsilon \), \( \|g^1_G\|_{L^2} \leq C\|g_\lambda\|_{H^{-N}} \) for some \( N_1 \), and
\[
T^1_G := E_G B^+_\lambda \varphi b^*_\omega \tilde{E}_G, \quad Q_G := E_G Q^+_\lambda b^*_\omega \tilde{E}_G.
\]

3. We now fix the parameters \( \alpha_{\pm} \) such that \(-\frac{1}{2} < \alpha_- < -\frac{1}{2} < \alpha_+ \) and take \( \delta > 0 \) small enough so that there exists \( \tau \) such that
\[
\max_{\pm} \sup_{\Omega^\pm(\delta)} \max(1, |\varphi|)(\partial_\theta b_\omega)^{1+\alpha_{\pm}} < \tau < 1,
\]
which is possible by (5.10) and since \( |\varphi| = 1 \) on \( \Sigma^\pm_\omega \).

Arguing similarly to the proof of Lemma 5.2, we get the bounds
\[
\|T^1_G v^1_G\|_{L^2} \leq \tau\|v^1_G\|_{L^2} + C\|v_\lambda\|_{H^{-N}}, \quad \|Q_G v^0_G\|_{L^2} \leq C\|v^0_G\|_{L^2}.
\]
Combining these with (5.34) and recalling (5.27) we get
\[
\|v^1_G\|_{L^2} \leq C\big(\|v^0_G\|_{L^2} + \|g_\lambda\|_{H^{N_1}} + \|v_\lambda\|_{H^{-N}}\big),
\]
\[
\|v^0_G\|_{L^2} \leq C\big(\|g_\lambda\|_{H^{N_1}} + \|v_\lambda\|_{H^{-N}}\big).
\]
By (5.16) this gives (5.32) for \( k = 1 \).

4. For \( k > 1 \) we argue by induction, writing similarly to (5.33)
\[
v_k = B^+_\lambda \varphi b^*_\omega v_k + \sum_{j=0}^{k-1} Q^+_{\lambda,k,j} b^*_\omega v_j + (\rho \partial_\theta)^k (B^+_\lambda v_\lambda + g^+_\lambda)
\]
where \( Q^+_{\lambda,k,j} \in \Psi^0 \) uniformly in \( \varepsilon \), defined inductively as follows:
\[
Q^+_{\lambda,k,j} := \left( [\rho \partial_\theta, B^+_\lambda \varphi^{k-1}] + B^+_\lambda \varphi^{k-1} \psi \right) \delta_{j,k-1} + Q^+_{\lambda,k-1,j-1} \varphi + [\rho \partial_\theta, Q^+_{\lambda,k-1,j}] + Q^+_{\lambda,k-1,j} \psi
\]
where \( \delta_{a,b} = 1 \) if \( a = b \) and 0 otherwise, and we define \( Q^+_{\lambda,k-1,j} = 0 \) when \( j \in \{-1, k-1\} \).

Choosing \( \delta \) small enough so that (5.35) holds with \( \max(1, |\varphi|) \) replaced by \( \max(1, |\varphi|^k) \), we repeat the argument in Step 3 of this proof to get (5.32) for any value of \( k \).

We will also need refinements concerning Lagrangian regularity and dependence on \( \omega \). We start with the former. Let \( B^\pm_{\omega+i0} \) be the operators \( B^\pm_\lambda \) from Lemma 5.1 with \( \varepsilon := 0 \).

**Lemma 5.5.** Suppose that \( v \in \mathcal{D}'(S^1; T^*S^1) \) satisfies (5.6) with \( \varepsilon = 0 \):
\[
v = B^+_\omega b^*_\omega v + B^-_\omega b^-_* v + g \quad \text{for some} \quad g \in C^\infty(S^1; T^*S^1).
\]
Similarly to (5.8) define \( v^\pm := \Pi^\pm v \). Then
\[
v^\pm \in I^\pm(\partial \Omega, N^+\Sigma^\pm_\omega) \quad \Longrightarrow \quad v^\pm \in I^\pm(\partial \Omega, N^+\Sigma^\pm_\omega).
\]
Proof. Let us consider the case \( + \). Suppose \( x_0 \in \Sigma^\sim_\chi \) and we choose coordinates in which \( x_0 = 0 \). We now identify \( v(x) = u(x)dx \) with \( u \in H^{-\frac{1}{2}-} \). If \( \chi \in C^\infty(\partial \Omega) \) is supported near 0 the assumption and (3.15) give

\[
\partial^k \hat{\chi} u(\xi) = \begin{cases}
\mathcal{O}(\langle \xi \rangle^{-k-\varepsilon}) & \xi \to +\infty \\
\mathcal{O}(\langle \xi \rangle^{-\infty}) & \xi \to -\infty,
\end{cases}
\]

(5.38)

for all \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). To obtain (5.37) we need to show that (5.38) holds with \( \varepsilon = 0 \).

Also (5.9) becomes

\[
u = C(b'(\bullet)b^* u) + f, \quad f \in C^\infty, \quad C \in \Psi^0, \quad \sigma(C) = 1.
\]

(5.39)

Since \( b'(\bullet)b^* (b'(\bullet)b^*) = (b^2)'(b^2)^* \) we can iterate this so that

\[
u = \tilde{C}((b^n)'(\bullet)(b^n)^*)u + \tilde{f}, \quad \tilde{f} \in C^\infty, \quad \tilde{C} \in \Psi^0, \quad \sigma(\tilde{C}) = 1.
\]

(5.40)

Now if 0 is a fixed point of \( B := b^n \), we can use elliptic parametrix for \( \tilde{C} \) in (5.40) to obtain

\[
u' B^* u = C_1 u + f_1, \quad C_1 \in \Psi^0, \quad \sigma(C_1) = 1, \quad f_1 \in C^\infty.
\]

(5.41)

We claim that this implies that

\[
\hat{\chi} u(B'(0)\xi) = \hat{\chi} u(\xi) + a(\xi), \quad \partial^k a(\xi) = \mathcal{O}(\langle \xi \rangle^{-k+\varepsilon}),
\]

(5.42)

for all \( \varepsilon > 0 \) and \( k \in \mathbb{N} \), that is \( a \in S^{-1+} \). In fact, if \( \chi_1 \) has the same properties as \( \chi \) and \( \chi_1 \equiv 1 \) on \( \text{supp} \chi \) then

\[
\hat{\chi} C_1 u(\xi) = \hat{\chi} C_1 \chi_1 u(\xi) + \mathcal{O}(\langle \xi \rangle^{-\infty})
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi(x)c_1(x, \eta)\hat{\chi} u(\eta)e^{ix(\eta-\xi)}d\eta + \mathcal{O}(\langle \xi \rangle^{-\infty}).
\]

We can apply the method stationary phase to the last integral (with the usual reduction to a compact range of integration – see for instance the proof of [17, Theorem 18.1.7]) which gives (noting that \( c_1 - 1 \in S^{-1} \) and that \( \hat{\chi} u - \hat{\chi} \hat{u} \in S^{-\infty} \)) to obtain

\[
\hat{\chi} C_1 u(\xi) = \hat{\chi} \hat{u}(\xi) + a_1(\xi), \quad a_1 \in S^{-1+}.
\]

We now consider

\[
\hat{\chi} B^* B^* u(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi(x) B'(x) \hat{\chi} u(\eta)e^{i\eta B'(0)\xi}d\xi d\eta.
\]

(Here we chose the supports small enough so that \( B^* \chi_1 \equiv 1 \) on \( \text{supp} \chi_1 \).) We can again apply the method of stationary phase and that gives \( \hat{\chi} B^* B^* u(\xi) = \hat{\chi} \hat{u}(B'(0)^{-1}\xi) + a_2(\xi), \quad a_2 \in S^{-1+} \). Comparison with (5.41) gives (5.42). We now claim that (5.42) implies that \( \hat{\chi} \hat{u} \) is bounded. It first implies that

\[
\hat{\chi} \hat{u}(B'(0)^k \xi) = \hat{\chi} \hat{u}(\xi) + \sum_{\ell=1}^k a(B'(0)^{\ell-1}\xi),
\]
so that (since \( B'(0) > 1 \)),

\[
\sup_{1 < \xi} |\hat{\chi}u(\xi)| = \sup_{k \in \mathbb{N}} \sup_{1 < \xi < B'(0)} |\hat{\chi}u(B'(0)^k \xi)|
\]

\[
\leq \sup_{1 < \xi < B'(0)} |\hat{\chi}u(\xi)| + C \sum_{\ell=1}^{\infty} B'(0)^{-1 - \varepsilon}(\ell - 1) = O(1).
\]

Higher derivative estimates follow by differentiating (5.42) \( k \) times and then applying the same argument with \( \hat{\chi}u \) and \( a \) replaced by \( \xi^k \partial^k \hat{\chi}u \) and \( \xi^k \partial^k a \in S^{-1+} \).

In addition, when \( v \) in Lemma 5.5 is given by \( u^\pm \) we have smooth dependence on \( \lambda \):

**Proposition 5.6.** Suppose that \( J \) is an open connected set such that conditions in Definition 2 are satisfied for \( \lambda \in J \), (5.1) holds for \( \lambda \in J \), and that \( v_\lambda^\pm \), are given in (5.1). Then

\[
v_\lambda^\pm \in C^\infty(J, I^+(\partial \Omega, N^* \Sigma^\pm_\lambda))
\]

in the following sense: let \( \Sigma^\pm_\lambda = \{ x_\ell^\pm(\lambda) \}_{\ell=1}^{L} \). Then

\[
v_\lambda^\pm(x) = \sum_{\ell=1}^{L} \int_{\mathbb{R}} \chi(x - x_\ell^\pm(\lambda)) e^{i(x - x_\ell^\pm(\lambda)) \xi} a^\pm(\lambda, \xi) d\xi + w^\pm_\lambda(x)
\]

(5.43)

\[
|\partial^k \partial^m a^\pm(\lambda, \xi)| \leq C_{km}(\xi)^{-k}, \quad \pm \partial_\lambda x_\ell^\pm(\lambda) > 0, \quad x_\ell^\pm \in C^\infty(J),
\]

and \( w \in C^\infty(J, C^\infty(\partial \Omega)) \).

