Introduction to Partial Differential Equations

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Math 126 | Partial Differential Equations

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Contents

Introduction to Differential Equations

Ordinary Differential Equations

Ordinary differential equations (ODEs) are differential equations with funtions of one variable. ODEs are a subset of partial differential equations (PDEs).

Linear ODEs

Let the notation $\partial_t u$ mean $u_t = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t}$ where $u : t \to \mathbb{C}$ where $t \in \mathbb{R}$. First order (constant coefficients and homogeneous) linear ODEs are in the form

$$\partial_t u = \alpha u$$

where α is a constant. This ODE can then be solved:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \alpha u \\
\frac{1}{u}\,\mathrm{d}u &= \alpha\,\mathrm{d}t \\
\int_{u(0)}^{u(t)} \frac{1}{u}\,\mathrm{d}u &= \int_0^t \alpha\,\mathrm{d}t \\
\ln\left(u(t)\right) - \ln(u(0)) &= \alpha t \\
\ln(u(t)) &= \ln(u(0)) + \alpha t \\
u(t) &= u(0)e^{\alpha t}
\end{aligned}$$

Linear ODEs satisfy the **superposition principle**.

- If $\partial_t u = \alpha u$ and $\partial_t v = \alpha v$, then $\partial_t(u+v) = \alpha(u+v)$.
- If u and v solve the system of equations, so does $u+v$.
- Similarly, cu also solves the system of equations for constant c.

Nonlinear ODEs

An example of a nonlinear ODE is $\partial_t u = u^3$, which can be solved.

$$\begin{aligned}
\partial_t u &= u^3 \\
\int_{u(0)}^{u(t)} \frac{1}{u^3}\,\mathrm{d}u &= \int_0^t \mathrm{d}t \\
-\frac{1}{2u(t)^2} + \frac{1}{2u(0)^2} &= t \\
-\frac{1}{2u(t)^2} &= -\frac{1}{2u(0)^2} + t \\
\frac{1}{u(t)^2} &= \frac{1}{u(0)^2} - 2t \\
u(t) &= \left(\frac{1}{u(0)^2} - 2t\right)^{-1/2} \\
&= \left(\frac{1}{u(0)^2}\left(1 - 2tu(0)^2\right)\right)^{-1/2} \\
&= u(0)\left(1 - 2tu(0)^2\right)^{-1/2}
\end{aligned}$$

As t approaches $\frac{1}{2(u(0))^2}$, the solution "blows up" since the denominator approaches 0.

ODEs with complex numbers

Let $i\partial_t u = |u|^2 u$ where now u is allowed to be complex valued. To solve it helps to find a **conserved quantity** as t varies, meaning a quantity does not change in time. In this example, $|u(t)|^2$ is a conserved quantity. To show this, we prove its derivative is 0. Recall the following properties of complex numbers: if $z \in \mathbb{C}$, $|z|^2 = z \cdot \overline{z}$ where $z = x + iy$ and $\overline{z} = x - iy$. Additionally, $z + \overline{z} = 2x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t) \cdot \overline{u(t)}\right] \\
&= u'(t)\overline{u(t)} + u(t)\overline{u'(t)} \\
&= 2\,\mathrm{Re}\left(u'(t)\overline{u(t)}\right)
\end{aligned}$$

Using the equation $\partial_t u = \frac{1}{i}|u|^2 u$ (and knowing $\frac{1}{i} = -i$ since $i^2 = -1$):

$$\begin{aligned}
&= 2\,\mathrm{Re}\left(\frac{1}{i}|u(t)|^2 u(t)\overline{u(t)}\right) \\
&= 2\,\mathrm{Re}\left(-i|u(t)|^2|u(t)|^2\right) \\
&= 2\,\mathrm{Re}\underbrace{\left(-i|u(t)|^4\right)}_{\text{imaginary}} \\
&= 0
\end{aligned}$$

So, $|u(t)|^2 = |u(0)|^2$ meaning $|u(t)|^2$ is the conserved quantity. Now, we can write the ODE as a linear ODE.

$$i\partial_t u = |u(0)|^2 u(t)$$

Using the solution from the earlier with $\alpha = \frac{|u(0)|^2}{i}$, we can solve for $u(t)$.

$$u(t) = u(0)e^{-i|u(0)|^2 t}$$

Partial Differential Equations

Partial differential equations (PDEs) are equations involving partial derivatives, meaning they involve functions of multiple variables.

Linear PDEs

Professor Zworski's favorite PDE is the Schrödinger Equation, specifically the free Schrödinger Equation on a line which describes a free particle in one dimension in quantum mechanics.

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$$

The notation $\partial_t u$ means $\frac{\partial u}{\partial t}$. Let P be an operator. As a linear PDE, this equation can be written as

$$P(u) = i\partial_t u + \frac{1}{2}\partial_x^2 u$$

with properties

$$\begin{aligned}
P(u+v) &= P(u) + P(v) \\
P(cu) &= cP(u) \qquad c \in \mathbb{C}
\end{aligned}$$

If $P(u) = 0$ and $P(v) = 0$, then $P(au + bv) = 0$ for $a, b \in \mathbb{C}$. *This is the superposition principle for this linear PDE.*

We want to find some solutions to

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0.$$

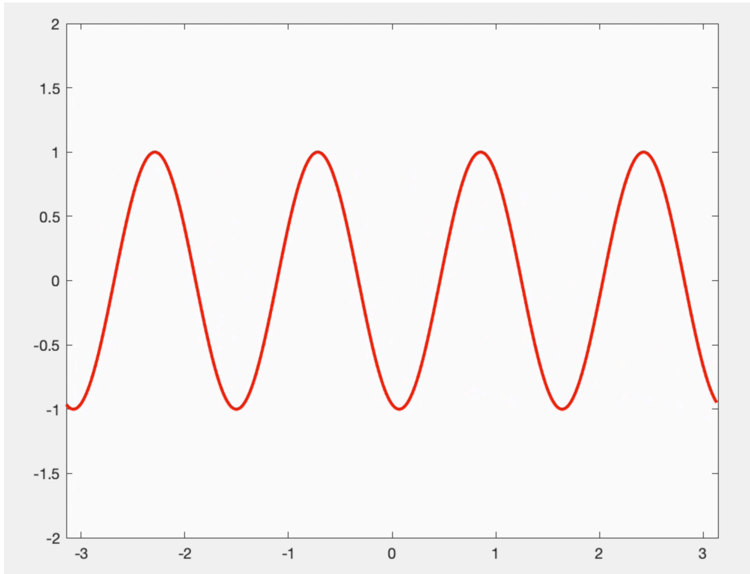


Figure 1: Plot of $\operatorname{Re} u(0,x)$ where u is given by (1) – you can see $\operatorname{Re} u(x,t)$ for $-$ here (click on "here") and for $+$ here. Which way is the wave moving?

Separation of variables is a technique that can be used to reduce PDEs to ODEs. In this case, we want to write $u(t,x) = v(t)w(x)$, meaning u is a product of single variable equations.

$$\begin{aligned} 0 &\stackrel{?}{=} i\partial_t u + \frac{1}{2}\partial_x^2 u \\ &= i\partial_t\left[v(t)w(x)\right] + \frac{1}{2}\partial_x^2\left[v(t)w(x)\right] \\ &= iv'(t)w(x) + \frac{1}{2}v(t)w''(x) \\ iv'(t)w(x) &= -\frac{1}{2}v(t)w''(t) \\ \underbrace{2i\frac{v'(t)}{v(t)}}_{\text{func. of } t} &= \underbrace{-\frac{w''(x)}{w(x)}}_{\text{func. of } x} \end{aligned}$$

Since both sides are equations of a single variable, both must be constant. Let this constant be μ.

$$2i\frac{v'(t)}{v(t)} = -\frac{w''(x)}{w(x)} = \mu$$

Let $\mu = \lambda^2 > 0$. Solve for $v(t)$ and $w(x)$ separately.

$$\begin{aligned} 2i\frac{v'(t)}{v(t)} &= \lambda^2 \\ v'(t) &= \frac{\lambda^2}{2i}v(t) \\ v(t) &= e^{-(i\lambda^2/2)t} \end{aligned} \qquad\qquad \begin{aligned} -\frac{w''(x)}{w(x)} &= \lambda^2 \\ w''(x) + \lambda^2 w(x) &= 0 \\ w(x) &= e^{\pm i\lambda x} \end{aligned}$$

Since Euler's Formula states $e^{i\xi} = \cos\xi + i\sin\xi$ and $e^{-i\xi} = \cos\xi - i\sin\xi$.

So, we have our solution for the linear Schr odinger Equation.

$$u(t,x) = v(t)w(x) = e^{-i\lambda(\lambda t/2 \pm x)} \tag{1}$$

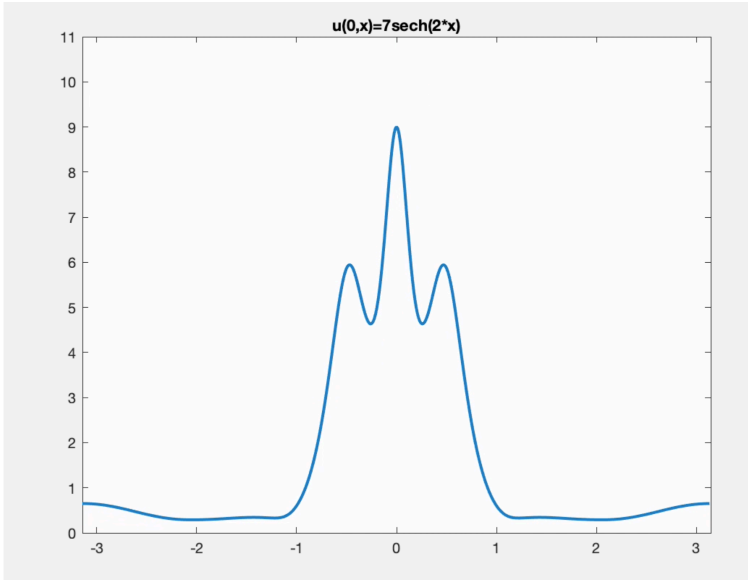


Figure 2: Plot of $|u(t,x)|$ where u is given by (2) – click on it to see a movie. Strictly speaking the movie shows a solution which is periodic rather than on $\mathbb{R}$ but the main features are the same.

Nonlinear PDEs

The nonlinear Schrödinger Equation is

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^2 u = 0,$$

We have already found solutions to $iu_t + |u|^2 u = 0$ and $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$. Our task is to find some interesting solutions of this *nonlinear PDE*. We will try

$$u(t,x) = \left(e^{i\lambda x + i\mu t}\right)\left(\psi(x - \sigma t)\right).$$

This is an example of an **ansatz**, or a guess of the form of a solution.

As an aside, we mention that there are many different looking solution: a crazy looking example is given by

$$u(x,t) = 2e^{it/2}\operatorname{sech} x \frac{4 + \operatorname{sech}^2\left(e^{4it} - 1\right)}{4 - e\operatorname{sech}^4 x \sin^2 2t} \tag{2}$$

where $\operatorname{sech} x = \frac{1}{\cosh x}$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$. For a picture of this solution and a movies see Figure 2

Going back to the ansatz, we want to find ψ, a function of one variable. To do this, we will use the original equation and find every term in it in terms of ψ.

$$\begin{aligned} i\partial_t u &= e^{i\lambda x + i\mu t}(-\mu\psi - i\sigma\psi') \\ \frac{1}{2}\partial_x^2 u &= \frac{1}{2}\partial_x\left(e^{i\lambda x + i\mu t}\left(i\lambda\psi + \psi'\right)\right) \\ &= \frac{1}{2}\left(-\lambda^2\psi + 2i\lambda\psi' + \psi''\right)e^{i\lambda x + i\mu t} \\ |u|^2 u &= |\psi|^2 e^{i\lambda x + i\mu t}\psi \end{aligned}$$

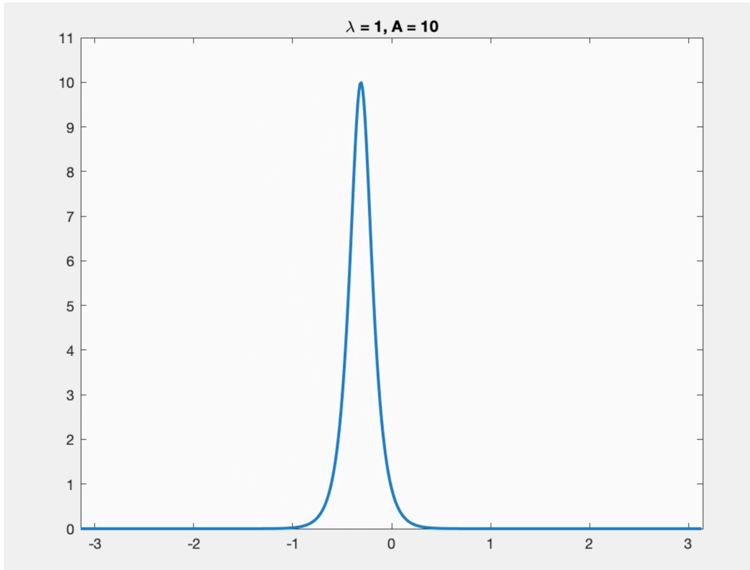


Figure 3: Plot of $|u(t,x)|$ where u is given by (3) – click on it to see a movie. You clearly see the profile given by sech.

So, we can rewrite the equation in terms of ψ.

$$
\begin{aligned}
e^{i\lambda x+i\mu t}(-\mu\psi - i\sigma\psi') + \frac{1}{2}\left(-\lambda^2\psi + 2i\lambda\psi' + \psi''\right)e^{i\lambda x+i\mu t} + |\psi|^2 e^{i\lambda x+i\mu t}\psi &= 0\\
-\mu\psi - i\sigma\psi' - \frac{1}{2}\lambda^2\psi + i\lambda\psi' + \frac{1}{2}\psi'' + |\psi|^2\psi &= 0\\
\left(-\mu - \frac{\lambda^2}{2}\right)\psi + (-i\sigma + i\lambda)\psi' + \frac{1}{2}\psi'' + |\psi|^2\psi &= 0
\end{aligned}
$$

We want to get rid of the ψ' term, so set $\sigma = \lambda$. Let $\gamma := \mu + \frac{\lambda^2}{2}$. We want a real solution, so $|\psi|^2 = \psi^2$.

$$
\begin{aligned}
\frac{1}{2}\psi'' - \gamma\psi + \psi^3 &= 0\\
\frac{1}{2}\psi''\psi' - \gamma\psi\psi' + \psi'\psi^3 &= 0
\end{aligned}
$$

Use the law $(u^p)' = pu'u^{p-1}$:

$$
\begin{aligned}
\frac{1}{4}\left((\psi')^2\right)' - \frac{\gamma}{2}\left(\psi^2\right)' + \frac{1}{4}\left(\psi^4\right)' &= 0\\
\frac{1}{4}\left((\psi')^2 + 2\gamma\psi^2 + \psi^4\right)' &= 0\\
(\psi')^2 - 2\gamma\psi^2 + \psi^4 &= \text{ constant} = 0
\end{aligned}
$$

(The constant has to be zero as we want ψ to go to zero at infinity). Now let $A^2 := 2\gamma$.

$$
\begin{aligned}
\psi' &= \left(A^2\psi^2 - \psi^4\right)^{1/2}\\
\frac{1}{\left(A^2\psi^2 - \psi^4\right)^{1/2}}\,\mathrm{d}\psi &= \mathrm{d}x
\end{aligned}
$$

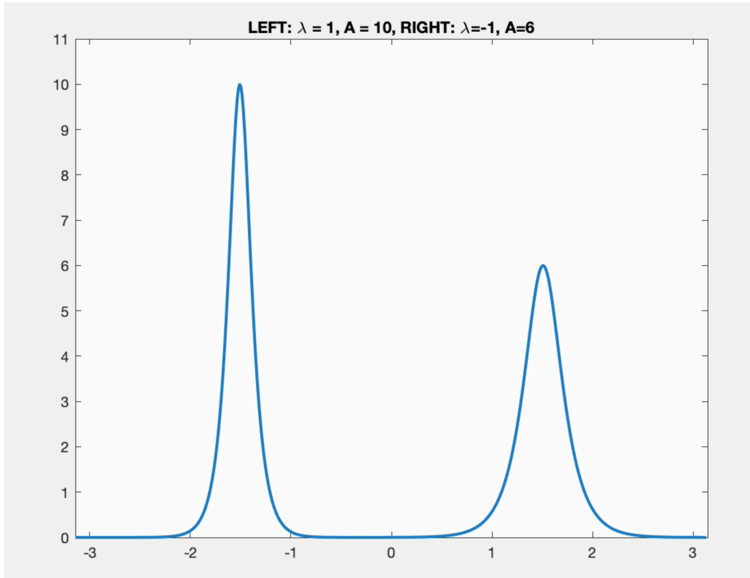


Figure 4: Plot of $|u(t,x)|$ at $t=0$ where u solves the nonlinear Schrödinger equation with the value at $t=0$ given by the sum of two different solitons – click on the figure to see the interaction of the two solitons (solution as time goes on) and the stability of the outcome.

