A MATHEMATICAL STUDY OF PERIODIC BAND INVERSION

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1. The Hamiltonian

We study an interesting Hamiltonian introduced by Tan–Devakul [TaDe24]. To describe it we let $V \in C^\infty(\mathbb{R}^2/\Gamma;\mathbb{R})$ be a Γ-periodic potential where we take $\Gamma = 2\pi\mathbb{Z}^2$. The specific potential considered in [TaDe24] was $V(x) = V_0(\cos x_1 + \cos x_2)$.

We use the following notation for the modified $\bar\partial$ and annihilation operators:

$$\mathcal{D} = 2D_{\bar z} = D_{x_1} + iD_{x_2}, \quad z = x_1 + ix_2, \quad A = \tfrac{1}{\sqrt{2}}(D_w - iw), \quad w \in \mathbb{R}, \quad D = \tfrac{1}{i}\partial.$$

The Hamiltonian in [TaDe24] is the following self-adjoint operator on $L^2(\mathbb{R}^2_x \times \mathbb{R}_w, dxdw)$ which models an electron in a 2D crystal coupled to a circularly polarized photon cavity mode [OnDe26]:

$$H_J := \mathcal{D}\mathcal{D}^* + J(A - \lambda\mathcal{D})^*(A - \lambda\mathcal{D}) + V(x). \tag{1.1}$$

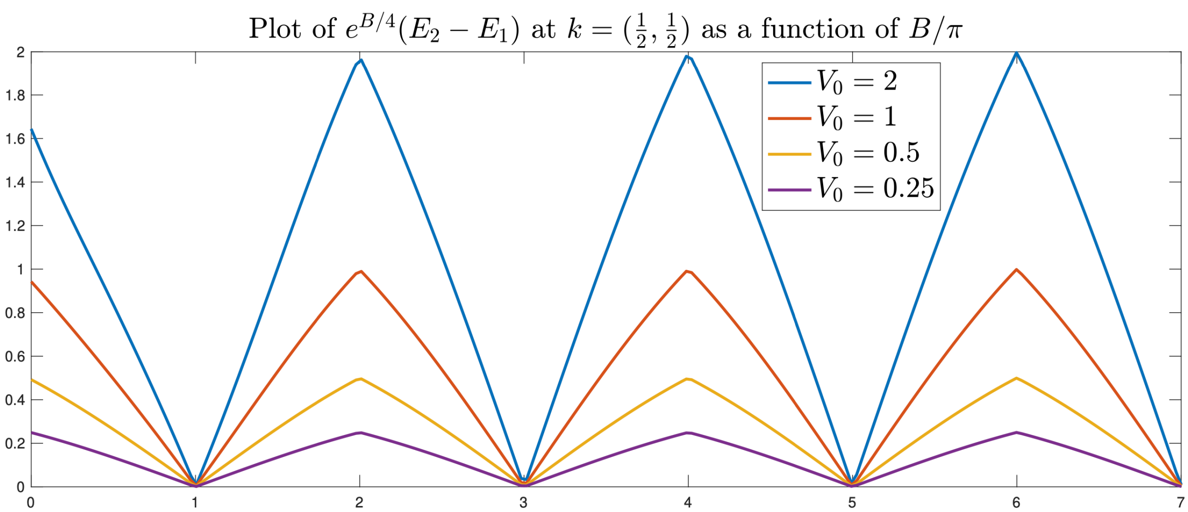

Figure 1. Reinterpretation of [TaDe24, Figure 2(b)]: after rescaling the gap for the effective Hamiltonian (1.5) displays an approximate periodicity: $e^{B/4}(E_2 - E_1) = V_0(\|\cos(B/4)| - |\sin(B/4)\|) + \mathcal{O}(V_0^2 e^{-B/4})$. For an animated version see `https://math.berkeley.edu/~zworski/gap_movie.mp4`.

The operator H_J is Γ-periodic in $z = x_1 + ix_2$ so it is natural to consider Bloch–Floquet spectrum,

$$\begin{gathered} E_1(k,B,J) \leq E_2(k,B,J) \leq \cdots \leq E_n(k,B,J) \to \infty, \\ k \in \mathbb{C}/\mathbb{Z}^2, \quad B = 2\lambda^2, \quad k = k_1 + ik_2 \in \mathbb{C}, \end{gathered} \tag{1.2}$$

given by the eigenvalues of

$$H_J(k) := (\mathcal{D} - k)(\mathcal{D} - k)^* + J(A - \lambda(\mathcal{D} - k))^*(A - \lambda(\mathcal{D} - k)) + V(x), \tag{1.3}$$

with periodic boundary conditions in x with respect to $\Gamma = 2\pi\mathbb{Z}^2$.

Tan–Devakul [TaDe24] considered the case of $J \to \infty$ and made striking observations about the bands and their topology: as a function of B the gap between the bands,

$$g(B) := \min_{k \in \mathbb{C}/\mathbb{Z}^2} (E_2(k,B) - E_1(k,B)), \tag{1.4}$$

oscillates with regularly spaced zeros and local maxima (see [TaDe24, Figure 2(b)] where $g((2\ell+1)\pi) = 0$ and with local maxima at $B = 2\ell\pi$, $\ell \in \mathbb{N}$) Every time the gap closes, the topology of the line bundle corresponding to $E_1(k,B)$ changes with the Chern number increasing by one. This process is referred to in physics as *band inversion*. That is done by considering an effective Hamiltonian ($E_j(k,B)$ are its bands) which provides an approximation in the $J \to \infty$ limit.

The purpose of this paper is to clarify and extend some of the results of [TaDe24] and our findings can be summarised as follows:

- Construction of an effective Hamiltonian (1.5) for an arbitrary periodic potential V and the analysis of the $J \to \infty$ approximation – see Theorem 1.

- Perturbative analysis of the gap $g(B)$ in (1.4), for the case $V(x) = V_0(\cos x_1 + \cos x_2)$ as $V_0\, e^{-B/4} \to 0$ – see Figure 1 and Theorem 2.

- Proof of $g((2\ell+1)\pi) = 0$, $\ell \in \mathbb{N}$ for $V(x) = V_0(\cos x_1 + \cos x_2)$; showing that the 1st and 2nd bands, and 3rd and 4th bands touch at conic (Dirac) points for $\gamma := V_0 e^{-(2\ell+1)\pi/4}$ small; we also show that $\gamma \in \mathbb{R} \setminus \mathcal{A}$, where $\mathcal{A}$ is a discrete set, the Dirac cone structure persists – see Figure 2 and Theorem 3.

- Proof that there is *no* gap for bands corresponding to $E_j(k,B)$, $j > 3$, for small $V_0^2 e^{-B/4}$. The gap here is understood as a minimum over k, not the gap between the unions over all k – see (1.17) and (1.16). We also estimate the number of bands that can be isolated for an arbitrary $V_0^2 e^{-B/4}$ – see Theorem 4.

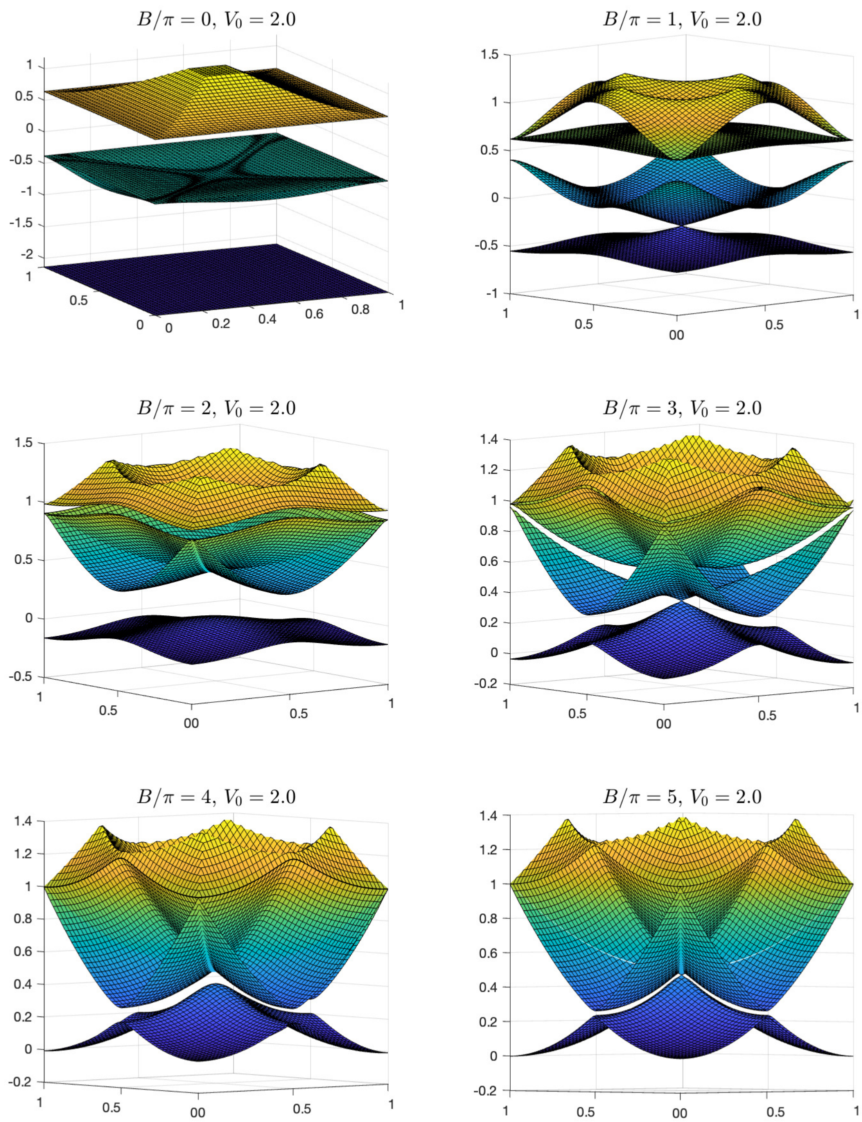

FIGURE 2. The first four bands for the effective Hamiltonian (1.5) plotted over the Brillouin torus. Perturbation theory shows that the bands for (1.3) are within $\mathcal{O}(1/J)$ of the bands of (1.5). That means that separated bands remain separated (for large J) and the change of Chern numbers computed in §6 shows that they have to touch. For an animated plot of $k \mapsto E_j((k,k),B)$ see `https://math.berkeley.edu/~zworski/TD_movie.mp4`.

- Computation of the Chern number, ℓ, of the line bundle of eigenvectors of $E_1(k, 2\pi\ell)$, and of the rank 2-bundle, 2ℓ, of eigenvectors of $E_j(k, 2\pi\ell)$, $j = 2, 3$ (guaranteed to be separated for *small* or *large* values $V_0^2 e^{-\pi\ell/2}$ and expected to be separated throughout) – see Theorem 5.

We also address a sign discrepancy emphasized in [TaDe24] – see §6.3. It concerns the sign of the well defined Berry curvature of the natural connection over the Brillouin torus and the curvature of a *parent* connection defined over the entire plane when $V \equiv 0$. Although we use different conventions we see the same sign difference between the two curvatures. The discrepancy is resolved by keeping track of the descent to the Brillouin torus. The curvature computed on the covering space does not by itself determine the Chern number: the transition functions have to be specified. In the unperturbed problem, $V \equiv 0$ (the *parent band* of [TaDe24]), the natural descent is given by magnetic translations and gives a trivial line bundle (for $B = 2\pi\ell$). After the periodic perturbation is introduced, the relevant descent is instead the Bloch quasi-periodicity of the effective eigenfunction – see (6.10) and (6.11) respectively. With our conventions this gives Chern number ℓ. The different signs (the corresponding Chern number in [TaDe24] is $-\ell$) come from different conventions: [TaDe24] considers bundles which are dual to ours – see (6.6). We follow the standard mathematical convention used in [TaZw23] (and many physics papers).

