Quantum decay rates in chaotic scattering

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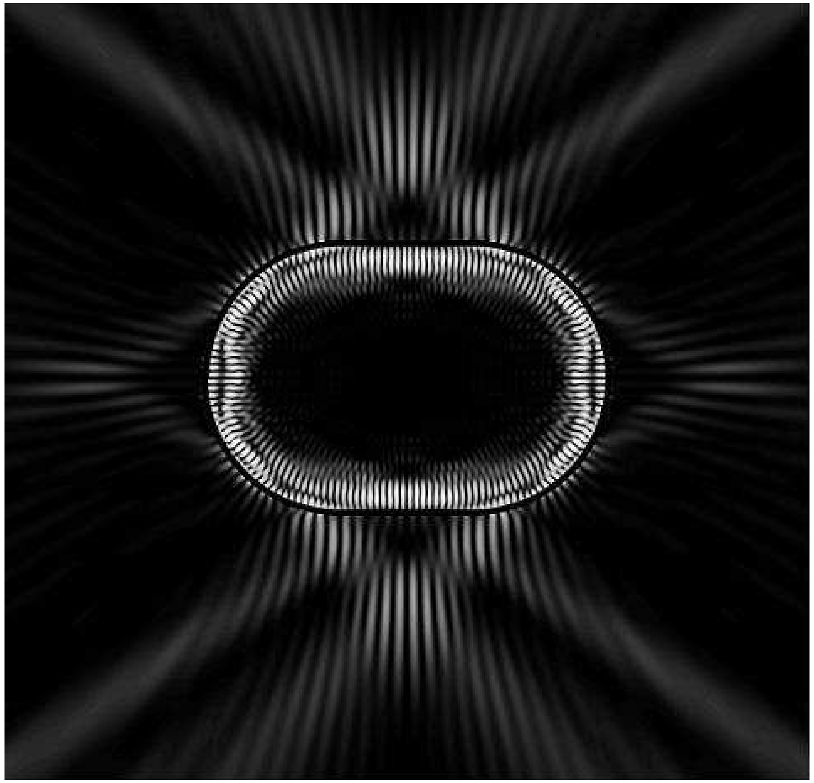
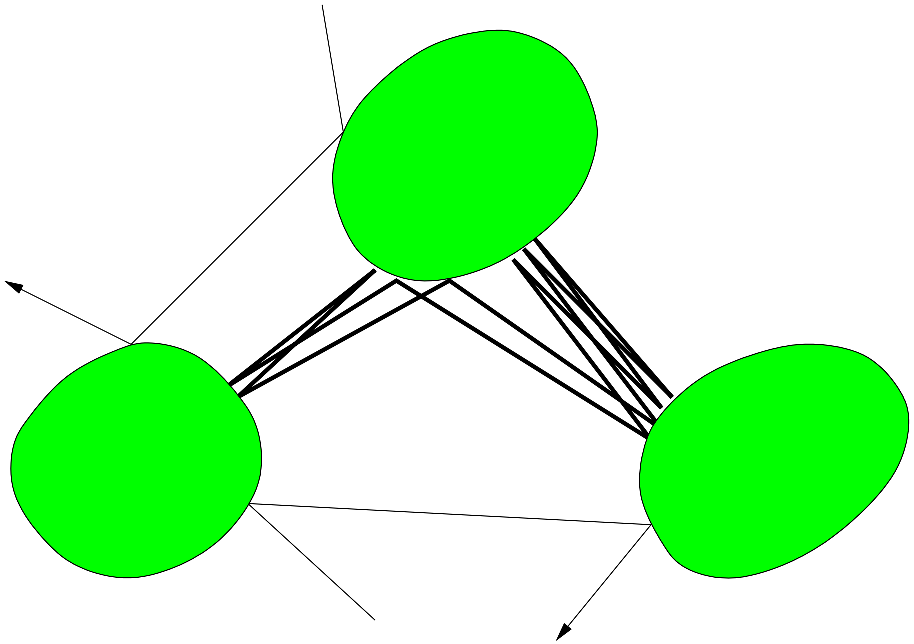


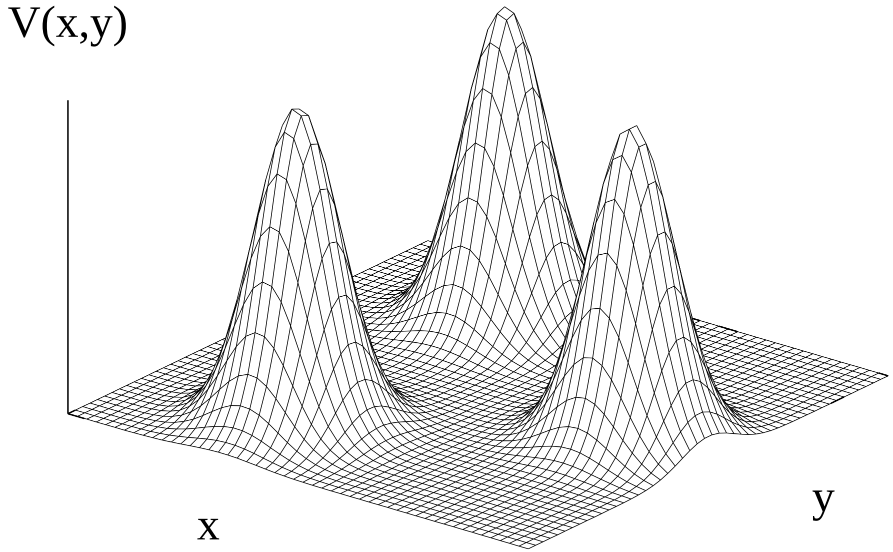
A resonant state for the partially open stadium billiard, computed by C.Schmit.

Outline

- Classical and quantum scattering, resonances.
- Chaotic régime: fractal trapped set
- Semiclassical limit: fractal Weyl law, eigenstates distribution
- **spectral gap** for thin trapped sets; link with the topological pressure
- Proof of the gap for a model with absorbing potential:
  - open covers of the trapped set to compute the pressure
  - a quantum partition of unity
  - decomposing the propagator up to logarithmic times
  - a hyperbolic estimate for the dominant terms