To evaluate the integral of the LHS, use the substitution $u = -\frac{1}{\psi}$ meaning $\mathrm{d}\psi = \frac{1}{u^2}\,\mathrm{d}u$.

$$\begin{aligned}
\int \frac{1}{\left(A^2\psi^2 - \psi^4\right)^{1/2}}\,\mathrm{d}\psi &= -\int \frac{1}{u^2\frac{1}{u}\left(A^2 - \frac{1}{u^2}\right)^{1/2}}\,\mathrm{d}u \\
&= -\int \frac{1}{u\left(A^2 - \frac{1}{u^2}\right)^{1/2}}\,\mathrm{d}u \\
&= -\int \frac{1}{\left(A^2u^2 - 1\right)^{1/2}}\,\mathrm{d}u
\end{aligned}$$

To find out this integral we recall cosh and sinh functions:

$$\begin{aligned}
\cosh y &= \frac{1}{2}\left(e^y + e^{-y}\right) \\
\sinh y &= \frac{1}{2}\left(e^y - e^{-y}\right) \\
\cosh^2 y - 1 &= \sinh^2 y \\
(\cosh y)' &= \sinh y
\end{aligned}$$

Substitute $u(x) = \frac{1}{A}\cosh(Ax)$. This yields

$$\psi(x) = A\,\mathrm{sech}(Ax)$$

Now, plug this back into the original equation along with the substitutions $\sigma = \lambda$, $\mu = -\frac{\lambda^2}{2} + \gamma$ and $\gamma = \frac{A^2}{2}$:

$$u(t,x) = e^{i\lambda x + i(A^2/2 - \lambda^2/2)t} A\,\mathrm{sech}(A(x - \lambda t)) \tag{3}$$

We obtained a very interesting family of solutions to the nonlinear Schrödinger equation. They are called **solitons** – see Figures 3, 5

Some Preliminaries

Real and Complex Numbers

The set of **real numbers** is $\mathbb{R}$. A real number $r \in \mathbb{R}$ is an element of $\mathbb{R}$. A set $A \subset \mathbb{R}$ is a subset of $\mathbb{R}$. $x_0 = \sup A$ is the smallest number $\geq$ all numbers in A, and $y_0 = \inf A$ is the largest number $\leq$ all numbers in A.

The set of **complex numbers** is $\mathbb{C}$. For $z \in \mathbb{C}$, $z = x + iy$ where $x, y \in \mathbb{R}$, $\overline{z} = x - iy$, and $i = \sqrt{-1}$.

$$\begin{aligned} x &= \operatorname{Re} z = \frac{1}{2}(z + \overline{z}) \\ y &= \operatorname{Im} z = \frac{1}{2i}(z - \overline{z}) \end{aligned}$$

For any operator $*$, $\overline{z * w} = \overline{z} * \overline{w}$. Additionally, the magnitude of a complex number is defined as

$$|z| = (z \cdot \overline{z})^{1/2} = \left(x^2 + y^2\right)^{1/2}.$$

Limits and Series

For $z_j \in \mathbb{C}$, $z = \lim_{j \to \infty} z_j$ (or $z_j \to z$) if $\forall \varepsilon > 0$, $\exists J$ such that for every $j \geq J$, $|z - z_j| < \varepsilon$. An **infinite series** is of the form

$$\sum_{k=0}^{\infty} a_k.$$

The series converges to s if each partial sum s_n converges to s, where

$$s_n := \sum_{k=0}^{n} a_k.$$

An infinite series converges absolutely if $\sum_{k=0}^{\infty} |a_k| < \infty$. Absolute convergence implies convergence, but the converse is not true.

Professor Zworski's favorite absolutely convergent series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for $x \in \mathbb{C}$.

$$\begin{aligned} e^{iy} &= \sum_{k=0}^{\infty} \frac{i^k y^k}{k!} \\ &= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!}}_{\cos y} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}}_{\sin y} \\ &= \cos y + i \sin y \end{aligned}$$

Taylor series of $\cosh y$ and $\sinh y$:

$$\cosh y = \frac{1}{2}\left(e^y + e^{-y}\right) \qquad \sinh y = \frac{1}{2}\left(e^y - e^{-y}\right)$$

$$\cosh y = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} \qquad \sinh y = \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!}$$

$\mathbb{R}^n$ and its Subsets

$\mathbb{R}^n$ is the set $\{(x_1, \dots, x_n) : x_j \in \mathbb{R}, 1 \le j \le n\}$. An element $x \in \mathbb{R}^n$ has the form $(x_1, \dots, x_n)$.

Let $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x \cdot y &= \sum_{j=1}^{n} x_j y_j \\ |x| &= \sqrt{x \cdot x} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2} \\ x + y &= (x_1 + y_1, \dots, x_n + y_n) \\ cx &= (cx_1, \dots, cx_n) \text{ for } c \in \mathbb{R} \end{aligned}$$

A **ball** of radius R centered at x_0 is $B(x_0, R) := \{x \mid |x - x_0| < R\}$. For set $A \subset \mathbb{R}^n$, x is a **boundary point** of A if $\exists$ a sequence $x_j \in A$ and $\tilde{x}_j \notin A$ such that both x_j and $\tilde{x}_j$ converge to x. The set of boundary points of A is ∂A.

A is **closed** if and only if the boundary of A is contained in A ($\partial A \subset A$). A is **open** if and only if A contains none of its boundary points ($\partial A \cap A = \emptyset$). The **closure** of A is $\overline{A} = A \cup \partial A$, which is the smallest closed set containing A. A is closed if and only if $A = \overline{A}$. The **interior** of A is $\operatorname{int}(A) = A \backslash \partial A$ (also written as A^0).

Example

1. Let $A = B(x_0, R) = \{x \mid |x - x_0| < R\}$. In this case, $\partial A = \{x \mid |x - x_0| = R\}$. This set is open since the set of points points $\{x \mid |x - x_0| = R\}$ has no elements in $A = B(x_0, R)$.
2. Let $A = \{x \mid 0 < |x - x_0| \le R\}$. In this case, $\partial A = \{x_0\} \cup \{x \mid |x - x_0| = R\}$. This set is neither open nor closed since although $\{x \mid |x - x_0| = R\}$ is contained in A, $\{x_0\}$ is not in A.
3. The closure of the ball $B(x_0, R)$ is $\overline{B(x_0, R)} = \{x \mid |x - x_0| \le R\}$.

A is **connected** if any 2 points in A can be connected by a continuous curve contained in A. A is **convex** if any $x, y \in A$, $tx + (1-t)y \in A$ where $0 \le t \le 1$ (the shortest path from x to y is contained in A). If A is convex, then it must be connected, but the converse is not true.

$\Omega \subset \mathbb{R}^n$ is a **domain** if it is open and connected. $K \subset \mathbb{R}^n$ is compact if K is closed and bounded, meaning $\exists x_0, R$ such that $K \subset B(x_0, R)$.

Differentiability

Let Ω be a domain in $\mathbb{R}^n$ and f be a function $f : \Omega \to \mathbb{R}$. The set of **continuous** functions in Ω is

$$C^0(\Omega) := \{f : \Omega \to \mathbb{R} \mid \text{ if } x_j, x \in \Omega \text{ exist such that } x_j \to x, \text{ then } f(x_j) \to f(x)\}.$$

If the derivative exists, the **partial derivative** is defined as

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, \dots, x_{j+h}, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}.$$

The set of continuously differentiable functions in Ω is

$$C^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \frac{\partial f}{\partial x_j} \in C^0(\Omega) \right\}.$$

The set of continuously m-times differentiable functions in Ω is

$$C^m(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \frac{\partial f}{\partial x_j} \in C^{m-1}(\Omega) \right\}.$$

The set of continuously infinitely differentiable functions (smooth functions) in Ω is

$$C^\infty(\Omega) := \bigcap_{m \ge 0} C^m(\Omega).$$

Clairaut's Theorem

Suppose all the second derivative exist and are continuous. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

The same is true for higher derivatives as long as everything is continuous.

Example

1. Let $f(x) = x^\alpha$ and $\Omega = (0, \infty)$. Find the values of α such that $f \in C^\infty(\Omega)$.

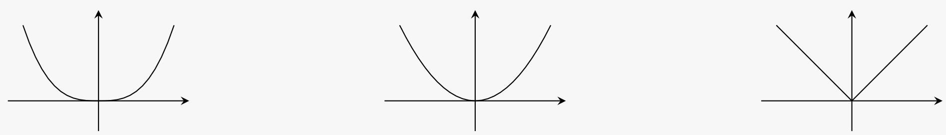
   We know that $f'(x) = \alpha x^{\alpha-1}$, so $f \in C^\infty(\Omega)$ for all values of α.

2. Let $f(x) = |x|^\alpha \in C^\infty(\Omega)$ where $\Omega = \mathbb{R}$. Find the values of α such that $f \in C^\infty(\Omega)$.

   For f to be in $\mathbb{C}^\infty(\Omega)$, α must be of the form $\alpha = 2n$ where $n \in \mathbb{Z}_{\geq 0}$. Otherwise, we will end up with some term with $|x|$ which is not differentiable.

3. Let $f(x) = |x|^{2k+1} \in C^n(\Omega)$. Find the values of n such that the statement is true.

   When $k = 0$, $f(x)$ is not differentiable and when $k = 1$, $f(x)$ is twice differentiable. So, $n = 2k$ since we always end up with $f^{(\alpha)}(x) = \beta|x|$, which is not differentiable.

   So, $f(x) = |x|^{2k+1}$ is $2k$ differentiable.

4. Let $f(x)$ be an even function. Since $f(-x) = f(x)$, $f(|x|) = f(x)$ meaning if $f(x) \in C^m(\Omega)$, then $f(|x|) \in C^m(\Omega)$. In particular, if $f(x) \in C^\infty(\Omega)$, then $f(|x|) \in C^\infty(\Omega)$.

Let $f \in C^m(\Omega)$ where $m \geq 0$. The **support** of f is

$$\operatorname{supp}(f) = \overline{\{x \mid f(x) \neq 0\}}$$

where as before, the line indicates the closure of the set.

Example

1. Let $m = 0$ and f be the function

   $$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

   Find $\operatorname{supp}(f)$.

   $$\begin{aligned} \operatorname{supp}(f) &= \overline{\{x \mid f(x) \neq 0\}} \\ &= \overline{\{x \mid |x < 1\}} \\ &= \{x \mid |x| \leq 1\} \end{aligned}$$

2. Let $\Omega = [0, \infty)$ and put

   $$f(x) = \sin\frac{1}{x}, \quad x > 0, \quad f(0) = 0.$$

   Find $\operatorname{supp}(f)$.

   $$\begin{aligned} \operatorname{supp}(f) &= \overline{\left\{x \;\middle|\; x > 0,\ x \neq \frac{1}{\pi k} \text{ where } k \in \mathbb{N}\right\}} \\ &= [0, \infty) \end{aligned}$$

The set of compact support functions is $C_c^\infty(\mathbb{R}^n)$ or $C_{\text{comp}}^\infty(\mathbb{R}^n)$, where

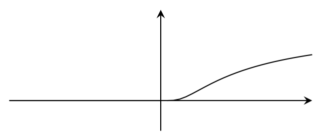
$$C_c^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp}(f) \text{ is compact}\}.$$

Recall that compact means closed and bounded. By definition, all supports are closed, but not all are bounded. A trivial example of one such function is $f(x) = 0$ for all values of x, so $\operatorname{supp}(f) = \emptyset$.

To obtain a nontrivial example we consider a function $h \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} h = [0, \infty)$:

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We can view the plot of this function.



To see that $h \in C^\infty(\mathbb{R})$ we first claim that

$$h^{(k)}(x) = \frac{q_k(x)}{x^{2k}} e^{-1/x}$$

for $x > 0$ where $q_k(x)$ is a polynomial of degree k. We can use proof by induction.

1. Base case: when $k = 0$, we have

$$\begin{aligned} h^{(0)}(x) &= e^{-1/x} \\ &= \frac{q_0(x)}{x^0} e^{-1/x} \end{aligned}$$

   where $q_0(x) = 1$.

2. Induction hypothesis: suppose $h^{(k)}(x) = \frac{q_k(x)}{x^{2k}} e^{-1/x}$ for some $k \geq 0$.

3. Inductive step: show that the formula holds for $k + 1$.

$$\begin{aligned} h^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q_k(x)}{x^{2k}} e^{-1/x}\right) \\ &= \frac{q_k'(x)}{x^{2k}} e^{-1/x} - \frac{2kq_k(x)}{x^{2k+1}} e^{-1/x} + \frac{q_k(x)}{x^{2k}x^2} e^{-1/x} \\ &= \frac{q_k'(x)x^2 - 2kq_k(x)x + q_k(x)}{x^{2k+2}} e^{-1/x} \\ &= \frac{q_{k+1}(x)}{x^{2(k+1)}} e^{-1/x} \end{aligned}$$

Now, to show that $h \in C^\infty(\mathbb{R})$, we must match the derivatives at $x = 0$. From the definition of h, we know that $h^{(k)}(x) = 0$ for $x < 0$, so for $x > 0$, we need $h^{(k)}(x) \to 0$ as $x \to 0$.

$q_k(x)$ is a polynomial of degree k, so for $|x| < 1$, it is bounded by a constant which we can show using the triangle inequality:

$$|q_k(x)| = |A_k x^k + \cdots + A_0| \leq |A_k| + \cdots + |A_0| \leq C_k,$$

for some constant C_k. We can assume $|x| < 1$ since x is approaching 0. So, we have the following bound for $h^{(k)}(x)$.

$$|h^{(k)}(x)| \leq \frac{|q_k(x)|}{x^{2k}} e^{-1/x} \leq \frac{C_k}{x^{2k}} e^{-1/x}$$

To show that this approaches 0, write it as $x^{-2k}e^{-1/x} = y^{2k}e^{-y}$ where $y = 1/x$. As $x \to 0$, $y \to \infty$.

$$e^y = \sum_0^\infty \frac{y^m}{m!} \geq \frac{y^m}{m!}$$

where $y > 0$. So, $e^{-y} \leq m! y^{-m}$.

$$x^{-2k}e^{-1/x} = y^{2k}e^{-y} \leq \frac{m!}{y^m} y^{2k} = m! y^{2k-m}$$

for $m > 2k$. This approaches 0 as $y \to \infty$. So, $C_k x^{-2k} e^{-1/x} \to 0$ as $x \to 0$. This proves that $h \in C^\infty(\mathbb{R})$.

Now, we can show that $f(x) = h(1 - |x|^2)$ also satisfies the required conditions. To show that $f \in C_c^\infty(\mathbb{R}^n)$, we must first show that $f \in C^\infty$. The chain rule can be used here.

Chain Rule

Let $F : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $F, h \in C^\infty$. We then define a new function $G(x) \in \mathbb{R}^n$ by $G(x) = h(F(x))$. Then

$$\frac{\partial}{\partial x_j} G(x) = \frac{\partial F(x)}{\partial x_j} h'(F(x))$$

Combined with the product rule, we can see that $G \in C^\infty$.

For higher order derivatives there is a more complicate Faà di Bruno formula. (We do not need to worry about it in this course but if you click here you will see it!).

For this case, we have $f(x) = h(g(x))$ where

$$\begin{aligned} g(x) &= 1 - |x|^2 \\ &= 1 - x_1^2 - x_2^2 - \cdots - x_n^2 \in C^\infty(\mathbb{R}) \end{aligned}$$

So, $f(x) \in C^\infty$.

Now, show that $\operatorname{supp}(f)$ is bounded. To do this, show that $f(x) = 0$ when $|x| > R$ for some R. But, $h(y) = 0$ when $y \leq 0$. $y = 1 - |x|^2 < 0$ when $x| > 1$, so $h(1 - |x|^2) = 0$ when $|x| > 1$.

Ordinary Differential Equations

We now review some facts about differential equations. The most important thing will be our *proof* of *local existence of solutions.*

For a function $y : t \mapsto y(t)$ where $y, t \in \mathbb{R}$, the general form of the scalar equation is

$$y^{(n)}(t) = F(t, y, y', \ldots, y^{(n-1)}).$$

Example

1. Let $y'' = y + y^3$. $F = F(y(t)) = y(t) + y(t)^3$.
2. Let $y' = \alpha y$. $F(y) = \alpha y$.
3. Let $y'' = -\omega^2 y + \beta y'$. $F(y, y') = -\omega^2 y + \beta y'$.

We can solve the euation $y'(t) = g(y(t))$ using integration:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(y) \implies \frac{\mathrm{d}y}{g(y)} = \mathrm{d}t \implies \int_{y(0)}^{y(t)} \frac{\mathrm{d}y}{g(y)} = \int_0^t \mathrm{d}t = t$$

The issue is that the last integral on the left may be difficult to evaluate, depending on g. However, we know that solutions exist!

First order ODEs

The general form for a first order ODE is $f \mapsto \underline{w}(t)$ where $t \in \mathbb{R}$ and $\underline{w} \in \mathbb{R}^n$. The notation $\underline{w}$ represents a vector.

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{w}(t) = \begin{pmatrix} w_1'(t) \\ \vdots \\ w_n'(t) \end{pmatrix} = G(t, \underline{w}(t))$$

or $\underline{w}'(t) = G(t, \underline{w}(t))$ where $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

The general form of an ODE of order n can be transformed into a first order ODE.

$$\begin{aligned} y^{(n)}(t) &= F(t, y, y', \dots, y^{(n-1)}) \\ \underline{w}(t) &:= (y(t), y'(t), \dots, y^{(n-1)}(t))^\intercal \in \mathbb{R}^n \text{ (transpose)} \\ \underline{w}'(t) &= (y'(t), y''(t), \dots, y^{(n)}(t))^\intercal \\ &= (w_2(t), w_3(t), \dots, F(t, w(t), \dots, w_n(t)))^\intercal \\ &=: G(t, \underline{w}(t)) \end{aligned}$$

Example

Let $y'' + \omega^2(y) = 0$, or $y'' = -\omega^2 y$.

$$\underline{w} = \begin{pmatrix} y \\ y' \end{pmatrix} \implies \underline{w}' = \begin{pmatrix} w_2 \\ -\omega^2 w_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{w}$$

First order ODEs can be written as integral equations.

$$\begin{aligned} \int_{t_0}^{t} \frac{\mathrm{d}\underline{w}}{\mathrm{d}\tau} &= \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \\ \underline{w}(t) - \underline{w}(t_0) &= \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \\ \underline{w}(t) &= \underline{w}(t_0) + \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \end{aligned}$$

This is an integral representation of $\underline{w}'(t) = G(t, \underline{w}(t))$.