We now provide precise statements of results. We start with the effective Hamiltonian. For a real valued $2\pi\mathbb{Z}$-periodic potential W we define an operator on $L^2(\mathbb{R}^2)$ with the domain given by $H^2(\mathbb{R}^2)$,

$$H = \mathcal{D}^*\mathcal{D} + W^{\rm w}(x_1 + BD_{x_2}, x_2), \tag{1.5}$$

where we consider x_1 as a parameter and (for $u \in \mathscr{S}(\mathbb{R}^2)$) define

$$W^{\rm w}(x_1 + BD_{x_2}, x_2)u := \operatorname{Op}^{\rm w}(W(x_1 + B\xi_2, x_2)),$$

where we used Weyl quantization, see [Zw12, Chapter 4],

$$\operatorname{Op}^{\rm w}(a)u := a^{\rm w}(x, D)u := \frac{1}{(2\pi)^2}\int\int a(\tfrac{1}{2}(x+y), \xi)e^{i\langle x-y,\xi\rangle}u(y)dyd\xi. \tag{1.6}$$

The operator (1.5) is unitarily equivalent to

$$\begin{aligned} H_\theta &:= e^{-i\theta BD_{x_1}D_{x_2}} H e^{i\theta BD_{x_1}D_{x_2}} \\ &= \mathcal{D}^*\mathcal{D} + W^{\rm w}(x_1 + (1-\theta)BD_{x_1}, x_2 - \theta BD_{x_1}), \end{aligned} \tag{1.7}$$

so that $\theta = \frac{1}{2}$ is the symmetric gauge.

The bands for H in (1.5) are defined as the spectrum,

$$E_1(k, B) \le E_2(k, B) \le \cdots \le E_n(k, B) \to \infty, \quad n \to \infty, \quad k = k_1 + ik_2 \in \mathbb{C}, \tag{1.8}$$

of $H(k) : H^2(\mathbb{C}/\Gamma) \to L^2(\mathbb{C}/\Gamma)$, where

$$H(k) = H(k,B) := (\mathcal{D}-k)^*(\mathcal{D}-k) + W^{\mathrm{w}}(x_1 + B(D_{x_2} - k_2), x_2), \tag{1.9}$$

The first result relates the bands of H_J to the bands of H with a specific W:

Theorem 1. *For any fixed E and definitions (1.2), (1.8) with*

$$W = \exp(-B\mathcal{D}^*\mathcal{D}/4)V, \tag{1.10}$$

we have, for $E_j(k,B,J) < E$,

$$E_j(k,B,J) = E_j(k,B) + \mathcal{O}(1/J), \quad J \to \infty. \tag{1.11}$$

Here $\exp(-t\mathcal{D}^*\mathcal{D})$ is the heat propagator for the Laplacian $\Delta = -\mathcal{D}^*\mathcal{D}$.

The potential considered in [TaDe24] was given by

$$V(x) = V_0(\cos x_1 + \cos x_2), \tag{1.12}$$

so that the potential W in (1.5) is given by

$$W(x) = V_0 e^{-B/4}(\cos x_1 + \cos x_2). \tag{1.13}$$

In this case we have a perturbative formula for the gap:

Theorem 2. *Suppose that V is given by (1.12) and that $|V_0|e^{-B/4} \ll 1$. Then with g in (1.4), the notation of Theorem 1, and $k_0 = (\frac{1}{2}, \frac{1}{2})$,*

$$\begin{aligned} g(B) &= E_2(k_0,B) - E_1(k_0,B) \\ &= |V_0|e^{-B/4}(||\cos(B/4)| - |\sin(B/4)||) + \mathcal{O}(V_0^2 e^{-B/2}). \end{aligned} \tag{1.14}$$

The remarkable accuracy of the approximation (1.14) is illustrated by Figure 1 and animation linked there. It suggests that at $B = (2\ell+1)\pi$, $\ell \in \mathbb{N}$, the bands touch. That is indeed the case for all V_0 with the bands touching at a Dirac point except possibly a discrete set of values of V_0 (numerically it seems to happen for all values). That is seen from perturbation theory for small values of V_0 and the results about persistence of Dirac points by Drouot–Lyman [DrLy26] (see that paper for references to earlier literature, in particular to the work of Fefferman–Weinstein [FW12]).

Theorem 3. *Suppose that V is given by (1.12) and $H(k)$ and $E_j(k,B)$ are given by (1.5), (1.8). Then for all V_0, the eigenvalues of $H(k_0)$, $k_0 = (\frac{1}{2}, \frac{1}{2})$ with $B_0 := (2\ell+1)\pi$, $\ell \in \mathbb{N}$ are double and in particular,*

$$E_{1+p}(k_0,B_0) = E_{2+p}(k_0,B_0) \quad p = 0,2, \quad k_0 = (\tfrac{1}{2}, \tfrac{1}{2}).$$

If $|V_0|e^{-B/4} \ll 1$ then there exist positive definite matrices $A_p(\alpha)$ such that for $p = 0,2$,

$$E_{2+p}(k,B_0) - E_{1+p}(k_0,B_0) = \langle A_p(V_0 e^{-B/4})(k-k_0), k-k_0\rangle^{\frac{1}{2}} + \mathcal{O}(|k-k_0|^2), \tag{1.15}$$

Moreover, there exists a discrete set $\mathcal{A} \subset \mathbb{R}$ such that, for $\alpha := V_0 e^{-B/4} \notin \mathcal{A}$, (1.15) for $p = p_j$, $j = 1,2$, for some $1 \le p_1 + 1 < p_2$.

We believe that we can take $p_1 = 0$ and $p_2 = 2$ in the second part of the theorem but, as in [FW12] and [DrLy26], the general arguments cannot exclude the possibility of the cone "going up". In forthcoming work we plan to show (1.15) when $|V_0| \gg 1$ (the semiclassical limit).

We show that the isolation conditions in Theorem 4, at least for small values of $V_0 e^{-B/4}$ (but most likely for all values) cannot hold except for the first three eigenvalues:

Theorem 4. *There exist $c_0, c_1 > 0$ such that if $M \geq c_0(1 + V_0^2 e^{-\pi\ell})$, then for any finite set $\mathcal{S} \subset \mathbb{N} \setminus [1, M]$,*

$$\inf_{k \in \mathbb{R}^2/\mathbb{Z}^2} \operatorname{dist}\Big(\{E_j(k, 2\pi\ell) : j \in \mathcal{S}\}, \{E_j(k, 2\pi\ell) : j \notin \mathcal{S}\} \Big) = 0. \tag{1.16}$$

Moreover, when $|V_0| e^{-\ell\pi/2} < c_1$ then (1.16) holds with $S = \mathbb{N} \setminus \{1, 2, 3\}$.

Remark. The proof gives c_0 and c_1 but we did not attempt to find optimal constants.

The next result confirms the calculation of the Chern number for the first band from [TaDe24] and provides a calculation of the Chern number of the vector bundle associated to the next two bands (assuming that they are isolated):

Theorem 5. *Suppose that $g(2\pi\ell) > 0$ (which, thanks to (1.14), holds for small or large $|V_0 e^{-\pi\ell/2}| > 0$). Then the Chern number of the line bundle L associated to the first band (see [TaZw23, (9.2)]) is given by $c_1(L) = \ell$.*

Assuming that for $E_j(k) := E_j(k, 2\pi\ell)$,

$$\inf_{k \in \mathbb{R}^2/\mathbb{Z}^2} \operatorname{dist}\Big(\{E_j(k) : j = 2, 3\}, \{E_j(k) : j \notin \{2, 3\}\} \Big) > 0. \tag{1.17}$$

(valid for $|V_0 e^{-\pi\ell/2}| > 0$ small or large), the Chern number of the corresponding rank-2 vector bundle V, is given by $c_1(V) = 2\ell$.

At the moment the detailed analysis is provided in the case of $V_0 e^{-\pi\ell/2}$ small. The semiclassical case $V_0 \to \infty$ will be addressed in detail later. The conclusions about large values of $|V_0 e^{-\pi\ell/2}|$, $B = 2\pi\ell$, in Theorem 5 follow from the semiclassical analysis of the one dimensional problem $-h^2 \partial_x^2 + \gamma_0 \cos x$ found in [TaZw23, §6.3]. It implies that for $|V_0| e^{-\pi\ell/2}$ large (which is the value of $h^{-\frac{1}{2}}$ after rescaling), the first band at $B = 2\pi\ell$ is separated from other bands. The same goes for the union of 2nd and 3rd band, and so on (governed by the multiplicities of the eigenvalues of the $2D$ harmonic oscillator), as long as $E_m \leq Ch^{-1} = C|V_0|^2 e^{-\pi\ell/2}$ (this is consistent with Theorem 4).

Notation. We use the *mathematics* inner product convention on L^2, $\langle u, v \rangle := \int u \bar{v}$ which is also the convention used in [TaZw23, (8.20), §§9.1, 9.4] which compares to the bra-ket notation (on the right) as follows $\langle u, v \rangle = \langle v | u \rangle$. The notation $f = \mathcal{O}_V(g)$ or $A = \mathcal{O}_{V \to W}(g)$ means that $\|f\|_V \leq C|g|$ and $\|A\|_{V \to W} \leq C|g|$, respectively.

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2. Effective Hamiltonian in the $J \to \infty$ limit.

In this section we derive the formula for the effective Hamiltonian and prove Theorem 1. We start with a unitary transformation which is implicit in [TaDe24, §III]. It is based on the observation (we denote by w the operator $f(w) \mapsto wf(w)$)

$$D_w - a = e^{iaw} D_w e^{-iaw}, \quad w - b = e^{-ibD_w} w e^{ibD_w}, \quad e^{ibD_w} f(w) = f(w+b),$$

so that

$$D_w - iw - (a + ib) = e^{iaw} e^{ibD_w} (D_w - iw) e^{-ibD_w} e^{-iaw}.$$

Putting $B := 2\lambda^2$,

$$A - \lambda(\mathcal{D} - k) = \tfrac{1}{\sqrt{2}}(D_w - iw - \sqrt{B}(D_{x_1} + iD_{x_2} - k)),$$

and motivated by this we define the following family of unitary operators:

$$U_k := e^{i\sqrt{B}w(D_{x_1}-k_1)} e^{i\sqrt{B}D_w(D_{x_2}-k_2)} : L^2(\mathbb{R}^2/\Gamma \times \mathbb{R}) \to L^2(\mathbb{R}^2/\Gamma \times \mathbb{R}),$$

$k = k_1 + ik_2 \in \mathbb{C}$. Then

$$A - \lambda(\mathcal{D} - k) = \tfrac{1}{\sqrt{2}} U_k (D_w - iw) U_k^*.$$

and

$$U_k^*(|\mathcal{D}-k|^2 + J(A-\lambda(\mathcal{D}-k))^*(A-\lambda(\mathcal{D}-k)))U_k = (\mathcal{D}-k)^*(\mathcal{D}-k) + \tfrac{1}{2}J(D_w^2 + w^2 - 1).$$

Remark. We can also use a more symmetric unitary operator described in terms of Weyl quantisation [Zw12, §4.2.4]:

$$\widetilde{U}_k = \mathrm{Op}^{\mathrm{w}}\left(e^{i\ell_k}\right) = e^{-\frac{iB}{2}k_1k_2} U_k, \tag{2.1}$$

where, in the notation of (1.6),

$$\ell_k(x, w; \xi, \eta) = \sqrt{B}\,[w(\xi_1 - k_1) + \eta(\xi_2 - k_2)] - \tfrac{1}{2}B\,(\xi_1\xi_2 - k_2\xi_1 - k_1\xi_2)\,.$$