Scattering in Hamiltonian systems





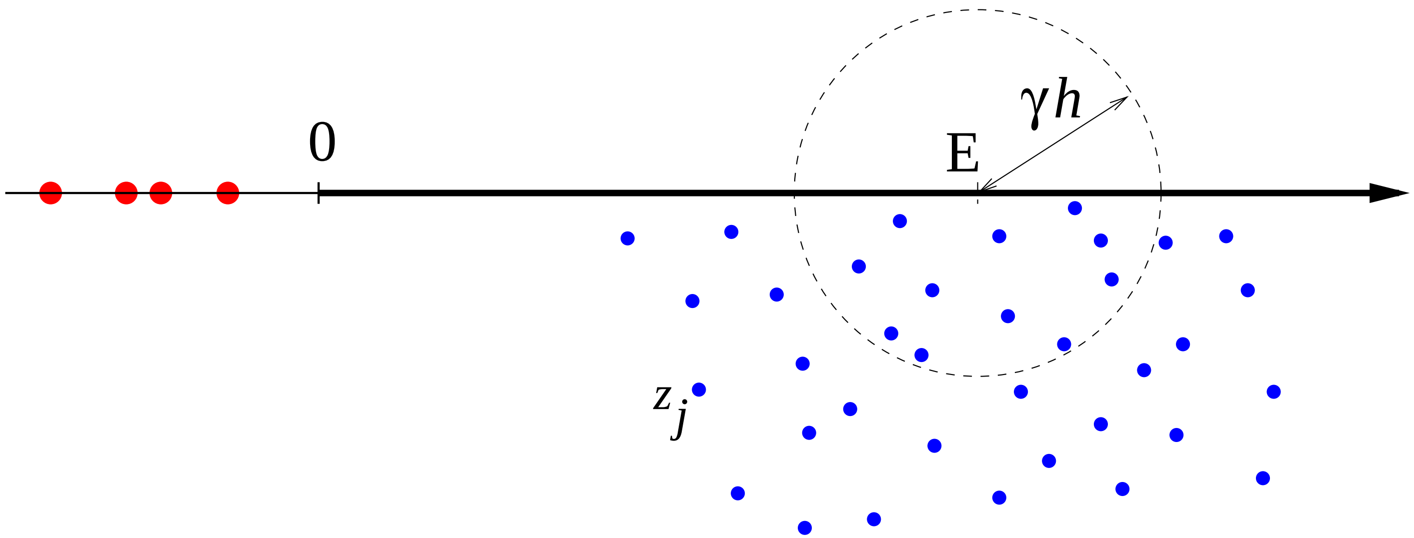
Hamiltonian scattering system

- Left: hard obstacles in $\mathbb{R}^2$

- Right: Hamiltonian $H = \frac{\xi^2}{2} + V(x)$, with potential $V(x)$ decaying at infinity

The flow at energies $E > 0$ is **unbounded**, and the quantum Hamiltonian $H_h = -\frac{h^2 \Delta_D}{2}$ (resp. $H_h = -\frac{h^2 \Delta}{2} + V(x)$) has a **purely continuous** spectrum on $\mathbb{R}^+$.

Quantum resonances



The resolvent $(z - H_h)^{-1}$ may be continued meromorphically from $\{\operatorname{Im} z > 0\}$ to $\{\operatorname{Im} z < 0\}$. Its poles $\{z_j(h)\}$ are the **resonances** of H_h.

Each $z_j(h)$ is associated with a **metastable** (non-normalizable) state of H_h, with **lifetime** $\tau_j = h(2|\operatorname{Im} z_j|)^{-1} \Longrightarrow$ long-living state if $\operatorname{Im} z_j = \mathcal{O}(h)$.

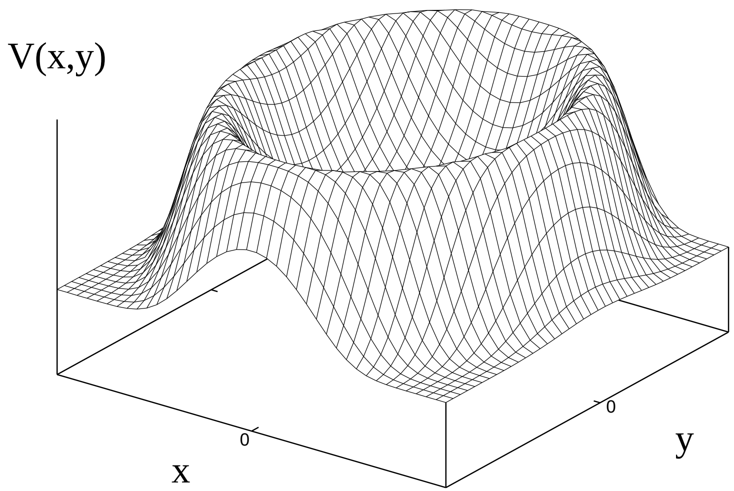
Questions in the semiclassical régime:

- Distribution of **long-living** resonances $z_j(h)$ with $\operatorname{Re} z_j \approx E$, $\operatorname{Im} z_j = \mathcal{O}(h)$.

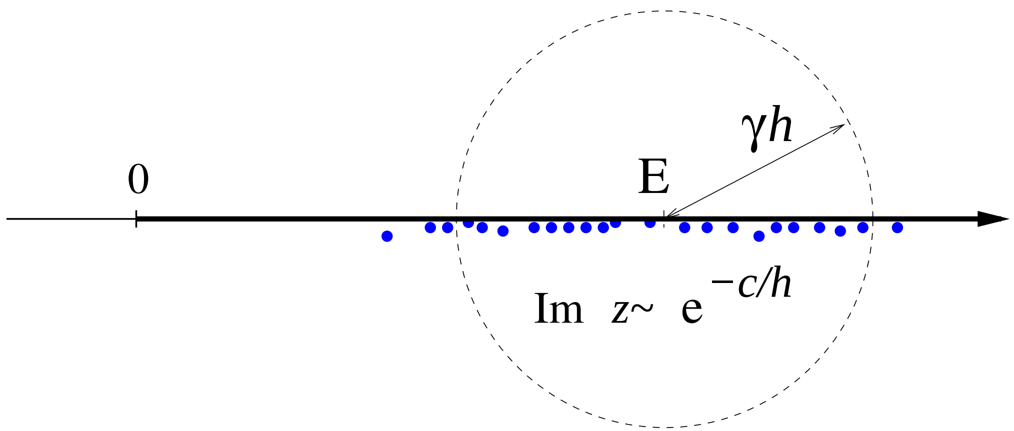
- Description of the metastable states ψ_h associated with these resonances.

Distribution of resonances - Trapped set

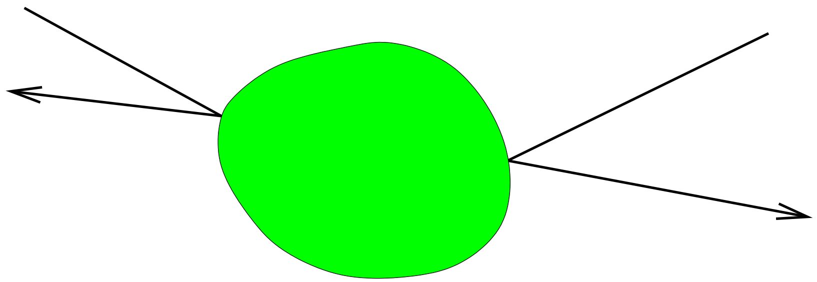
Main semiclassical idea: the semiclassical distribution of resonances near E depends on the set $K_E = \Gamma_E^+ \cap \Gamma_E^-$ of **trapped trajectories** at energy E.
We always assume that K_E is contained in a bounded interaction region.



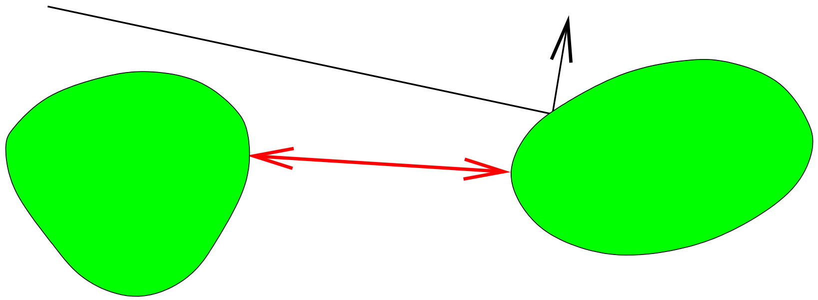
Ex. 1: trapped set of **positive volume**. Resonances are very close from being real $(\operatorname{Im} z_j = \mathcal{O}(h^\infty))$, and from eigenvalues of an associated closed system.