Theorem: Picard's Theorem

Suppose we have the following conditions:

1. $(t, \underline{w}) \to G(t, \underline{w})$ is in $C^0((t_0 - c, t_0 + c) \times \Omega; \mathbb{R}^n)$ where $\Omega \subset \mathbb{R}^n$ is a domain (connected open set).
2. G is Lipschitz continuous, that is,

$$|G(t, \underline{w}) - G(t, \underline{v})| \leq M|\underline{w} - \underline{v}|, \quad |t - t_0| < c, \quad \underline{w}, \underline{v} \in \Omega.$$

   for some constant $M > 0$.

Then, $\forall \underline{w}_0 \in \Omega$, $\exists \varepsilon$ such that the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{w}(t) = G(t, \underline{w}(t)) \quad \underline{w}(t_0) = \underline{w}_0 \in \mathbb{R}^n$$

has a unique solution for t such that $|t - t_0| < \varepsilon$.

Note that Picard's Theorem does not solve the differential equation, it merely proves the *existence* and *uniqueness* of a solution.

Example

Let $w \in \mathbb{R}$ and $w' = 1 + w^2$. In this case $n = 1$ and $G(t, w) = 1 + w^2 = G(w)$ since G is not a function of t.

We want to show that this satisfies the condition for Picard's Theorem.

First, show that $w \mapsto 1 + w^2$ is continuous. This is intuitive since it is a polynomial.

Next, show that G is Lipschitz continuous.

$$\begin{aligned} |G(w) - G(v)| &= |w^2 - v^2| \\ &= |w + v||w - v| \\ &\leq (|w| + |v|)\,|w - v| \end{aligned}$$

Let $\Omega = (-R, R)$. If $w, v \in \Omega$, then $|w|, |v| \leq R/$ So, $|w| + |v| \leq 2R$.

$$|G(w) - G(v)| \leq M|v - w|$$

where $M = 2R$. So, the assumption of Picard's Theorem is satisfied as long as $|w|, |v| < R$.

This example is quite simple, so we can solve the differential equation exactly:

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= 1 + w^2 \\ \int_0^{w(t)} \frac{1}{1 + w^2}\,\mathrm{d}w &= \int_0^t \mathrm{d}t \\ \arctan w(t) &= t \\ w(t) &= \tan t \end{aligned}$$

By Picard's Theorem, this solution is unique. Note however that this solution blows up $t \to \pm\frac{\pi}{2}\mp$. There is no contradiction with the theorem: at some point we violate the condition that $|w| < R$ for any R!

Example

Let's apply Picard's iteration to the previous example: $w' = 1 + w^2$, $w(0) = 0$.

$$\begin{aligned} w_0 &= 0 \\ w_1(t) &= 0 + \int_0^t \underbrace{G(s, w_0(s))}_{1 + w_0(s)^2}\,\mathrm{d}s = \int_0^t 1\,\mathrm{d}s = t \\ w_2(t) &= 0 + \int_0^t (1 + s^2)\,\mathrm{d}s = t + \frac{t^3}{3} \\ w_3(t) &= 0 + \int_0^t \left(1 + \left(s + \frac{s^3}{3}\right)^2\right)\mathrm{d}s = \int_0^t \left(1 + s^2 + \frac{2s^4}{3} + \frac{s^6}{9}\right)\mathrm{d}s \\ &= t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63} \end{aligned}$$

We can continue this process for higher values of k. This is the Taylor expansion of $\tan t$.

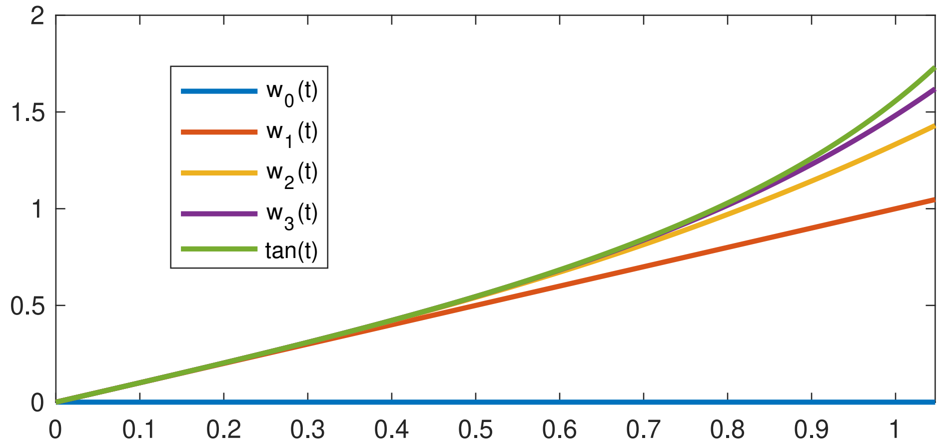


Figure 5: Plots of the iterates in the example $w' = 1 + w^2$, $w(0) = 0$. Although they look encouraging Picard iteration is not the way to solve ODE numerically.

Proving Picard's Theorem requires a method known as Picard's iteration.

$$\begin{aligned}
\underline{w}_0(t) &:= \underline{w}_0 \\
\underline{w}_1(t) &:= \underline{w}_0 + \int_{t_0}^{t} G(s, \underline{w}_0(s))\,\mathrm{d}s \\
&\vdots \\
\underline{w}_{k+1}(t) &:= \underline{w}_0 + \int_{t_0}^{t} G(s, \underline{w}_k(s))\,\mathrm{d}s \\
&\vdots ? \\
\underline{w}(t) &:= \underline{w}_0 + \int_{t_0}^{t} G(s, \underline{w}(s))\,\mathrm{d}s
\end{aligned}$$

But for this to work we need to show that there exists w such that as $k \to \infty$, $\underline{w}_k$ converges to $\underline{w}$. (That is why we put ? above.)
To prove that Picard's Theorem works, we must show that the interation above converges. Let $g : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$ be a function. We define

$$\|g\| := \sup_{|t-t_0| \le \varepsilon} |g(t)|.$$

(Note that $g(t) = (g_1(t), \cdots, g_n(t))^\intercal \in \mathbb{R}^n$ and $|g(t)| = (g_1(t)^2 + \cdots g_n(t)^2)^{\frac{1}{2}}$.)

If g and h are continuous functions from $[t_0 - \varepsilon, t_0 + \varepsilon]$ to $\mathbb{R}^n$, the distance between g and h is defined to be $\|g - h\|$. Why does Picard's iteration converge? Let sup represent $\sup_{|t-t_0| \le \varepsilon}$.

$$\begin{aligned}
\|\underline{w}_{k+1} - \underline{w}_k\| &= \sup \left| \int_{t_0}^{t} G(s, \underline{w}_k(s))\,\mathrm{d}s - \int_{t_0}^{t} G(s, \underline{w}_{k-1}(s))\,\mathrm{d}s \right| \\
&= \sup \left| \int_{t_0}^{t} G(s, \underline{w}_k(s)) - G(s, \underline{w}_{k-1}(s))\,\mathrm{d}s \right|
\end{aligned}$$

Recall that by triangle inequality, $\left|\int f(t)\,\mathrm{d}t\right| \le \int |f(t)|\,\mathrm{d}t$.

$$\|\underline{w}_{k+1} - \underline{w}_k\| \le \sup \int_{t_0}^{t} \left|G(s, \underline{w}_k(s)) - G(s, \underline{w}_{k-1}(s))\right|\,\mathrm{d}s$$

We know that $|G(s,v)_G(s,w)| \le M|v-w|$ for some constant $M>0$.

$$\|\underline{w}_{k+1}-\underline{w}_k\| \le \sup \int_{t_0}^{t} M \left|\underline{w}_k(s)-\underline{w}_{k-1}(s)\right| \mathrm{d}s$$

Since $|t-t_0|<\epsilon$ we note that $\sup \int_{t_0}^{t} |F(s)| \,\mathrm{d}s \le \varepsilon \sup F(t)|$ when we take $\sup_{|t-t_0|<\epsilon}$. Hence,

$$\begin{aligned}
\|\underline{w}_{k+1}-\underline{w}_k\| &\le \varepsilon \sup M \left|\underline{w}_k(t)-\underline{w}_{k-1}(t)\right| \\
&= \varepsilon M \|\underline{w}_k-\underline{w}_{k-1}\| \\
&\le \varepsilon M \left(\varepsilon M \|\underline{w}_{k-1}-\underline{w}_{k-2}\|\right) \\
&\vdots \\
&\le (\varepsilon M)^k \|\underline{w}_1-\underline{w}_0\| \\
&\le A(\varepsilon M)^k
\end{aligned}$$

where $A = \sup \int_{t_0}^{t} G(s,\underline{w}_0)\,\mathrm{d}s$ is constant. For $\varepsilon = \frac{1}{2M}$, we have $\|\underline{w}_{k+1}-\underline{w}_k\| \le A2^{-k}$.

$$\underline{w}(t) = \underline{w}_0 + \sum_{k=0}^{\infty} \underbrace{\left(\underline{w}_{k+1}(t)-\underline{w}_k(t)\right)}_{\le A2^{-k} \text{ when } |t-t_0|\le\varepsilon}$$

By the comparison test, the sum converges. So, we have a telescoping series:

$$\underline{w}_0 + \underline{w}_1 - \underline{w}_0 + \underline{w}_2 - \underline{w}_1 + \cdots + \underline{w}_{k+1} - \underline{w}_k = \underline{w}_{k+1}$$

So, this converges to $\underline{w}$.

Comment. We have shown that $\|\underline{w}_k - \underline{w}| \to 0$ which implies that $\underline{w}_k(t) \to \underline{w}(t)$ for every $t \in [t_0-\epsilon, t_0+\epsilon]$ (since $|\underline{w}_k(t)-\underline{w}(t)| \le \|\underline{w}_k-\underline{w}\|$). What we did *not* prove is that $\underline{w}(t)$ is a continuous function. This is however true whenever $\underline{w}_k(t)$'s are continuous and $\|\underline{w}_k-\underline{w}\| \to 0$.

Example

Using an example from earlier, let $y''+\omega y=0$. $y(0)=y_0$ and $y'(0)=y_1$.

$$\underline{w}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \qquad G(t,\underline{w}) = G(\underline{w}) = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \underline{w}$$

$$\begin{aligned}
\underline{w}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} & |G(v)-G(w)| &= |Av-Aw| \\
&= \begin{pmatrix} w_2(t) \\ -\omega^2 w_1(t) \end{pmatrix} & &= |A(v-w)| \\
&= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{w}(t) & &\le M|v-w|
\end{aligned}$$

So, we have the following:

$$\underline{w}_0(t) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \underline{w}_0, \qquad \underline{w}_1(t) = \underline{w}_0 + \int_0^t A\underline{w}_0 \,\mathrm{d}s = \underline{w}_0 + tA\underline{w}_0$$

We want to prove that $\underline{w}_{k+1}(t) = \sum_{j=0}^{k+1} \frac{t^j}{j!} A^j \underline{w}_0$. We can use induction to prove this.

1. Base case: this is shown above.

2. Induction hypothesis: suppose that for some $k \geq 0$, we have $\underline{w}_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!} A^j \underline{w}_0$.

3. Inductive step: show that the statement holds for $k+1$.

$$
\begin{aligned}
\underline{w}_{k+1}(t) &= \underline{w}_0 + \int_0^t A \underline{w}_k \,\mathrm{d}s \\
&= \underline{w}_0 + \int_0^t A \sum_{j=0}^{k} \frac{s^j}{j!} A^j \underline{w}_0 \,\mathrm{d}s \\
&= \underline{w}_0 + \sum_{j=0}^{k} \int_0^t \frac{s^j}{j!} A^{j+1} \underline{w}_0 \,\mathrm{d}s \\
&= \underline{w}_0 + \sum_{j=0}^{k} \frac{s^{j+1}}{(j+1)!} A^{j+1} \underline{w}_0 \\
&= \sum_{\ell=0}^{k+1} \frac{t^\ell}{\ell!} A^\ell \underline{w}_0 \text{ where } \ell = j+1
\end{aligned}
$$

So, the statement holds for $k+1$.

We have the following:

$$
\begin{aligned}
A^j &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}^j \\
&= \begin{cases} \begin{pmatrix} (-\omega^2)^\ell & 0 \\ 0 & (-\omega^2)^\ell \end{pmatrix} & j = 2\ell \\ \begin{pmatrix} 0 & (-\omega^2)^\ell \\ (-\omega^2)^{\ell+1} & 0 \end{pmatrix} & j = 2\ell + 1 \end{cases}
\end{aligned}
$$

So, $y(t) = \cos(t\omega) y_0 + \frac{1}{\omega} \sin(t\omega) y_1$.

Vector Calculus

Differentiation

Let $f : \Omega \to \mathbb{R}$ where Ω is a domain and $f \in C^1$. We define the **gradient** of f as

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^n$$

where $\nabla f \in C^0(\Omega; \mathbb{R}^n)$. Note that $\Omega; \mathbb{R}^n$ implies that ∇f is a function $\Omega \to \mathbb{R}^n$.

Let $\underline{v} : \Omega \to \mathbb{R}^n$ where $\underline{v}(x) = (v_w(x), \ldots, v_n(x))$. The **divergence** of $\underline{v}(x)$ is

$$\boldsymbol{\nabla} \cdot \underline{v}(x) = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} \in C^0(\Omega : \mathbb{R})$$

Note that the divergence of $\underline{v}(x)$ sends a vector to a scalar, while the gradient of f sends a scalar to a vector.

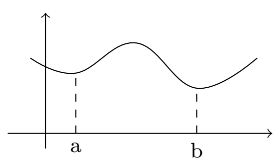
$$\begin{array}{ccc} f \in C^m(\Omega; \mathbb{R}) & \to & \nabla f(x) \in C^{m-1}(\Omega; \mathbb{R}^n) \\ \underline{v} \in C^m(\Omega; \mathbb{R}^n) & \to & \boldsymbol{\nabla} \cdot \underline{v} \in C^m(\Omega; \mathbb{R}) \end{array}$$

Note that we can set $m = \infty$ meaning $m - 1$ is also ∞. This is the case that we care about the most for the purpose of this class.

Integration

Let $f \in C^0([a, b]; \mathbb{R})$. The **Riemann integral** of f is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N \to \infty} \frac{1}{N} \sum_{j/N \in [a,b]} f\left(\frac{j}{N}\right), \quad j \in \mathbb{Z}, \quad \mathbb{Z} := \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$



For a more general $f \in C^0(\overline{\Omega}; \mathbb{R})$, the Riemann integral is

$$\int_\Omega f(x)\,\mathrm{d}^n x = \lim_{N \to \infty} \frac{1}{N^n} \sum_{\mathbf{j}/N \in \Omega} f\left(\frac{\mathbf{j}}{N}\right), \quad \mathbf{j} \in \mathbb{Z}^n := \{(j_1, \cdots, j_n) : j_k \in \mathbb{Z}\}.$$

where $\mathrm{d}^n x$ is the n dimensional element of integration $\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n$.

Change of Variables

Let Ω be a domain such that $F(\Omega) = \Omega'$. $f \in C^0(\Omega'; \mathbb{R})$ and $f \circ F(y) = f(F(y))$.

$$\int_{F(\Omega)} f(x)\,\mathrm{d}^n x = \int_\Omega f(F(y)) \left|\frac{\partial f}{\partial y}\right| \mathrm{d}y$$

where

$$\left|\frac{\partial F}{\partial y}\right| = \left|\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}\right|$$

where $F(x) = (F_1(x), \ldots, F_n(x))$. Note that the matrix that we calculate the determinant of is called the Jacobian.

Example

Let $u(t,x)$ represent the density at time t and position x where $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$.

$$V(t) = \int_\Omega u(t,x)\,\mathrm{d}^n x$$

where V is the mass in Ω at time t.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_\Omega \partial_t u(t,x)\,\mathrm{d}^n x$$

represents how mass changes over time.

Let $f \in C_c^\infty(\mathbb{R}^n)$, meaning $f \in C^\infty(\mathbb{R}^n)$ and $f = 0$ for $|x| > R$, for some R (this means that the support of f is *compact*.) Then put

$$\begin{aligned}
u(t,x) &= f(tx) \\
V(t) &= \int_{\mathbb{R}^n} f(tx)\,\mathrm{d}^n x \\
V'(t) &= \int_{\mathbb{R}^n} \partial_t[f(tx)]\,\mathrm{d}^n x
\end{aligned}$$

Apply the chain rule: $\partial_t[f(G(t,x))] = \left(\frac{\partial G_1}{\partial t}\frac{\partial f}{\partial x_1} + \cdots + \frac{\partial G_n}{\partial t}\frac{\partial f}{\partial x_n}\right)(G(t,x))$ where $G = (G_1, \dots, G_n)$ and $G_i(t,x) \in \mathbb{R}$.

$$\begin{aligned}
V'(t) &= \int_{\mathbb{R}^n} x \cdot \nabla f(tx)\,\mathrm{d}^n x \\
F(y) &= \frac{y}{t} \\
g(x) &= f(x,t) \\
V(t) &= \int_{\mathbb{R}^n} g(x)\,\mathrm{d}^n x \\
&= \int g(F(y)) \left|\frac{\partial F}{\partial y}\right| \mathrm{d}^n y \\
g(F(y)) &= f(tF(y)) = f(t(y/t)) = f(y) \\
\left|\frac{\partial F}{\partial y}\right| &= \left|\det \begin{bmatrix} \frac{1}{t} & 0 & \cdots & 0 \\ 0 & \frac{1}{t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{t} \end{bmatrix}\right| = t^{-n} \\
V(t) &= \int_{\mathbb{R}^n} f(y) t^{-n}\,\mathrm{d}^n y \\
V(t) &:= \int_{\mathbb{R}^n} f(tx)\,\mathrm{d}^n x = \frac{1}{t^n}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x \\
\frac{\mathrm{d}}{\mathrm{d}t} V(t) &= -nt^{-n-1}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x \\
&= \int_{\mathbb{R}^n} x \cdot \nabla f(tx)\,\mathrm{d}^n x
\end{aligned}$$

We can set $t = 1$, which yields

$$\int_{\mathbb{R}^n} x \cdot \nabla f(x)\,\mathrm{d}^n x = -n \int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x\,.$$

Divergence theorem

The divergence theorem relies on **surface integrals** and it relates integrals over different dimensions.