**Proof.** Let \( \Phi(x, \lambda) \) be a smooth function such that \( \Phi(x, \lambda) = \partial_\lambda x_\ell^\pm(\lambda) \) in a neighbourhood of \( x_\ell^\pm(\lambda) \) and suppose that \( u \) is written as (5.43). Since, \( (\partial_\lambda + \Phi(x, \lambda) \partial_x)e^{i(x - x_\ell^\pm(\lambda))} = 0 \)

\forall \ L, M, \ell \quad (\rho \partial_\xi)^L(\partial_\lambda + \Phi(x, \lambda) \partial_x)^M u \in H^{-\frac{1}{2}}, \quad \chi_\mp u \in H^M
\]

(5.44) implies that

\[
|\partial^k_\lambda \partial^m_\xi a(\lambda, \xi)| \leq C_{km}(\xi)^{-k+},
\]

(5.45)

see the beginning of the proof of Lemma 5.5.

To obtain (5.44) we proceed as in the proof of Lemma 5.5 using the iterated equation (5.40), where \( b \) is replaced by \( B := b^a \), with \( n \) is the primitive period. The key observation is this: with \( B = B(x, \lambda) \), and for \( x \) near \( x_\ell^\pm(\lambda) \),

\[
(\partial_\lambda + \partial_\lambda x_\ell^\pm \partial_x)[u(B, \lambda)] = [\partial_\lambda u](B, \lambda) + (\partial_\lambda x_\ell^\pm \partial_x B + \partial_\lambda f) [\partial_x u](B, \lambda)
\]

\[
= [\partial_\lambda + \partial_\lambda x_\ell^\pm \partial_x u](B, \lambda) + \alpha_\ell(x) [\rho \partial_\xi u](B, \lambda), \quad \alpha_\ell \in C^\infty.
\]

Here we used the fact that \( B(x_\ell^\pm(\lambda), \lambda) = x_\ell^\pm(\lambda) \) so that by differentiation with respect to \( \lambda, \partial_\lambda x_\ell^\pm \partial_x B(x_\ell^\pm, \lambda) + \partial_\lambda B(x_\ell^\pm, \lambda) = \partial_\lambda x_\ell^\pm \). We can then use the fact that we know (5.44) for \( M = 0 \) and proceed by an inductive argument as in the second part \((K > 0)\) of the proof of Lemma 5.5. \( \square \)
6. Microlocal properties of Morse–Smale maps

Here we prove properties of distributions invariant under Morse–Smale maps (see Definition 2). We start with a stand alone local result about distributions invariant under contracting maps. The quantum flux defined below (6.3) is reminiscent of similar quantities appearing in scattering theory – see [11, (3.6.17)]. The wave front condition (6.16) is an analogue of the outgoing condition in scattering theory – see [11, Theorem 3.37]. Although technically very different, Lemma 6.1 and Proposition 6.3 are analogous to [10, Lemma 2.3] and play the role of that lemma in showing the absence of embedded eigenvalues – see [12, §3.2].

6.1. Local analysis. In this section we assume that \( f : [-1, 1] \to (-1, 1) \) is a \( C^\infty \) map such that
\[
    f(0) = 0, \quad 0 < f'(x) < 1.
\]
We also assume that
\[
    u \in \mathcal{D}'((-1, 1)), \quad \text{singsupp } u \subset \{0\}, \quad f^* u = u \text{ on } (-1, 1).
\]
For \( \chi \in C^\infty_c(f(-1, 1)) \), \( \chi = 1 \) near 0 we then define the flux of \( u \) (understood as an integral of a differential 1-form):
\[
    F(u) := i \int_{(-1, 1)} (f^* \chi - \chi) \bar{u} \, du.
\]

The integral is well defined since \( u \) is smooth on \( \text{supp}(f^* \chi - \chi) \subset (-1, 1) \setminus \{0\} \).

We note that \( F(u) \) is independent of \( \chi \). In fact, if \( \chi_j \in C^\infty_c(f(-1, 1)) \), \( \chi_1 = \chi_2 \) near 0, then the difference of the two fluxes is given by (6.3) with \( \tilde{\chi} = \chi_1 - \chi_2 \in C^\infty_c(f(-1, 1) \setminus \{0\}) \). Since \( \tilde{\chi} \) is supported away from 0 we can split the integral:
\[
    \int_{(-1, 1)} (f^* \tilde{\chi} - \tilde{\chi}) \bar{u} \, du = \int (f^* \tilde{\chi}) \bar{u} \, du - \int f^* (\tilde{\chi} \bar{u}) \, du = \int (f^* \tilde{\chi})(\bar{u} \, du - f^* (\bar{u} \, du)) = 0.
\]

Here in the first equality we made a change of variables by \( f : (-1, 1) \to f(-1, 1) \) and in the last equality we used (6.2). In fact, this argument shows that we could take \( \chi \) in (6.3) to be the indicator function of some interval \( f(a_-, a_+) \) with \(-1 < a_- < 0 < a_+ < 1\), obtaining
\[
    F(u) = i \int_{[a_-, f(a_-)] \cup [f(a_+), a_+]} \bar{u} \, du.
\]

Similarly we see that \( F(u) \) is real. For that we take \( \chi \) real valued so that
\[
    2 \text{Im } F(u) = 2 \int_{(-1, 1)} (f^* \chi - \chi) \Re(\bar{u} \, du) = \int (f^* \chi - \chi) d(|u|^2)
\]
\[
    = \int |u|^2 \, d(\chi - f^* \chi) = \int |u|^2 d\chi - \int |u|^2 f^* d\chi = 0,
\]
where in the last line we used (6.2) and the fact that \( \chi' = 0 \) near 0.
The key local result is given in

**Lemma 6.1.** Suppose that (6.1) and (6.2) hold. Then
\[ \WF(u) \subset \{0\} \times \mathbb{R}_+, \quad \mathbf{F}(u) \geq 0 \quad \Rightarrow \quad u = \text{const}. \] (6.5)

**Remark.** The wavefront set restriction to positive frequencies is crucial: for example, if \( u \) is the Heaviside function, then (6.2) holds and \( \mathbf{F}(u) = 0 \). A nontrivial example when (6.1), (6.2), and the wavefront set condition in (6.5) hold is \( f(x) = e^{-2\pi x}, \ u(x) = (x + i0)^{ik}, \ k \in \mathbb{Z} \setminus \{0\}, \) where \( \mathbf{F}(u) = 2\pi k(e^{-2\pi k} - 1) < 0 \).

To prove Lemma 6.1 we use a standard one dimensional linearization result [28]. For the reader’s convenience we present a variant of the proof from [34, Appendice 4].

**Lemma 6.2.** Assume that \( f \) satisfies (6.1). Then there exists a unique \( C^\infty \) diffeomorphism \( h : [-1, 1] \to h([−1, 1]) \subset \mathbb{R} \) such that for all \( x \in [-1, 1] \)
\[ h(f(x)) = f'(0)h(x), \quad h(0) = 0, \quad h'(0) = 1. \] (6.6)

**Proof.** 1. We first note that any \( C^1 \) diffeomorphism satisfying (6.6) is unique. In fact, suppose that \( h_j, \ j = 1, 2 \) are two such diffeomorphisms. With \( a = f'(0) \in (0, 1), \) \( ah_j = h_j \circ f \) we have \( ah_1 \circ h_2^{-1}(x) = h_1 \circ f h_2^{-1}(x) = h_1 \circ h_2^{-1}(ax) \) for all \( x \in h_2([-1, 1]), \) so that
\[ h_1 \circ h_2^{-1}(x) = a^{-n} h_1 \circ h_2^{-1}(a^n x) = \lim_{n \to \infty} a^{-n} h_1 \circ h_2^{-1}(a^n x) = (h_1 \circ h_2^{-1})'(0) x = x. \]
Hence it is enough to show that for every \( n \) there exists a \( C^n \) diffeomorphism satisfying (6.6).

Using the fact that \( a = f'(0) \in (0, 1) \) we can construct a formal power series such that (6.6) holds for the Taylor series of \( f \) as an asymptotic expansion. Using Borel’s Lemma [16, Theorem 1.2.6] we can then construct a diffeomorphism \( h_0 \) of \([-1, 1]\) onto itself with that formal series as Taylor series at 0. Then \( h_0 \circ f \circ h_0^{-1} = ax(1 + g(x)) \) where \( g \in C^\infty \) vanishes to infinite order at 0. Hence we can assume that
\[ f(x) = ax(1 + g(x)). \]
We might no longer have \( f' < 1 \) but \( f \) is still eventually contracting: there exists \( m > 0 \) such that the \( m \)-th iterate \( f^m \) satisfies
\[ \partial_x(f^m(x)) < 1 \quad \text{for all} \ x \in [-1, 1]. \] (6.7)

2. We are now looking for \( h(x) = x(1 + \varphi(x)), \ \varphi(0) = 0 \) such that \( h(ax(1 + g(x))) = ah(x), \) that is \( ax(1 + g(x))(1 + \varphi(f(x))) = ax(1 + \varphi(x)), \) or
\[ (1 + g(x))(1 + \varphi(f(x))) = 1 + \varphi(x). \]
A formal solution is then given by \( 1 + \varphi(x) = \prod_{k=0}^{\infty}(1 + g(f^k(x))). \) Rather than analyse this expression, we follow [34, Appendice 4] and use the contraction mapping principle.
for Banach spaces, $B_n$, of $C^n$ functions on $[-\delta, \delta]$ vanishing to order $n \geq 2$ at 0: we look for $\varphi \in B_n$ such that

$$g(x) + (1 + g(x))\varphi(f(x)) = \varphi(x), \quad x \in [-\delta, \delta].$$

(6.8)