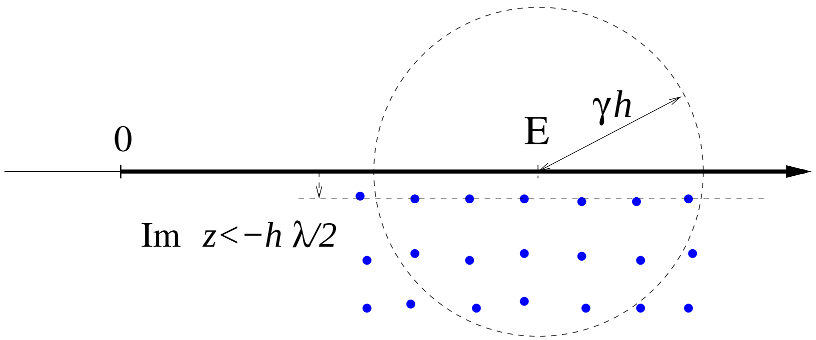
Ex. 2: **empty** trapped set for a convex body: resonances are "far" from the real axis, $|\operatorname{Im} z_j| \sim h^{\frac{2}{3}}$, and satisfy a Weyl law, $\sim h^{-n+1}$ [Sjöstrand-Z,'99].



Ex 3: the trapped set = a **single unstable orbit** [IKAWA'83,GÉRARD,SJÖSTRAND..]



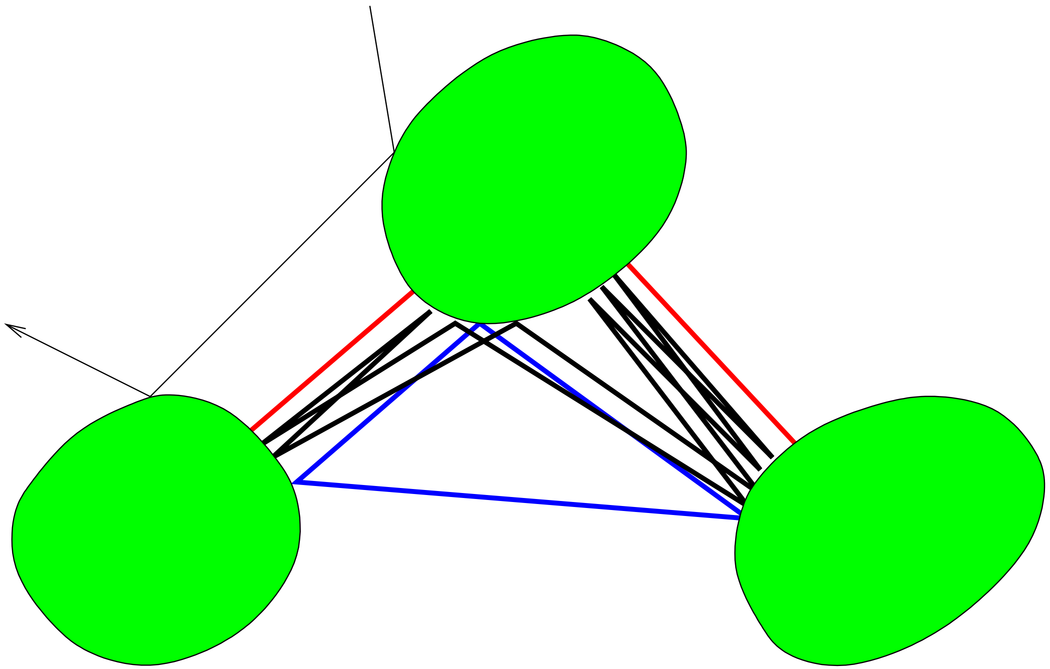
The resonances form a quasi-lattice (Bohr-Sommerfeld + inverted harm. osc.).

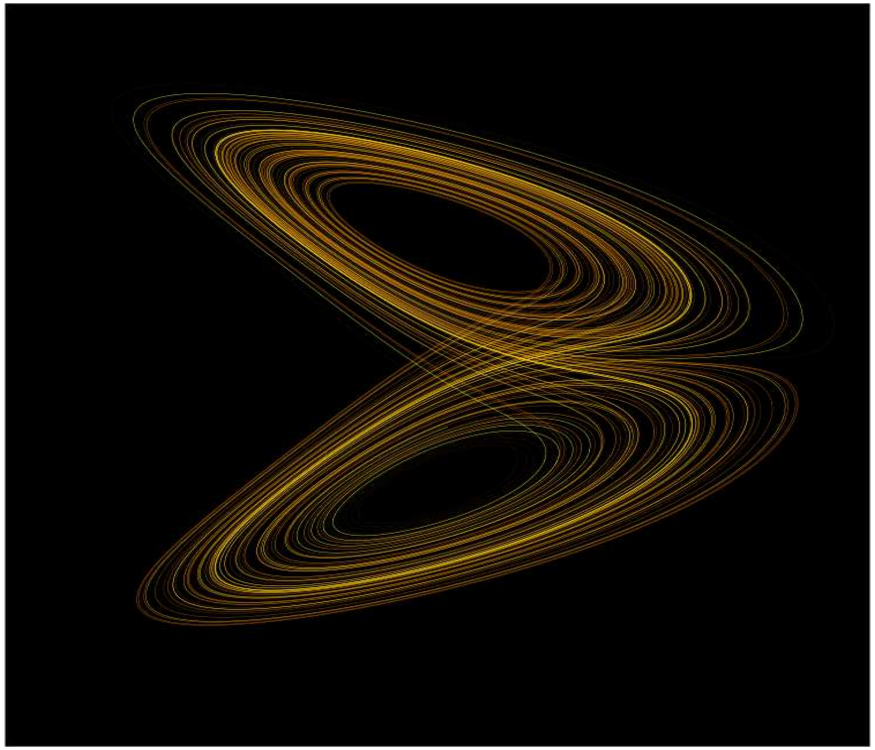


Chaotic scattering

We will consider chaotic systems, for which K_E is a **hyperbolic repeller**, with a **fractal geometry**.

Ex. 4: $n \geq 3$ convex obstacles in $\mathbb{R}^2$





$K_E = \Gamma_E^+ \cap \Gamma_E^-$ has a fractal (Hausdorff/box) dimension

$$\dim(K_E) = 2\mu_E + 1 \qquad (\mu_E < 1).$$

Fractal Weyl law

Theorem. [Sjöstrand-Z'05] *In the limit $h \to 0$, the number of resonances near E is given by a* ***fractal Weyl law***

$$\# \{|z_j - E| < \gamma h\} \leq C_{E,\gamma} h^{-\mu_E}, \qquad h \to 0 .$$

A bold conjecture would say that $\leq$ should be replaced by $\sim$.

An asymptotic relation of that kind was proven for a special open quantum map [Nonnenmacher-Z'05].

Distribution of resonant states

The metastable states associated with long-living resonances have specific phase space distributions:

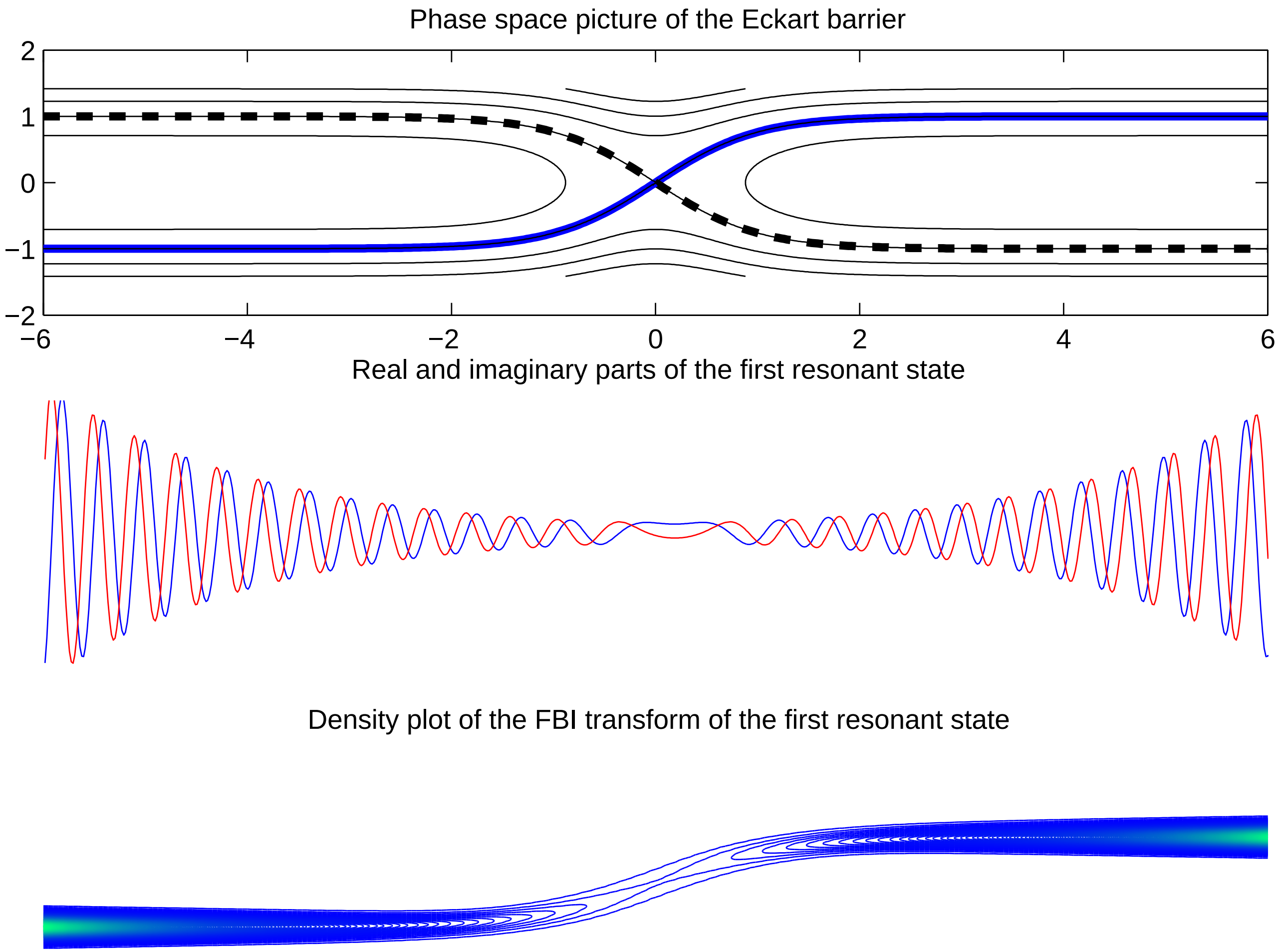
Theorem. [Nonnenmacher-Rubin'05, Nonnenmacher-Z '06] *Consider a family of resonant states* $(\psi_h)_{h\to 0}$ *s.t.* $\operatorname{Re} z(h) = E + o(1)$, $\operatorname{Im} z(h) = O(h)$ *and* $\|\psi_h\|_{L^2(inter)} = 1$.
Suppose a semiclassical measure μ *is associated with* (ψ_h)*:*

$$\forall f \in C_c^\infty(T^*\mathbb{R}^d),\ \chi_{|\pi \operatorname{supp} f} = 1, \qquad \langle \chi\psi_h, \operatorname{Op}_h(f)\chi\psi_h\rangle \xrightarrow{h\to 0} \int f(\rho)\, d\mu(\rho)\,.$$

Then $\operatorname{supp}\mu \in \Gamma_E^+$*, and for some* $\lambda \geq 0$ *we have*

$$\frac{\operatorname{Im} z(h)}{h} \xrightarrow{h\to 0} \lambda/2 \quad \textit{and} \quad \mathcal{L}_{X_H}\mu = \lambda\mu\,.$$

Ex: resonant state for a single hyperbolic point



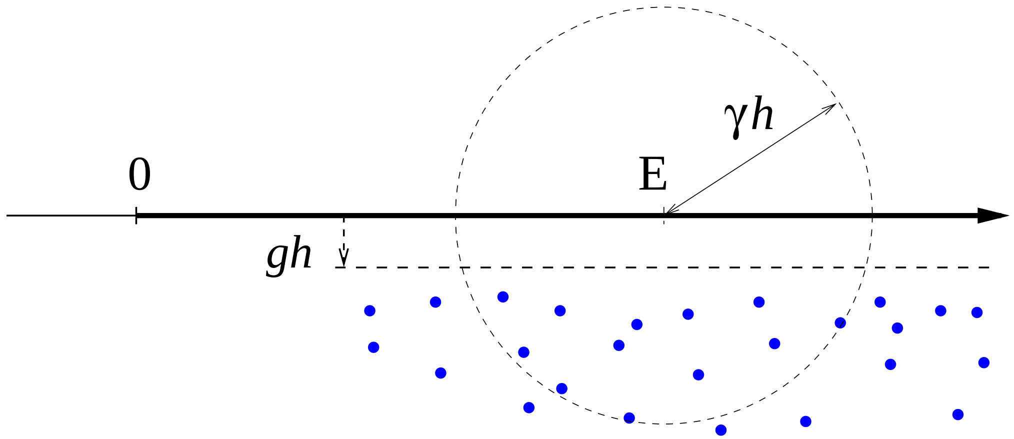
Top: the phase portrait for $p(x,\xi) = \xi^2 + \cosh^{-2}(x)$, with $\Gamma_1^{\pm}$ highlighted. Middle: the "first" resonant state, $h = 1/16$. Bottom: squared modulus of its FBI tranform. The resonant state was computed by D. Bindel, the FBI transform was provided by L. Demanet. As predicted in the above Theorem, the mass of the FBI transform is concentrated on Γ_1^+, with an exponential mass growth in the outgoing direction.

Resonance gap for "filamentary" repellers

[IKAWA'88, BURQ'93]: if n convex obstacles in $\mathbb{R}^d$ are *far enough from e.o.* $\Rightarrow$ **gap** in the semiclassical ($\equiv$ high-energy) resonance spectrum:

$\exists g > 0,$ for h small enough, any resonance with $\mathrm{Re}\, z_j \approx 1/2$ satisfies $\quad \mathrm{Im}\, z_j \leq -g\, h.$

Equivalently, the quantum lifetimes $\tau_j \leq (2g)^{-1}$.



[GASPARD-RICE'89] (3 disks in $\mathbb{R}^2$):
If the dimension $\mu_E < 1/2$, which is equivalent with $P_E(1/2) < 0$, then there is a gap, and one can take $g = -P_E(1/2)$. Here $P_E(1/2)$ is the topological pressure.

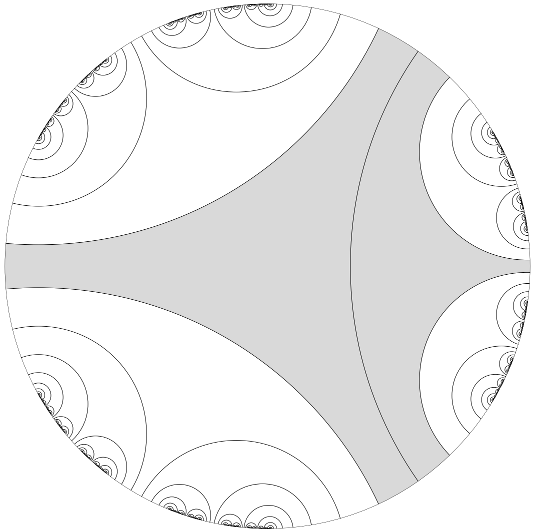
Their argument assumes that, in the semiclassical limit, resonances $z_j = (hk_j)^2/2$ are approximately given by zeros of the Gutzwiller-Voros ($\sim$ Selberg) zeta function:

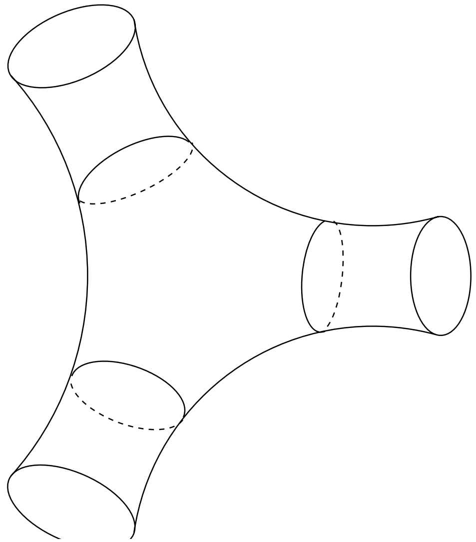
$$Z(k) = \prod_{\omega} \prod_{j\geq 0} \left(1 - \mathrm{e}^{-ikl_\omega - (1/2+j)\lambda_\omega}\right) \quad \text{converges abs. for } \mathrm{Im}\, k > P(1/2)\,.$$

Scattering on convex co-compact quotients

The geodesic flow in the infinite-volume manifold $X = \Gamma\backslash\mathbb{H}^{n+1}$ is uniformly hyperbolic ($\kappa = -1$). For any $E > 0$ the trapped set K_E has dimension $2\delta + 1$, where δ is the dim. of the limit set $\Lambda(\Gamma)$.

The resonances $z = s(n-s) = \frac{n^2}{4} + k^2$ of Δ_X are given by the zeros of $Z_{Selberg}(s)$.





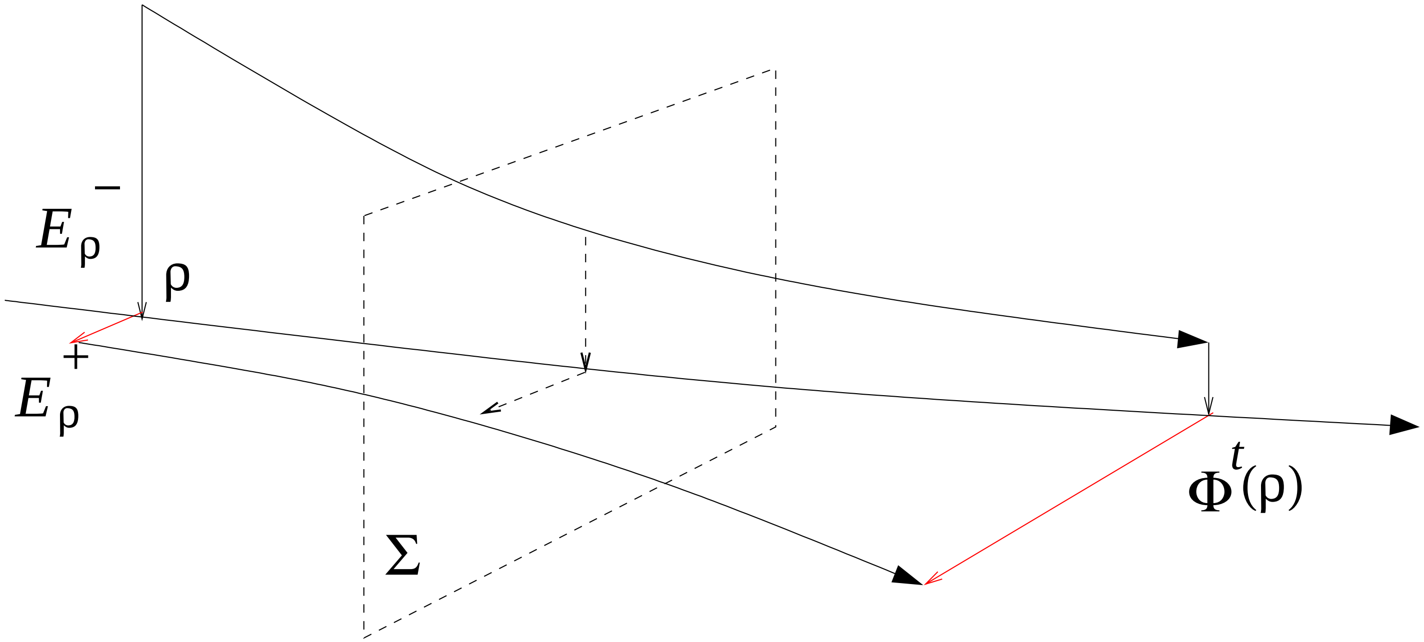
[PATTERSON'76, SULLIVAN'79, PATTERSON-PERRY'01]: All the zeros are in the half-plane
$\operatorname{Re} s \leq \delta \Longleftrightarrow \operatorname{Im} k \leq \delta - n/2$ (their density is bounded by r^δ [Guillopé-Lin-Z'04]).

Here we also have $\delta - n/2 = P(1/2)$ the topological pressure of the geodesic flow. Therefore, if $P(1/2) < 0$ there is a resonance gap $g = |P(1/2)|$.

Actually, in the semiclassical limit $|\operatorname{Re} k| \to \infty$, the gap is $|P(1/2)| + \varepsilon$, due to phase cancellations [NAUD'05].

Resonance gap in Euclidean potential scattering

We consider Euclidean scattering by a potential $V \in C_c^\infty(\mathbb{R}^d)$. We assume that near some noncritical $E > 0$ the Hamiltonian flow Φ^t is uniformly hyperbolic on K_E.



Let $P_E(s)$ be the topological pressure of the flow Φ^t on K_E, associated with the unstable Jacobian $J_t^+(\rho)^{-1} = \det\left(d\Phi^t_{|E_\rho^+}\right)^{-1}$.

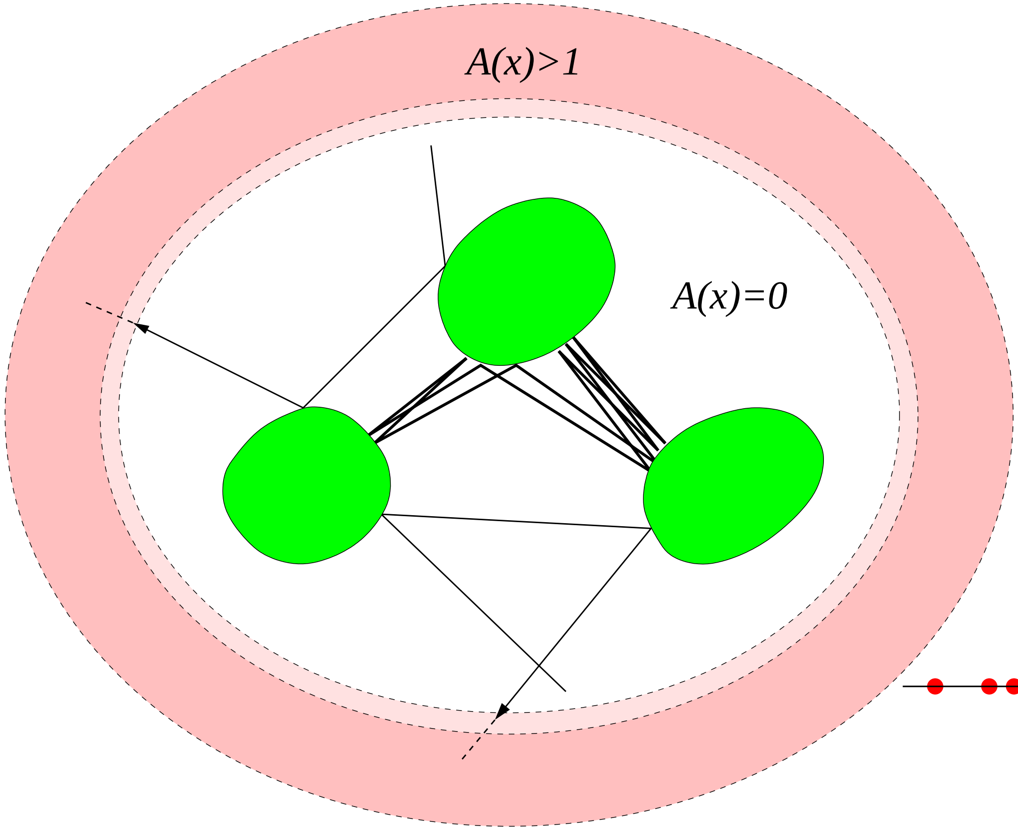
Theorem. [Nonnenmacher-Z'06] *Assume the topological pressure $P_E(1/2) < 0$, and take any $0 < g < -P_E(1/2)$.*
Then, for h small enough, the resonances of H_h such that $\mathrm{Re}\, z_j = E + o(1)$ satisfy $\mathrm{Im}\, z_j \leq -gh$.