We parametrize a surface Σ by a function, σ defined on $U \in \mathbb{R}^{n-1}$, $\sigma : U \to \mathbb{R}^n$. Then $\Sigma = \sigma(U) \subset \mathbb{R}^n$ is an $n-1$ dimensional surface. We then want to integrate with respect to the *surface element of integration*, dS. For $n = 3$, we have

$$\int_\Sigma f \mathrm{d}S = \int_U f(\sigma(w))|\sigma_{w_1} \times \sigma_{w_2}| \,\mathrm{d}u_1 \,\mathrm{d}u_2$$

For the general case, we have a formula involving the *unit normal vector* ν: to any point of the surface we assign a vector which is normal to the surface. Then

$$\int_\Sigma f \,\mathrm{d}S = \int_U f(\sigma(w)) \left|\det\left(\sigma_{w_1}\middle|\sigma_{w_2}\middle|\cdots\middle|\sigma_{w_{n-1}}\middle|\nu\right)\right| \mathrm{d}^{n-1}w$$

Comment. You should not worry too much about this formula. What is important that you know that you can integrate over surfaces in any dimension.

Example

Let $\sigma(w) = (w, g(w))$ where $g : \mathbb{R}^{n-1} \to \mathbb{R}$. The unit normal vector is given by (this is a formula from Math53 when $n = 3$ and it looks exactly the same; note that $(\nabla g(w), -1)$ is a vector in $\mathbb{R}^n$)

$$\nu(w) = \frac{(\nabla g(w), -1)}{(1 + |\nabla g(w)|^2)^{1/2}}$$

Now, solve for the determinant:

$$\det \left[\begin{array}{cccc|c} 1 & 0 & 0 & \cdots & \frac{\partial g}{\partial w_1} \\ 0 & 1 & 0 & \cdots & \frac{\partial g}{\partial w_2} \\ 0 & 0 & 1 & \cdots & \frac{\partial g}{\partial w_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \frac{\partial g}{\partial w_1} & \frac{\partial g}{\partial w_2} & \cdots & \frac{\partial g}{\partial w_n} & -1 \end{array}\right] = \det \left[\begin{array}{c|c} \mathcal{I}_{n-1} & \nabla g^\intercal \\ \hline \nabla g & -1 \end{array}\right]$$

$$= \det \left[\begin{array}{c|c} \mathcal{I}_{n-1} & (v_1 \cdots v_{n-1})^\intercal \\ \hline v_1 \cdots v_{n-1} & -1 \end{array}\right] \text{ where } v_j = g w_j$$

$$= \det \begin{bmatrix} 1 & 0 & \cdots & v_1 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & v_{n-1} \\ 0 & 0 & \cdots & -1 - |v|^2 \end{bmatrix}$$

$$= 1 + |\nabla g|^2$$

$$\int_\Sigma f \,\mathrm{d}S = \int_U f(w, g(w)) \left(1 + |\nabla g(w)|^2\right)^{1/2} \mathrm{d}^{n-1}w$$

A special case of the surface integral is where

$$\Sigma = \left\{(u, g(u))\middle| u \in U, g \in C^1(\overline{U}, \mathbb{R})\right\}.$$

In this case, we have

$$\int_\Sigma f \mathrm{d}s = \int \left(1 + |\nabla g(u)|^2\right)^{1/2} f(u, g(u)) \,\mathrm{d}^{n-1}u\,.$$

To formulat the divergence theorem we need the concept of *piecewise C^1 surfaces*. These are surfaces which can be written as unions of surfaces defined using C^1 functions:

$$\Sigma = \bigcup_{k=1}^{K} \sigma_k(U_k), \quad U_k \subset \mathbb{R}^{k-1}, \quad \sigma_k \in C^1(\mathbb{R}^{n-1}; \mathbb{R}^n).$$

(Do not worry about this too much – the point is that we can built a surface from nice pieces but it can have kinks and corners.)

The Divergence Theorem

Let $\Omega \subset \mathbb{R}^n$ and suppose that the boundary of Ω, $\partial\Omega$ is a piecewise C^1 surface. Let $\underline{F} \in C^1(\overline{\Omega}; \mathbb{R}^n)$. Then

$$\int_\Omega \operatorname{div} \underline{F} \, \mathrm{d}^n x = \int_{\partial\Omega} \underline{\nu} \cdot \underline{F} \, \mathrm{d}s$$

where $\underline{\nu}$ is an outward pointing unit normal vector.

Unit Balls and Spheres

We can define the n dimensional **unit ball** as

$$B_n(0,1) = B_n := \{x \in \mathbb{R}^n | |x| < 1\}$$

and the n dimensional **unit sphere** as

$$\mathbb{S}^n(0,1) = \mathbb{S}^n := \partial B_{n+1}(0,1) = \left\{x \in \mathbb{R}^{n+1} \middle| |x| = 1\right\}.$$

We are interested in the volume of B_n, V_n and the surface area (in the sense of an n-dimensional surface in $\mathbb{R}^n$) of $\mathbb{S}^n$, , S_n.

We have

$$V_n = \int_{B_n} \mathrm{d}^n x \,, \qquad S_n = \int_{\mathbb{S}^n} \mathrm{d}S \,.$$

We first want to relate V_n and S_n. Calculate the first few values of each:

$$\begin{aligned} V_1 &= 2 & S_0 &= 2 \\ V_2 &= \pi & S_1 &= 2\pi \\ V_3 &= \frac{4\pi}{3} & S_2 &= 4\pi \end{aligned}$$

Proposition

$V_{n+1} = \frac{S_n}{n+1}$.

- **Proof**: we want to relate $\int_{B_{n+1}} \mathrm{d}^{n+1}x$ and $\int_{\mathbb{S}^n} \mathrm{d}S$.

$$\begin{aligned} F(x) &= (x_1, \ldots, x_{n+1}) \\ \operatorname{div} F(x) &= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_{n+1}}{\partial x_{n+1}} \text{ where } F_i = x_i \\ &= 1 + \cdots + 1 \\ &= n + 1 \end{aligned}$$

The normal unit vector at a point x on the unit sphere is equal to the vector defined by that point, so $\underline{\nu}(x) = x$.

$$\begin{aligned} F(x) \cdot x &= x^2 + \cdots + x_{n+1}^2 \\ &= 1 && |x| = (x_1^2 + \cdots + x_{n+1}^2)^{1/2} = 1 \\ \int_{B_{n+1}} \underbrace{\operatorname{div} F(x)}_{n+1} \mathrm{d}^{n+1}x &= \int_{\mathbb{S}^n} \underbrace{F \cdot \underline{\nu}}_{1} \, \mathrm{d}S \\ (n+1)V_{n+1} &= S_n \end{aligned}$$

So, $V_{n+1} = \frac{S_n}{n+1}$. □

Suppose we have a sphere of radius r. We define spheres of radius r.

$$\mathbb{S}_r^k = \left\{x \in \mathbb{R}^{k+1} \middle| |x| = r\right\} \qquad \int_{\mathbb{S}_r^k} 1 \, \mathrm{d}S = r^k S_k.$$

We will think of the *top hemisphere* $\mathbb{S}_n$ as a graph of $g(y) = (1 - |y|^2)^{1/2}$, $y = (x_1, \ldots, x_n)$. Since $x_1^2 + \cdots + x_{n+1}^2 = 1$, $x_{n+1} = (1 - x_1^2 - \cdots - x_n^2)^{1/2} = (1 - |y|^2)^{1/2}$. Since our equation is only for the top hemisphere, to get the area of $\mathbb{S}_r^{n-1}$ we must multiply by 2:

$$\begin{aligned} \int_{\mathbb{S}^n} \mathrm{d}S &= 2 \int_{B_n} \left(1 + |\nabla g(y)|^2\right) \mathrm{d}^n y \\ &= 2 \int_{B_n} \left(1 - |y|^2\right)^{-1/2} \mathrm{d}^n y \\ &= 2 \int_0^1 \int_{\mathbb{S}_r^{n-1}} (1 - r^2)^{-1/2} dS dr \\ &= 2 S_{n-1} I_n \\ I_n &= \int_0^1 (1 - r^2)^{-1/2} r^{n-1} \, \mathrm{d}r \end{aligned}$$

This is the analogue of polar coordinates: check it when $n = 1$ and $n = 2$:

Let $r = |y|$.

$$\int_{B_n} f(|y|) \, \mathrm{d}^n y = \int_0^1 f(r) \left(\int_{S_r^{n-1}} 1 \, \mathrm{d}S \right) \mathrm{d}r = \int_0^1 f(r) r^{n-1} S_{n-1} \, \mathrm{d}r \,, \qquad f(r) = (1 - r^2)^{-1/2}$$

Substitute: $r = \sin\theta$ so that $(1 - r^2)^{-1/2} dr = d\theta$:

$$I_n = \int_0^{\pi/2} \sin^{n-1} \theta \mathrm{d}\theta$$

When $n = 1$ and $n = 2$, we can easily solve for $I_1 = \pi/2$ and $I_2 = 1$. For $n > 2$, we use integration by parts.

$$\begin{aligned}
I_n &= \int_0^{\pi/2} \sin^{n-2}\theta(-\cos\theta)' \mathrm{d}\theta \\
&= (n-2)\int_0^{\pi/2} \sin^{n-3}\theta\cos^2\theta \mathrm{d}\theta \\
&= (n-2)\int_0^{\pi/2} \sin^{n-3}\theta(1-\sin^2\theta)\mathrm{d}\theta \\
&= (n-2)\underbrace{\int_0^{\pi/2} \sin^{n-3}\theta \mathrm{d}\theta}_{I_{n-2}} - (n-2)\underbrace{\int_0^{\pi/2} \sin^{n-1}\theta\, \mathrm{d}\theta}_{I_n} \\
&= (n-2)I_{n-2} - (n-2)I_n \\
I_n &= \frac{n-2}{n-1} I_{n-2}
\end{aligned}$$

We have solved for I_1 and I_2, so we can solve for some higher values of I_n:

$$\begin{aligned}
I_3 &= \frac{3-2}{3-1} I_1 = \frac{\pi}{4} \\
I_4 &= \frac{4-2}{4-1} I_2 = \frac{2}{3} \\
I_5 &= \frac{5-2}{5-1} I_3 = \frac{3\pi}{16} \\
I_6 &= \frac{6-2}{6-1} I_4 = \frac{8}{15}
\end{aligned}$$

Now, solve for values of S_n using $S_n = 2S_{n-1}I_n$ and V_n using $V_{n+1} = \frac{S_n}{n+1}$.

$$\begin{aligned}
S_0 &= 4 & V_1 &= 2 \\
S_1 &= 2\pi & V_2 &= \pi \\
S_2 &= 4\pi & V_3 &= \frac{4\pi}{3} \\
S_3 &= 2\pi^2 & V_4 &= \frac{\pi^2}{2} \\
S_4 &= \frac{8}{3}\pi^2 & V_5 &= \frac{8}{15}\pi^2 \\
S_5 &= \pi^3 & V_6 &= \frac{\pi^3}{6}
\end{aligned}$$

So, we have the general formula

$$V_{n+2} = \frac{2\pi}{n+2} V_n.$$

Observe the volumes look different for odd and even dimensions.

First Order Semilinear PDEs

First Order PDEs on $\mathbb{R}$

We will consider the following general class of equations:

$$\begin{aligned} \partial_t u(t,x) + v(t,x)\partial_x u(t,x) + w(t,x,u(t,x)) &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

for $t, x \in \mathbb{R}$ where $(t,x) \mapsto v(t,x)$ is given, $(t,x,u) \mapsto w(t,x,u)$ is given, and $u(t,x)$ is unknown. $u(0,x) = f(x)$ is the initial condition and is given. Semilinear refers to the fact that derivatives $\partial_t u$ and $\partial_x u$ appear linearly but u could appear non-linearly depending on w.

The simplest case of this PDE is if $v(t,x)$ is constant and $w(t,x,u) \equiv 0$. (This means that w is *identically* 0.)

This gives us the system of equations

$$\begin{aligned} \partial_t u + v\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

Proposition. The system of equations

$$\begin{aligned} \partial_t u + v\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

has unique solution

$$u(t,x) = f(x - tv).$$

- **Proof**: first, verify that $u(t,x) = f(x - tv)$ is a solution of our PDE. For that, compute the values of $\partial_t u$ and $\partial_x u$.

  Note that this requires use of the chain rule. For any function $f(x)$, if we want to evaluate $\partial_t f(g(t,x))$, we have $\partial_t f(g(t,x)) = f'(g(t,x)) \cdot \partial_t g(t,x)$.

$$\begin{aligned} \partial_t u &= f'(x - tv) \cdot \partial_t [x - tv] \\ &= -vf'(x - tv) \\ \partial_x u &= f'(x - tv) \cdot \partial_x [x - tv] \\ &= f'(x - tv) \\ \partial_t u + v\partial_x u &= 0 \end{aligned}$$

  So, $u(t,x) = f(x - tv)$ is a solution to the PDE.

  Now, verify that this is the only solution. We can use change of variables.

$$\begin{aligned} s &= t & t &= s \\ y &= x - vt & x &= y + vs \end{aligned}$$

  By the chain rule, we have

$$\begin{aligned} \partial_s &= \frac{\partial t}{\partial s}\frac{\partial}{\partial t} + \frac{\partial x}{\partial s}\frac{\partial}{\partial x} \\ &= \partial_t + v\partial x \end{aligned}$$

  So, we have

$$\begin{aligned} \partial_s u(s, y + vs) &= \partial_t u + v\partial_x u \\ &= 0 \end{aligned}$$

If we put $U(s,y) = u(s, y+vs)$, this means that $\partial_s U = 0$ and $U(0,y) = f(y)$. Since $\partial_s U = 0$, U does not change when s varies and $U(s,y) = f(y)$. This is the unique constant which solves the equation. But this means that $u(t,x) = U(t, x-tv) = f(x-tv)$. So, the only solution is $u(t,x) = f(x-tv)$. $\square$

Now, for a less simple case. Suppose $w = 0$ but v is no longer constant but depends on t. We are solving the system of equations:

$$\begin{aligned} \partial_t u + v(t)\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

Proposition

The system of equations $\partial_t u + v(t)\partial_x u = 0$ with initial condition $u(0,x) = f(x)$ has unique solution

$$u(t,x) = f(x - X(t))$$

where $X(t) = \int_0^t v(s)\mathrm{d}s$.

- **Proof**: first, show correctness of the solution $u(t,x) = f(x-X(t))$. Note that $X'(t) = v(t)$.

$$\begin{aligned} \partial_t u &= \partial_t\left[x - X(t)\right]\cdot f'(x-X(t)) \\ &= -X'(t)f'(x-X(t)) \\ &= -v(t)f'(x-X(t)) \end{aligned} \qquad\qquad \begin{aligned} \partial_x u &= \partial_x\left[x - X(t)\right]\cdot f'(x-X(t)) \\ &= f'(x-X(t)) \end{aligned}$$

So, $\partial_t u + v(t)\partial_x u = 0$ meaning $u(x,t) = f(x - X(t))$ is a solution to the PDE. Once again, to determine if the solution is unique, we can use a similar change of variables $s = t$ and $y = x - X(t)$. Note that for $t = s$, $X'(s) = v(s)$ and $X(0) = 0$. Here again is chain rule:

$$\begin{aligned} \partial_s = \frac{\partial t}{\partial s}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} &= \partial_t + X'(s)\partial_x = \partial_t + v(t)\partial_x \\ \partial_s u(s, y + X(s)) &= \partial_t u + v(t)\partial_x u = 0 \\ u(0, y + X(0)) &= f(y) \\ w(s,y) = u(s, y+X(s)) & \end{aligned}$$

$\partial_s w = 0$, so w is constant as s varies and that constant is given by $f(y)$: $w(s,y) = f(y)$ meaning the solution $u(t,x) = w(t, x - X(t)) = f(x - X(t))$ is unique. $\square$

Now for the general case:

$$\partial_t u + v(t,x)\partial_x u + w(t,x,u(t,x)) = 0$$

$u = u(t,x)$ is the unknown. We want to convert this into a (nonlinear) ODE. To achieve that we will consider the following ODE for "trajectories" of x:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t, x(t)).$$

The solution $t \mapsto x(t)$ is called a **characteristic**. We can think of it as a trajectory of the point $x(t)$ as time t varies and call it a *characteristic curve* (it really becomes a curve if x is in higher dimensions).

When u is restricted to the characteristic curve the derivative in t is called the **Lagrangian derivative**:

$$\frac{\mathrm{D}u}{\mathrm{D}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{u(t,x(t))}_{h(t)} = h'(t), \quad h(t) := u(t,x(t)).$$

Theorem

On each characteristic $t \mapsto x(t)$ which solves $\frac{\mathrm{d}x}{\mathrm{d}t} = v(t,x)$, the 1^{st} order PDE

$$\frac{\partial u}{\partial t} + v(t,x)\frac{\partial u}{\partial x} + w(t,x,u(t,x)) = 0$$

reduces to an ODE

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \widetilde{w} = 0 \iff \frac{d}{dt}h(t) + \widetilde{w}(t) = 0, \quad h(t) := u(t,x(t))$$

where $\widetilde{w} := w(t,x(t),u(t,x(t)))$ is a function of t only.