We claim that for $\delta > 0$ small enough,

$$\varphi(x) \mapsto (T \varphi)(x) := (1 + g(x))\varphi(f(x))$$

is a contraction on $B_n$. The norm on $B_n$ is given by

$$\|\varphi\|_{B_n} := \sup_{|x| \leq \delta} |\partial^n \varphi(x)|, \quad \sup_{|x| \leq \delta} |\partial^j \varphi(x)| \leq C_n \delta^{n-j} \|\varphi\|_{B_n}, \quad \varphi \in B_n, \quad j \leq n,$$

(6.9)

where the last inequality follows from Taylor’s formula. Since $f(x) = ax(1 + g(x))$, we have $f'(x) = a + O(x^\infty)$ and $f^{(j)}(x) = O(x^\infty)$ for $j > 1$. Hence, we obtain for $|x| \leq \delta$, using (6.9) and with homogenous polynomials $Q_j$,

$$\partial^n[\varphi(f(x))] = \partial^n \varphi(f(x))(\partial f(x))^n + \sum_{j=1}^{n-1} \partial^j \varphi(f(x))Q_j(\partial f(x), \ldots, \partial^{n-j+1} f(x))$$

$$= \partial^n \varphi(f(x))a^n(1 + O_n(\delta)) + \sum_{j=1}^{n-1} O_n(\delta^{n-j})\|\varphi\|_{B_n}.$$ 

It follows that $\|T \varphi\|_{B_n} \leq (a^n + O(\delta)\|\varphi\|_{B_n}$, which for $\delta$ small enough (depending on $n$) shows that $T$ is a contraction on $B_n$. That gives a solution $\varphi$ to (6.8). Consequently, we have shown that for every $n$ there exist $\delta > 0$ and $\varphi \in C^n([-\delta, \delta])$ such that for $h(x) = x(1 + \varphi(x))$,

$$h(f(x)) = ah(x), \quad |x| \leq \delta, \quad h \in C^n([-\delta, \delta]).$$

By (6.7), there exists $N > 0$ such that $f^N([-1, 1]) \subset [-\delta, \delta]$. We extend $h$ to $[-1, 1]$ by putting $h(x) := a^{-N}h(f^N(x))$, to obtain a $C^n$ diffeomorphism $h : [-1, 1] \to h([-1, 1])$ satisfying (6.6).

Proof of Lemma 6.1. 1. We first note that if $u \in C^\infty((-1, 1))$ then $u$ is constant as follows from (6.2): for each $x \in (-1, 1)$ we have $u(x) = u(f^N(x)) \to u(0)$ as $N \to \infty$. Since we assumed that $\text{singsupp} u \subset \{0\}$ it suffices to show that $u$ is smooth in a neighborhood of 0.

Making the change of variable given by Lemma 6.2, we may assume that $f(x) = ax$ for small $x$, where $a := f'(0) \in (0, 1)$. Restricting to a neighborhood of 0, rescaling, and using (6.4) we reduce to the following statement: if

$$u \in D'((-a^{-1}, a^{-1})), \quad \text{WF}(u) \subset \{0\} \times \mathbb{R}_+, \quad u(ax) = u(x), \quad |x| < a^{-1},$$

(6.10)

$$\mathbf{F}(u) := i \int_{[-1,-a] \cup [a,1]} \mathbf{n} \, du \geq 0$$

(6.11)

then $u \in C^\infty((-1, 1))$. 
2. We next extend \( u \) to a distribution on the entire \( \mathbb{R} \). Fix
\[
\psi \in C_c^\infty((-a^{-1}, a^{-1}) \setminus [-a, a]), \quad \sum_{k \in \mathbb{Z}} \psi(a^{-k}x) = 1, \quad x \neq 0.
\]
Then \( \psi u \in C_c^\infty(\mathbb{R} \setminus \{0\}) \). Define
\[
v(x) := \sum_{k \in \mathbb{Z}} (\psi u)(a^{-k}x) \in C^\infty(\mathbb{R} \setminus \{0\}). \tag{6.12}
\]
Since \( u(ax) = u(x) \) for \( |x| < a^{-1} \), we have \( u = v \) on \((-a^{-1}, a^{-1}) \setminus \{0\} \). Thus we may extend \( v \) to an element of \( \mathcal{D}'(\mathbb{R}) \) so that \( u = v|_{(-a^{-1}, a^{-1})} \). We note that
\[
v \in \mathcal{S}'(\mathbb{R}), \quad v(ax) = v(x), \quad x \in \mathbb{R},
\]
\[
\text{WF}(v) \subset \{0\} \times \mathbb{R}_+, \quad \mathbf{F}(v) = \mathbf{F}(u) \geq 0. \tag{6.13}
\]
It remains to show that \( v \in C^\infty \); in fact, we will show that \( v \) is constant.

3. Fix \( \chi \in C_c^\infty(\mathbb{R}) \) such that \( \chi = 1 \) near 0 and write
\[
v = v_1 + v_2, \quad v_1 := \chi v, \quad v_2 := (1 - \chi)v.
\]
From (6.12) we obtain uniformly in \( x \neq 0 \),
\[
\partial_x^\ell v(x) = x^{-\ell} \sum_{k \in \mathbb{Z}} ([\chi]^\ell(\psi u)(a^{-k}x))(a^{-k}x) = \mathcal{O}(x^{-\ell}),
\]
since \( ([\chi]^\ell(\psi u)(a^{-k}x))(a^{-k}x) \in C_c^\infty(\mathbb{R} \setminus \{0\}) \) and the sum is locally finite with a uniformly bounded number of terms. Hence \( \partial_x^\ell v_2(x) = \mathcal{O}(x^{-\ell}) \) which implies that \( \widehat{v}_2(\xi) \) (and thus \( \widehat{v}(\xi) \)) is smooth when \( \xi \in \mathbb{R} \setminus \{0\} \) and
\[
\widehat{v}_2(\xi) = \mathcal{O}(\langle \xi \rangle^{-\infty}), \quad |\xi| \to \infty.
\]
On the other hand the assumption on \( \text{WF}(v) \) and [16, Proposition 8.1.3] shows that \( \widehat{v}_1(\xi) = \mathcal{O}(\langle \xi \rangle^{-\infty}) \), as \( \xi \to -\infty \). From (6.13) we obtain for \( \xi < 0 \) and \( k \in \mathbb{N} \),
\[
\widehat{v}(\xi) = a^{-1}\widehat{v}(a^{-1}\xi) = a^{-k}\widehat{v}(a^{-k}\xi) = \mathcal{O}_\xi(a^k) \quad \Rightarrow \quad \widehat{v}|_{\mathbb{R}_-} \equiv 0. \tag{6.14}
\]
4. It follows from (6.14) that the distributional pairing
\[
V(z) := \widehat{v}(e^{iz\bullet})/2\pi, \quad \text{Im } z > 0 \quad \tag{6.15}
\]
is well defined and holomorphic in \( \{\text{Im } z > 0\} \) and \( |V(z)| \leq C(z)^N/(\text{Im } z)^M, \text{ Im } z > 0 \) (with more precise bounds possible). We also have \( v(x) = V(x + i0) \) for \( x \in \mathbb{R} \setminus \{0\} \), and \( V(az) = V(z) \) when \( \text{Im } z > 0 \) which follows from (6.15). We will now use \( V \) to calculate \( \mathbf{F}(v) \). We have
\[
\mathbf{F}(v) = i \int_{\gamma_0} \overline{V(z)} \partial_z V(z) \, dz, \quad \gamma_0 := [-1, -a] \cup [a, 1],
\]
where the curve $\gamma_0$ is positively oriented. Let $\gamma_\alpha$, $\alpha > 0$, be the half circle $|z| = \alpha$, $\text{Im } z > 0$ oriented counterclockwise. Since $V(az) = V(z)$,

$$
\int_{\gamma_1} V(z) \partial_z V(z) dz = \int_{\gamma_1} V(az)(\partial_z V)(az)d(az) = \int_{\gamma_0} V(z) \partial_z V(z) dz.
$$

If $\Gamma$ is the semi-annulus bounded by $\partial \Gamma := \gamma_0 + \gamma_1 - \gamma_\alpha$ (see Figure 11) it follows from the Cauchy–Pompeiu formula [16, (3.1.9)] that (with $z = x + iy$)

$$
F(v) = i \oint_{\partial \Gamma} V(z) \partial_z V(z) dz = -2 \int_{\Gamma} \partial_z (V(z) \partial_z V(z)) dxdy = -2 \int_{\Gamma} |\partial_z V(z)|^2 dxdy.
$$

Since we assumed $F(v) \geq 0$ it follows that $V$ is constant on $\Gamma$ and thus on the entire upper half-plane, which implies that $v$ is constant on $\mathbb{R} \setminus \{0\}$. Since functions supported at 0 are linear combinations of derivatives of the delta function and cannot solve the equation $v(ax) = v(x)$, we see that $v$ is constant on $\mathbb{R}$, which finishes the proof. □

6.2. A global result. We now use the local result in Lemma 6.1 to obtain a global result for Morse–Smale diffeomorphisms of the circle.

**Proposition 6.3.** Let $b : \partial \Omega \to \partial \Omega$ be a Morse–Smale diffeomorphism (see Definition 2). Denote by $\Sigma^+, \Sigma^- \subset \partial \Omega$ the sets of attractive, respectively repulsive, periodic points of $b$, and define $N^*_+ \Sigma^\pm \subset T^* \partial \Omega$ by (3.17). Suppose that $u \in \mathcal{D}'(\partial \Omega)$ satisfies

$$
b^* u = u, \quad \text{WF}(u) \subset N^*_+ \Sigma^+ \cup N^*_- \Sigma^-.
$$

Then $u$ is constant.