Denoting by V the operator $u(x, w) \mapsto V(x)u(x, w)$ on $L^2(\mathbb{R}^2/\Gamma \times \mathbb{R})$, we now obtain

$$\begin{aligned} U_k^* H_J(k) U_k &= \widetilde{H}_J(k) \\ &:= (\mathcal{D} - k)^*(\mathcal{D} - k) + \tfrac{1}{2}J(D_w^2 + w^2 - 1) + U_k^* V U_k. \end{aligned} \tag{2.2}$$

We analyse $U_k^* V U_k$ using the Fourier expansion: $V(x) = \sum_{n\in\mathbb{Z}^2} \widehat{V}(n) e^{i(n_1x_1+n_2x_2)}$, $\widehat{V}(n) = \mathcal{O}(\langle n\rangle^{-\infty})$, where the rapid decay of coefficients follows from the smoothness assumption. Then

$$U_k^* V U_k = \sum_{n\in\mathbb{Z}^2} \widehat{V}(n) e^{in_1(x_1-\sqrt{B}w+B(D_{x_2}-k_2))} e^{in_2(x_2-\sqrt{B}D_w)}.$$

We will define the effective Hamiltonian by projecting $\widetilde{H}_J(k)$ to the ground state of the harmonic oscillator, $\psi_0(w) := \pi^{-\frac14} e^{-\frac{w^2}{2}}$:

$$H(k) := (D_{x_1} - k_1)^2 + (D_{x_2} - k_2)^2 + \mathcal{W}(k),$$

where

$$\begin{aligned}\mathcal{W}(k) &:= \Big\langle \sum_{n\in\mathbb{Z}^2} \widehat{V}(n) e^{in_1(x_1-\sqrt{B}w+B(D_{x_2}-k_2))}\, e^{in_2(x_2-\sqrt{B}D_w)}\psi_0, \psi_0\Big\rangle_{L^2(\mathbb{R}_w)} \\ &= \sum_{n\in\mathbb{Z}^2} \widehat{V}(n) e^{in_1(x_1+B(D_{x_2}-k_2))} e^{in_2x_2} \Big\langle e^{-in_1\sqrt{B}w} e^{-in_2\sqrt{B}D_w}\psi_0, \psi_0\Big\rangle_{L^2(\mathbb{R}_w)} \\ &= \pi^{-\frac12} \sum_{n\in\mathbb{Z}^2} \widehat{V}(n) \Big(\int_{\mathbb{R}} e^{-\frac12 w^2} e^{-in_1\sqrt{B}w} e^{-\frac12(w-n_2\sqrt{B})^2} \mathrm{d}w \Big) e^{in_1(x_1+B(D_{x_2}-k_2))} e^{in_2x_2} \\ &= \sum_{n\in\mathbb{Z}^2} \widehat{V}(n)\, e^{-\frac14 B|n|^2} e^{-\frac12 iBn_1n_2}\, e^{in_1(x_1+B(D_{x_2}-k_2))} e^{in_2x_2}.\end{aligned}$$

We let $\xi := (\xi_1, \xi_2)$, $x^* := n$, $\xi^* := (0, Bn_1)$ and define the linear symbol

$$\ell_n(x,\xi) := \langle x^*, x\rangle + \langle \xi^*, \xi\rangle = n_1x_1 + n_2x_2 + Bn_1\xi_2.$$

We have

$$\begin{aligned}\mathrm{Op}^{\mathrm{w}}(e^{i\ell_n}) &= e^{i\langle x^*,x\rangle} e^{\frac{i}{2}\langle x^*,\xi^*\rangle} e^{i\langle \xi^*, D_x\rangle} = e^{\frac{i}{2}Bn_1n_2} e^{in_1x_1} e^{in_2x_2} e^{in_1BD_{x_2}} \\ &= e^{-\frac{i}{2}Bn_1n_2} e^{in_1x_1} e^{in_1BD_{x_2}} e^{in_2x_2}.\end{aligned}$$

If we define $W := e^{\frac{B}{4}\Delta} V$, then

$$\begin{aligned}\mathcal{W}(k) &= \sum_{n\in\mathbb{Z}^2} \widehat{W}(n) e^{-\frac{i}{2}Bn_1n_2} e^{-in_1Bk_2} e^{in_1(x_1+BD_{x_2})} e^{in_2x_2} \\ &= \sum_{n\in\mathbb{Z}^2} \widehat{W}(n)\, \mathrm{Op}^{\mathrm{w}}(e^{i(n_1x_1+n_2x_2+Bn_1(\xi_2-k_2))}) = \mathrm{Op}^{\mathrm{w}}(a_k),\end{aligned}$$

where the symbol $a_k(x,\xi) := W(x_1 + B(\xi_2 - k_2), x_2)$. This shows that

$$H(k) = e^{i\langle x,k\rangle} H e^{-i\langle x,k\rangle},$$

where H is given in (1.5) with W in (1.10). We can now give

Proof of Theorem 1. We put

$$\mathcal{H} := L^2(\mathbb{T}^2_{x,y} \times \mathbb{R}_w) \cong L^2(\mathbb{T}^2) \otimes L^2(\mathbb{R}), \quad \mathcal{D} := H^2(\mathbb{T}^2) \otimes \mathcal{D}(D_w^2 + w^2),$$
$$H_0 := \tfrac{1}{2}(D_w^2 + w^2 - 1), \quad \widetilde{V}(k) := U_k^* V U_k.$$

Let $\Pi_0 : L^2(\mathbb{R}) \ni u \mapsto \langle u, \psi_0 \rangle \psi_0 \in \ker H_0$ be the orthogonal projection onto the ground state. We define

$$\Pi := I_{L^2(\mathbb{T}^2)} \otimes \Pi_0, \quad \Pi^\perp := I_{L^2(\mathbb{T}^2)} \otimes (1 - \Pi_0),$$
$$R_+ u := \Pi u, \quad R_- u_- := -u_-,$$

and set up the Grushin problem

$$\mathcal{Q}^J(k,z) := \begin{pmatrix} \widetilde{H}_J(k) - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D} \times \operatorname{ran} \Pi \to \mathcal{H} \times \operatorname{ran} \Pi.$$

We fix an arbitrary $E > 0$ and assume $z \le E$ and $J \to \infty$. We have for all $u \in \mathcal{D}$

$$\langle \widetilde{H}_J(k) \Pi^\perp u, \Pi^\perp u \rangle \ge J \langle H_0 \Pi^\perp u, \Pi^\perp u \rangle + \langle \widetilde{V}(k) \Pi^\perp u, \Pi^\perp u \rangle \ge (J - \|\widetilde{V}(k)\|_{L^2 \to L^2}) \|\Pi^\perp u\|_{L^2}^2.$$

This implies that $\Pi^\perp \widetilde{H}_J(k) \Pi^\perp - z$ is invertible on $\operatorname{ran} \Pi^\perp$ and

$$\|(\Pi^\perp \widetilde{H}_J(k) \Pi^\perp - z)^{-1}\|_{L^2 \to L^2} \le (J - \|\widetilde{V}(k)\|_{L^2 \to L^2} - E)^{-1} = \mathcal{O}(\frac{1}{J}).$$

Hence, we can write

$$\mathcal{Q}^J(k,z)^{-1} = \begin{pmatrix} E^J(k,z) & E_+^J(k,z) \\ E_-^J(k,z) & \mathcal{F}_{-+}^J(k,z) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{F}_{-+}^J(k,z) &:= (\Pi \widetilde{H}_J(k) \Pi - z) - \Pi \widetilde{H}_J(k) \Pi^\perp (\Pi^\perp \widetilde{H}_J(k) \Pi^\perp - z)^{-1} \Pi^\perp \widetilde{H}_J(k) \Pi \\ &= (\Pi \widetilde{H}_J(k) \Pi - z) - \Pi \widetilde{V}(k) \Pi^\perp (\Pi^\perp \widetilde{H}_J(k) \Pi^\perp - z)^{-1} \Pi^\perp \widetilde{V}(k) \Pi. \end{aligned}$$

We have

$$z \in \operatorname{Spec}(\widetilde{H}_J(k)) \iff 0 \in \operatorname{Spec}(\mathcal{F}_{-+}^J(k,z))$$

and the eigenvalues have the same multiplicity.

We identify $\Pi \widetilde{H}_J(k) \Pi$ as the effective Hamiltonian $H(k)$ and consider the set

$$\Sigma_E := \{(k,\lambda) \in \mathbb{R}^2/\mathbb{Z}^2 \times [-C, E] \,:\, \lambda \in \operatorname{Spec}(H(k))\},$$

where $C > 0$ is chosen such that $\operatorname{Spec}(H(k)) \subseteq [-C, \infty)$ for all k. For each point $(k_*, \lambda_*) \in \Sigma_E$, choose a small disc $D_* \subseteq \mathbb{C}$ centered at λ_* such that $\partial D_* \cap \operatorname{Spec}(H(k_*)) = \emptyset$. After shrinking to a neighborhood $U_* \ni k_*$, D_* contains exactly m_* eigenvalues of $H(k_*)$ counted with multiplicity, for every $k \in U_*$.

We let $\{U_k \times (D_k \cap \mathbb{R})\}_{k=1}^{n(E)}$ be a finite open cover of Σ_E satisfying the conditions above, and set up a local Grushin problem on each U_*. We define

$$\Pi_*(k) := -\frac{1}{2\pi i}\int_{\partial D_*} (H(k)-\zeta)^{-1} \mathrm{d}\zeta, \quad \operatorname{rank}\Pi_*(k) = m_* \text{ on } U_*.$$

We now choose a local orthonormal frame

$$\varphi_1(k), \dots, \varphi_{m_*}(k) \in \operatorname{Ran}\Pi_*(k),$$

and define

$$R_{-,*}(k) : \mathbb{C}^{m_*} \ni c \mapsto \sum_{j=1}^{m_*} c_j \varphi_j(k) \in L^2(\mathbb{T}^2),$$
$$R_{+,*}(k) : H^2(\mathbb{T}^2) \ni u \mapsto (\langle u, \varphi_1(k)\rangle, \dots, \langle u, \varphi_{m_*}(k)\rangle) \in \mathbb{C}^{m_*}.$$

With this choice, the Grushin problem

$$\mathcal{P}_*^0(k,z) := \begin{pmatrix} H(k)-z & R_{-,*}(k) \\ R_{+,*}(k) & 0 \end{pmatrix}$$

is well-posed for $z \in D_*$ with

$$\mathcal{P}_*^0(k,z)^{-1} = \begin{pmatrix} E_*^0(k,z) & R_{-,*}(k) \\ R_{+,*}(k) & E_{-+,*}^0(k,z) \end{pmatrix},$$
$$E_*^0(k,z) := \left((H(k)-z)\big|_{\operatorname{Ran}(I-\Pi_*(k))}\right)^{-1}(I-\Pi_*(k)),$$
$$E_{-+,*}^0(k,z) := zI_{\mathbb{C}^{m_*}} - \Pi_*(k)H(k)\Pi_*(k).$$