- **Proof**: apply the chain rule where $x = x(t)$.

$$\begin{aligned} \frac{\mathrm{D}u}{\mathrm{D}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t,x(t))\right] \\ &= \partial_t u + x'(t)\partial_x u \\ &= \partial_t u + v(t,x)\partial_x u \end{aligned}$$

Substitute this back into the original equation.

$$\begin{aligned} \partial_t u + v(t,x)\partial_x u + w(t,x,u(t,x)) &= 0 \\ \frac{\mathrm{D}u}{\mathrm{D}t} + w(t,x,u(t,x)) &= 0 \end{aligned}$$

So, for $\widetilde{w} = w(t,x,u(t,x(t)))$ and $x = x(t)$, we have $\frac{\mathrm{D}u}{\mathrm{D}t} + \widetilde{w} = 0$. $\square$

Outline of Steps for solving 1st order semilinear PDE

There are 3 main steps to solving a first order PDE of the form

$$\begin{aligned} \frac{\partial u}{\partial t} + v(t,x)\frac{\partial u}{\partial x} + w(t,x,u(t,x)) &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

1. Solve $x'(t) = v(t,x(t))$ for characteristic $x(t)$.
2. Solve $h'(t) + w(t,x(t),h(t)) = 0$ for $h(t)$.
3. Match $u(t,x(t)) = h(t)$ and $h(0) = f(x(0))$ where $x(t) = x$. Solve for any constants of integration and plug it back in to $u(t,x) = u(t,x(t)) = h(t)$.

Example

Let $v = a + bx$ and $w = bu$ where $a, b > 0$ and $x \geq 0$. We want to solve the following equation with the condition at $x = 0$ (rather than $t = 0$):

$$\begin{aligned} \partial_t u + (a + bx)\partial_x u + bu = 0 \\ u(t, 0) = g(t) \end{aligned}$$

First, solve for the characteristic.

$$\begin{aligned} x'(t) &= v(t, x) \\ &= ax(t) + b \\ \int \frac{1}{a + bx} \mathrm{d}x &= \int \mathrm{d}t \\ \frac{1}{b} \ln(a + bx) &= t + C \\ x(t) &= \frac{1}{b}\left(e^{b(t+C)} - a\right) \end{aligned}$$

As initial condition take $x(t_0) = 0$ at some t_0 (remember – we are starting at $x = 0$). This yields $C = -t_0 - \frac{\ln(-a)}{b}$, so the equation can be rewritten as

$$x(t) = \frac{a}{b}\left(e^{b(t-t_0)} - 1\right).$$

From the theorem we know that $(d/dt)[u(t, x(t))] + bu(t, x(t)) = 0$ and $u(t_0, x(t_0)) = g(t_0)$. This gives,

$$u(t, x(t)) = g(t_0)e^{-b(t-t_0)}.$$

Using our equation for $x(t)$ from above:

$$u\left(t, \frac{a}{b}\left(e^{b(t-t_0)} - 1\right)\right) = g(t_0)e^{-b(t-t_0)}$$

Solve for t_0 in terms of t and x from the equation of $x(t)$ using $x = x(t)$.

$$\frac{a}{b}\left(e^{b(t-t_0)} - 1\right) \Longrightarrow t_0 = t + \frac{1}{b}\ln\left(\frac{a}{a + bx}\right)$$

So, our final solution for $u(t, x)$ is

$$u(t, x) = \frac{a}{a + bx} g\left(t + \frac{1}{b}\ln\left(\frac{a}{a + bx}\right)\right).$$

As always, you should double check and make sure that your answer solves the equation!

Example

Solve the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \gamma u$$
$$u(x,0) = f(x)$$

where c and γ are constant. Note that this is a linear PDE since if we have $\frac{\partial u_j}{\partial t} + c\frac{\partial u_j}{\partial x} = \gamma u_j$ for $j = 1, 2$, $u = u_1 + u_2$ then is also a solution.
Here, we have $v(t, x) = c$ and $w(t, x, u(t, x)) = -\gamma u$. Follow the steps outlined above.

1. Solve $x'(t) = v(t, x(t))$.

$$x'(t) = v(x,t) = c$$
$$x(t) = ct + A$$

2. Solve $h'(t) + w(t, x(t), h(t)) = 0$.

$$h'(t) + w(t, x(t), h(t)) = h'(t) - \gamma h(t) = 0$$
$$h(t) = Be^{\gamma t}$$

3. Match $u(t, x(t)) = h(t)$ and $h(0) = f(x(0))$ where $x(t) = x$.

$$u(t, x(t)) = h(t)$$
$$u(t, \underbrace{ct + A}_{x}) = Be^{\gamma t}$$
$$x = ct + A$$
$$A = x - ct$$

Use the initial condition.

$$h(0) = g(x(0))$$
$$B = g(A) = f(x - ct)$$

So, we have our final answer:

$$u(t, x) = g(x - ct)e^{\gamma t}$$

Check that our answer satisfies the original equation and initial condition:

$$\partial_t u = -cg'(x - ct)e^{\gamma t} + \gamma g(x - ct)e^{\gamma t}$$
$$c\partial_x u = cg'(x - ct)e^{\gamma t}$$
$$\partial_t u + c\partial_x u = \gamma g(x - ct)e^{\gamma t} = \gamma u \qquad \checkmark$$
$$u(0, x) = g(x - c(0))e^{\gamma \cdot 0} = g(x) \qquad \checkmark$$

So, our solution $u(t, x) = g(x - ct)e^{\gamma t}$ is a solution to the PDE.

Example

Solve the PDE

$$\frac{\partial u}{\partial t} + 2t\frac{\partial u}{\partial x} + u^2 = 0$$
$$u(0,x) = g(x)$$

Note that this is a nonlinear PDE since $w(t,x,u) = u^2$. Here, $v(t,x) = 2t$. Follow the same steps:

1. Solve $x'(t) = v(t, x(t))$.
$$x'(t) = 2t \Longrightarrow x(t) = t^2 + A$$
2. Solve $h'(t) + w(t, x(t), h(t)) = 0$.
$$\begin{aligned} h'(t) + h(t)^2 &= 0 \\ \frac{1}{h^2}\mathrm{d}h &= -\mathrm{d}t \\ -\frac{1}{h} = -t - B &\Longrightarrow h(t) = \frac{1}{t+B} \end{aligned}$$
3. Match $u(t, x(t)) = h(t)$ and $h(0) = f(0)$ where $x(t) = x$.
$$\begin{aligned} u(t, t^2 + A) &= \frac{1}{t+B} \\ x &= t^2 + A \\ A &= x - t^2 \end{aligned}$$
Use the initial condition.
$$\begin{aligned} h(0) &= g(x(0)) \\ \frac{1}{B} = g(A) &\Longrightarrow B = \frac{1}{g(x-t^2)} \end{aligned}$$

So, we have our final answer:

$$\begin{aligned} u(t,x) &= \frac{1}{t + \frac{1}{g(x-t^2)}} \\ &= \frac{g(x-t^2)}{1 + tg(x-t^2)} \end{aligned}$$

Check that our answer satisfies the original equation and initial condition. Here, let $g = g(x - t^2)$ and $g' = g'(x - t^2)$.

$$\begin{aligned} \partial_t u &= \frac{-2tg' \cdot (1+tg) - g \cdot (g - 2t^2 g')}{(1+tg)^2} \\ &= \frac{-2tg' - g^2}{(1+tg)^2} \\ \partial_x u &= \frac{g' \cdot (1+tg) - g \cdot (tg')}{(1+tg)^2} \\ &= \frac{g'}{(1+tg)^2} \\ \partial_t u + 2t\partial_x u + u^2 &= \frac{-tg' - g^2}{(1+tg)^2} + \frac{2tg'}{(1-tg)^2} + \frac{g'}{(1+tg)^2} = 0 && \checkmark \\ u(0,x) &= \frac{g(x)}{1+0} = g(x) && \checkmark \end{aligned}$$

So, our solution $u(t,x) = \frac{g(x-t^2)}{1+tg(x-t^2)}$ is a solution to the PDE.

Higher Dimension PDEs

A general PDE of dimension n is

$$\begin{aligned}\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w = 0 \\ u(0, \underline{x}) = f(\underline{x})\end{aligned}$$

where u is unknown, ∇u is in terms of x, and $\underline{v}, w$ are given:

$$\begin{aligned}
&u : [0,T] \times \Omega \to \mathbb{R}, && \Omega \in \mathbb{R}^n \\
&\underline{v} : [0,T] \times \Omega \to \mathbb{R}^n, && \underline{v} = \underline{v}(t, \underline{x}) \\
&w : \underbrace{[0,T]}_{t} \times \underbrace{\Omega}_{\underline{x}} \times \underbrace{\mathbb{R}}_{u} \to \mathbb{R}, && w = w(t, \underline{x}, u)
\end{aligned}$$

Here $\Omega \subset \mathbb{R}^n$ is a domain, that is an open and connected subset of $\mathbb{R}^n$.

The steps for solving higher dimensional PDEs are exactly the same as solving a first order PDE. We start by solving the equation for characteristic $t \mapsto \underline{x}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{x}(t) = \underline{v}(t, \underline{x}(t))$$

where $t \mapsto \underline{x}(t)$ is a **characteristic**. The Lagrangean derivative is again defined as

$$\frac{\mathrm{D}u}{\mathrm{D}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{u(t, \underline{x}(t))}_{h(t)} = h'(t)$$

Theorem

If $u \in C^1([0,T] \times \Omega; \mathbb{R})$ solves the PDE

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w = 0,$$

then $h(t) := u(t, \underline{x}(t))$ solves $h'(t) + w(t, \underline{x}(t), h(t)) = 0$.

- **Proof**: the proof of this is the same as that for first order PDEs.

$$\begin{aligned}
h'(t) + w(t, \underline{x}(t), h(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}[u(t, \underline{x}(t))] + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= \frac{\partial u}{\partial t} + \frac{\mathrm{d}\underline{x}}{\mathrm{d}t} \cdot \nabla u + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= \frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= 0
\end{aligned}$$

since by assumption, this u solves the PDE. □

Outline of Steps

Like with first dimension PDEs, there are 3 main steps to solving a higher dimension first order PDE of the form

$$\begin{aligned}\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w(t, \underline{x}, u) = 0 \\ u(0, \underline{x}) = f(\underline{x})\end{aligned}$$

1. Solve $\underline{x}'(t) = \underline{v}(t, \underline{x}(t))$ for characteristic $\underline{x}(t)$.

2. Solve $h'(t) + w(t, \underline{x}(t), h(t)) = 0$ for $h(t)$.
3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$. Solve for any constants of integration and plug it back in to $u(t, \underline{x}) = w(t, \underline{x}(t)) = h(t)$.

Example

Let $\Omega = \mathbb{R} \times (-1, 1)$. Solve the PDE

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u = 0$$
$$u(0, \underline{x}) = f(\underline{x})$$

where $\underline{x} = (x_1, x_2)$ and $\underline{v}(t, \underline{x}) = (1 - x_2^2, 0)$. Note that this is a linear PDE. Follow the same steps as the first order PDEs when x is in one dimension:

1. Solve $\underline{x}'(t) = \underline{v}(t, \underline{x}(t))$. Let $\underline{x}(t) = (x_1(t), x_2(t))$.

$$\begin{aligned} \underline{x}'(t) &= (x_1'(t), x_2'(t)) \\ &= \underline{v} = (1 - x_2^2, 0) \\ x_1'(t) &= 1 - x_2(t)^2 \\ x_2'(t) &= 0 \\ x_2(t) &= A_2 \end{aligned}$$

   Use this to solve for $x_1(t)$.

$$\begin{aligned} x_1'(t) &= 1 - A_2^2 \\ x_1(t) &= A_1 + (1 - A_2^2)t \end{aligned}$$

   So, we have

$$\underline{x}(t) = (x_1(t), x_2(t)) = (A_1 + (1 - A_2^2)t, A_2)$$

2. Solve $h'(t) + w(t, \underline{x}(t), h(t)) = 0$.

$$\begin{aligned} h'(t) + 0 &= 0 \\ h(t) &= B \end{aligned}$$

3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.

$$\begin{aligned} u(t, \underline{x}(t)) &= B \\ \underline{x} &= (x_1, x_2) = (A_1 + (1 - A_2^2)t, A_2) \\ x_1 &= A_1 + (1 - A_2^2)t \\ &= A_1 + (1 - x_2^2)t \\ A_2 &= x_2 \\ A_1 &= x_1 - (1 - x_2^2)t \end{aligned}$$

   Use the initial condition.

$$\begin{aligned} h(0) &= B = f(\underline{x}(0)) \\ \underline{x}(t) &= (A_1 + (1 - A_2^2)t, A_2) \\ \underline{x}(0) &= (A_1, A_2) \\ B &= f(A_1, A_2) \\ &= f(x_1 - (1 - x_2^2)t, x_2) \end{aligned}$$

So, we have our final answer:

$$u(t,\underline{x}) = f(x_1 - (1-x_2^2)t, x_2)$$

Check that our answer satisfies the original equation and initial condition.

$$\begin{aligned}
\partial_t u &= -(1-x_2^2)\cdot f_{x_1}(x_1-(1-x_2^2)t, x_2) \\
\underline{v}\cdot\nabla u &= (1-x_2^2, 0)\cdot\nabla u \\
&= (1-x_2^2)\cdot\partial_{x_1}u \\
&= (1-x_2^2)f_{x_1}(x_1(1-x_2^2)t, x_2)\partial_t u + \underline{v}\cdot\nabla u = 0 && \checkmark \\
u(0,x) &= f(x_1-(1-x_2^2)\cdot 0, x_2) \\
&= f(x_1,x_2) = f(\underline{x}) && \checkmark
\end{aligned}$$

So, our solution $u(t,\underline{x}) = f(x_1-(1-x_2^2)t, x_2)$ is a solution to the PDE.

Example

Solve the PDE

$$x_1\frac{\partial u}{\partial x_1} - 2x_2\frac{\partial u}{\partial x_2} - u^2 = 0, \quad \Omega := \{(x_1,x_2) : x_j > 0,\ j=1,2\},$$
$$u(y,y) = y^3, \quad y>0.$$

Here $\underline{x} = (x_1,x_2)$. Also, $y = x_1 = x_2$ if $\underline{x}$ falls on the line $x_1 = x_2$ and then $u(\underline{x}) = y^3$. Note that in this problem we do not have dependence of $u = u(x)$ on t. This can be rewritten as

$$(x_1, -2x_2)\cdot\nabla u - u^2 = 0$$

where $\underline{v}(t,\underline{x}) = (x_1,-2x_2)$ and $w(t,\underline{x},u) = -u^2$. Follow the steps outlined previously.

1. Solve $\underline{x}'(t) = (x_1(t), x_2(t)) = \underline{v}(t,\underline{x}(t))$.

$$\begin{aligned}
\underline{x}'(t) &= (x_1'(t), x_2'(t)) = \underline{v}(t,\underline{x}(t)) = (x_1(t), -2x_2(t)) \\
x_1'(t) &= x_1(t) \implies x_1(t) = A_1e^t \\
x_2'(t) &= -2x_2(t) \implies x_2(t) = A_2e^{-2t}
\end{aligned}$$

2. Solve $h'(t) + w(t,\underline{x}(t),h(t)) = 0$ for $h(t)$.

$$\begin{aligned}
h'(t) - h(t)^2 &= 0 \\
\frac{1}{h^2}\mathrm{d}h &= \mathrm{d}t \\
-\frac{1}{h} &= t - B \\
h(t) &= \frac{1}{B-t}
\end{aligned}$$

3. Match $u(t,\underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.

$$\begin{aligned}
u(\underline{x}(t)) &= \frac{1}{B-t} \\
\underline{x}(t) &= (x_1(t), x_2(t)) \\
&= (A_1e^t, A_2e^{-2t})
\end{aligned}$$

Use the initial condition $u(y, y) = y^3$. Set $A_1 = A_2 = y$.

$$\begin{aligned}
\underline{x}(t) &= (ye^t, ye^{-2t}) \\
h(0) &= \frac{1}{B} = y^3 \\
B &= \frac{1}{y^3} \\
u(x(t)) &= \frac{1}{\frac{1}{y^3} - t} = \frac{y^3}{1 - y^3 t} \\
x_1 &= x_1(t) = ye^t \\
x_2 &= x_2(t) = ye^{-2t}
\end{aligned}$$

We do not want y in our final answer, so solve y and t in terms $\underline{x}$:

$$\begin{aligned}
y &= x_1 e^{-t} = x_2 e^{2t} \\
x_2 &= x_1 e^{-3t} \\
e^{3t} &= \frac{x_1}{x_2}
\end{aligned}$$

This gives:

$$\begin{aligned}
t &= \frac{1}{3} \ln\left(\frac{x_1}{x_2}\right) \\
y^3 &= e^{-3t} x_1^3 = x_1^2 x_2 \\
u(\underline{x}) &= \frac{y^3}{1 - y^3 t} = \frac{x_1^2 x_2}{1 - \frac{1}{3}\ln\left(\frac{x_1}{x_2}\right) x_1^2 x_2}
\end{aligned}$$

So, our final answer is $u(t, \underline{x}) = u(\underline{x}) = \frac{x_1^2 x_2}{1 - \frac{1}{3}\ln\left(\frac{x_1}{x_2}\right) x_1^2 x_2}$. We will not check this answer since it will take longer than solving the problem!

Classifying 1st order linear PDEs

Suppose we have the following PDE:

$$Lu = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + Fu = 0$$

where $F = f(x, t)$ and $v = v(x, t)$. This is a **1$^{\text{st}}$ order linear homogeneous PDE**.

- **1$^{\text{st}}$ order**: the highest order derivative is of the 1$^{\text{st}}$ order.
- **linear**: solutions can be added together to form another solution:

$$\begin{aligned}
L(u_1, u_2) &= Lu_1 + Lu_2 \\
\frac{\partial(u_1 + u_2)}{\partial t} + v\frac{\partial(u_1 + u_2)}{\partial x} + F(u_1 + u_2) &= \left(\frac{\partial u_1}{\partial t} + v\frac{\partial u_1}{\partial x} + Fu_1\right) + \left(\frac{\partial u_2}{\partial t} + v\frac{\partial u_2}{\partial x} + Fu_2\right)
\end{aligned}$$

- **homogeneous**: if u_1 and u_2 are solutions of Lu, then $Lu_1 = Lu_2 = 0$ meaning $L(u_1 + u_2) = 0$.
- **PDE**: the equation uses partial differential equations.