**Proof.** We introduce fluxes associated to $g := b^n$, where $n$ is the minimal period of periodic points of $b$. For that we take two arbitrary cutoff functions

$$
\chi_\pm \in C^\infty(\partial \Omega), \quad \text{supp}(1 - \chi_\pm) \cap \Sigma^\pm = \emptyset, \quad \text{supp } \chi_\pm \cap \Sigma^\pm = \emptyset.
$$

**Figure 11.** The domain $\Gamma$ used in the proof of Lemma 6.1.
Assume that \( u \) satisfies (6.16) and define the fluxes (where we again use positive orientation on \( \partial \Omega \) to define the integrals of 1-forms):

\[
F_+(u) := i \int_{\partial \Omega} (g^* \chi_+ - \chi_+) \, \bar{\nu} \, du,
\]

\[
F_-(u) := i \int_{\partial \Omega} ((g^{-1})^* \chi_- - \chi_-) \, \bar{\nu} \, du.
\]

The integrals above are well-defined since \( g^* \chi_+ - \chi_+ \) and \((g^{-1})^* \chi_- - \chi_- \) are supported in \( \partial \Omega \setminus (\Sigma^+ \cup \Sigma^-) \), where \( u \) is smooth. Moreover as in the case of \( F(u) \) defined in (6.3), \( F_\pm(u) \) are real and do not depend on the choice of \( \chi_\pm \). We also note that (by taking \( \chi_\pm \) real valued)

\[
F_\pm(\bar{u}) = -F_\pm(u) = -F_\pm(u). \tag{6.17}
\]

Since \( F_\pm(u) \) are independent of \( \chi_\pm \), we may choose \( \chi_+ := 1 - (g^{-1})^* \chi_- \) to get the identity

\[
F_+(u) = F_-(u). \tag{6.18}
\]

Let \( \Sigma^+ = \{x_1^+, \ldots, x_m^+\} \). By taking \( \chi_+ = \chi_1^1 + \cdots + \chi_m^+ \) where each \( \chi_j^+ \) is supported near \( x_j^+ \), we can write \( F_+(u) = F_{1+}(u) + \cdots + F_{m+}(u) \). We may apply Lemma 6.1 with \( f \) defined by \( g \) in local coordinates near \( x_j^+ \simeq 0 \) to see that \( F_{2+}(u) \leq 0 \) with equality only if \( u \) is constant near \( x_j^+ \). Adding these together, we see that \( F_+(u) \leq 0 \) with equality only if \( u \) is locally constant near \( \Sigma^+ \).

Arguing similarly near \( \Sigma^- \), using \( f := g^{-1} \) and replacing \( u \) by \( \bar{\pi} \), with \( WF(\pi) = \{(x, -\xi) \mid (x, \xi) \in WF(u)\} \), we see that \( F_-(u) \geq 0 \) with equality only if \( u \) is locally constant near \( \Sigma^- \). By (6.18) we then see that \( u \) is locally constant near \( \Sigma^+ \cap \Sigma^- \) and hence \( u \in C^\infty(\partial \Omega) \). Since for \( x \in \partial \Omega \setminus \Sigma^- \), \( g^n(x) \to x_0 \), for some \( x_0 \) in \( \Sigma^+ \), we conclude that \( u \in C^\infty \) takes finitely many values and hence is constant. \( \square \)

7. Limiting absorption principle

In this section we consider operators

\[
P := \partial^2_{x_2} \Delta_{\Omega}^{-1} : H^{-1}(\Omega) \to H^{-1}(\Omega),
\]

\[
P(\lambda) := \partial^2_{x_2} - \lambda^2 \Delta = (P - \lambda^2) \Delta_{\Omega} : H^1_0(\Omega) \to H^{-1}(\Omega). \tag{7.1}
\]

We prove the limiting absorption principle for \( P \) in the form presented in Theorem 2. To do this we follow §4.4 to reduce the equation \( P(\lambda)u_\lambda = f \) to the boundary \( \partial \Omega \). We next analyze the resulting ‘Neumann data’ \( v_\lambda \) uniformly as \( \varepsilon = \Im \lambda \to 0^+ \), using the high frequency estimates of §5 and the absence of embedded spectrum following from the results of §6. This is slightly non-standard since the boundary has characteristic points and the problem changes from elliptic to hyperbolic as \( \Im \lambda \to 0^+ \).
7.1. Poincaré spectral problem. We recall (see for instance [8, Chapter 6]) that \( \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 \) with the domain \( H^2(\Omega) \cap H_0^1(\Omega) \), \( (H_0^1(\Omega) \) is the closure of \( C_c^\infty(\Omega) \) with respect to the norm \( \| \cdot \|_{H_0^1(\Omega)} \) below) is a negative definite unbounded self-adjoint operator on \( L^2(\Omega) \). Its inverse is an isometry,

\[
\Delta_\Omega^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega),
\]

with inner products on these Hilbert spaces given by

\[
\langle u, w \rangle_{H_\Omega^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla w \, dx, \quad \langle U, W \rangle_{H^{-1}(\Omega)} := \langle \Delta_\Omega^{-1} U, \Delta_\Omega^{-1} W \rangle_{H_\Omega^1(\Omega)}.
\]

Since \( \partial_{x_2} : H_0^1(\Omega) \to H^{-1}(\Omega) \) the operator \( P \) in (7.1) is indeed bounded on \( H^{-1}(\Omega) \).

Let \( \{e_\alpha\}_{\alpha \in A} \) be an \( L^2(\Omega) \)-orthonormal basis of eigenfunctions of \( -\Delta_\Omega \):

\[
-\Delta_\Omega e_\alpha = \mu_\alpha^2 e_\alpha, \quad e_\alpha|_{\partial \Omega} = 0, \quad \langle e_\alpha, e_\beta \rangle_{L^2(\Omega)} = \delta_{\alpha,\beta}.
\]

Then \( \{\mu_\alpha e_\alpha\}_{\alpha \in A} \) is an orthonormal basis of the Hilbert space \( H^{-1}(\Omega) \). The matrix elements of \( P \) in this basis are given by

\[
\langle P \mu_\alpha e_\alpha, \mu_\beta e_\beta \rangle_{H^{-1}} = \langle \Delta_\Omega^{-1} \partial_{x_2}^2 \mu_\alpha^2 e_\alpha, \mu_\beta^{-1} e_\beta \rangle_{H_\Omega^1(\Omega)} = -\mu_\alpha^{-1} \mu_\beta^{-1} \langle \partial_{x_2} e_\alpha, e_\beta \rangle_{L^2(\Omega)} = \mu_\alpha^{-1} \mu_\beta^{-1} \langle \partial_{x_2} e_\alpha, \partial_{x_2} e_\beta \rangle_{L^2(\Omega)},
\]

where the last integration by parts is justified as \( e_\beta|_{\partial \Omega} = 0 \). This shows that \( P \) is a bounded self-adjoint operator on \( H^{-1}(\Omega) \). This representation is particularly useful in numerical calculations needed to produce Figure 1. Testing \( P \) against \( \Delta^2(\psi(x)e^{i(n,x)}) \), \( \psi \in C_c^\infty(\Omega), n \in \mathbb{Z}^2 \), shows that

\[
\text{Spec}(P) = [0, 1],
\]

see [25, Theorem 2]. In particular, for \( \lambda^2 \in \mathbb{C} \setminus [0, 1] \),

\[
\|P(\lambda)^{-1}\|_{H^{-1}(\Omega) \to H_0^1(\Omega)} = \|(P - \lambda^2)^{-1}\|_{H^{-1}(\Omega) \to H^{-1}(\Omega)} = \frac{1}{d(\lambda^2, [0, 1])}. \tag{7.2}
\]

Limiting absorption principle in its most basic form means showing we have limiting operators acting on smaller spaces with values in larger spaces: for \( \omega \in (0, 1) \) satisfying the Morse–Smale conditions

\[
(P - \omega^2 - i0)^{-1} : C_c^\infty(\Omega) \to H^{-\frac{3}{2}}(\Omega), \quad P(\omega + i0)^{-1} : C_c^\infty(\Omega) \to H^\frac{1}{2}(\Omega). \tag{7.3}
\]

7.2. Analysis of the boundary data. In this section we analyze the behavior of the ‘Neumann data’ \( \nu_\lambda \) of \( P(\lambda)^{-1}f \) (see (7.11)) as \( \lambda \) approaches the real line. Let

\[
\lambda = \omega + i\varepsilon, \quad 0 < \varepsilon \ll 1, \tag{7.4}
\]

where \( \omega \in (0, 1) \) satisfies the Morse–Smale conditions of Definition 2. We start with a uniqueness result for the limiting operator \( P(\omega) \) whose proof uses the analysis of §6:
Lemma 7.1. Let $P(\omega)$ be defined in (7.1), $\Lambda^+(\omega) \subset N^*\Gamma_\omega(\Sigma_\omega)$ be given in (1.9), and conormal spaces up to the boundary be as defined in §3.2. Then we have

$$u \in I^{-1+}(\overline{\Omega}, \Lambda^+(\omega)), \quad u|_{\partial \Omega} = 0, \quad P(\omega)u = 0 \implies u \equiv 0. \quad (7.5)$$

Remark. The restriction to the boundary $u|_{\partial \Omega}$ is well defined as, under our assumptions, $\Gamma_\omega(\Sigma_\omega)$ is transverse to the boundary $\partial \Omega$, as follows from (2.2).