We define

$$W^J(k,z) := -\Pi\widetilde{V}(k)\Pi^\perp(\Pi^\perp\widetilde{H}_J(k)\Pi^\perp - z)^{-1}\Pi^\perp\widetilde{V}(k)\Pi,$$

and replace $H(k)-z$ by

$$\mathcal{F}_{-,+}^J(k,z) = H(k) - z + W^J(k,z).$$

Define the perturbed Grushin problem

$$\mathcal{P}_*^J(k,z) := \begin{pmatrix} \mathcal{F}_{-+}^J(k,z) & R_{-,*}(k) \\ R_{+,*}(k) & 0 \end{pmatrix}.$$

Since as $J \to \infty$

$$\|E_*^0(k,z)W^J(k,z)\|, \|W^J(k,z)E_*^0(k,z)\| \le \frac{C_*}{J}$$

uniformly for $k \in U_*$ and $z \in D_*$, [TaZw23, Proposition 2.12] gives the well-posedness of $\mathcal{P}_*^J(k,z)$ with

$$\mathcal{P}_*^J(k,z)^{-1} = \begin{pmatrix} E_*^J(k,z) & E_{+,*}^J(k,z) \\ E_{-,*}^J(k,z) & E_{-+,*}^J(k,z) \end{pmatrix},$$

where

$$E^J_{-+,*}(k,z) := E^0_{-+,*}(k,z) + \sum_{l=1}^{\infty}(-1)^l R_{+,*}(k)W^J(k,z)(E^0_*(k,z)W^J(k,z))^{l-1}R_{-,*}(k).$$

Hence, we have for $z \in D_*$, $k \in U_*$

$$z \in \operatorname{Spec}(\widetilde{H}_J(k)) \iff 0 \in \operatorname{Spec}(\mathcal{F}^J_{-+}(k,z)) \iff 0 \in \operatorname{Spec}(E^J_{-+,*}(k,z)),$$

and as $J \to \infty$

$$\|E^J_{-+,*}(k,z) - (zI_{\mathbb{C}^{m_*}} - \Pi_*(k)H(k)\Pi_*(k))\| \leq \frac{C_*}{J}$$

holds uniformly. Thus, (1.11) follows by taking C to be the maximum of C_* over finitely many patches. □

3. Perturbative formula for the gap

In this section we consider the specific potential (1.13) from [TaDe24] using perturbation theory as $V_0 e^{-B/4} \to 0$. To prove Theorem 2 we construct a Grushin problem (see the previous section and [TaZw23, §2.6]) for the operator $P_k(B) := P(x, D-k; B)$, $P(x,D;B) := H$ where H is the effective Hamiltonian (1.5) (the notation is chosen for consistency with [TaZw23, Example 13] which we will use).

Putting $P_k := (D_{x_1} - k_1)^2 + (D_{x_2} - k_2)^2$, we see that

$$\operatorname{Spec}_{L^2(\mathbb{R}^2/\Gamma)} P_k = \{|m-k|^2 \,:\, m \in \mathbb{Z}^2\}.$$

Inside the fundamental domain $(-\frac{1}{2}, \frac{1}{2}]^2$, the first eigenvalue $\lambda_1(k)$ is not simple if and only if $k_1 = \frac{1}{2}$ or $k_2 = \frac{1}{2}$. More precisely, if we denote the orthonormal eigenfunctions by $e_m(x) := (2\pi)^{-1}\exp(i\langle m, x\rangle)$, $m \in \mathbb{Z}^2$, then we have the following 3 cases:

(1) At the horizontal edge: $-\frac{1}{2} < k_1 < \frac{1}{2}$, $k_2 = \frac{1}{2}$, $\lambda_1(k) = k_1^2 + \frac{1}{4}$,

$$\ker(P_k - \lambda_1(k)) = \operatorname{span}(e_{0,0}, e_{0,1}).$$

(2) At the vertical edge: $k_1 = \frac{1}{2}, -\frac{1}{2} < k_2 < \frac{1}{2}$, $\lambda_1(k) = \frac{1}{4} + k_2^2$,

$$\ker(P_k - \lambda_1(k)) = \operatorname{span}(e_{0,0}, e_{1,0}).$$

(3) At the corner: $k_1 = k_2 = \frac{1}{2}$, $\lambda_1(k) = \frac{1}{2}$,

$$\ker(P_k - \lambda_1(k)) = \operatorname{span}(e_{0,0}, e_{1,0}, e_{0,1}, e_{1,1}).$$

To simplify notation, we denote the perturbation parameter and the effective potential by

$$\gamma := e^{-\frac{B}{4}}V_0, \quad V_B := \gamma(\cos(x_1 + B(D_{x_2} - k_2)) + \cos x_2).$$

We first consider the case where $-\frac{1}{2} < k_1 < \frac{1}{2}$, $k_2 = \frac{1}{2}$, and define

$$R_- : \mathbb{C}^2 \ni u_- \mapsto u_{-,1}e_{0,0} + u_{-,2}e_{0,1}, \quad R_+ := R_-^* : H^2(\mathbb{R}^2/\Gamma) \to \mathbb{C}^2.$$

(Here and below we take the standard inner product of $\mathbb{C}^n$.)

For a sufficiently small $\varepsilon > 0$, $|z - k_1^2 - \frac14| < \varepsilon$, $|\gamma| < \varepsilon$, implies that the Grushin problem for $P_{(k_1,\frac12)} - z$ with these $R_\pm$ is well posed and, with the standard notation for the inverse,

$$\begin{gathered} Ev := \sum_{m \neq (0,0),(0,1)} ((m_1 - k_1)^2 + (m_2 - \tfrac12)^2 - z)^{-1} \langle v, e_m\rangle e_m, \\ E_- = R_+, \quad E_+ = R_-, \quad E_{-+} := (z - k_1^2 - \tfrac14) I_{\mathbb{C}^2}. \end{gathered} \tag{3.1}$$

The perturbed Grushin problem, with $P_{k,1/2}$ replaced by $P_{(k,1/2)}(B)$, is also well posed. The corresponding E_{-+}^γ term is given by [TaZw23, Proposition 2.12] using (3.1):

$$E_{-,+}^\gamma := E_{-+} + \sum_{l=1}^\infty (-1)^l R_+ V_B (E V_B)^{l-1} R_-. \tag{3.2}$$

From

$$\cos(x_1 + B(D_{x_2} - k_2)) e_{m_1,m_2} = \tfrac12 (e^{iB(m_2-k_2)} e_{m_1+1,m_2} + e^{-iB(m_2-k_2)} e_{m_1-1,m_2}),$$

$$\cos(x_2) e_{m_1,m_2} = \tfrac12 (e_{m_1,m_2+1} + e_{m_1,m_2-1}),$$

we obtain

$$E_{-,+}^\gamma = \begin{pmatrix} z - k_1^2 - \frac14 & -\frac12\gamma \\ -\frac12\gamma & z - k_1^2 - \frac14 \end{pmatrix} + \mathcal{O}(\gamma^2).$$

Since $E_j((k_1, 1/2))$, $j = 1,2$ are the values of z for which $\det E_{-+}^\gamma(z) = 0$ we obtain

$$E_2((k_1, \tfrac12); B) - E_1((k_1, \tfrac12); B) = |\gamma| + \mathcal{O}(\gamma^2).$$

A similar argument applies to the case $k = (\frac12, k_2)$, and gives

$$E_{-,+}^\gamma = \begin{pmatrix} z - k_2^2 - \frac14 & -\frac12\gamma e^{iBk_2} \\ -\frac12\gamma e^{-iBk_2} & z - k_2^2 - \frac14 \end{pmatrix} + O(\gamma^2),$$

so that

$$E_2((\tfrac12, k_2); B) - E_1((\tfrac12, k_2); B) = |\gamma| + \mathcal{O}(\gamma^2).$$

The interesting case is given by $k_0 := (\frac12, \frac12)$. We define

$$\begin{aligned} &R_- : \mathbb{C}^4 \ni u_- \mapsto u_{-,1} e_{0,0} + u_{-,2} e_{1,0} + u_{-,3} e_{0,1} + u_{-,4} e_{1,1} \in L^2(\mathbb{R}^2/\Gamma), \\ &R_+ = R_-^* : H^2(\mathbb{R}^2/\Gamma) \to \mathbb{C}^4. \end{aligned} \tag{3.3}$$

If $z \in B(\frac12, \varepsilon)$, $|\gamma| < \varepsilon$, $0 < \varepsilon \ll 1$, the corresponding Grushin problem for P_{k_0} is well posed, and $E : L^2(\mathbb{R}^2/\Gamma) \to H^2(\mathbb{R}^2/\Gamma)$, $E_{-+} : \mathbb{C}^4 \to \mathbb{C}^4$ are given by

$$Ev := \sum_{\substack{m \neq (0,0),(0,1) \\ (1,0),(1,1)}} \big((m_1 - \tfrac12)^2 + (m_2 - \tfrac12)^2 - z\big)^{-1} \langle v, e_m\rangle\, e_m, \quad E_{-+} := (z - \tfrac12) I_{\mathbb{C}^4}.$$

The perturbed Grushin problem, with P_{k_0} replaced by $P_{k_0}(B)$, is also well posed and the corresponding E^{γ}_{-+} term is given by the analogue of (3.2). A computation then shows that

$$E^{\gamma}_{-,+} = (z - \tfrac{1}{2}) I_{\mathbb{C}^4} - \tfrac{1}{2}\gamma \begin{pmatrix} 0 & e^{\frac{iB}{2}} & 1 & 0 \\ e^{-\frac{iB}{2}} & 0 & 0 & 1 \\ 1 & 0 & 0 & e^{-\frac{iB}{2}} \\ 0 & 1 & e^{\frac{iB}{2}} & 0 \end{pmatrix} + \mathcal{O}(\gamma^2),$$

and that the spectrum on the 4-by-4 matrix above is given by $\pm 2\cos(\frac{1}{4}B)$, $\pm 2\sin(\frac{1}{4}B)$.

Hence, up to $O(V_0^2 e^{-\frac{B}{2}})$, the first four eigenvalues of $P_{k_0}(B)$ are given by

$$\left\{ \tfrac{1}{2} \pm V_0 e^{-\frac{B}{4}} \cos(\tfrac{1}{4}B), \tfrac{1}{2} \pm V_0 e^{-\frac{B}{4}} \sin(\tfrac{1}{4}B) \right\}.$$

This gives (1.14) proving Theorem 2.

4. Dirac cones at $B = (2\ell + 1)\pi$

This section is devoted to the proof of Theorem 3. We first show that bands always touch when B is an odd multiple of π (as already suggested by (1.14)). Then we show that for small values of $V_0 e^{-(2\ell+1)\pi/4}$ the first two bands touch at a Dirac point. Finally, we apply results of [DrLy26] to see that Dirac points persist for all but (possibly) a discrete set of coupling constants.

4.1. **Symmetry induced degeneracy.** Let $k_0 = (\frac{1}{2}, \frac{1}{2})$. We claim that for the operator (1.9) with the potential (1.13),

$$\forall\, E \in \mathbb{R} \quad \dim \ker_{H^2(\mathbb{R}^2/\Gamma)}(H(k_0, (2\ell+1)\pi) - E) \in 2\mathbb{N}. \tag{4.1}$$

To see this we introduce the following two unitary operators defined on $L^2(\mathbb{R}^2/\Gamma)$:

$$(S_0 u)(x_1, x_2) := e^{ix_2} u(x_1 + \pi, -x_2), \qquad (S_1 u)(x_1, x_2) := e^{i(x_1+x_2)} u(-x_1, -x_2). \tag{4.2}$$

We then check that

$$S_j^2 = I, \quad [H(k_0, (2\ell+1)\pi), S_j] = 0, \quad j = 0, 1,$$

and that

$$S_1 S_0 = -S_0 S_1.$$

This shows that every eigenspace of $H(k_0, (2\ell+1)\pi)$ is even dimensional. In fact, since eigenvalues of S_j are ± 1, for any E,

$$V_E := \ker(H(k_0, (2\ell+1)\pi) - E) = V_E^+ + V_E^-, \quad V_E^{\pm} := \{u \in V_E : S_0 u = \pm u\}.$$

But then $S_1 : V_E^{\pm} \to V_E^{\mp}$ and hence $\dim V_E^+ = \dim V_E^-$.