If $Lu = f \neq 0$, then it is a 1$^{\text{st}}$ order linear **nonhomogeneous** PDE.

If u_1 and u_2 solve $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + w(t, x, u) = 0$ but $w(t, x, u_1 + u_2) \neq w(t, x, u_1) + w(t, x, u_2)$, then it is a 1$^{\text{st}}$ order **semilinear** PDE (linear in w, but not outside w).

First Order Quasilinear PDEs

Quasilinear 1^{st} order PDEs are of the form

$$\begin{aligned} \partial_t u + \underline{a}(u) \cdot \nabla u &= 0 \\ u(0, x) &= f(z) \end{aligned}$$

where $u = u(t, x)$ is unknown, ∇u is in terms of x, and $\underline{a} \in C^1(\mathbb{R}; \mathbb{R}^n)$ is a given function of u.

Theorem

Suppose $u \in C^1([0, t] \times \Omega; \mathbb{R})$ solves the PDE

$$\begin{aligned} \partial_t u + \underline{a}(u) \cdot \nabla u &= 0 \\ u(0, x) &= f(z) \end{aligned}$$

Then, $\forall x_0 \in \Omega$, u is constant along the characteristics of the system given by

$$\underline{x}(t) = \underline{x}_0 + \underline{a}(u(0, \underline{x}_0))t.$$

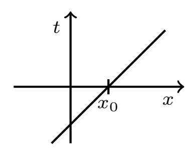
This is equivalent to the statement $u(t, \underline{x}_0 + t\underline{a}(f(\underline{x}_0))) = f(\underline{x}_0)$.

So, if we have a solution to the system, the solution is constant on the line

$$\begin{aligned} \underline{x}(t) &= \underline{x}_0 + \underline{a}(u(0, \underline{x}_0))t \\ &= \underline{x}_0 + \underline{a}_0(f(\underline{x}_0))t \end{aligned}$$

The theorem states that $u(t, \underline{x}_0 + t\underline{a}(f(\underline{x}_0))) = f(\underline{x}_0)$. The line can be seen in the plot in the case when $n = 1$.

When $n = 1$ (in which case $\underline{a} = a$ and $\underline{x}_0 = x$ are scalars!), the slope of the line is $1/a(f(x_0)$ and the equation is given by $x = x_0 + ta(f(x_0))$.



Example (this is a repeat of the linear case but with this perspective)

Let $\underline{a} = (a_1, \ldots, a_n)$ be a constant vector. Solve the PDE

$$\begin{aligned} \partial_t u + \underline{a} \cdot \nabla u &= 0 \\ u(0, \underline{x}) &= f(\underline{x}) \end{aligned}$$

This is a simple case that follows the steps outline previously for semilinear PDEs.

1. Solve $\underline{x}'(t) = \underline{a}(u(t, \underline{x}(t))$.
$$\underline{x}'(t) = \underline{a} \Longrightarrow \underline{x}(t) = \underline{a}t + A$$
2. Solve $h'(t) = w(t, \underline{x}(t), h(t))$ for $h(t)$. In this form of quasilinear PDEs, $w(t, x, u) = 0$.
$$h'(t) = 0 \Longrightarrow h(t) = B$$
3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.
$$\begin{aligned} u(t, \underline{x}(t)) &= B \\ \underline{x} &= \underline{a}t + A \\ h(0) &= B = f(\underline{x}(0)) \\ \underline{x}(t) &= \underline{a}t + A \\ \underline{x}(0) &= A = \underline{x} - \underline{a}t \\ B &= f(\underline{x} - \underline{a}t) \end{aligned}$$

So, we have our final answer:

$$u(t, \underline{x}) = f(\underline{x} - \underline{a}t)$$

- **Proof of Theorem**: We prove the theorem from the beginning of this section. Assuming that $u(t, x)$ solves the equation, consider first the following ODE: $\underline{x}'(t) = \underline{a}(u(t, \underline{x}(t)))$ where $\underline{x}(0) = \underline{x}_0$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(t, \underline{x}(t)) &= \partial_t u(t, \underline{x}(t)) + \underline{a}(u(t, \underline{x})) \cdot \nabla u(t, \underline{x}(t)) \\ &= \partial_t u(t, \underline{x}(t)) + \underline{a}(u(t, \underline{x}(t))) \cdot \nabla u(t, \underline{x}(t)) \\ &= 0\end{aligned}$$

by assumption that u is a solution to the PDE. This implies

$$u(t, \underline{x}(t)) = u(0, \underline{x}(0)) = u(0, \underline{x}_0)$$

is constant. Back to the ODE $\underline{x}'(t) = \underline{a}(u(t, \underline{x}(t))$ where $\underline{x}(0) = \underline{x}_0$, we see that we can replace $u(t, \underline{x}(t))$ by $u(0, \underline{x}(0)) = u(0, \underline{x}_0)$:

$$\begin{aligned}\underline{x}'(t) &= \underline{a}(u(0, \underline{x}_0)) \\ &= a(f(\underline{x}_0)) \\ x(t) &= \underline{x}_0 + \underline{a}(f(\underline{x}_0))t \\ \Rightarrow \quad u(t, \underline{x}_0 + a(f(\underline{x}_0))t) &= f(x_0)\end{aligned}$$

So, we have $u(t, \underline{x}_0 + \underline{a}(f(\underline{x}_0))t) = f(\underline{x}_0)$. $\square$

Notable Examples

Burger's Equation

Burger's equation is a simple quasilinear PDE related to models of gas dynamics. We have $n = 1$ and $\underline{a}(u) = u$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Let us first consider a case of a simple initial condition.

$$u(0, x) = ax + b,$$

where a and b are constants. The theorem says that the solution has to satisfy $u(t, x(t)) = f(x_0)$ where

$$x(t) = x_0 + t(ax_0 + b)$$

Solve for x_0.

$$\begin{aligned}x &= x_0 + t(ax_0 + b) \\ x_0 &= \frac{x - tb}{1 + ta} \\ u(t, x) &= f\left(\frac{x - tb}{1 + ta}\right) \\ &= a\frac{x - tb}{1 + ta} + b \\ &= \frac{ax - tab + b + tab}{1 + ta} \\ &= \frac{ax + b}{1 + ta}\end{aligned}$$

If $a < 0$, then the solution exists when $t < -\frac{1}{a}$. If $a > 0$, the solution exists for all $t > 0$.

Now for an arbitrary initial condition, we have

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0 \\ u(0,x) &= f(x) \\ u(t, y + tf(y)) &= f(y) \end{aligned}$$

We want to solve $x = y + tf(y)$ for y with t, x given.

This cannot be done explicitly for most functions u. However, we can do it using

The Implicit Function Theorem Suppose that

$$\begin{aligned} F(x,y,t) &= 0 \\ F(x_0,y_0,t_0) &= 0 \end{aligned}$$

If $\frac{\partial}{\partial y}F(x_0,y_0,t_0) \neq 0$ then can we find y as a function of x and t, defined for x and t *near* x_0 and t_0 such that

$$F(x, y(t,x), t) = 0, \quad y(t_0, x_0) = y_0.$$

In addition

$$\begin{aligned} \frac{\partial y}{\partial t} &= -\frac{F_t(x_0,y_0,t_0)}{F_y(x_0,y_0,t_0)} \\ \frac{\partial y}{\partial x} &= -\frac{F_x(x_0,y_0,t_0)}{F_y(x_0,y_0,t_0)} \end{aligned}$$

So, we have $F(x,y,t) = y + tf(y) - x = 0$ where $x_0 = x$, $y_0 = x$, and $t_0 = 0$.

$$\frac{\partial}{\partial y}F(x_0,y_0,t_0) = \frac{\partial}{\partial y}F(x,x,0) = 1 + 0f'(y) = 1 \neq 0.$$

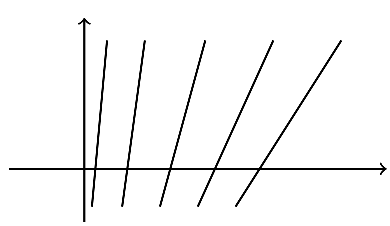
This means that we can find $y = y(x,t)$ such that for small t, $y + tf(y) = x$.

Hence,

$$u(t,x) = u(t, \underbrace{y + tf(y)}_{x}) = f(y) = f(y(t,x)).$$

If we want to go to longer times we we need $\partial_y F = 1 + tf'(y) \neq 0$.

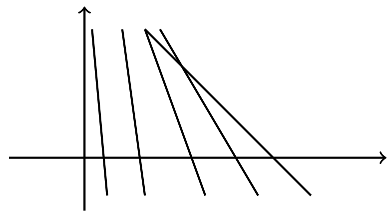
If $f' \geq 0$, f is non decreasing meaning the slope $\frac{1}{f(x)}$ is non increasing. In that case we can solve for y for all times!



Example

Suppose $f(x) = x^7$. Then, $y + ty^7 - x = 0$. We want to solve for y. Although we cannot find an equation for y explicitly, by the Implicit Function Theorem, we know that $y(t,x)$ exists. So, we have our solution in the form $u(t,x) = y(t,x)^7$.

Now, suppose $f' < 0$. We have the reverse of the plot we had earlier, so at some point, the lines must intersect.



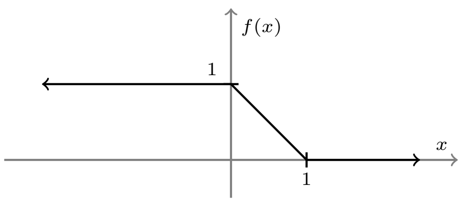
We stop at the first intersection since at this point, we cannot determine $y = x(0)$ uniquely. However, by the Implicit Function Theorem with $F(t, x, y) = y + tf(y) - x = 0$ meaning $F_y = 1 + tf'(y) \neq 0$ if t is small. So, some solution $u(t, x)$ must exist.

Telegrapher's Equation

The Telegrapher's Equation is another famous quasilinear PDE. It takes the form

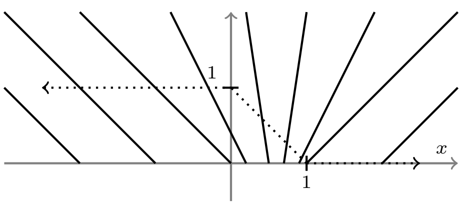
$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0$$
$$u(0, x) = f(x)$$

with $a(u) = 1 - 2u$. We solve this for



$$f(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

f is differentiable everywhere except for $x = 0$ and $x = 1$. To solve this, we follow the same procedure. We have $a(u) = 1 - 2u$ and $x(t) = y + ta(f(y))$. Since $f(y)$ varies depending on the value of $y = x(0)$, we have the following $x(t)$:



$$x(t) = \begin{cases} x(0) - t & x(t) \leq -t \\ x(0) + t(2x(0) - 1) & -t \leq x(t) \leq 1 + t \\ x(0) + t & x(t) \geq 1 + t \end{cases}$$

We have $u(t, x(t)) = f(x(0))$. We want to find $x(0)$ such that $x(t) = x$, which is given.

$$\begin{aligned} &\text{For } x \leq -t: & x(0) &= x + t \\ &\text{For } -t \leq x \leq t + 1: & x(0) + t(2x(0) - 1) &= x \\ & & x(0) &= \frac{x + t}{1 + 2t} \\ &\text{For } x \geq 1 + t: & x(0) &= x - t \end{aligned}$$

So, we can calculate each $f(x(0))$. For $x \leq -t$, $f(x) = 1$ and for $x \geq 1 + t$, $f(x) = 0$. For $-t \leq x \leq t + 1$:

$$\begin{aligned} f(x) = 1 - x \implies f(x(0)) &= 1 - x(0) \\ &= 1 - \frac{x + t}{1 + 2t} = \frac{-x + t + 1}{1 + 2t} \end{aligned}$$

So, we have our final answer for $u(t, x)$:

$$u(t, x) = \begin{cases} 1 & x \leq -t \\ \frac{-x+t+1}{1+2t} & -t \leq x \leq t + 1 \\ 0 & x \geq 1 + t \end{cases}$$

Note that Telegrapher's Equation is simply Burger's Equation "in disguise". Let $v(t,x) := 1 - 2u(t,x)$. If $u(t,x)$ solves the Telegrapher's Equation, $v(t,x)$ solves Burger's Equation. We have $\partial_t u + (1-2u)\partial_x u = 0$, and we can calculate $\partial_t u$ and $\partial_x u$ in terms of $\partial_t v$ and $\partial_x v$: $\partial_t v = -2\partial_t u$ and $\partial_x v = -2\partial_x u$, so $\partial_t u = -\frac{1}{2}\partial_t v$ and $\partial_x u = -\frac{1}{2}\partial_x v$. So:

$$-\frac{1}{2}\partial_t v + \underbrace{(1-2u)}_{v}\left(-\frac{1}{2}\partial_x v\right) = 0 \iff \partial_t v + v\partial_x v = 0$$

So, $v(t,x)$ solves Burger's Equation. Observe that our answer is always of the form $\underline{y} + t\underline{a}(f(\underline{y})) - \underline{x} = 0$, so it will always have some solution for some t.

Eikonal Equation

Let $u = u(x)$ and $x \in \mathbb{R}^2$. We consider $|\nabla u(x)|^2 = 1$. Note this is note quite an equation of the form we studied. It is a fully nonlinear equation since the dependence on derivatives is non-linear. But the method of characteristic can still help us here. Here, u is a function of x only, so we can we will have $\frac{\partial u}{\partial t} = 0$. This equation has many solutions, such as

$$\begin{aligned} u(x) &= \cos(\theta)x_1 + \sin(\theta)x_2 \\ &= \underline{\omega} \cdot \underline{x} \end{aligned}$$

where

$$\underline{\omega} = (\cos(\theta), \sin(\theta)).$$

For this PDE, we have $\nabla u(x) \cdot \nabla u(x) = 1$ so by analogy $\underline{a}(u) = \nabla u(x)$. We now calculate the analogue of the equation of characteristics (note some cool chain rule along the way; we use here the fact that $|\nabla u(x)|^2 = 1$):

$$\begin{aligned} \underline{x}'(t) = \nabla u(\underline{x}(t)) \implies \underline{x}''(t) &= (\partial_t[\partial_{x_1} u(\underline{x}(t)), \cdots, \partial_t[\partial_{x_n} u(\underline{x}(t))]) \\ &= (\underline{x}'(t) \cdot \partial_{x_1} \nabla u(x), \cdots, \underline{x}'(t) \cdot \partial_{x_n})\nabla u(x)) \\ &= (\nabla u(\underline{x}) \cdot \partial_{x_1} \nabla u(\underline{x}), \cdots, \nabla u(\underline{x}) \cdot \partial_{x_n} \nabla u(\underline{x})) = \tfrac{1}{2}\nabla \cdot |\nabla u(x)|^2 = 0 \\ \underline{x}'(t) = \underline{v}_0 &\implies \underline{x}(t) = \underline{x}(0) + \underline{v}_0 t. \end{aligned}$$

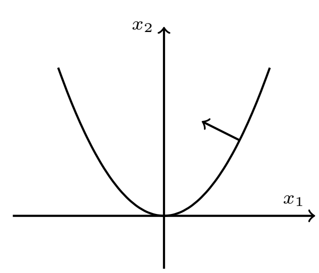
Here we used the fact that $\underline{x}''(t) = 0$ implies that $\underline{x}'(t) = \text{const} = v_0$.

Let the initial condition be given on a parabola: $u(y, y^2) = 1$. We note that

$$0 = \partial_y[u(y,y^2)] = (y, 2y) \cdot [\nabla u](y, y^2).$$

The vector $(y, 2y)$ is tangent to the parabola at (y, y^2) so $[\nabla u](y, y^2)$ has to be normal to the parabola. And it has to have unit length. That means that, for $\underline{x}(0) = (y, y^2)$ (note that we have two choices of the unit normal and choose the upward pointing one):

$$\underline{v}_0 = \underline{x}'(0) = [\nabla u](\underline{x}(0)) = [\nabla u](y, y^2) = \frac{(-2y, 1)}{(1+4y^2)^{1/2}} \implies \underline{x}(t) = (y, y^2) + t\frac{(-2y, 1)}{(1+4y^2)^{1/2}}.$$



Now,

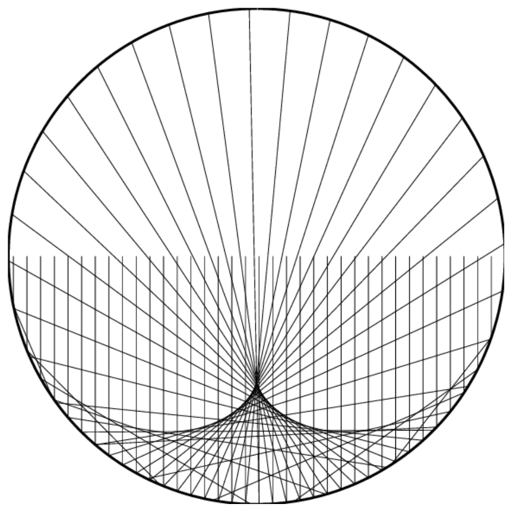
$$\partial_t u(\underline{x}(t)) = \underline{x}'(t) \cdot \nabla u(\underline{x}(t)) = |\nabla u(\underline{x}(t))|^2 = 1, \quad u(\underline{x}(0)) = u(y, y^2) = 1 \implies u(\underline{x}(t)) = 1 + t.$$

In other words,

$$u(\underline{x}) = 1 + t, \quad \underline{x} = (x_1, x_2) = (y - 2yt/\sqrt{1+4y^2}, y^2 + t/\sqrt{1+4y^2})).$$

For small t we can solve for (y, t) in terms of $x = (x_1, x_2)$ using the implicit function theorem. But the solution is not explicit.

As time grows, the rays $\underline{x}(t)$ *focus* and we do not have a nice solution $u(\underline{x})$ anymore. This focusing corresponds to a formation of *caustic*.