Proof. As $\omega \in (0, 1)$, $P(\omega)$ is a constant coefficient hyperbolic operator. In view of (4.3) and (4.4) we then have, denoting $\ell^\pm(x) := \ell^\pm(x, \omega)$, $\ell^\pm_{\min} := \ell^\pm(x_{\min})$, $\ell^\pm_{\max} := \ell^\pm(x_{\max})$

$$u(x) := u_+(\ell^+(x)) - u_-(\ell^-(x)), \quad x \in \Omega, \quad u_\pm \in D'(((\ell^\pm_{\min}, \ell^\pm_{\max})). \quad (7.6)$$

Since $u \in I^{-1+}(\overline{\Omega}, \Lambda^+(\omega))$, we see that $u^\pm$ are smooth near the boundary points $\ell^\pm_{\min}, \ell^\pm_{\max}$ up to the boundary. Moreover, $\text{WF}(u) \subset \Lambda^+(\omega)$. We then define $w_\pm = u_\pm(\ell^\pm(x)|_{\partial \Omega}$ which are equal since $u|_{\partial \Omega} = 0$ and obtain a distribution satisfying

$$w = w_+ = w_- \in D'(\partial \Omega), \quad (\gamma^\pm)^*w = w, \quad \text{WF}(w) \subset N^*\Sigma^+_\omega \sqcup N^*\Sigma^-_\omega. \quad (7.7)$$

This implies that $b^*w = w$ and we can apply Proposition 6.3 (with $u := \overline{w}$) to see that $w$ is constant. But then $u_\pm$ are constant and $u \equiv 0$. \hfill \Box

Lemma 7.1 implies the following uniqueness statement featuring the operator $R_{\omega+i0} : g \mapsto E_{\omega+i0} * g$ defined similarly to (4.20) and the restricted single layer potential operator $C_{\omega+i0}$ studied in §6.6:

Lemma 7.2. Assume that $v \in I^{1+}_{\partial \Omega}(N^*_\omega \Sigma^+_\omega \sqcup N^*_\omega \Sigma^-_\omega)$ is a section of $T^*\Omega$. Then

$$C_{\omega+i0}v = 0, \quad \text{supp}(R_{\omega+i0}T_v) \subset \overline{\Omega} \implies v = 0. \quad (7.8)$$

Proof. 1. Put $U := R_{\omega+i0}T_v \in D'(\mathbb{R}^2)$ and $u := S_{\omega+i0}v = U|_{\Omega}$ where $S_{\omega+i0}$ is the limiting single layer potential defined in (4.27). Since $P(\omega)E_{\omega+i0} = \delta_0$ we have

$$P(\omega)U = T_v. \quad (7.9)$$

In particular, $P(\omega)u = 0$ since $\text{supp}T_v \subset \partial \Omega$. From (4.31), (4.32), and the first assumption in (7.8) we see that $u$ satisfies (7.5). Lemma 7.1 then shows that $u \equiv 0$.

By the second assumption in (7.8) we have $\text{supp}U \subset \overline{\Omega}$. Therefore

$$\text{supp}U \subset \partial \Omega. \quad (7.10)$$

2. We now show that $v = 0$ away from the characteristic set $\mathcal{C}_\omega$ of $P(\omega)$ on $\partial \Omega$ (see (2.2)). For each $x_0 \in \partial \Omega \setminus \mathcal{C}_\omega$ we can find a neighbourhood $V \subset \mathbb{R}^2$ of $x_0$ and coordinates $(y_1, y_2)$ on $V$ such that for some open interval $\mathcal{I} \subset \mathbb{R}$

$$\partial \Omega \cap V = \{y_1 = 0, y_2 \in \mathcal{I}\}, \quad P(\omega)|_V = \sum_{|\alpha| \leq 2} a_\alpha(y)\partial_\alpha y, \quad a_{2,0} \neq 0.$$
Then we have for each $\beta > 1$. We first use the high frequency estimates of $\sum_{|\alpha|=2} a_{\alpha} \eta^{\alpha}$. Now, by [16, Theorem 2.3.5] we see that (7.10) implies $U|_V = \sum_{k \leq K} u_k(y_2)\delta^{(k)}(y_1)$, $u_k \in \mathcal{D}'(\mathcal{I})$. Hence, for some $\tilde{u}_k \in \mathcal{D}'(\mathcal{I})$,

$$P(\omega)U|_V = a_{2,0}(y)u_K(y_2)\delta^{(K+2)}(y_1) + \sum_{k \leq K+1} \tilde{u}_k(y_2)\delta^{(k)}(y_1).$$

By (7.9) we have $P(\omega)U|_V = \mathcal{I}v|_V = a(y_2)v(y_2)\delta(y_1), a \neq 0$. Thus $u_K = 0$. (Here we use $y_2$ as a local coordinate on $\partial \Omega$ to identify $v|_{\partial \Omega \cap V}$ with a distribution on $\mathcal{I}$.) Iterating this argument shows that $U|_V = 0$ which means $v|_{V \cap \partial \Omega} = 0$.

3. We have shown that supp $v$ is contained in the finite set $\mathcal{C}_\omega$. On the other hand, $v \in H^{\frac{1}{2}+}(\partial \Omega, N_+^+ \Sigma_+ \cup N_-^+ \Sigma_-)$ is smooth away from $\Sigma_\omega$. Since $\Sigma_\omega \cap \mathcal{C}_\omega = \emptyset$ by (2.2), we get $v = 0$. \hfill $\square$

We now use (5.1) to obtain a uniform description of the ‘Neumann data’ $v_\lambda$ as $\varepsilon = \Im \lambda \to 0+$:

**Proposition 7.3.** Suppose that $\lambda = \omega + i\varepsilon$ as in (7.4) and $u_\lambda \in C^\infty(\overline{\Omega})$ is the solution to the boundary value problem (4.16) with $\lambda$-independent right-hand side $f \in C^\infty_c(\Omega)$. As in (4.21), define

$$v_\lambda := -2\lambda\sqrt{1-\lambda^2}j^*(L_\lambda u_\lambda d\ell^+(\bullet, \lambda)) \in C^\infty(\partial \Omega; T^* \partial \Omega).$$

(7.11)

Then we have for each $\beta > 0$

$$v_\lambda \to v_{\omega + i0} \quad \text{in} \quad H^{\frac{1}{2}-\beta}(\partial \Omega; T^* \partial \Omega) \quad \text{as} \quad \varepsilon \to 0^+$$

(7.12)

where $v_{\omega + i0} \in H^{\frac{1}{2}}(\partial \Omega; T^* \partial \Omega)$ is the unique distribution such that

$$C_{\omega + i0} v_{\omega + i0} = (R_{\omega + i0} f)|_{\partial \Omega}, \quad \text{supp} \, R_{\omega + i0} (f - \mathcal{I}v_{\omega + i0}) \subset \overline{\Omega}.\quad (7.13)$$

Moreover, $v_{\omega + i0} \in H^{\frac{1}{2}}(\partial \Omega, N_+^+ \Sigma_+ \cup N_-^+ \Sigma_-)$.

**Proof.** 1. We first use the high frequency estimates of §5 to show that if

$$\lambda_\varepsilon = \omega + i\varepsilon_\varepsilon, \quad \varepsilon_\varepsilon \to 0^+, \quad u_\varepsilon \in C^\infty(\overline{\Omega}), \quad f_\varepsilon \in C^\infty_c(\Omega), \quad P(\lambda_\varepsilon)u_\varepsilon = f_\varepsilon,$$

$$u_\varepsilon|_{\partial \Omega} = 0, \quad v_\varepsilon := -2\lambda_\varepsilon\sqrt{1-\lambda_\varepsilon^2}j^*(L_{\lambda_\varepsilon} u_\varepsilon d\ell^+(\bullet, \lambda_\varepsilon)) \in C^\infty(\partial \Omega; T^* \partial \Omega)\quad (7.14)$$

are sequences such that for some $N$, $v_0 \in H^{-N}(\partial \Omega; T^* \partial \Omega)$, and $f_0 \in C^\infty_c(\Omega)$ we have

$$v_\varepsilon \to v_0 \quad \text{in} \quad H^{-N}(\partial \Omega; T^* \partial \Omega), \quad f_\varepsilon \to f_0 \quad \text{in} \quad C^\infty_c(\Omega),$$

(7.15)

then we have the stronger convergence statement

$$v_\varepsilon \to v_0 \quad \text{in} \quad H^{-\frac{1}{2}-\beta}(\partial \Omega; T^* \partial \Omega) \quad \text{for all} \quad \beta > 0. \quad (7.16)$$
To see this we note that \( v_\ell \) solves the equations (5.1) and (5.6):

\[
C_{\lambda_\ell} v_\ell = G_\ell := (R_{\lambda_\ell} f_\ell)|_{\partial \Omega},
\]

\[
v_\ell = B_{\lambda_\ell}^+ b_\omega^* v_\ell + B_{\lambda_\ell}^- b_{\omega}^* v_\ell + g_\ell, \quad g_\ell = (I - A_{\lambda_\ell}) E_{\lambda_\ell} dG_\ell.
\]

By (7.15) and Lemma 4.3 we have

\[
G_\ell \to G_0 := (R_{\omega + i 0} f_0)|_{\partial \Omega} \text{ in } C^\infty(\partial \Omega).
\]

This implies that \( \|g_\ell\|_{H^{N_0}} \) is bounded in \( \ell \) for each \( N_0 \).

Take arbitrary \( 0 < \beta' < \beta \). Then Proposition 5.3 shows that \( \|v_\ell\|_{H^{-\frac{1}{2} - \beta'}} \) is bounded in \( \ell \). Using compactness of the embedding \( H^{-\frac{1}{2} - \beta'} \hookrightarrow H^{-\frac{1}{2} - \beta} \) we see that each subsequence of \( \{v_\ell\} \) has a subsequence converging in \( H^{-\frac{1}{2} - \beta} \); the limit of this further subsequence has to be equal to \( v_0 \). This implies (7.16).