4.2. **Perturbative calculations near** $(2\ell+1)\pi$**.** We let $\gamma := V_0 e^{-\frac{B}{4}}$, $R_\mp$ be as in (3.3). Then for $|z-\frac{1}{2}|<\varepsilon$, $|k-(\frac{1}{2},\frac{1}{2})|<\varepsilon$, $\varepsilon \ll 1$ the corresponding Grushin problem for $P_k - z$ is well posed. In the inverse $E_\pm = R_\mp$, and

$$Ev := \sum_{\substack{m\neq(0,0),(0,1)\\(1,0),(1,1)}} \left((m_1-k_1)^2+(m_2-k_2)^2-z\right)^{-1}\langle v, e_m\rangle\, e_m,$$

$$E_{-+} := zI_{\mathbb{C}^4} - \begin{pmatrix} k_1^2+k_2^2 & & & \\ & (1-k_1)^2+k_2^2 & & \\ & & k_1^2+(1-k_2)^2 & \\ & & & (1-k_1)^2+(1-k_2)^2 \end{pmatrix}.$$

Using (3.2), we obtain

$$\begin{aligned} E^\gamma_{-,+} &= E_{-,+} - \frac{\gamma}{2}\begin{pmatrix} 0 & e^{iBk_2} & 1 & 0 \\ e^{-iBk_2} & 0 & 0 & 1 \\ 1 & 0 & 0 & e^{-iB(1-k_2)} \\ 0 & 1 & e^{iB(1-k_2)} & 0 \end{pmatrix} + O(\gamma^2) \\ &=: \widetilde{E}^\gamma_{-,+} + O(\gamma^2). \end{aligned}$$

We let $B_0 := (2\ell+1)\pi$, $\kappa := k-(\frac{1}{2},\frac{1}{2})$, $\delta := B - B_0$. A direct calculation shows

$$\det(\widetilde{E}^\lambda_{-,+}(z,k,B)) = 0 \iff z \in \{\widetilde{z}_1(\kappa,B), \widetilde{z}_2(\kappa,B), \widetilde{z}_3(\kappa,B), \widetilde{z}_4(\kappa,B)\},$$

where $\widetilde{z}_1 \le \widetilde{z}_2 \le \widetilde{z}_3 \le \widetilde{z}_4$ are given by

$$\widetilde{z}_{1,2,3,4}(\kappa,B) = \tfrac{1}{2} + |\kappa|^2 \pm \sqrt{|\kappa|^2 + \tfrac{\gamma^2}{2} \pm 2\sqrt{\kappa_1^2\kappa_2^2 + \tfrac{\gamma^2}{4}|\kappa|^2 + \tfrac{\gamma^4}{16}\sin^2(\tfrac{\delta}{2})}}.$$

Hence, if $|\gamma|$ is sufficiently small and fixed, then near $\kappa = 0$ and $B = B_0$ the first four eigenvalues of $P_k(B)$ have the following approximate formulas

$$z(\kappa,B) = \tfrac{1}{2} \pm 2^{-\frac{1}{2}}\gamma \pm 2^{-\frac{1}{2}}\sqrt{|\kappa|^2 + \tfrac{1}{4}\gamma^2\sin^2(\tfrac{\delta}{2})} + \mathcal{O}_\gamma(|\kappa|^2+\delta^2) + \mathcal{O}(\gamma^2). \tag{4.3}$$

4.3. **Dirac cones for small values of** $V_0 e^{-B/4}$**.** We fix

$$k_0 := (\tfrac{1}{2},\tfrac{1}{2}),\ \ B_0 := (2\ell+1)\pi,\ \ \gamma := V_0 e^{-\frac{B_0}{4}}.$$

Assume $0 < |\gamma| \ll 1$, so that by (4.1), (4.3) the first eigenvalue E_0 of $H(k_0,B_0)$ has multiplicity 2. To justify the first two bands $E_{1,2}(k,B_0)$ touch at a Dirac point, we need to remove the correction term γ^2 from (4.3). More precisely, we prove

Proposition 4.1. *Let $\rho := |\kappa| + |\delta|$ be sufficiently small. There exist constants $a,b,c,\alpha,\beta \in \mathbb{R}$ such that the first two eigenvalues of $H(k,B)$ are given by the approximate formula*

$$E_{1,2}(k,B) = E_0 + a\delta \pm \sqrt{(\alpha\kappa_1+\beta\kappa_2)^2 + c^2\kappa_2^2 + b^2\delta^2} + \mathcal{O}(\rho^2). \tag{4.4}$$

Proof. We define

$$\begin{aligned} W(k,B) &:= H(k,B) - H(k_0,B_0) \\ &= \kappa_1 K_1 + \kappa_2 K_2 + \delta Q + \mathcal{O}_{H^2\to L^2}(\rho^2), \end{aligned}$$

where

$$K_1 := -2(D_{x_1} - \tfrac{1}{2}),$$
$$K_2 := -2(D_{x_2} - \tfrac{1}{2}) + \gamma B_0 \sin(x_1 + B_0(D_{x_2} - \tfrac{1}{2})),$$

and

$$\begin{aligned} Q &:= -\gamma(Q_1 + Q_2) \\ &= -\gamma\Big(\tfrac{1}{4}\big(\cos(x_1 + B_0(D_{x_2} - \tfrac{1}{2})) + \cos x_2\big) + (D_{x_2} - \tfrac{1}{2})\sin(x_1 + B_0(D_{x_2} - \tfrac{1}{2}))\Big). \end{aligned}$$

Let Π_0 be the spectral projector onto $\ker(H(k_0,B_0) - E_0)$. By (4.1), we can choose an orthonormal basis $\{\psi_+, \psi_-\} \subseteq \operatorname{Ran}\Pi_0$ such that

$$S_0\psi_\pm = \pm\psi_\pm, \quad S_1\psi_\pm = \psi_\mp. \tag{4.5}$$

We put

$$R_+ : H^2(\mathbb{R}^2/\Gamma) \ni u \mapsto \begin{pmatrix} \langle u, \psi_+\rangle \\ \langle u, \psi_-\rangle \end{pmatrix} \in \mathbb{C}^2,$$

$$R_- : \mathbb{C}^2 \ni \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \mapsto c_+\psi_+ + c_-\psi_- \in L^2(\mathbb{R}^2/\Gamma).$$

If we let $|z - E_0| + |\delta| + |\kappa| < \varepsilon \ll 1$, then the Grushin problem

$$\begin{pmatrix} H(k,B) - z & R_- \\ R_+ & 0 \end{pmatrix} : H^2(\mathbb{R}^2/\Gamma) \times \mathbb{C}^2 \to L^2(\mathbb{R}^2/\Gamma) \times \mathbb{C}^2$$

is invertible with

$$E_{-+}(z,k,B) = (z - E_0)I_{\mathbb{C}^2} - \kappa_1 A_1 - \kappa_2 A_2 - \delta M + \mathcal{O}(\rho^2),$$
$$A_1 := R_+K_1R_-, \; A_2 := R_+K_2R_-, \; M := R_+QR_-.$$

We denote

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A computation shows

$$[K_1, S_0] = 0, K_1S_1 = -S_1K_1,$$
$$(D_{x_2} - \tfrac{1}{2})S_0 = -S_0(D_{x_2} - \tfrac{1}{2}), \; (D_{x_2} - \tfrac{1}{2})S_1 = -S_1(D_{x_2} - \tfrac{1}{2}),$$
$$[\sin(x_1 + B_0(D_{x_2} - \tfrac{1}{2})), S_0] = 0, \; \sin(x_1 + B_0(D_{x_2} - \tfrac{1}{2}))S_1 = -S_1\sin(x_1 + B_0(D_{x_2} - \tfrac{1}{2})),$$
$$[Q_1, S_0] = [Q_1, S_1] = [Q_2, S_1] = 0, \; Q_2S_0 = -S_0Q_2.$$

It follows that there exist $\alpha, \beta, a, b, c \in \mathbb{R}$ such that

$$A_1 =: \alpha\sigma_3, \; A_2 =: c\sigma_2 + \beta\sigma_3, \; M =: aI_{\mathbb{C}^2} + b\sigma_1.$$

Hence, the two eigenvalues of $E_{-+}(z,k,B)$ are given by the approximate formula

$$z - E_0 - a\delta \pm \sqrt{(\alpha\kappa_1 + \beta\kappa_2)^2 + c^2\kappa_2^2 + b^2\delta^2} + \mathcal{O}(\rho^2).$$

This completes the proof of the Proposition. □

Taking $\delta = 0$ in (4.4), we obtain

$$E_{1,2}(k, B_0) = E_0 \pm \sqrt{(\alpha\kappa_1 + \beta\kappa_2)^2 + c^2\kappa_2^2} + \mathcal{O}(|\kappa|^2). \tag{4.6}$$

To prove Theorem 3 for $|\gamma|$ sufficiently small, it remains to show

$$\alpha = \langle -2(D_{x_1} - \tfrac{1}{2})\psi_+, \psi_+\rangle \neq 0 \quad\text{and}\quad c = i\langle -2(D_{x_2} - \tfrac{1}{2})\psi_-, \psi_+\rangle \neq 0 \tag{4.7}$$

We use Fourier modes to approximate the eigenfunctions $\psi_\pm$ defined in (4.5) to compute the constants $\alpha, c \in \mathbb{R}$ up to leading order in γ.

We denote

$$P_{k_0} := (D_{x_1} - \tfrac{1}{2})^2 + (D_{x_2} - \tfrac{1}{2})^2, \quad H := H(k_0, B_0),$$
$$V := \cos(x_1 + B_0(D_{x_2} - \tfrac{1}{2})) + \cos x_2.$$

Let Π be the projector onto $K_0 := \operatorname{Ker}(P_{k_0} - \frac{1}{2})$ and $\Pi^\perp := 1 - \Pi$. Since $\psi_\pm$ solve

$$(H - E_0)\psi_\pm = 0$$

writing $\psi_\pm = \Pi\psi_\pm + \Pi^\perp\psi_\pm =: u_\pm + u_{\pm,\perp}$ gives

$$F(E_0)u_\pm = 0, \quad u_{\pm,\perp} = -\gamma(\Pi^\perp(H - E_0)\Pi^\perp)^{-1}\Pi^\perp V \Pi u_\pm, \tag{4.8}$$

where

$$\begin{aligned} F(E_0) &= \Pi(H - E_0)\Pi - \Pi H \Pi^\perp(\Pi^\perp(H - E_0)\Pi^\perp)^{-1}\Pi^\perp H \Pi \\ &= (\tfrac{1}{2} - E_0)I_{K_0} + \gamma \Pi V \Pi + \mathcal{O}(\gamma^2). \end{aligned} \tag{4.9}$$

We decompose

$$K_0 = K_{0,+} \oplus K_{0,-}, \quad K_{0,\pm} := \{u \in K_0 \mid S_0 u = \pm u\} = \operatorname{span}(f_{1,\pm}, f_{2,\pm}),$$
$$f_{1,\pm} = \tfrac{e_{0,0} \pm e_{0,1}}{\sqrt{2}}, \quad f_{2,\pm} = \tfrac{e_{1,0} \mp e_{1,1}}{\sqrt{2}}.$$