In dimension $d = 2$, the gap condition $P_E(1/2) < 0$ is equivalent with $\mu_E < 1/2$, so we recover the criterium of [Gaspard-Rice] for a smooth flow.

A simpler model: scattering with absorbing potential

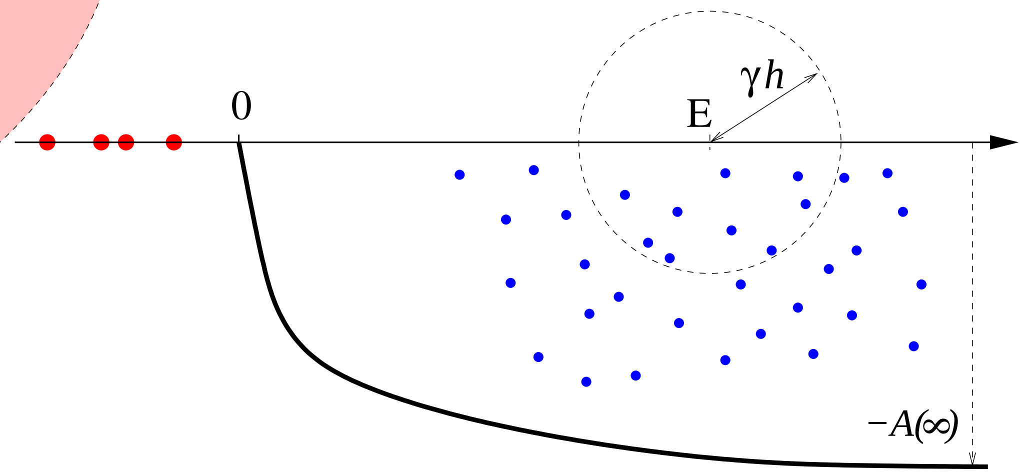
To avoid the trouble of complex dilation, it is convenient to add a complex potential $-iA(x)$ to the quantum Hamiltonian, obtaining the nonselfadjoint operator

$$H_{h,A} = -\frac{h^2\Delta}{2} + V(x) - iA(x).$$

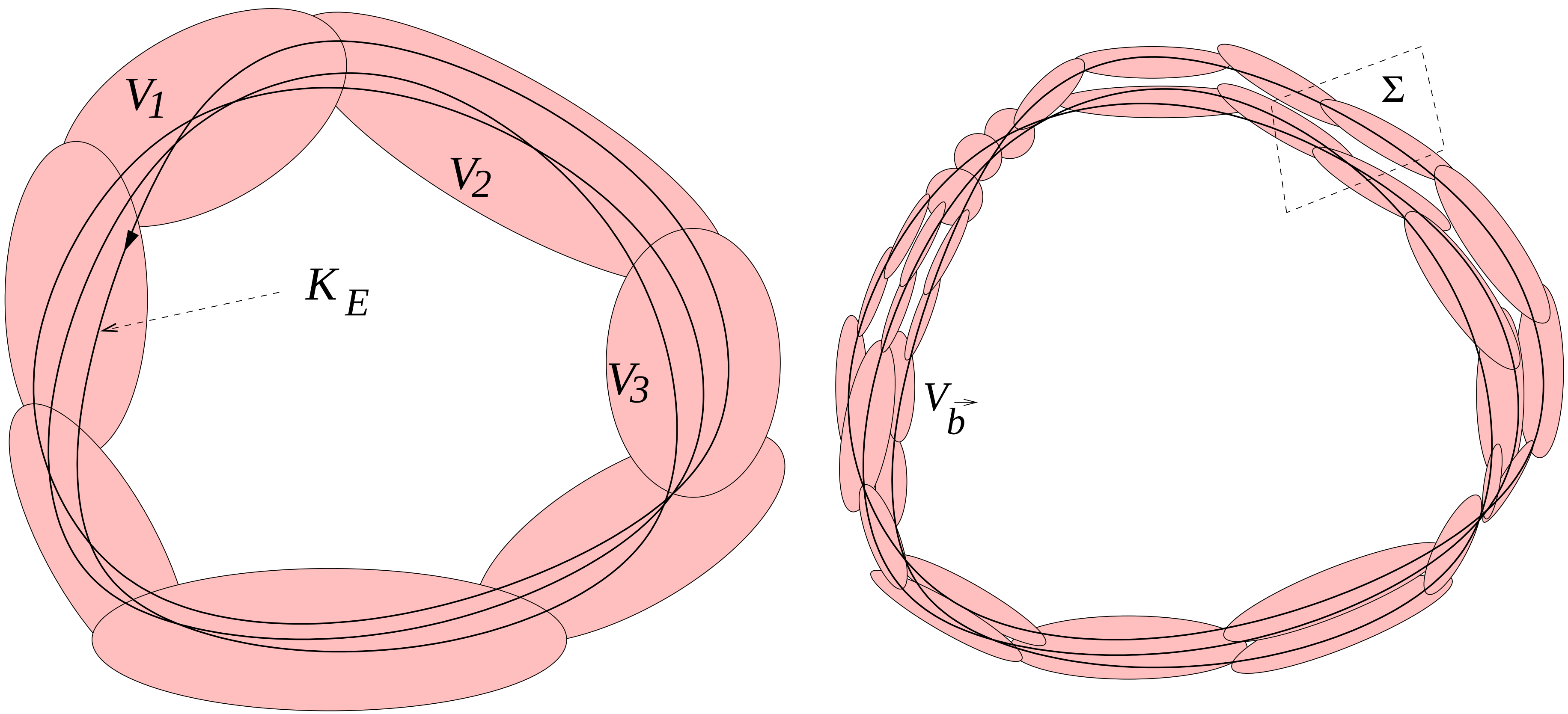


This potential vanishes in the interaction region. Its role is to absorb (kill) the outgoing wavepackets.

As a result, the spectrum of $H_{h,A}$ near the real axis is made of L^2 eigenvalues instead of resonances. These long-living eigenvalues are expected to behave like the resonances of H_h.



An alternative definition of the topological pressure



To define the pressure of Φ^t on K_E, one starts from an open cover $(V_b)_{b\in B}$ of K_E. This cover is then refined T times through the flow, producing sets

$$V_{\vec{b}} = V_{b_0} \cap \Phi^{-1}V_{b_1} \cap \Phi^{-2}V_{b_2} \cap \cdots \cap \Phi^{-T+1}V_{b_{T-1}}\,, \qquad \vec{b} \in B^T\,.$$

Keep the $V_{\vec{b}}$ intersecting K_E.