We now study the properties of the limit \( v_0 \). Using a similar argument with the estimate (5.26) we see that

\[
\Pi^\pm (v_\ell - v_0) \to 0 \quad \text{in } C^\infty(\partial \Omega \setminus \Sigma^+_\omega; T^* \partial \Omega).
\]

In particular, \( \text{WF}(v_0) \subset N^+_\omega \Sigma^+_\omega \cup N^+_\omega \Sigma^-_\omega \). Similarly, Proposition 5.4 implies that for each function \( \rho \in C^\infty(\partial \Omega; \mathbb{R}) \) vanishing simply on \( \Sigma_\omega \) we have

\[
\| (\rho \partial_\theta)^K (v_\ell - v_0) \|_{H^{-\frac{1}{2} - \beta}} \to 0 \quad \text{for all } \beta > 0, \ K \in \mathbb{N}_0.
\]

In particular, \( (\rho \partial_\theta)^K v_0 \in H^{-\frac{1}{2} - \beta} \) for all \( K \). Together with the wavefront set statement above this implies by (3.16) that

\[
v_0 \in I^{\frac{1}{2}+}(\partial \Omega, N^+ \Sigma^+_\omega \cup N^+_\omega \Sigma^-_\omega).
\]

Moreover, taking the limit in (7.17) using Lemma 4.16 we have

\[
C_{\omega + i 0} v_0 = G_0.
\]

Finally, if we put \( U_\ell := u_\ell \mathbb{1}_{\Omega} \in \mathcal{E}'(\mathbb{R}^2) \), then by (4.23), in the notation of (4.20),

\[
U_\ell = R_{\lambda_\ell}(f_\ell - I v_\ell).
\]

Passing to the limit (using Lemma 4.3 and the fact that convolution is a continuous operator \( \mathcal{D}'(\mathbb{R}^2) \times \mathcal{E}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2) \)), and since \( \text{supp} U_\ell \subset \overline{\Omega} \), we get

\[
U_\ell \to U_0 := R_{\omega + i 0}(f_0 - I v_0) \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \quad \text{supp} U_0 \subset \overline{\Omega}.
\]

2. We next show that for each \( \beta > 0 \) there exists a constant \( C \) (depending on \( f \) and \( \beta \)) such that for all small \( \varepsilon > 0 \)

\[
\|v_\lambda\|_{H^{-\frac{1}{2} - \beta}(\partial \Omega; T^* \partial \Omega)} \leq C.
\]
We proceed by contradiction. Assume that there exists a sequence \( \varepsilon_\ell \to 0^+ \) such that \( \|v_{\lambda_\ell}\|_{H^{-\frac{1}{2}-\beta}} \to \infty \) where \( \lambda_\ell := \omega + i\varepsilon_\ell \). We then put
\[
v_\ell := v_{\lambda_\ell}/\|v_{\lambda_\ell}\|_{H^{-\frac{1}{2}-\beta}}, \quad u_\ell := u_{\lambda_\ell}/\|v_{\lambda_\ell}\|_{H^{-\frac{1}{2}-\beta}}, \quad f_\ell := f/\|v_{\lambda_\ell}\|_{H^{-\frac{1}{2}-\beta}}.
\]
We see that (7.14) holds. By the compactness of the embedding \( H^{-\frac{1}{2}-\beta} \hookrightarrow H^{-N} \), where we fix \( N > \frac{1}{2} + \beta \), we may pass to a subsequence to make (7.15) hold as well, where \( f_0 = 0 \).

The limiting distribution \( v_0 = \lim_{\ell \to \infty} v_\ell \) satisfies by (7.16), (7.21), (7.22), and (7.23)
\[
v_0 \in I_\Omega^+(\partial \Omega, N^*_\omega \Sigma^+_\omega \sqcup N^*_\omega \Sigma^-_\omega), \quad \mathcal{C}_{\omega+i0} v_0 = 0,
\]
\[
\text{supp}(R_{\omega+i0} I_\Omega v_0) \subset \overline{\Omega}, \quad \|v_0\|_{H^{-\frac{1}{2}-\beta}} = 1.
\]
By Lemma 7.2 we get from the first three properties that \( v_0 = 0 \) which contradicts the last property.

3. Assume that \( \beta > 0 \) and we have a sequence \( \lambda_\ell = \omega + i\varepsilon_\ell, \varepsilon_\ell \to 0^+ \), such that \( v_{\lambda_\ell} \) converges to some \( v_0 \) in \( H^{-\frac{1}{2}-\beta} \). In this case (7.14) and (7.15) hold with \( u_\ell := u_{\lambda_\ell}, v_\ell := v_{\lambda_\ell}, \) and \( f_0 = f \). By (7.21), (7.22), and (7.23) we see that \( v_0 \) satisfies (7.13). On the other hand, by Lemma 7.2 there is at most one distribution satisfying (7.13). It follows that each sequence \( v_{\lambda_\ell} \) which converges in \( H^{-\frac{1}{2}-\beta} \) has to have the same limit.

Now, by (7.24) and compactness of the embedding \( H^{-\frac{1}{2}-\beta'} \hookrightarrow H^{-\frac{1}{2}-\beta} \) for \( 0 < \beta' < \beta \) we see that the family \( v_\lambda \) is precompact in each \( H^{-\frac{1}{2}-\beta}, \beta > 0, \varepsilon \to 0^+ \). It follows that \( v_\lambda \to v_{\omega+i0} \) in \( H^{-\frac{1}{2}-\beta} \) as \( \varepsilon \to 0^+ \) where \( v_{\omega+i0} \) is the unique solution to (7.13).

Finally, from (7.13) we have \( \mathcal{C}_{\omega+i0} v_{\omega+i0} \in C^\infty(\partial \Omega) \). Similarly to (5.6) we then have
\[
v_{\omega+i0} = B_{\omega+i0}^+ b_\omega v_{\omega+i0} + B_{\omega+i0}^- b_\omega^* v_{\omega+i0} + g \quad \text{for some} \quad g \in C^\infty(\partial \Omega; T^* \partial \Omega).
\]
Now Lemma 5.5 and (7.13) imply that \( v_{\omega+i0} \in I_\Omega^+(\partial \Omega, N^*_\omega \Sigma^+_\omega \sqcup N^*_\omega \Sigma^-_\omega) \).

**Remark.** The proof would be simpler if we knew that for \( \omega \)-simple \( \Omega \)'s, the limiting single layer potential operators \( S_{\omega+i0} \) were injective acting on the conormal spaces (4.31) (and even simpler if we knew that the restricted single layer potentials \( C_{\omega+i0} \) were injective). However, that is not clear. Under the dynamical assumptions made here, the proof of Proposition 5.3 shows that \( \ker S_{\omega+i0} \subset \ker C_{\omega+i0} \) is finite dimensional but injectivity seems to be a curious open problem.

7.3. **Proof of Theorem 2.** Fix \( f \in C^\infty_\Omega(\Omega) \), let \( \lambda = \omega + i\varepsilon \) where \( \omega \in (0,1) \) satisfies the Morse–Smale conditions of Definition 2 and \( 0 < \varepsilon \ll 1 \). Let \( u_\lambda \in C^\infty(\overline{\Omega}) \) be the solution to the boundary-value problem (4.16). Recalling (7.1) we see that
\[
(P - \lambda^2)^{-1} f = \Delta u_\lambda \in C^\infty(\overline{\Omega}).
\]
Next, by (4.24) we have
\[
u_\lambda = (R_\lambda f)|_{\Omega} - S_\lambda v_\lambda \quad (7.25)
\]
where the ‘Neumann data’ $v_\lambda$ is given by (7.11). By Proposition 7.3 we have $v_\lambda \to v_{\omega+i0}$ in \( H^{-\frac{1}{2}} \) as $\varepsilon \to 0^+$. Then similarly to (7.23) we pass to the limit in (7.25) to get
\[
\mathcal{R} u_\lambda \to \mathcal{R} u_{\omega+i0} := (\mathcal{R}_{\omega+i0} f)_{\Omega} - S_{\omega+i0} v_{\omega+i0} \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{as} \quad \varepsilon \to 0^+.
\]
This gives the convergence statement (1.15) with
\[
(P - \omega^2 - i0)^{-1} f = \Delta u_{\omega+i0}.
\]
Next, since $R_{\omega+i0} f \in C^\infty(\mathbb{R}^2)$, $v_{\omega+i0} \in I^{-1}(\partial \Omega, N^+ \Sigma^+_{\omega} \cup N^- \Sigma^-_{\omega})$, and by (4.31) we have $u_{\omega+i0} \in I^{-1}(\Omega, \Lambda^+(\lambda))$ which implies
\[
(P - \omega^2 - i0)^{-1} f \in I^1(\Omega, \Lambda^+(\lambda)).
\]
Lemma 5.5 and Proposition 5.6 then provide smooth dependence of $v_{\lambda+i0}$ on $\lambda$ which translates to the smooth dependence of $u_{\lambda+i0}$ as a family of Lagrangian distributions.

Since $C^\infty_c(\Omega)$ is dense in $H^{-1}(\Omega)$ (see for instance [25, Lemma 5]), it is then standard (see for instance [6, Proposition 4.1]) that the spectrum of $P$ in $J$ is absolutely continuous. □

8. Large time asymptotic behaviour

We will now follow [12, §§5,6] and use (1.13) to describe asymptotic behaviour of solutions of (1.1) when $f \in C^\infty_c(\Omega)$ and the chess billiard for $\lambda$ and $\Omega$ has the Morse–Smale property of Definition 2. We start with the Stone formula (see for instance [11, Theorem B.10]) to rewrite (1.13) as follows. Let $\varphi \in C^\infty_c((0,1))$ be equal to 1 near $\lambda$. Then
\[
\mathcal{R} u = \Delta^{-1}_\Omega \text{Re} w, \quad w = w_1 + b_1,
\]
where
\[
b_1(t) = e^{i\lambda t} \frac{1}{2\sqrt{P}} \sum_{\pm} \pm (1 - \varphi(\sqrt{P}))/\lambda \pm \sqrt{P})^{-1} (1 - e^{-it(\lambda \pm \sqrt{P})}) f,
\]
and $\|b_1(t)\|_{H^{-1}} = O(1)$ as $t \to \infty$. The asymptotically singular term is given by
\[
w_1(t) = \int_0^t \sqrt{P}^{-1} \sin((t - s)\sqrt{P}) \varphi(\sqrt{P}) f e^{i\lambda s} ds
\]
\[
= \frac{1}{2\pi} e^{i\lambda t} \int_0^t \varphi(\omega)(u^+(\omega) - u^-(\omega))(e^{i(\omega - \lambda)s} - e^{-i(\omega + \lambda)s}) d\omega ds,
\]
where $u^\pm(\omega) := (P - \omega^2 \pm i0)^{-1} f$. To understand the behaviour of $w_1$ as $t \to \infty$ we present a more concrete version of Lemma 4.9:

Lemma 8.1. Suppose that for $\omega \in J \Subset (0,1)$, Morse–Smale assumptions hold (Definition 2). Suppose that $\Sigma^\pm_{\omega} = \{x^\pm_{\ell}(\omega)\}_{\ell=1}^L$, and for a positively oriented ordering on
\[ \partial \Omega \setminus x_1^+, x_2^+ < x_1^-, x_2^+ < x_{\ell+1}^+, \ell < L \text{ (see Figure 12)}. \] We also put \( \nu_\ell^\pm := \nu^\pm(x_\ell^\pm) \).