We let $\Pi_\pm$ be the projector onto $K_{0,\pm}$ and take

$$v_\pm := \tfrac{1}{\sqrt{1 + (\sqrt{2} \mp 1)^2}}(f_{2,\pm} - (-1)^l i(\sqrt{2} \mp 1)f_{1,\pm}) \in K_{0,\pm}$$

to be the normalized eigenvectors of $\Pi_\pm V \Pi_\pm$ for the lowest eigenvalue $\lambda = -\frac{\sqrt{2}}{2}$. Then (4.8), (4.9) give

$$|\alpha| = 2|\langle (D_{x_1} - \tfrac{1}{2})v_+, v_+\rangle| + \mathcal{O}(\gamma) = \tfrac{\sqrt{2}}{2} + \mathcal{O}(\gamma),$$
$$|c| = 2|\langle (D_{x_2} - \tfrac{1}{2})v_+, v_-\rangle| + \mathcal{O}(\gamma) = \tfrac{\sqrt{2}}{2} + \mathcal{O}(\gamma).$$

4.4. **Persistence of Dirac cones.** We denote

$$H_\gamma(k) := (D_{x_1} - k_1)^2 + (D_{x_2} - k_2)^2 + \gamma\big(\cos(x_1 + B_0(D_{x_2} - k_2)) + \cos x_2\big).$$

Proposition 4.2. *There exist real analytic families $\gamma \mapsto \mu_\pm(\gamma)$ of eigenvalues of $H_\gamma(k_0)$, such that near $\gamma = 0$,*

$$\mu_\pm(\gamma) = \tfrac{1}{2} \pm \tfrac{\gamma}{\sqrt{2}} + \mathcal{O}(\gamma^2).$$

and a discrete set $D_\pm \subseteq \mathbb{R}$ such that for $\gamma \in \mathbb{R} \backslash D_\pm$:

(1) *$\mu_\pm(\gamma)$ is an eigenvalue of $H_\gamma(k_0)$ of multiplicity exactly 2;*
(2) *for $E_{\pm,1}(k,\gamma) \le E_{\pm,2}(k,\gamma)$, eigenvalues of $H_\gamma(k)$ such that*

$$E_{\pm,1}(k_0,\gamma) = E_{\pm,2}(k_0,\gamma) = \mu_\pm(\gamma),$$

we have

$$E_{\pm,1}(k_0+\kappa,\gamma) = \mu_\pm(\gamma) - \sqrt{\langle A_\pm(\gamma)\kappa,\kappa\rangle} + \mathcal{O}(|\kappa|^2),$$
$$E_{\pm,2}(k_0+\kappa,\gamma) = \mu_\pm(\gamma) + \sqrt{\langle A_\pm(\gamma)\kappa,\kappa\rangle} + \mathcal{O}(|\kappa|^2),$$

where the real analytic families of 2-by-2 matrices, $\gamma \mapsto A_\pm(\gamma)$ are positive definite for $\gamma \notin D_\pm$.

Proof. The family

$$\gamma \mapsto H_\gamma(k_0)$$

satisfies the assumption of [DrLy26, Proposition 2.1]. This gives discrete sets $D_\pm \subseteq \mathbb{R}$ and analytic families of orthogonal projections

$$\Pi_\pm(\gamma) : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$$

of constant rank, such that for $\gamma \notin D_\pm$, $\Pi_\pm(\gamma)$ is the spectral projector of $H_\gamma(k_0)$ associated to $\mu_\pm(\gamma)$, and $\mu_\pm(\gamma)$ has constant multiplicity on $\mathbb{R}\backslash D_\pm$.

Since every eigenspace of $H_\gamma(k_0)$ is even dimensional, the perturbative approximations at the end of §4.2 for small γ and then the smoothness of $\Pi_\pm(\gamma)$ shows

$$\dim \operatorname{Ran} \Pi_\pm(\gamma) = 2, \quad \gamma \in \mathbb{R} \quad \text{and} \quad \operatorname{mult}_{H_\gamma(k_0)}(\mu_\pm(\gamma)) = 2, \quad \gamma \notin D_\pm.$$

This proves part (1) of the proposition.

We now fix $\gamma \notin D_\pm$ so that $\mu_\pm(\gamma)$ is an isolated eigenvalue of $H_\gamma(k_0)$ of multiplicity 2. We choose contour $\Gamma_\pm$ around $\mu_\pm(\gamma)$ so that $\Gamma_\pm \cap \operatorname{Spec} H_\gamma(k_0) = \emptyset$. For $|\kappa|$ sufficiently small, we then have $\Gamma_\pm \cap \operatorname{Spec} H_\gamma(k_0+\kappa) = \emptyset$. Using this fact we define

$$P_{\gamma,\pm}(\kappa) := -\frac{1}{2\pi i}\int_{\Gamma_\pm} (H_\gamma(k_0+\kappa) - z)^{-1}\, \mathrm{d}z,$$

and choose a local analytic orthonormal basis $u_{1,\pm}(\kappa), u_{2,\pm}(\kappa)$ of $\operatorname{Ran} P_{\gamma,\pm}(\kappa)$. (That the rank of the projection $P_{\gamma,\pm}(\kappa)$ is 2 follows from the fact that $P_{\gamma,\pm}(0) = \Pi_\pm(\gamma)$ and the continuity in κ.) Let $C_\pm(\gamma,\kappa)$ be the 2×2 matrix representing $P_{\gamma,\pm}(\kappa)H_\gamma(k_0+\kappa)P_{\gamma,\pm}(\kappa)$ defined by

$$\big(C_\pm(\gamma,\kappa)\big)_{ij} := \Big\langle H_\gamma(k_0+\kappa)u_{j,\pm}(\kappa), u_{i,\pm}(\kappa)\Big\rangle.$$

Taylor expansion gives

$$C_\pm(\gamma,\kappa) = \mu_\pm(\gamma)I_{\mathbb{C}^2} + L_\pm(\gamma,\kappa) + \mathcal{O}(|\kappa|^2),$$

where

$$L_\pm(\gamma,\kappa) = \kappa_1\Pi_\pm(\gamma)\partial_{k_1}H_\gamma(k_0)\Pi_\pm(\gamma) + \kappa_2\Pi_\pm(\gamma)\partial_{k_2}H_\gamma(k_0)\Pi_\pm(\gamma).$$

If $l_\pm(\gamma,\kappa) := \frac{1}{2}\mathrm{tr}\, L_\pm(\gamma,\kappa)$, $q_\pm(\gamma,\kappa) := l_\pm(\gamma,\kappa)^2 - \det L_\pm(\gamma,\kappa) \geq 0$ then the eigenvalues of $L_\pm(\gamma,\kappa)$ are given by $l_\pm(\gamma,\kappa) \pm \sqrt{q_\pm(\gamma,\kappa)}$. Moreover, a direct calculation shows that for S_1 in (4.2),

$$S_1 H_\gamma(k) S_1^{-1} = H_\gamma(1-k), \tag{4.10}$$

Differentiating (4.10) at $k_0 = (\frac{1}{2}, \frac{1}{2})$ gives

$$S_1\Pi_\pm(\gamma)\partial_{k_j}H_\gamma(k_0)\Pi_\pm(\gamma)S_1^{-1} = -\Pi_\pm(\gamma)\partial_{k_j}H_\gamma(k_0)\Pi_\pm(\gamma) \quad j = 1,2,$$

which implies $l_\pm(\gamma,\kappa) = 0$.

To finish the proof of the proposition, it suffices to show the positive definiteness of $q_\pm(\gamma,\kappa)$ fails only on a discrete set. To see this, we write $q_\pm(\gamma,\kappa) = \kappa^T G_\pm(\gamma)\kappa$. The function $\gamma \mapsto \det G_\pm(\gamma)$ is real analytic on $\mathbb{R}$. (4.6), (4.7) and similar computations for $E_{3,4}(k, B_0)$ shows $\det G_\pm(\gamma)$ is not identically zero. ☐

5. Overlaps for higher eigenvalues

In this section we prove Theorem 4. We let $B = 2\pi l$, $l \in \mathbb{Z}$ and $\gamma := e^{-\frac{\pi l}{2}}V_0$. We have

$$\begin{aligned} H_\gamma(k) &= (D_{x_1} - k_1)^2 + (D_{x_2} - k_2)^2 + \gamma\Big(\cos(x_1 + 2\pi l(D_{x_2} - k_2)) + \cos x_2\Big) \\ &= (D_{x_1} - k_1)^2 + \gamma\cos(x_1 - 2\pi l k_2) + (D_{x_2} - k_2)^2 + \gamma\cos x_2. \end{aligned}$$

Thus,

$$\mathrm{Spec}(H_\gamma(k)) = \{E^\gamma_{m,n}(k_1,k_2) \,:\, m,n \in \mathbb{N}_+\} = \{\varepsilon^\gamma_m(k_1) + \varepsilon^\gamma_n(k_2) \,:\, m,n \in \mathbb{N}_+\},$$

where $\varepsilon^\gamma_j(\kappa)$ denotes the j-th eigenvalues of the 1D operator $(D_t - \kappa)^2 + \gamma\cos t$. We let $r \geq 2$ and set

$$A_r := \{(m,n) \in \mathbb{N}_+^2 \,:\, m + n = r\}.$$

Proposition 5.1. *If $|\gamma|$ is sufficiently small, then a subset $S \subseteq \mathbb{N}_+^2$ satisfying*

$$|S| < \infty \quad \text{and} \quad \inf_{k\in\mathbb{R}^2/\Gamma^*} \mathrm{dist}\Big(\{E^\gamma_{m,n}(k) \,:\, (m,n) \in S\}, \{E^\gamma_{m,n}(k) \,:\, (m,n) \notin S\}\Big) > 0. \tag{5.1}$$

implies

$$S \subseteq A_2 \cup A_3. \tag{5.2}$$

Proof. If $0 \le \kappa \le \frac{1}{2}$, then

$$\varepsilon^0_{2a-1}(\kappa) = (a-1+\kappa)^2, \quad \varepsilon^0_{2a}(\kappa) = (a-\kappa)^2, \quad a \in \mathbb{N}_+.$$

If we define $d_j(\kappa) := \varepsilon^0_{j+1}(\kappa) - \varepsilon^0_j(\kappa)$, then

$$d_{2a-1}(\kappa) = (2a-1)(1-2\kappa), \quad d_{2a}(\kappa) = 4a\kappa.$$

Hence, if we set

$$\sigma_j := \begin{cases} 0, & j \text{ odd}, \\ \frac{1}{2}, & j \text{ even}, \end{cases} \tag{5.3}$$

then

$$d_j(\sigma_j) = j, \quad d_j(\tfrac{1}{2} - \sigma_j) = 0. \tag{5.4}$$

Suppose $m \ge 2$. Then

$$\begin{aligned} E^0_{m,n}(k_1,k_2) - E^0_{m-1,n+1}(k_1,k_2) &= (\varepsilon^0_m(k_1) - \varepsilon^0_{m-1}(k_1)) - (\varepsilon^0_{n+1}(k_2) - \varepsilon^0_n(k_2)) \\ &= d_{m-1}(k_1) - d_n(k_2). \end{aligned}$$

We have by (5.3)

$$\begin{aligned} E^0_{m,n}(\sigma_{m-1}, \tfrac{1}{2} - \sigma_n) - E^0_{m-1,n+1}(\sigma_{m-1}, \tfrac{1}{2} - \sigma_n) = m-1 > 0, \\ E^0_{m,n}(\tfrac{1}{2} - \sigma_{m-1}, \sigma_n) - E^0_{m-1,n+1}(\tfrac{1}{2} - \sigma_{m-1}, \sigma_n) = -n < 0. \end{aligned} \tag{5.5}$$

Therefore, for $\gamma = 0$, if S satisfies condition (5.1) and contains one element of A_r, then $A_r \subseteq S$.