Weigh each $V_{\vec{b}}$ using the coarse-grained unstable Jacobian:

$$w_T(V_{\vec{b}}) \stackrel{\text{def}}{=} \sup_{\rho \in V_{\vec{b}} \cap K_E} \big(J_T^+(\rho)\big)^{-1/2} \sim \mathrm{e}^{-T(d-1)\bar{\lambda}/2},$$

where $\bar{\lambda}$ is an "average" stretching exponent for initial points in $V_{\vec{b}}$.
One then considers the partition function

$$\mathcal{Z}_T \stackrel{\text{def}}{=} \inf\{\sum_{\vec{b}\in\mathcal{B}_T} w_T(V_{\vec{b}}) \; : \; \mathcal{B}_T \subset B^T, \; K_E \subset \bigcup_{\vec{b}\in\mathcal{B}_T} V_{\vec{b}}\}.$$

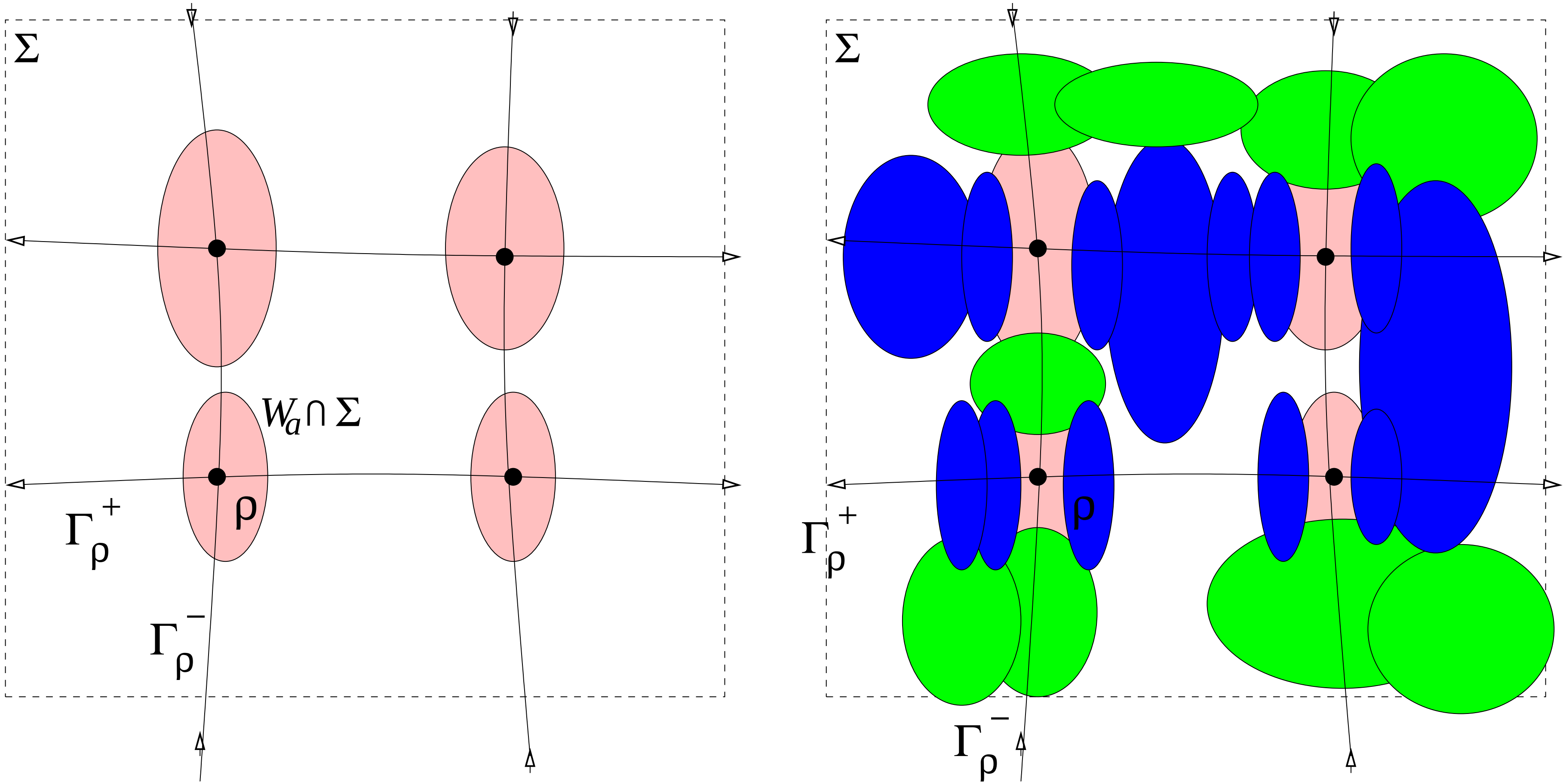
The pressure $P_E(1/2)$ is finally given by

$$P_E(1/2) = \lim_{\operatorname{diam}(V_b)\to 0} \lim_{T\to\infty} \frac{1}{T} \log \mathcal{Z}_T .$$

For a given $\epsilon > 0$, we may select a (fine) partition (V_b), a time $t_0 \in \mathbb{N}$ and a cover $(V_{\vec{b}})_{\vec{b}\in\mathcal{B}_{t_0}} \stackrel{\text{def}}{=} (W_a)_{a\in A_1}$, such that that

$$\sum_{a\in A_1} w_{t_0}(W_a) \leq \exp\big\{t_0\big(P_E(1/2) + \epsilon\big)\big\}.$$

Completing the cover



We need to complete $(W_a)_{a\in A_1}$ in order to cover a thin energy layer $H^{-1}([E-\delta, E+\delta])$. One set lies in the forbidden zone: $W_\infty = \{A(x) > 1\}$. Each of the remaining sets $(W_a)_{a\in A_2}$ must have escaped to W_∞ at the time $N_0 t_0$ or $-N_0 t_0$, for some $N_0 \in \mathbb{N}$.

We thus get a cover

$$H^{-1}([E-\delta, E+\delta]) \subset \bigcup_{a\in A_1\cup A_2\cup\infty} W_a \,.$$

A quantum partition of unity

To this open cover we associate a smooth partition of unity:

$$1 = \sum_{a \in A_1 \cup A_2} \pi_a + \pi_\infty + \pi_E \,,$$

where $\pi_a \in C_c^\infty(W_a)$, π_∞ is supported in W_∞, and π_E vanishes in $H^{-1}([E \pm \delta/2])$.
This partition may be h-quantized into a sum of bounded ΨDOs:

$$Id_{L^2} = \sum_{a \in A_{tot}} \Pi_a \,, \qquad A_{tot} = A_1 \cup A_2 \cup \infty \cup E, \quad \Pi_a = \mathrm{Op}_h^w(\pi_a)\,.$$

We use this quantum partition to split the “absorbing propagator” $U = \mathrm{e}^{-it_0 H_{h,A}/h}$:

$$U = \sum_{a \in A_{tot}} U_a \,, \qquad U_a = U\,\Pi_a$$

Decomposition of the propagator

Let ψ_h be a normalized eigenstate of $H_{h,A}$ with eigenvalue $z(h)$ (assuming $\mathrm{Re}\, z = E + o(1)$, $\mathrm{Im}\, z = O(h)$). To prove the gap, our aim is to bound from above

$\|U^N \psi_h\| = \mathrm{e}^{N t_0 \,\mathrm{Im}\, z(h)/h}$ and we will need to go up to $N \sim M|\log h|$, $\quad M >> 1\,$.