Then, in the notation of (8.2), with \( \varepsilon = \pm \), and for \((\omega, x) \in \Omega \times J\),

\[ u_\varepsilon(\omega, x) = \sum_{\ell = 1}^L \sum_{\pm} g_{\ell, \pm}(x, \omega) + u_0^\varepsilon(x, \omega), \quad u_0^\varepsilon \in C^\infty(J \times \overline{\Omega}), \quad g_{\ell, \pm} \in D'(\mathbb{R}), \]

(8.3)

where \( a_{\ell, \pm}^\varepsilon \in C^\infty(J, S^1(\mathbb{R})), a_{\ell, \pm}^\varepsilon = \mathcal{O}((t)^{-\infty}), \forall \varepsilon t > 0. \)

**Proof.** We consider the case of \( \varepsilon = + \). We then have \( u^+(\omega) = \Delta u \), where \( u \) is given by (4.24). We then need to analyse singularities of \( \Delta S_{J1}^{-1}v \) where the structure of \( v \) is given in Proposition 5.6 (see the proof of Theorem 2 in §7.2). Then in the notation of (4.30) we have and with \( x_0 = x_\ell^\pm, g_\pm \equiv 0 \) and

\[ g_\pm(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\varepsilon \omega(t - t^\pm(x_\ell^\pm(\omega), \omega)) - \ell^\pm(x_\ell^\pm(\omega), \omega))} a_{\ell, \pm}^\varepsilon(\omega, t) dt, \]

where \( a_{\ell, \pm}^\varepsilon \in C^\infty(J, S^1(\mathbb{R})), a_{\ell, \pm}^\varepsilon = \mathcal{O}((t)^{-\infty}), \forall \varepsilon t > 0. \)

Summing contributions from these terms we obtain (8.3). \( \square \)

The first lemma disposes of the term \( u_0^\varepsilon \):

**Lemma 8.2.** Suppose that \( u^\pm(\omega, x) \in C^\infty(J \times \overline{\Omega}) \). If \( w_1 \) is defined by (8.2) then for any \( k \), there exists \( C_k \) such that \( \|w_1(t)\|_{C^k(\overline{\Omega})} \leq C_k \).

**Proof.** We can put \( u^- = u \) and \( u^+ = 0 \), the other case being similar. Let \( U \in C^\infty(J \times \mathbb{R}^2) \) be an extension of \( u \) such that \( \text{supp} U \subset K \subset \mathbb{R}^2 \) for a fixed set \( K \). We
then define $W$ by (8.2) with $u^- = U$, $u^+ = 0$, so that

$$
2\pi e^{-i\lambda t} \overline{W}(t, \xi) = \int_0^t \int_{\mathbb{R}} \varphi(\omega) \hat{U}(\omega, \xi) [(1 + s^2)^{-1}(1 + D_{\omega}^2)](e^{i(\omega - \lambda)s} - e^{-i(\omega + \lambda)s})d\omega ds
$$

$$
= \int_0^t \int_{\mathbb{R}} (D_{\omega}^2 + 1) \varphi(\omega) \hat{U}(\omega, \xi) [(e^{i(\omega - \lambda)s} - e^{-i(\omega + \lambda)s})(1 + s^2)^{-1}]d\omega ds,
$$

which is uniformly bounded in $C^\infty(J_\omega; S^{-\infty}(\mathbb{R}^2))$. \hfill \square

**Proof of Theorem 1.** We rewrite (8.2) as follows: $w_1(t) = \sum_{\pm} \pm w_1^\pm(t)$,

$$
w_1^\pm(t) := \frac{1}{2\pi} e^{i\lambda t} \int_0^t \int_{\mathbb{R}} \varphi(\omega)(u^-(\omega) - u^+(\omega)) e^{i(\pm \omega - \lambda)s}d\omega ds,
$$

where $u^\pm(\omega) := (P - \omega^2 \pm i0)^{-1}f$.

In view of Lemma 8.2 we only need to analyse the long time behaviour of (8.4) for $u^\varepsilon$, $\varepsilon = \pm$, replaced by

$$
v_{\pm}^\varepsilon(\omega, x) := \psi_{\pm}(\ell^\pm(x, \omega))g_{\pm}^\varepsilon(\ell^\pm(x, \omega)), \; \psi_{\pm} \in C^\infty_c(\mathbb{R}), \; \psi_{(\ell^\pm_{\min-1, \ell^\pm_{\max+1}})} \equiv 1. \tag{8.5}
$$

(We have $\psi_{\pm}(\ell^\pm(x, \omega)) = 1$ on a neighbourhood of $\Omega$ and we can then treat $u^\varepsilon$ globally on $\mathbb{R}^2$.) We can also assume that $\text{supp } a_{\ell^\pm \varepsilon} \subset \pm \varepsilon[1, \infty)$. Thinking of $L_{\pm}(\omega)$ as elements of the dual of the dual of $(\mathbb{R}^2)^*$ we have

$$
(\mathbb{R}^2)^* \ni \xi = (L_+(\omega), \xi)\ell^+(\omega) + (L_-(\omega), \xi)\ell^-(\omega), \; \ell^\pm \in (\mathbb{R}^2)^*, \; \ell^\pm(x) := \ell^\pm(x, \omega).
$$

Hence, since $\partial(x_1, x_2)/\partial(\ell^+, \ell^-) = \omega \sqrt{1 - \omega^2}/2$,

$$
F_{x_1, x_2}(f(\ell^+(x))g(\ell^-(x))) = \int_{\mathbb{R}^2} e^{-i((L_+(\omega), \xi)\ell^+(x) + (L_-(\omega), \xi)\ell^-(x))} f(\ell^+(x))g(\ell^-(x))dx
$$

$$
= \frac{1}{2} \omega \sqrt{1 - \omega^2} \hat{f}(\langle L_+(\omega), \xi \rangle) \hat{g}(\langle L_-(\omega), \xi \rangle).
$$

Consequently,

$$
\hat{v}_\pm^\varepsilon(\omega, \xi) = \hat{\psi}_{\pm}(\langle L_+(\omega), \xi \rangle) e^{-i\nu^\varepsilon x^\pm_{\ell^\pm}(\omega, \xi)} a_{\ell^\pm \varepsilon} \langle \omega, \langle L_\pm(\omega), \xi \rangle \rangle. \tag{8.6}
$$

We also need a lemma similar to [12, Lemma 5.1]. Since it is slightly different, we give a proof.

**Lemma 8.3.** Suppose that $F \in C^\infty([\lambda - \delta, \lambda + \delta])$, $|\partial_\omega F(\omega)| > 0$, $\varphi \in C^\infty_c((\lambda - \delta, \lambda + \delta))$, and $a_j \in C^\infty([\delta, \delta] \times \mathbb{R})$ satisfy

$$
\forall k, \ell, p \in \mathbb{N}, \exists C_{k\ell} \; |\partial_\xi^\ell \partial_\omega^k a_1(\omega, \xi)| \leq C_{k\ell} \xi^{-k}, \; |\partial_\xi^\ell \partial_\omega^m a_2(\omega, \xi)| \leq C_{k\ell} \xi^{-p}.
$$

Then, if $\delta$ is small enough, with $\xi \in \mathbb{R}^2$,

$$
J(t, \xi) := \int_0^t \int_{\mathbb{R}} e^{-iF(\omega)(\xi, L_{\pm}(\omega)) + is(\omega - \lambda)} a_1(\omega, \xi, L_{\pm}(\omega)) a_2(\langle L_{\pm}(\omega), \xi \rangle) \varphi(\omega)d\omega ds
$$

decomposes as follows

$$
J(t, \xi) = J_\infty(\xi) + c(t, \xi) + d(t, \xi), \tag{8.7}
$$
where
\[ c(t, \xi) = \mathcal{O}(\langle L^\pm(\lambda), \xi \rangle^{-1} \langle L_\mp(\lambda), \xi \rangle^{-\infty}), \quad \langle \xi \rangle^{-m-\frac{1}{2}} d(t, \xi) \xrightarrow{L^2(\mathbb{R}^2)} 0, \quad t \to +\infty, \]
and
\[ J_\infty(\xi) := 2\pi \varphi(\lambda) a_1(\lambda, \langle L_\pm(\lambda), \xi \rangle) a_2(\langle L_\mp(\lambda), \xi \rangle) e^{-iF(\lambda)\langle L_\pm(\lambda), \xi \rangle}, \]
when
\[ \partial_\omega F(\lambda) \langle L_\pm(\lambda), \xi/|\xi| \rangle > 0, \]
and 0 otherwise.