The unit vectors focus to a caustic, which can be seen in the figure above. This figure was obtained from a paper on wave propagation.

The Wave Equation

The wave equation takes the form

$$\left(\partial_t^2 - c^2\Delta\right) u = 0$$

where the Laplace operator is

$$\begin{aligned} \Delta &= \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 \\ &= \nabla \cdot \nabla \end{aligned}$$

c is a constant known as the "speed of propagation" and $c > 0$.

Example

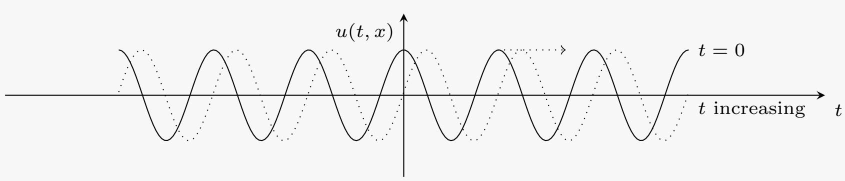
We will look at an important solution to the wave equation, known as the standing wave solution. The function

$$u(t, x) = e^{ic\lambda t + i\lambda x \cdot \omega}$$

solves the wave equation where $\omega \in \mathbb{R}^n$ and $|\omega| = 1$. We write $x \cdot \omega = x_1\omega_1 + \cdots + x_n\omega_n$.
Let $\mu(t, x) := c\lambda t + \lambda x \cdot \omega$.

$$\begin{aligned} \partial_t^2 u(t,x) &= \partial_t \left[ic\lambda e^{\mu(t,x)}\right] \\ &= -c^2\lambda^2 e^{\mu(t,x)} \\ \partial_{x_j}^2 u(t,x) &= \partial_{x_j}\left[i\lambda\omega_j e^{\mu(t,x)}\right] \\ &= -\lambda^2\omega_j^2 e^{\mu(t,x)} \\ \partial_t^2 u - c^2\Delta u &= \partial_t^2 u - c^2\sum_{j=1}^n \partial_{x_j}^2 u \\ &= -c^2\lambda^2 e^{\mu(t,x)} - c^2(-\lambda^2 \underbrace{(\omega_1^2 + \cdots + \omega_n^2)}_{=1 \text{ since } |\omega|=1}) e^{\mu(t,x)} \\ &= -c^2\lambda^2 e^{\mu(t,x)} + c^2\lambda^2 e^{\mu(t,x)} \\ &= 0 \end{aligned}$$

In dimension 1, $\omega = \pm 1$. Let $\omega = -1$. Then, we have $u(t, x) = e^{i\lambda(ct-x)}$. We have $\mathrm{Re}(u) = \cos(\lambda(ct - x)) = \cos(\lambda(x - ct))$. We can plot this solution and observe the effect as t increases.



The function $\varphi(x) = \omega \cdot x$ is a solution to the eikonal equation, $|\nabla\varphi(x)|^2 = 1$. Try $u(t,x) = e^{i\lambda t + i\lambda\varphi(x)}$ as a solution (take $c = 1$ for simplicity).

$$\begin{aligned} \partial_t^2 u &= -\lambda^2 e^{it + i\lambda\varphi(x)} \\ \partial_{x_j}^2 u &= \partial_{x_j}\left[i\lambda\partial_{x_j}\varphi(x)e^{i\lambda t + i\lambda\varphi(x)}\right] \\ &= \left(-\lambda^2\left(\partial_{x_j}\varphi(x)\right)^2 + i\lambda\partial_{x_j}^2\varphi(x)\right)e^{-\lambda t + i\lambda\varphi(x)} \\ \left(\partial_t^2 - \nabla\right)\left(e^{i\lambda t + i\lambda\varphi(x)}\right) &= \left[-\lambda^2 + \lambda^2|\nabla\varphi|^2 + i\lambda\Delta\varphi\right]e^{i\lambda t + i\lambda\varphi(x)} \end{aligned}$$

We now consider high frequency oscilations, that is λ large. Hence if φ solves the eikonal equation, $|\nabla\varphi|^2 = 1$, we kill the largest term on the right-hand side and we get an approximate solution. It can be refined to eliminate $i\lambda\Delta\varphi e^{i\lambda t + i\lambda\varphi(x)}$ and to get a better approximate solution.

One Dimensional Wave Equation

Theorem: d'Alembert's Formula

Suppose $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then, $\exists$ a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$ of the following initial value problem:

$$\begin{aligned} \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right) u = \left(\partial_t^2 - c^2\partial_x^2\right) u = 0 \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{aligned}$$

It is given by

$$u(t,x) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\mathrm{d}s$$

In other words, we have a formula for the unique solution.

- **Proof**: note that the wave equation can be written as

  $$(\partial_t^2 - c^2\partial_x^2) = (\partial_t + c\partial_x)(\partial_t - c\partial_x).$$

  Let

  $$w(t,x) := (\partial_t - c\partial_x)u(t,x).$$

  Then, if u exists, we have

  $$(\partial_t + c\partial_x)w = (\partial_t^2 - c^2\partial_x^2)u(t,x) = 0$$

  So, we are solving the first order system

  $$(\partial_t + c\partial_x)w = 0.$$

  To do this, we can use the method of characteristics.

  $$\begin{aligned} x'(t) &= c \\ x(t) &= ct + x_0 \\ x_0 &= x - ct \\ w(t,x) &= w_0(x-ct) \text{ for some function } w_0 \\ (\partial_t - c\partial_x)u(t,x) &= w(t,x) \\ &= w_0(x-ct) \end{aligned}$$

  So, we want to solve the PDE

  $$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w_0(x-ct)$$

  for some function $w_0(x-ct)$.

  Note: for a general PDE

  $$\partial_t u + c\partial_x u = f(t,x)$$

  where $u(0,x) = g(x)$, we can solve for $u(t,x)$.

  1. Solve $x'(t) = v(t, x(t))$ for $x(t)$.

     $$\begin{aligned} x'(t) &= c \\ x(t) &= ct + A \end{aligned}$$

2. Solve $h'(t) + w(t, x(t), h(t)) = 0$ for $h(t)$.

$$\begin{aligned} h'(t) - f(t, x(t)) &= 0 \\ h'(t) &= f(t, x(t)) \\ h(t) &= B + \int_0^t f(s, x(s))\mathrm{d}s \end{aligned}$$

3. Match $u(t, x(t)) = h(t)$ and $h(0) = g(x(0))$.

$$\begin{aligned} u(t, x(t)) &= B + \int_0^t f(s, x(s))\mathrm{d}s \\ x(t) = x &= ct + A \\ A &= x - ct \\ x(0) &= A = x - ct \\ x(s) &= A + cs \\ &= x - ct + cs \\ &= x + c(s - t) \\ h(0) &= B + \int_0^0 f(s, x(s))\mathrm{d}s \\ &= B \\ &= g(x(0)) \\ &= g(x - ct) \end{aligned}$$

So, we have $u(t, x) = g(x - ct) + \int_0^t f(s, x + c(s - t))\mathrm{d}s$.

In this case, we have $-c$ instead of $+c$, so we reverse the signs:

$$u(t, x) = g(x + ct) + \int_0^t f(s, s + c(t - s))\mathrm{d}s$$

Apply this function where $f(t, x) = w_0(x - ct)$.

$$u(t, x) = g(x + ct) + \int_0^t w_0(\underbrace{x - 2cs + ct}_{\tau})\mathrm{d}s$$

Write the integral in terms of τ.

$$\begin{aligned} \tau &= x - 2cs + ct \\ c &= \frac{1}{2c}(x - \tau + ct) \\ \mathrm{d}s &= -\frac{1}{2c}\mathrm{d}\tau \end{aligned}$$

To convert the bounds, note that at $s = 0$, $\tau = x + ct$ and at $s = \tau$, $\tau = x - ct$.

$$\begin{aligned} u(t, x) &= g(x + ct) - \frac{1}{2c}\int_{x+ct}^{x-ct} w_0(\tau)\mathrm{d}\tau \\ &= g(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau \end{aligned}$$

Find w_0 using the initial conditions.

$$\begin{aligned}
u(0,x) &= g(x) + \frac{1}{2c}\int_x^x w_0(\tau)\mathrm{d}\tau \\
&= g(x) \\
\partial_t u(0,x) &= cg'(x) + \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau\right]
\end{aligned}$$

The chain rule can be used to calculate the derivative of the integral.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(\tau)\mathrm{d}\tau = b'(t)F(b(t)) - a'(t)F(a(t))$$

In our case, $a(t) = x - ct$, $b(t) = x + ct$, and $F(\tau) = w_0(\tau)$.

$$\begin{aligned}
\partial_t u &= cg'(x) + \frac{1}{2c}(cw_0(x)) - \frac{1}{2c}(-cw_0(x)) \\
&= cg'(x) + w_0(x) \\
&= h(x) \\
w_0(x) &= h(x) - cg'(x)
\end{aligned}$$

Plug this back into the equation for $u(t,x)$.

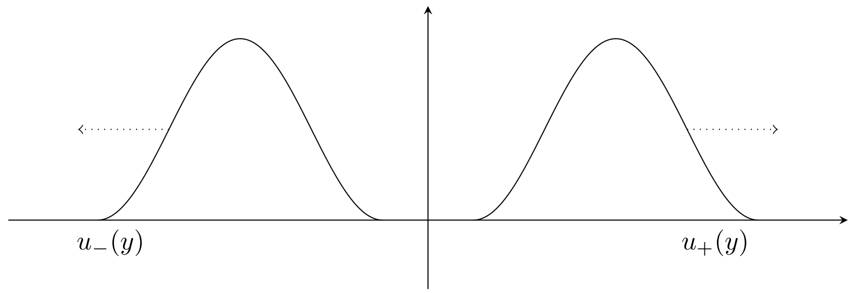
$$\begin{aligned}
u(t,x) &= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau \\
&= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left[h(\tau) - cg'(\tau)\right]\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}\int_{x-ct}^{x+ct} g'(\tau)\mathrm{d}\tau + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}[g(\tau)]_{x-ct}^{x+ct} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}g(x+ct) + \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau
\end{aligned}$$

So, we obtained a formula for the solution. The solution is unique since from assuming that u exists we derived the formula for u. □

Note that we can write $u(t,x)$ as a sum of two parts:

$$\begin{aligned}
u(t,x) &= u_-(x+ct) + u_+(x-ct) \\
u_-(y) &= \frac{1}{2}g(y) + \frac{1}{2}\int_0^y h(\tau)\mathrm{d}\tau \\
u_+(y) &= \frac{1}{2}g(y) - \frac{1}{2}\int_0^y h(\tau)\mathrm{d}\tau
\end{aligned}$$

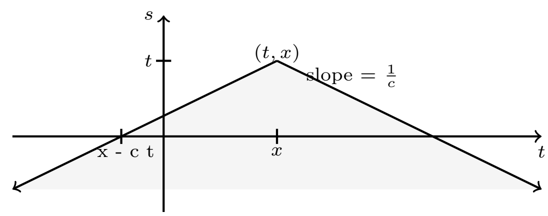
This solution can be plotted:



The two profiles move away from each other.

Huygens' Principle

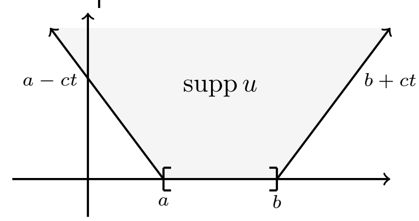
$x(0) = x + ct$ is constant on the line $x = x(0) - ct$. $u(t, x)$ depends only on the data in the "backwards cone" shown in the diagram.



Huygens' Principle implies that if $\operatorname{supp} g, \operatorname{supp} h \subset [a, b]$, then

$$\operatorname{supp} u(t, \bullet) \subseteq [a - ct, b + ct]$$

for fixed $t \geq 0$ and varying $\bullet$.

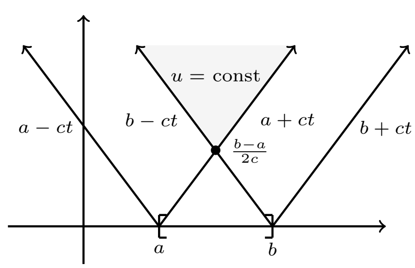


- **Proof**: if $x < a - ct$, then $x + ct < a$. Also that means that $x - ct < a - 2ct < a$, so $g(x \pm ct) = 0$ and $h(s) = 0$ for $x - ct < s < x + ct < a$. Hence $u(t, x) = 0$. Similarly for $x > b + ct$: $x - ct > b$ and $x + ct > b + 2ct > b$ so that $g(x \pm ct) = 0$, and $h(s) = 0$ for $b < x - ct < s < x + ct$.

Strong Huygens' Principle states that for $\operatorname{supp} g, \operatorname{supp} h \subset [a, b]$,

$$u(t, x) = \frac{1}{2c} \int_a^b h(\tau) \mathrm{d}\tau = \text{constant}$$

for $b - ct < x < a + ct$.



Boundary Value Problems

First order wave equation boundary problems have form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\text{Initial conditions} \begin{cases} u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

$$\text{Boundary conditions} \begin{cases} u(t,0) = 0 \\ u(t,L) = 0 \end{cases}$$

This type of boundary condition is called a Dirichlet boundary condition. We only care about $u(t,x)$ between $x = 0$ and $x = L$.

Using linearity of the PDE, we can solve for u_1 and u_2 separately where $u = u_1 + u_2$ solves the PDE.

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} &= 0 \\ u_1(0,x) &= g(x) \\ \partial_t u_1(0,x) &= 0 \\ u_1(t,0) = u_1(t,L) &= 0 \end{aligned} \qquad \begin{aligned} \frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} &= 0 \\ u_2(0,x) &= 0 \\ \partial_t u_2(0,x) &= h(x) \\ u_2(t,0) = u_2(t,L) &= 0 \end{aligned}$$

We split it up so that as solving for u_1 and u_2 separately is easier. Solve for u_1. Motivated by the equation from the previous theorem when $h \equiv 0$, we try

$$u_1(t,x) = \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct))$$

for some $\tilde{g}$, satisfying $u(0,x) = \tilde{g}(x) = g(x)$ for $0 \le x \le L$. Note that we have $\partial_t^2 u_1 - c^2 \partial_x^2 u_1 = 0$.

$$\partial_t u_1(0,x) = \frac{1}{2}\left.\left(c\tilde{g}'(x+ct) - c\tilde{g}'(x-ct)\right)\right|_{t=0} = 0$$

We need $\tilde{g} \in C^2(\mathbb{R})$ such that

$$\begin{aligned} u_1(t,0) &= \frac{1}{2}(\tilde{g}(tc) + \tilde{g}(-tc)) = 0 \\ u_1(t,L) &= \frac{1}{2}(\tilde{g}(L+tc) + \tilde{g}(L-tc)) = 0 \end{aligned}$$

Since $t \ge 0$ is arbitrary, we can set $tc = y$ for arbitrary y.

$$\tilde{g}(y) + \tilde{g}(-y) = 0 \implies \tilde{g}(-y) = -\tilde{g}(y)$$

So, $\tilde{g}$ is an odd function. The second condition reads:

$$\tilde{g}(L+y) + \tilde{g}(L-y) = 0.$$

We now write $-w = L - y$ so that $L + y = 2L + w$.

$$\begin{aligned} -\tilde{g}(-w) &= \tilde{g}(2L+w) \\ \tilde{g}(w) &= \tilde{g}(w+2L). \end{aligned}$$

So, $\tilde{g}$ is both odd and periodic with period $2L$. If we have $\tilde{g}$ defined on 0 to L, we can extend $\tilde{g}$ to $[-L,0]$ since $\tilde{g}$ is odd and then extend it in both directions to $[-3L,-L]$ and $[L,3L]$ and so on since $\tilde{g}$ is periodic on $2L$, meaning it repeats itself every $2L$.

$\exists u_1 \in C^2$ if the odd $2L$-periodic extension of g, called $\tilde{g}$, is C^2.

$$u_1(t,x) = \frac{1}{2}\left(\tilde{g}(x-ct) + \tilde{g}(x+ct)\right)$$

By the same argument as u_1, $\exists u_2 \in C^2$ if the odd $2L$-periodic extension of h, called $\tilde{h}$, is C^1. Here, the g function is 0, so we only have the integral.

$$u_2(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\mathrm{d}s$$

So, we can solve the initial PDE with $u \in C^2$ if g and h have C^2 and C^1 odd $2L$-periodic extensions respectively. This u has the form

$$u(t,x) = \frac{1}{2}\left(\tilde{g}(x-ct) + \tilde{g}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\mathrm{d}s$$

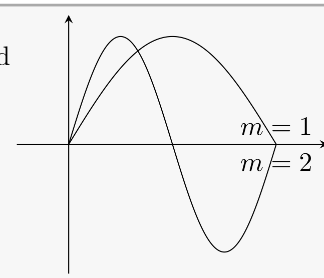
Example

Let $g(x) = \sin\left(\frac{\pi}{L}mx\right)$ and $h(x) = 0$. In this case, g is already odd and $2L$-periodic since sin is and odd function and

$$g(x+2L) = \sin\left(\frac{\pi}{L}mx + 2\pi mx\right) = \sin\left(\frac{\pi}{L}mx\right)$$

So, $\tilde{g}(x) = g(x)$ and we have our formula for u:

$$u(t,x) = \frac{1}{2}\left[\sin\left(\frac{\pi}{L}m(x-ct)\right) + \sin\left(\frac{\pi}{L}m(x+ct)\right)\right]$$



Nonhomogeneous Problems

Nonhomegenous problems are where the right hand side in the wave equation is no longer equal to 0. The general nonhomogeneous wave equation is given by

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} &= f(t,x)\\ u(0,x) &= g(x)\\ \partial_t u(0,x) &= h(x)\end{aligned}$$

where $f \in C^1(\mathbb{R}^2)$, $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$.