To do this, we use the above quantum partition of unity:

$$U^N \psi_h = \sum_{a_i \in A_{tot},\ 1 \leq i \leq N} U_{a_N} U_{a_{N-1}} \cdots U_{a_2} U_{a_1} \psi_h$$

- ψ_h is microlocalized in $H^{-1}(E) \Longrightarrow U_{\vec{a}} \psi_h = O(h^\infty)$ if any of the $a_i = E$.
- wavepackets are absorbed in $W_\infty \Longrightarrow \|U_\infty\| = O(h^\infty)$.
- from the escape properties of W_a, $a \in A_2$, we have $\|U_{b_n} \ldots U_{b_1} U_a\| = O(h^\infty)$ or $\|U_a U_{b_n} \ldots U_{b_1}\| = O(h^\infty)$ if $n \geq N_0$.
- As a result, the above sum is dominated by $\vec{a}$ such that $a_i \in A_1$ for $N_0 < i \leq N - N_0$.

Using hyperbolicity

The following bound (valid for h small enough) relies on the hyperbolicity of Φ^t:

$$\forall n = O(|\log h|),\ \forall \vec{a} \in A_1^n, \qquad \boxed{\|U_{a_n} \cdots U_{a_1}\| \le h^{-d/2}(1+\epsilon)^{n t_0} \prod_{j=1}^{n} w_{t_0}(W_{a_j})}\,.$$

The last product behaves as $\mathrm{e}^{-n t_0 (d-1)\bar{\lambda}/2}$. Due to the prefactor $h^{-d/2}$, the LHS starts to decay exponentially only for $n \ge \frac{d-1}{d}\frac{|\log h|}{\bar{\lambda}}$ (Ehrenfest time).

Take $N \sim M|\log h|$ and sum over all contributions:

$$\|U^N \psi_h\| \le C\, h^{-d/2}(1+\epsilon)^{N t_0}\Big(\sum_{a\in A_1} w_{t_0}(W_a)\Big)^N \le C\, \exp\Big\{N t_0\Big(\frac{d}{2M t_0} + P_E(1/2) + 2\epsilon\Big)\Big\}\,.$$

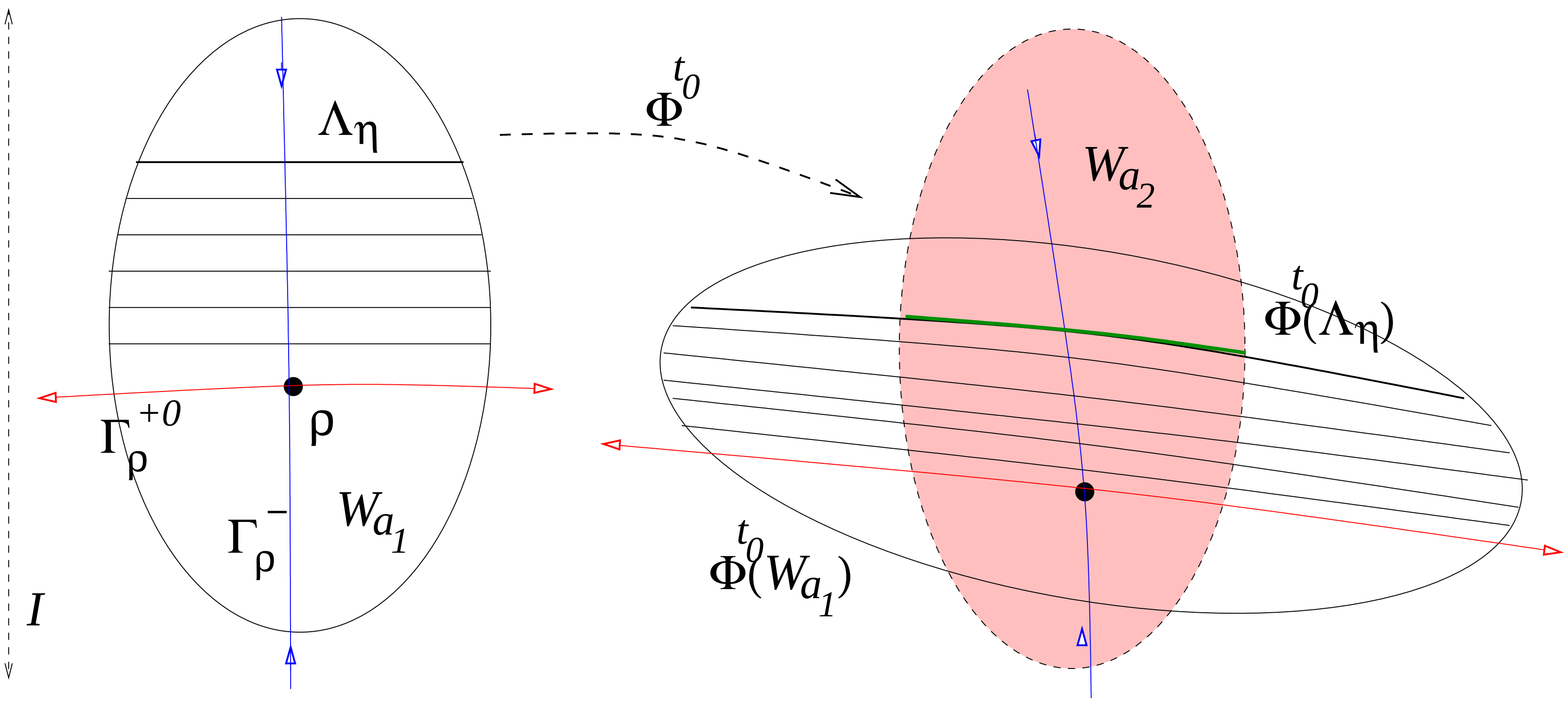
Choosing $M \ge d/(2\epsilon t_0)$, we finally get

$$\operatorname{Im} z(h)/h \le P_E(1/2) + 3\epsilon\,.$$

$\square$

Proof of the hyperbolic bound

The bound on $\|U_{\vec{a}}\|$ is similar as the one proven by [ANANTHARAMAN'06] in the case of the geodesic flow on Anosov manifolds.



For any normalized $\Psi \in L^2$, the state $\Pi_{a_1}\Psi$ can be decomposed as $\Pi_{a_1}\Psi = h^{-d/2}\int_I d\eta\, f(\eta)\, e_\eta + O(h^\infty)$, where $\|f\|_{L_2} \leq C$ and each e_η is a normalized WKB state supported on the Lagrangian Λ_η (close to Γ^{+0}). $U\, e_\eta$ is also Lagrangian, supported on $\Phi^{t_0}(\Lambda_\eta)$. After the truncation by Π_{a_2}, one has

$$\|\Pi_{a_2} U\, e_\eta\| \lesssim J^+_{t_0}(\rho)^{-1/2} \approx w_{t_0}(W_{a_1})\,.$$

By iterating this procedure, we get

$$\|U_{a_n} \cdots U_{a_2} U\, e_\eta\| \lesssim \prod_{j=1}^{n-1} w_{t_0}(W_{a_j}) + O(h^\infty)\,.$$

Since the RHS is independent of η, we may integrate these norms over $\eta \in I$:

$$\|U_{a_n} \cdots U_{a_1} \Psi\| \lesssim h^{-d/2} \|f\|_{L_1} \prod_{j=1}^{n-1} w_{t_0}(W_{a_j})\,,$$

and finally use $\|f\|_{L_1} \leq \sqrt{|I|}\, \|f\|_{L_2} \leq C$.

$\square$

Final remarks

- the same procedure can be used for actual resonances of H_h. One needs to complex-dilate the operator, to obtain a nonselfadjoint $H_{h,\theta}$ with properties similar with $H_{h,A}$.

- can one show that the semiclassical gap $g \geq |P(1/2)| + \varepsilon$?
  Need to control the *relative phases* of the components $U_{\vec{a}}\,\psi_h$.

- proof of the fractal Weyl law asymptotics?

- better study semiclassical measures associated with resonant states

- a nice "experimental" model is provided by open quantum maps