**Proof.** Let us assume for simplicity that \( m = 0 \) and put
\[ a(\xi, \omega) := a_1(\omega, \langle \xi, L_\pm(\omega) \rangle) a_2(\langle \xi, L_\mp(\omega) \rangle). \]
We then put \( h = 1/|\xi|, s = r/h, \eta = h\xi \) and
\[ G(\omega, \eta) := F(\omega) \langle \eta, L_\pm(\omega) \rangle. \]
Then
\[ J(t, \eta/h) = h^{-1} \int_0^t \int_{\mathbb{R}} e^{\frac{i}{h}(-G(\omega, \eta) + r(\omega - \lambda))} a(\omega, \eta/h) \varphi(\omega) d\omega dr. \]
We then observe that there exist \( c, \gamma > 0 \) such that
\[ |r + \partial_\omega G(\omega, \eta)| > c(r), \quad r \in \begin{cases} \mathbb{R}_+ \setminus (\gamma, 1/\gamma), & \partial_\omega G(\omega, \eta) > 0 \\ \mathbb{R}_+, & \partial_\omega G(\omega, \eta) \leq 0. \end{cases} \]
Let \( \chi_+ \) be supported in \((\gamma/2, 2/\gamma)\) and equal to 1 on \((\gamma, 1/\gamma)\), and let \( \chi_- \equiv 0 \). Since
\[ h^N((r - \partial_\omega G(\omega, \eta))^{-1} D_\omega)^N e^{\frac{i}{h}(-G(\omega, \eta) + ir(\omega - \lambda))} = e^{\frac{i}{h}(-G(\omega, \eta) + ir(\omega - \lambda))}, \]
integrations by part in \( \omega \) show that
\[ h^{-1} \int_0^t \int_{\mathbb{R}} e^{\frac{i}{h}(-G(\omega, \eta) + r(\omega - \lambda))} a(\omega, \eta/h) \varphi(\omega)(1 - \chi_\varepsilon(r)) d\omega ds = h^{-1} \int_0^t \mathcal{O}(\langle r \rangle^{-\infty} h^\infty) dr, \]
which gives a contribution which is uniformly \( \mathcal{O}(h^\infty) = \mathcal{O}(\langle \xi \rangle^{-\infty}) \). We now consider
\[ \tilde{J}(t, \eta/h) = h^{-1} \int_{-\infty}^t \int_{\mathbb{R}} \chi(r) e^{\frac{i}{h}(-G(\omega, \eta) + r(\omega - \lambda))} \chi_\varepsilon(r) a(\omega, \eta/h) \varphi(\omega) d\omega dr, \quad (8.8) \]
(which is identically 0 for \( \partial_\omega G(\omega, \eta) \leq 0 \); we can replace the left limit by \(-\infty \) thanks to the cut-off \( \chi_\varepsilon \)). For \( th > 2/\gamma, \tilde{J}(t, \eta/h) = J(\infty, \eta/h) \) and we can apply the method of stationary phase in \( r \) and \( \omega \) to the integral. That gives, with \( J_\infty \) from (8.7)
\[ \tilde{J}(\infty, \xi) = J_\infty(\xi) + c_\infty(\xi), \quad \partial_\xi^2 c_\infty(\xi, |\xi|) = \mathcal{O}(\langle \xi \rangle^{-1-|\alpha|} \langle L_\pm(\lambda), \xi \rangle^{-\infty}). \]
This shows that for every \( \xi \in \mathbb{R} \), \( \lim_{t \to +\infty} \tilde{J}(t, \xi) = J_\infty(\xi) + c_\infty(\xi) \) and it remains to show that
\[ \langle \xi \rangle^{-\frac{1}{2}} (\tilde{J}(t, \xi) - J_\infty(x) - c_\infty(\xi)) \xrightarrow{L^2(\mathbb{R}^2)} 0. \]
(We then put $c(t, \xi) = J(t, \xi) - \bar{J}(t, \xi) + c_\infty(\xi)$.) This follows from the dominated convergence theorem once we prove that for all $\delta > 0$,

$$\|\langle \xi \rangle^{-\frac{1}{2} - \delta}\bar{J}(t, \xi)\|_{L^2(B_\delta^2)} \leq C_\delta. \tag{8.9}$$

In fact, we put $\tau = th = t/|\xi|$, $\rho = \omega - \lambda$ and rewrite (8.8) as

$$\bar{J}(t, \xi) = |\xi| \int_{\mathbb{R}} A(\rho|\xi|, \tau)a(\lambda + \rho, \xi)\varphi(\lambda + \rho)e^{-iG(\omega, \xi)}d\rho,$$

$$A(r, \tau) := \int_{0}^{\infty} \chi_\epsilon(\tau - s)e^{-ir(\tau - s)}ds = \mathcal{O}(\langle t \rangle^{-1}).$$

(The last bound comes from the bound on the support of $\chi_\epsilon$ which gives $A(t) = \mathcal{O}(1)$ and an integration by parts which shows that $A(t) = it^{-1}A(0) + \mathcal{O}(t^{-2})$, the estimates uniform in $\tau$.) Hence, noting that $|\xi| > c > 0$ on the support of $\bar{J}(t, \xi)$ (due to the support properties of $a_j$'s), and making a change of variables $\zeta = \langle L_\pm(\lambda + \rho), \xi \rangle$, $\langle L_{\mp}(\lambda + \rho), \xi \rangle$ in the double integral,

$$\|\langle \xi \rangle^{-\frac{1}{2} - \delta}\bar{J}(t, \xi)\|_{L^2(B_\delta^2)}^2 \leq C \int_{|\rho| < \delta} \int_{|\xi| > c} |\xi|^{-\delta}|A(\rho|\xi|, \tau)|^2(1 + |\langle L_{\pm}(\lambda + \rho), \xi \rangle|)^{-2}d\xi d\rho \leq C_N \int_{|\xi| > c'} |\zeta|^{-\delta}|\langle \zeta_2 \rangle|^{-2} \int_{|\rho| < \delta} \langle \rho|\xi| \rangle^{-2}d\rho d\zeta \leq C'' \int_{|\xi| > c'} |\zeta|^{-1 - \delta}|\langle \zeta_2 \rangle|^{-2}d\zeta = C_\delta.$$

This is (8.9), completing the proof. \hfill \Box

**Remark.** If we replace $is(\omega - \lambda)$ in the definition of $J(t, \xi)$ by $is(-\omega - \lambda)$, then we see that (8.7) is valid with $J_{\infty}(\xi) \equiv 0$.

We now apply the lemma to analyse $w_1^+$ in (8.4) with $u^\pm$ given by a sum of terms of the form $\hat{\omega}^\pm$ (see (8.6)) with

$$F(\omega) = \nu_1^\pm \ell^\pm(x_1^\pm(\omega), \omega), \quad a_1(\omega, t) = a_\hat{\omega}^\pm(\omega, t).$$

Let $\theta \mapsto x(\theta) \in \partial\Omega$ be a positive parametrization near $x_1^\pm(\omega)$ (with $\partial\Omega \ni x \mapsto \theta(x)$ denoting its inverse) and put $f := b^\omega$ with $n$ the primitive period. Then

$$f(x_1^\pm(\omega), \omega) = x_1^\pm(\omega), \quad \pm(1 - \partial_\theta f(x_1^\pm(\omega))) > 0, \quad \partial_\omega[\theta(f(x, \omega))] > 0, \quad \tag{8.10}$$

where we used (2.14) to get the last inequality. Differentiating the first equality in $\omega$ gives

$$\partial_\omega[\theta(x_1^\pm(\omega))] = \partial_\omega[\partial_\theta f(x_1^\pm(\omega), \omega)]/(1 - \partial_\theta[\theta(f(x_1^\pm(\omega), \omega))]).$$

Then the inequalities in (2.14) give $\text{sgn} \partial_\omega[\theta(x_1^\pm(\omega))] = \pm 1$ and hence

$$\text{sgn} \partial_\omega F(\omega) = \text{sgn} [\nu_1^\pm \partial_\theta \ell^\pm(x_1^\pm(\omega), \omega) \partial_\omega[\theta(x_1^\pm(\omega))] = \text{sgn} \partial_\omega[\theta(x_1^\pm(\omega))] = \pm 1.$$


Hence, the sign condition in Lemma 8.3 combined with the support condition in (8.3) become
\[ \pm \langle L_{\pm}(\lambda), \xi \rangle > 0, \quad \pm \varepsilon \langle L_{\pm}(\lambda), \xi \rangle > 0, \]
which implies that the only contribution to the leading term comes from \( \varepsilon = + \). (We knew it had to come out this way because of the heuristic argument in (1.14).)

Hence,
\[ w^+_1(t) = e^{i\lambda t} u^+(\lambda) + \tilde{b}_1^+(t) + \tilde{e}^+(t), \quad \| \tilde{b}_1^+(t) \|_{H^{-1}} = \mathcal{O}(1), \quad \| \tilde{e}^+(t) \|_{H^{-\frac{3}{2}}-1} = o(1), \quad t \to \infty. \]

The remark after the proof of the lemma shows also that
\[ w^-_1(t) = \tilde{b}_1^-(t) + \tilde{e}^-(t), \quad \| \tilde{b}_1^-(t) \|_{H^{-1}} = \mathcal{O}(1), \quad \| \tilde{e}^-(t) \|_{H^{-\frac{3}{2}}-1} = o(1), \quad t \to \infty. \]

Going back to (8.1) gives the statement of Theorem 1. \( \square \)

**Acknowledgements.** We would like to thank Peter Hintz and Hart Smith for helpful discussions concerning §6 and injectivity of \( \mathcal{C}_\lambda \) (see Remark in §7.2), respectively, and for their interest in this project. Special thanks are also due to Leo Maas for his comments on an early version of the paper and help with the introduction and references. SD was partially supported by NSF CAREER grant DMS-1749858 and a Sloan Research Fellowship, while JW and MZ were partially supported by NSF grant DMS-1901462.

**References**


Email address: dyatlov@math.mit.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139

Email address: wangjian@berkeley.edu

Email address: zworski@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720