Now suppose $r \ge 4$. Consider $(1, r-1) \in A_r$ and $(3, r-2) \in A_{r+1}$. For $k_1 = 0$,

$$\begin{aligned} E^0_{1,r-1}(0,\kappa) - E^0_{3,r-2}(0,\kappa) &= (\varepsilon^0_{r-1}(\kappa) - \varepsilon^0_{r-2}(\kappa)) - (\varepsilon^0_3(0) - \varepsilon^0_1(0)) \\ &= d_{r-2}(\kappa) - 1. \end{aligned}$$

Using (5.3), we have

$$\begin{aligned} E^0_{1,r-1}(0, \sigma_{r-2}) - E^0_{3,r-2}(0, \sigma_{r-2}) = r-3 > 0, \\ E^0_{1,r-1}(0, \tfrac{1}{2} - \sigma_{r-2}) - E^0_{3,r-2}(0, \tfrac{1}{2} - \sigma_{r-2}) = -1 < 0. \end{aligned} \tag{5.6}$$

This together with (5.5) shows that every $E^0_{m,n}$ with $m+n \ge 4$ belongs to an infinite chain of intersections. The estimate

$$\|\varepsilon^\gamma_j - \varepsilon^0_j\| \le |\gamma|, \quad j \in \mathbb{N}_+ \tag{5.7}$$

shows the same conclusion remains true for $|\gamma|$ sufficiently small. $\square$

Proposition 5.2. *If $\gamma \in \mathbb{R}$ is fixed and*

$$m + n \geq 8\lceil 4|\gamma| \rceil + 12,$$

then every $E^\gamma_{m,n}$ belongs to an infinite chain of intersections, that is, there is an infinite sequence (m_j, n_j) such that $(m,n) = (m_0, n_0)$ and

$$\forall\, j \in \mathbb{N}\ \exists\, k_j \in \mathbb{R}^2/\mathbb{Z}^2 \quad E_{m_j,n_j}(k_j) = E_{m_{j+1},n_{j+1}}(k_j).$$

Proof. We use the notation

$$(m,n) \sim (m', n')$$

to mean that $E^\gamma_{m,n}$ and $E^\gamma_{m',n'}$ intersect somewhere on $\mathbb{R}^2/\Gamma^*$. We let $K := \lceil 4|\gamma| \rceil + 1$, $H := 2K + 1$.

We claim that

$$(m,n) \sim (m-1, n+1) \text{ whenever } \min(m-1, n) \geq K. \tag{5.8}$$

Indeed, (5.5) and (5.7) give

$$E^\gamma_{m,n}(\sigma_{m-1}, \tfrac{1}{2} - \sigma_n) - E^\gamma_{m-1,n+1}(\sigma_{m-1}, \tfrac{1}{2} - \sigma_n) \geq (m-1) - 4|\gamma| > 0,$$
$$E^\gamma_{m,n}(\tfrac{1}{2} - \sigma_{m-1}, \sigma_n) - E^\gamma_{m-1,n+1}(\tfrac{1}{2} - \sigma_{m-1}, \sigma_n) \leq 4|\gamma| - n < 0.$$

Thus, the two bands intersect.

We now show every $E^\gamma_{H,N}$ with $N \geq K + H + 1$ belongs to an infinite chain of intersections. We compute

$$E^\gamma_{H,N}(0, \sigma_{N-1}) - E^\gamma_{H+2,N-1}(0, \sigma_{N-1}) \geq N - 1 - H - 4|\gamma| > 0,$$
$$E^\gamma_{H,N}(0, \tfrac{1}{2} - \sigma_{N-1}) - E^\gamma_{H+2,N-1}(0, \tfrac{1}{2} - \sigma_{N-1}) \leq 4|\gamma| - H < 0.$$

This shows $(H, N) \sim (H+2, N-1)$. Using (5.8), we obtain an infinite chain

$$(H,N) \sim (H+2, N-1) \sim (H+1, N) \sim (H, N+1) \sim \cdots \sim (H, N+2) \sim \cdots .$$

It remains to connect every (m,n) with $r := m + n \geq 4H$ to a pair of the form (H, N).

If $\min(m,n) \geq K$, then (5.8) shows

$$(m,n) \sim \cdots \sim (H, m+n-H).$$

Now suppose $1 \leq m \leq K$. We compute

$$\begin{aligned} E^0_{m,n}(k_1,k_2) - E^0_{H,r-H}(k_1,k_2) &= (\varepsilon^0_n(k_2) - \varepsilon^0_{r-H}(k_2)) - (\varepsilon^0_H(k_1) - \varepsilon^0_m(k_1)) \\ &\geq \frac{(r-K-1)^2 - (r-H)^2}{4} - \frac{H^2}{4} > \frac{H^2}{2}. \end{aligned}$$

Hence,

$$E^\gamma_{m,n}(k_1,k_2) - E^\gamma_{H_\gamma, r-H_\gamma}(k_1,k_2) \geq \frac{H^2}{2} - 4|\gamma| > 0.$$

For s sufficiently large, we have $E^\gamma_{H,r-H+s} > E^\gamma_{m,n} > E^\gamma_{H,r-H}$. Thus, $E^\gamma_{m,n}$ intersects

$$(H, r-H) \sim \cdots \sim (H, r-H+s).$$

The case $1 \leq n \leq K$ follows by symmetry. □

Proposition 5.2 shows if $S \subseteq \mathbb{N}_+^2$ satisfies (5.1), then

$$S \subseteq \bigcup_{r=2}^{4H-1} A_r.$$

We then obtain

$$|S| \leq \sum_{r=2}^{8K+3} (r-1) \leq 32K^2 + 20K + 3 = \mathcal{O}(1 + |\gamma|^2),$$

which completes the proof of Theorem 4.

6. Chern numbers at $B = 2\pi\ell$.

We will now consider the case of $B = 2\pi\ell$ and compute the Chern number of the line bundle of Bloch–Floquet eigenfunctions corresponding to the lowest (isolated) band. It is given by ℓ and agrees with the result presented in [TaDe24, §III]. Our argument is similar to the computation of the Chern number in Thouless pumping [TaZw23, §9.1].

6.1. **The first band.** We consider the *smallest* Bloch eigenvalue, $\varepsilon(\kappa, \gamma)$, for the one dimensional potential $\gamma \cos t$. From [TaZw23, Theorem 6, (10.1)] we know that there exists an eigenstate $u \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\begin{gathered} ((D_t - \kappa)^2 + \gamma \cos t) u(\kappa, t) = \varepsilon(\kappa, \gamma) u(\kappa, t), \\ u(\kappa + p, t + 2\pi\ell) = e^{ipt} u(\kappa, t), \quad p, \ell \in \mathbb{Z}, \quad \int_0^{2\pi} |u(\kappa, t)|^2 dt = 1. \end{gathered} \tag{6.1}$$

Expressing the operator using the Fourier expansion in x_2 we see that

$$\cos(x_1 + 2\pi\ell(D_{x_2} - k_2)) = \cos(x_1 - 2\pi\ell k_2).$$

It then follows from (1.5) that

$$E_1(k, 2\pi\ell) = \varepsilon(k_1, e^{-\pi\ell/2} V_0) + \varepsilon(k_2, e^{-\pi\ell/2} V_0),$$

with the eigenfunction given by

$$w(k, x) = u(k_1, x_1 - 2\pi\ell k_2) u(k_2, x_2), \quad x = (x_1, x_2), \quad k = (k_1, k_2). \tag{6.2}$$

The Chern number of the Bloch–Floquet line bundle, L, (see [TaZw23, (9.2)] and (6.11) below) is given by [TaZw23, (9.7)]:

$$c_1(L) = \frac{i}{2\pi} \int_{\mathbb{R}^2/\Gamma^*} d\eta, \quad \eta = \langle d_k w(k, \bullet), w(k, \bullet) \rangle_{L^2(\mathbb{R}^2/\Gamma)}. \tag{6.3}$$

The curvature $\Theta := d\eta$ is a well defined 2-form, the 1-form η is defined on $F := [0,1)\times[0,1)$, a fundamental domain of $\Gamma^* = \mathbb{Z}^2$. That gives,

$$c_1(L) = \frac{i}{2\pi}\int_{\partial F} \eta,$$

where the boundary of the square, ∂F is oriented counter-clockwise. We have

$$\partial_{k_1} w(k,x) = \partial_\kappa u(k_1, x_1 - 2\pi\ell k_2) u(k_2, x_2),$$
$$\partial_{k_2} w(k,x) = -2\pi\ell \partial_t u(k_1, x_1 - 2\pi\ell k_2) u(k_2,x_2) + u(k_1, x_1 - 2\pi l k_2)\partial_\kappa u(k_2, x_2).$$

Only the first term in $\partial_{k_2} w$ contributes to (6.3) as the other term and $\partial_{k_1} w$ give the Chern number of the trivial bundle $k \mapsto \mathbb{C}u(k_1,x_1)u(k_2,x_2)$ (which is simple to verify directly). Hence,

$$\begin{aligned} c_1(L) &= -i\ell \int_{\partial F}\left(\int_0^{2\pi} \partial_t u(k_1,t)\overline{u(k_1,t)}dt\right) dk_2 \\ &= -i\ell \int_0^{2\pi}\left(\partial_t u(1,t)\overline{u(1,t)} - \partial_t u(0,t)\overline{u(0,t)}\right) dt. \end{aligned}$$

We now use (6.1) to see that

$$u(1,t) = e^{it}u(0,t), \quad \partial_t u(1,t) = \partial_t(e^{it}u(0,t)) = ie^{it}u(0,t) + e^{it}\partial_t u(0,t).$$

Since $\int_0^{2\pi} |u(0,t)|^2 dt = 1$ we conclude that

$$c_1(L) = \ell. \tag{6.4}$$

6.2. **Rank-2 bundle corresponding to $E_2(k, 2\pi\ell)$ and $E_3(k, 2\pi\ell)$.** The computations in [TaZw23, Example 13] show that for $\gamma > 0$ sufficiently small, A_2, A_3 satisfy condition (5.1). We therefore denote $V \to \mathbb{R}^2/\mathbb{Z}^2$ the Bloch bundle associated to bands $\{E_{1,2}(k), E_{2,1}(k)\}$. The Chern number of V is given by

$$\begin{aligned} c_1(V) := \frac{i}{2\pi}\int_{\mathbb{R}^2/\mathbb{Z}^2} \operatorname{tr}\Theta = \frac{1}{\pi}\int_{\mathbb{R}^2/\mathbb{Z}^2} & \operatorname{Im}\langle \partial_{k_1} w_{1,2}(k,\bullet), \partial_{k_2} w_{1,2}(k,\bullet)\rangle \\ & + \operatorname{Im}\langle \partial_{k_1} w_{2,1}(k,\bullet), \partial_{k_2} w_{2,1}(k,\bullet)\rangle \, dk_1 dk_2. \end{aligned}$$

We have, for each m, n

$$\langle \partial_{k_1} w_{m,n}, \partial_{k_2} w_{m,n}\rangle = -2\pi\ell\langle \partial_\kappa u_m, \partial_t u_m\rangle_{L^2_{x_1}} + \langle \partial_\kappa u_m, u_m\rangle_{L^2_{x_1}} \langle u_n, \partial_\kappa u_n\rangle_{L^2_{x_2}},$$

and (6.1) shows that the second term is real-valued. Computing this as in the previous section gives $c_1(V) = 2\ell$.

6.3. **Comparison with the parent bundle of** [TaDe24]**.** Here we discuss the computation in [TaDe24, §II.B] and the relation to §6. We use the notation of that section and §2 and the line-bundle convention of [TaZw23, §9.1, (9.5)–(9.7)]. In particular, for a local normalized frame $\Phi(k)$ of a Hermitian line bundle write the connection and curvature as

$$\eta := \langle d_k\Phi, \Phi\rangle, \qquad \Theta := d\eta.$$

Thus

$$\begin{aligned} \Theta &= -iB(k)dk_1 \wedge dk_2, \quad B(k) := 2\,\mathrm{Im}\langle \partial_{k_1}\Phi, \partial_{k_2}\Phi\rangle, \\ c_1(L) &= \frac{i}{2\pi}\int_{\mathbb{R}^2/\mathbb{Z}^2} \Theta = \frac{1}{2\pi}\int_{\mathbb{R}^2/\mathbb{Z}^2} B(k)dk_1dk_2, \end{aligned} \tag{6.5}$$

where we choose the standard orientation of the torus. We recall that although η is locally defined, the curvature Θ is a 2-form on $\mathbb{R}^2/\mathbb{Z}^2$.

The conjugation (2.2) shows that the spectrum of (1.1) with $V \equiv 0$ is given by the union of $[0,\infty) + Jm$, $m = 0, 1, \cdots$. As $J \to \infty$, the bottom of the spectrum is given by a *parent band*, $|k|^2$, with generalised eigenfunctions of $\widetilde{H}_J(k)$ in (2.2) given by $(2\pi)^{-1}e^{i\langle k,x\rangle}\psi_0(w)$, $k \in \mathbb{R}$. After the conjugation (1.3), the generalised eigenfunctions are given by $U_k((2\pi)^{-1} \otimes \psi_0)$. This suggests the following definition of the *parent line bundle* which we will compare below to the bundle implicit in [TaDe24]. For $k = (k_1, k_2) \in \mathbb{R}^2$ we use the symmetric unitary transformation (2.1) to define

$$\Phi_0(k) := \frac{1}{2\pi}\widetilde{U}_k(1 \otimes \psi_0), \qquad \Phi_0(k, x, w) = (2\pi)^{-1}e^{-i\sqrt{B}\,k_1w + \frac{iB}{2}k_1k_2}\psi_0(w - \sqrt{B}\,k_2).$$

We also note that with the $L^2(\mathbb{T}^2_x \times \mathbb{R}_w)$ inner product,

$$\langle \Phi_0(k), \Phi_0(k')\rangle = \exp\left(-\tfrac{1}{4}B\left(|k-k'|^2 - 2i\sigma(k,k')\right)\right), \tag{6.6}$$

where $\sigma(k, k') := k_1'k_2 - k_1k_2'$. This should be compared to [TaDe24, (6)] suggesting that, in the notation of that paper, $s_k^{\mathcal{B}} = \overline{\Phi_0(k)}$. That means that the *parent bundle* of [TaDe24] is dual to the bundle defined by $\Phi(k)$. From the explicit formula for $\Phi_0(k)$ we obtain in the notation of (6.5),

$$\eta_{\mathrm{par}} = \langle d_k\Phi_0, \Phi_0\rangle = -iBk_2\,dk_1, \quad \Theta_{\mathrm{par}} = -iB\,dk_1 \wedge dk_2, \quad B(k) = -B. \tag{6.7}$$

For $n = (n_1, n_2) \in \mathbb{Z}^2$ we define magnetic translations by

$$(T_nu)(x, w) := e^{-i\sqrt{B}\,n_1w}u(x, w - \sqrt{B}\,n_2). \tag{6.8}$$

Then

$$\Phi_0(k + n) = T_n\Phi_0(k), \qquad T_nT_{n'} = e^{iBn_1'n_2}T_{n+n'} = e^{iB(n_1'n_2 - n_1n_2')}T_{n'}T_n. \tag{6.9}$$

Hence for

$$B = 2\pi\ell, \qquad \ell \in \mathbb{Z},$$

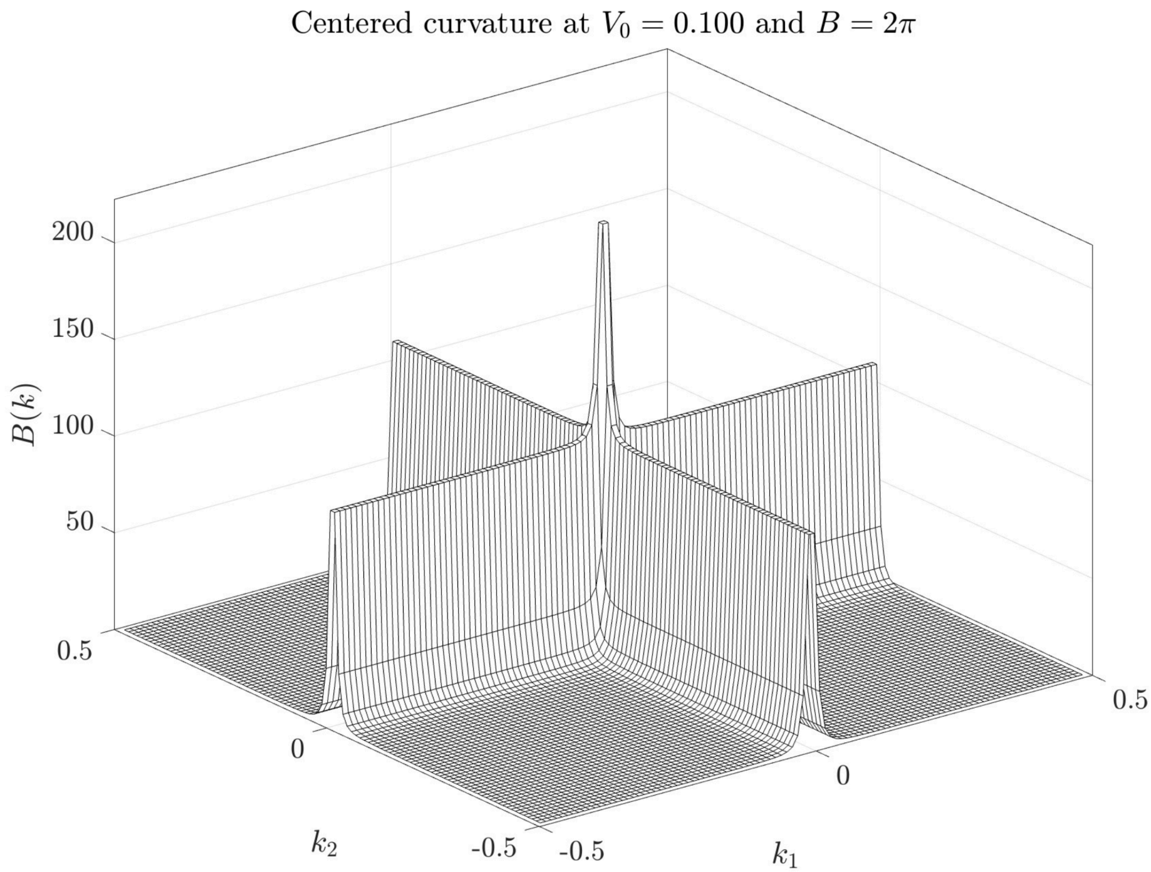

FIGURE 3. The curvature, $B(k)$, of the line bundle L in (6.11) for $B = 2\pi$ and $V_0 = 0.1$. We use the symmetric gauge here: (1.7) with $\theta = \frac{1}{2}$ illustrating (6.12). For an animated version see `https://math.berkeley.edu/~zworski/curv_movie.mp4`.

the cocycle is trivial and the lines $\widetilde{P}_k := \mathbb{C}\Phi_0(k)$, $k \in \mathbb{R}^2/\mathbb{Z}^2$, define a Hermitian line bundle $P \to \mathbb{R}^2/\mathbb{Z}^2$, as in [TaZw23, (9.2)]:

$$\begin{gathered} P := \{(k, v) : k \in \mathbb{R}^2, v \in \mathbb{C}\Phi_0(k)\}/\sim, \\ (k, v) \sim (k', v') \implies \exists\, n \in \mathbb{Z}^2,\ k' = k + n,\ v' = T_n v. \end{gathered} \tag{6.10}$$

The line bundle P is *trivial* as the equivalence class $[(k, \Phi_0)]$ defines a non-vanishing section. In particular $c_1(P) = 0$. This does *not* contradict (6.7) as $\eta_{\rm par}$ does *not* define a connection on P. The connection $\eta_{\rm par}$ is defined on the parent line bundle over $\mathbb{C}$, $\widetilde{P}$, has constant curvature $-B$ in our convention but does *not* have a well defined Chern number. However, it has a non-trivial holonomy – see [TaZw23, §7.1.5,(7.13)].

We next compare this with the first band of the effective Hamiltonian. In the notation of (6.2) we put

$$\widetilde{L}_k := \mathbb{C}w(k,\bullet) \subset L^2(\mathbb{T}^2_x), \qquad \Phi(k) := \widetilde{U}_k(w(k,\bullet)\otimes\psi_0).$$

The line bundle considered in §6.1 is given by

$$\begin{gathered} L := \left\{(k,v) : k\in\mathbb{R}^2, v\in\mathbb{C}w(k,\bullet)\right\}/\sim, \\ (k,v)\sim(k',v') \iff \exists\, n\in\mathbb{Z}^2,\ k'=k+n,\ v'(x)=e^{i\langle x,n\rangle}v(x). \end{gathered} \tag{6.11}$$

(See [TaZw23, Lemma 9.1] for a proof that this defines a line bundle.) This is the genuine spectral line bundle over the Brillouin torus, while $\Phi(k)$ is its lift to the U_k-gauge over $\mathbb{R}^2_k$. They should be compared as families on $\mathbb{R}^2$, but the line bundle descending to $\mathbb{R}^2/\mathbb{Z}^2$ on the effective side is $\widetilde{L}$.

We now recall the point responsible for the apparent sign discrepancy. The computation (6.7) is made on the covering space. To obtain a Chern number one must also specify the descent under $k\mapsto k+n$. If the descent is made using the magnetic translations T_n (see (6.10)), then, for $B=2\pi\ell$, the corresponding parent line bundle is trivial. For the perturbed state $\Phi(k)$, the relevant line bundle is $\widetilde{L}$, Its descent is the Bloch line bundle $L\to\mathbb{R}^2/\mathbb{Z}^2$, and the connection is represented locally by $\eta_L := \langle d_k w, w\rangle$ As computed in Section 6.1, this bundle has $c_1(L)=\ell$. Thus, the curvature of the parent state and the Chern number of the perturbed band refer to different descent data; the Chern number is not determined by the local curvature form alone.

An additional point should be emphasized when comparing the bundles defined by $\Phi_0(k)$ and $\Phi(k)$. The eigenfunctions $w(k,x)$ *cannot* be chosen so that $w(k,x)\to(2\pi)^{-1}$ as $V_0\to 0$. The Bloch periodicity condition prevents that from happening. For the symmetric gauge ($\theta=\frac{1}{2}$ in (1.7)), the curvature of L, $B_{V_0}(k)$ (see [TaZw23, §9.1] and (6.5)) can be shown to satisfy

$$B_{V_0}(k)\to\pi\ell\sum_{n\in\mathbb{Z}^2}\left(\delta(k_1+n_1)+\delta(k_2+n_2)\right), \tag{6.12}$$

in the sense of distributions on $\mathbb{R}^2$. We illustrate this in Figure 3.

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