Like with solving boundary problems, we can use linearity of u to split this PDE into two parts.

$$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right)u_1 &= 0\\ u_1(0,x) &= g(x)\\ \partial_t u_1(0,x) &= h(x)\end{aligned} \qquad \begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right)u_2 &= 0\\ u_2(0,x) &= 0\\ \partial_t u_2(0,x) &= 0\end{aligned}$$

Once again, $u = u_1 + u_2$. So, we can solve

$$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right)u &= f(t,x)\\ u(0,x) &= 0\\ \partial_t u(0,x) &= 0\end{aligned}$$

Theorem: Duhamel's Formula

For $f \in C^1(\mathbb{R}^2)$, $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$, $\exists$ a unique $u \in C^2$ which solves the nonhomegenous system

$$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right) u &= f(t,x)\\ u(0,x) &= g(x)\\ \partial_t u(0,x) &= h(x)\end{aligned}$$

which is given by

$$u(t,x) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\mathrm{d}s + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

- **Proof**: As stated previously, we have already proven this to be true for $f = 0$, so by the superposition principle, it suffices to consider $g = h = 0$. So, we must show that

  $$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

  is the unique solution to the system where $g = h = 0$.

  First prove uniqueness. Suppose $\tilde{u}$ also solves the system.

  $$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right) \tilde{u} &= f(t,x)\\ \tilde{u}(0,x) &= 0\\ \partial_t \tilde{u}(0,x) &= 0\end{aligned}$$

  We want to show that $u = \tilde{u}$. Let $U = u - \tilde{u}$.

  $$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right) U &= f(t,x) - f(t,x) = 0\\ U(0,x) &= 0\\ \partial_t U(0,x) &= 0\end{aligned}$$

  We already know that the unique solution for this system. Since $g(x) = h(x) = 0$, we have $U = 0$ meaning $u = \tilde{u}$.

  Now, we can prove the formula by verification. We want to show that

  $$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

  solves the system where $g = h = 0$. To simplify the formula, let

  $$F(t,s,x) := \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y$$

  So, we want to show

  $$u(t,x) = \frac{1}{2c}\int_0^t F(t,s,x)\mathrm{d}s$$

solves the system. We note that this $u(t,x)$ satisfies $u(0,x)=0$. We now differentiate:

$$\begin{aligned}
\partial_t u(t,x) &= \partial_t \left(\frac{1}{2c}\int_0^t F(t,s,x)\mathrm{d}s\right) \\
&= \frac{1}{2c}F(t,t,x) + \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s \\
&= \frac{1}{2c}\int_x^x f(s,y)\mathrm{d}s + \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s \\
&= \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s
\end{aligned}$$

This is 0 when $t=0$, so $\partial_t u(0,x)=0$ as needed.

Now, find $\partial_t^2 u(t,x)$.

$$\begin{aligned}
\partial_t^2 u(t,x) &= \partial_t \left(\frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s\right) \\
&= \frac{1}{2c}\partial_t F(t,t,x) + \frac{1}{2c}\int_0^t \partial_t^2 F(t,s,x)\mathrm{d}s
\end{aligned}$$

We must find $\partial_t F(t,s,x)$ and $\partial_t^2 F(t,s,x)$.

$$\begin{aligned}
\partial_t F(t,s,x) &= \partial_t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\right) \\
&= c\left(f(s,x+c(t-s)) + f(s,x-c(t-s))\right) \\
\partial_t^2 F(t,s,x) &= c^2\left(\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right)
\end{aligned}$$

Plug this back into the formula for $\partial_t^2 u(t,x)$.

$$\begin{aligned}
\partial_t^2 u(t,x) &= \frac{1}{2c}\left[c\left(f(s,x+c(t-s)) + f(s,x-c(t-s))\right)\right]|_{s=t} \\
&\quad + \frac{1}{2c}(c^2)\left(\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s\right) \\
&= \tfrac{1}{2}\left(f(s,x) + f(s,x)\right) \\
&\quad + \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s \\
&= f(t,x) + \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s
\end{aligned}$$

Now, find $c^2\partial_x^2 u(t,x)$.

$$c^2\partial_x^2 u(t,x) = c^2\frac{1}{2c}\int_0^t \partial_x^2 F(t,s,x)\mathrm{d}s$$

We must find $\partial_x^2 F(t,s,x)$.

$$\begin{aligned}
\partial_x F(t,s,x) &= \partial_x \left(\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\right) \\
&= f(s,x+c(t-s)) - f(s,x-c(t-s)) \\
\partial_x^2 F(t,s,x) &= \partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))
\end{aligned}$$

Plug this back into the equation for $c^2\partial_x^2 u(t,x)$.

$$c^2\partial_x^2 u(t,x) = \frac{c}{2}\int_0^t \left[\partial_x f(s, x + c(t-s)) - \partial_x f(s, x - c(t-s))\right] \mathrm{d}s$$

So, we can plug these values back into the original equation.

$$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right) u(t,x) &= f(t,x) + \frac{c}{2}\int_0^t \left[\partial_x f(s, x + c(t-s)) - \partial_x f(s, x - c(t-s))\right] \mathrm{d}s \\ &\quad - \frac{c}{2}\int_0^t \left[\partial_x f(s, x + c(t-s)) - \partial_x f(s, x - c(t-s))\right] \mathrm{d}s \\ &= f(t,x)\end{aligned}$$

Thus, $u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$ solves the system $\left(\partial_t^2 - c^2\partial_x^2\right) u = f(t,x)$ where $u(0,x) = \partial_t u(0,x) = 0$. This verifies Duhamel's formula. $\square$

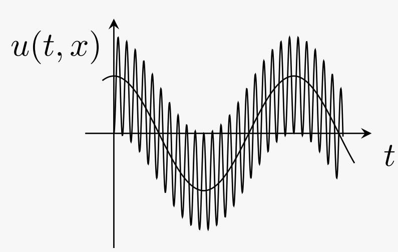
Example

Let $g = h = 0$ and $f(x,t) = (\sin(\omega_0 x))(\cos(\omega t))$. Suppose $c = 1$.

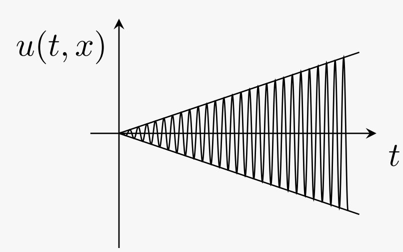
$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \sin(\omega_0 y)\mathrm{d}y \cos(\omega s)\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \left[-\frac{\cos(\omega_0 y)}{\omega_0}\right]_{x-(t-s)}^{x+(t-s)} \cos(\omega s)\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \frac{(\cos(\omega_0(x-(t-s))) - \cos(\omega_0(x+(t-s))))\cos(\omega s)}{w_0}\mathrm{d}s \\ &= \begin{cases} \frac{\sin(\omega_0 x)}{\omega_0^2-\omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right] & \omega \neq \pm\omega_0 \\ \frac{t}{2\omega_0}\sin(\omega_0 t)\sin(\omega_0 x) & \omega = \pm\omega_0 \end{cases}\end{aligned}$$

Let's plot these cases for fixed x.

For $\omega \ll \omega_0$:



For $\omega = \omega_0$:



Three dimensional Wave Equation

The homogeneous third dimension wave equation for $c = 1$ is given by

$$\left(\partial_t^2 - \Delta\right) u = 0$$

where $u \in C^2$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$. To solve this, we want to reduce this to an equation in one dimension.

Lemma: Darboux's Lemma

Suppose $u(t,x)$ solves $\left(\partial_t^2 - \Delta\right) u = 0$. Define

$$\tilde{u}(t,x,\rho) := \frac{1}{4\pi\rho} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w) \tag{4}$$

where $\partial B(x,\rho)$ is the boundary of the ball of radius ρ centered at x and $dS(w)$ is the surface element of integration. Then,

$$\left(\partial_t^2 - \partial_\rho^2\right) \tilde{u}(t,x,\rho) = 0 \tag{5}$$

meaning $\tilde{u}$ solves the 1-D wave equation. Conversely, if (5) holds then

$$u(t,x) := \lim_{\rho \to 0+} \frac{1}{\rho} \tilde{u}(t,x,\rho) \tag{6}$$

solves the wave equation.

Note that

$$\frac{1}{\rho}\tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w).$$

We also recall that $4\pi\rho^2$ is the area of the sphere of radius ρ. We can use change of coordinates to take the integral over the boundary of the ball of radius ρ centered at 0 instead of x.

$$\tilde{u}(t,x,\rho) = \frac{\rho}{4\pi\rho^2} \int_{\partial B(0,\rho)} u(t,x+w) \mathrm{d}S(w).$$

Since $u(t,x+w) \to u(t,x)$ as $|w| = \rho \to 0$, we see that

$$\frac{1}{\rho}\tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho^2} \int_{\partial B(0,\rho)} u(t,x+w) \mathrm{d}S(w) \to \frac{1}{4\pi\rho^2} \int_{\partial B(0,\rho)} u(t,x) \mathrm{d}S(w) = u(t,x), \quad \rho \to 0. \tag{7}$$

This is so as we have $\frac{1}{4\pi\rho^2} \int_{\partial B(0,\rho)} \mathrm{d}S = 1$. We also note that

$$\tilde{u}(t,x,0) = \lim_{\rho \to 0} \rho \lim_{\rho \to 0} \frac{1}{4\pi\rho^2} \int_{\partial B(0,\rho)} u(t,x+w) \mathrm{d}S(w) = 0 \times u(t,x) = 0.$$

- **Proof of Darboux's Lemma**: we want to show that $\partial_\rho^2 \tilde{u} = \partial_t^2 \tilde{u}$. First, calculate $\partial_\rho \tilde{u}(t,x,\rho)$. Use the change of variables $w = x + \rho y$ where $y \in \partial B(0,1)$. So, $y = \frac{w-x}{y}$.

$$\begin{aligned}
\partial_\rho \left[\frac{1}{\rho}\tilde{u}(t,x,\rho)\right] &= \partial_\rho \left[\frac{1}{4\pi\rho^2}\int_{\partial B(x,\rho)} u(t,w)\mathrm{d}S(w)\right] \\
&= \partial_\rho \left[\frac{1}{4\pi}\int_{\partial B(0,1)} u(t,x+\rho y)\mathrm{d}S(y)\right] \\
&= \frac{1}{4\pi}\int_{\partial B(0,1)} y\cdot(\nabla u(t,x+\rho y))\,\mathrm{d}S(y) \\
&= \frac{1}{4\pi\rho^2}\int_{\partial B(x,\rho)} \left(\frac{w-x}{\rho}\right)\cdot(\nabla u(t,w))\,\mathrm{d}S(w) \\
&= \frac{1}{4\pi\rho^2}\int_{\partial B(x,\rho)} \nu(w)\cdot\nabla u(t,w)\mathrm{d}S(w)
\end{aligned}$$

where $\nu(w) = \frac{w-x}{\rho}$ is the outward pointing vector that is normal to $\partial B(x,\rho)$. To evaluate this integral we use the divergence theorem. For a surface Ω with outward pointing normal vector $\nu(w)$, let $\vec{F} = (a,b,c)$. We have

$$\int_{\partial\Omega} \nu(w)\cdot\vec{F}(w)\mathrm{d}S(w) = \int_\Omega \nabla\cdot\vec{F}(w)\,\mathrm{d}^3w$$

We know

$$\nabla\cdot\vec{F}(w) = \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_2} + \frac{\partial c}{\partial x_3}$$

In our equation, we have $\vec{F}(w) = \nabla u(t,w)$. So,

$$\begin{aligned}
\nabla\cdot\vec{F}(w) &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \\
&= \Delta u
\end{aligned}$$

So, we have

$$\partial_\rho\left[\frac{1}{\rho}\tilde{u}(t,x,\rho)\right] = \frac{1}{4\pi\rho^2}\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w\,.$$

To find $\partial_\rho u$ we use this and the product rule:

$$\begin{aligned}
\partial_\rho\tilde{u}(t,x,\rho) &= \partial_\rho\left[\rho\frac{1}{4\pi\rho^2}\int_{\partial B(x,\rho)} u(t,w)\mathrm{d}S(w)\right] \\
&= \frac{1}{4\pi\rho^2}\int_{\partial B(x,\rho)} u(t,w)\mathrm{d}S(w) + \rho\cdot\frac{1}{4\pi\rho^2}\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w \\
\partial_\rho^2\tilde{u}(t,x,\rho) &= \frac{1}{4\pi\rho^2}\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w - \frac{1}{4\pi\rho^2}\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w \\
&\quad + \frac{1}{4\pi\rho}\partial_\rho\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w \\
&= \frac{1}{4\pi\rho}\partial_\rho\int_{B(x,\rho)} \Delta u(t,w)\,\mathrm{d}^3w
\end{aligned}$$

To evaluate the derivative with respect to ρ , we claim that for any function $f(w)$ we claim that

$$\partial_\rho\int_{B(x,\rho)} f(w)\,\mathrm{d}^3w = \int_{\partial B(x,\rho)} f(w)\mathrm{d}S(w).$$

To see this we use the change of variables $w = x + \rho y$ where $y \in B(0,1)$.

$$\begin{aligned}\partial_\rho \int_{B(x,\rho)} f(w)\,\mathrm{d}^3 w &= \partial_\rho \left[\rho^3 \int_{B(0,1)} f(x+\rho y)\,\mathrm{d}^3 y\right] \\ &= 3\rho^2 \int_{B(0,1)} f(x+\rho y)\,\mathrm{d}^3 y + \rho^3 \int_{B(0,1)} y \cdot \nabla f(x+\rho y)\,\mathrm{d}^3 y\end{aligned}$$

We have $y \cdot \nabla f(x+\rho y) = \frac{1}{\rho}\nabla_y \cdot (y f(x+\rho y)) - \frac{3}{\rho} f(x+\rho y)$ since $\nabla_y \cdot (y f(x+\rho y)) = 3f + \rho y \cdot \nabla f(x+\rho y)$ (chain rule!).

$$\begin{aligned}\partial_\rho \int_{B(x,\rho)} f(w)\,\mathrm{d}^3 w &= 3\rho^2 \int_{B(0,1)} f(x+\rho y)\,\mathrm{d}^3 y - \rho^3 \left(\frac{3}{\rho}\right) \int_{B(0,1)} f(x+\rho y)\,\mathrm{d}^3 y \\ &\quad + \rho^2 \int_{B(0,1)} \nabla \cdot (y f(x+\rho y))\,\mathrm{d}^3 y \\ &= \rho^2 \int_{B(0,1)} \nabla \cdot (y f(x+\rho y))\,\mathrm{d}^3 y\end{aligned}$$

Let $\vec{F}(y) = y f(x+\rho y)$. By the divergence theorem, we have

$$= \rho^2 \int_{\partial B(0,1)} \nu(y) \cdot y f(x+\rho y) \mathrm{d}S(y) = \int_{\partial B(x,\rho)} f(w) \mathrm{d}S(w)$$

where $\nu(y) = y$.

So, we have

$$\partial_\rho^2 \tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho} \int_{\partial B(x,\rho)} \Delta u(t,w) \mathrm{d}S(w)$$

We know that $u(t,x)$ solves $\left(\partial_t^2 - \Delta\right) u = 0$, or $\Delta u(t,x) = \partial_t^2 u(t,x)$.

$$\begin{aligned}&= \frac{1}{4\pi\rho} \int_{\partial B(x,\rho)} \partial_t^2 u(t,w) \mathrm{d}S(w) \\ &= \partial_t^2 \left[\frac{1}{4\pi\rho} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w)\right] \\ &= \partial_t^2 \tilde{u}(t,x,\rho)\end{aligned}$$

So, $\left(\partial_t^2 - \partial_\rho^2\right) \tilde{u}(t,w,\rho) = 0$.

To see the converse, that is to see that $\left(\partial_t^2 - \partial_\rho^2\right) \tilde{u}(t,w,\rho) = 0$ for $\widetilde{u}$ defined as

$$\tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho} \int_{\partial B(0,\rho)} u(t,x+w) \mathrm{d}S(w)$$

implies $(\partial_t^2 - \Delta)u = 0$ we note that we have actually shown that

$$(\partial_t^2 - \partial_\rho^2)\widetilde{u}(t,x,\rho) = \frac{1}{4\pi\rho} \int_{\partial B(0,\rho)} (\partial_t^2 - \Delta u)(t,x) \mathrm{d}S(w).$$

But now (7) shows that

$$\frac{1}{\rho}(\partial_t^2 - \partial_\rho^2)\widetilde{u}(t,x,\rho) \to (\partial_t^2 - \Delta)u(t,x),$$

proving the converse. $\square$

We can use Darboux's Lemma to solve for the wave equation in 3-D.

Theorem: Kirchhoff's Formula

The system

$$\begin{aligned}\left(\partial_t^2 - \Delta_x\right) u &= 0\\ u(0,x) &= g(x) \in C^2(\mathbb{R}^3)\\ \partial_t u(0,x) &= h(x) \in C^1(\mathbb{R}^3)\end{aligned}$$

is solved by $u(t,x)$ given by

$$u(t,x) = \partial_t \left[\frac{1}{4\pi t}\int_{\partial B(x,t)} g(w)\mathrm{d}S(w)\right] + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(w)\mathrm{d}S(w)$$

Here, $B(x,t) := \{y \in \mathbb{R}^3 : |y-x| < t\}$ and $\partial B(x,t) := \{y \in \mathbb{R}^3 : |y-x| = t\}$.

- **Proof**: use Darboux's Lemma to rewrite the wave equation as $\left(\partial_t^2 - \partial_\rho^2\right)\tilde{u}(t,x,\rho) = 0$ with initial conditions coming from g and h:

$$\begin{aligned}\tilde{u}(0,x,\rho) = \tilde{g}(x,\rho) &= \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} g(w)\mathrm{d}S(w)\\ \partial_t\tilde{u}(0,x,\rho) = \tilde{h}(x,\rho) &= \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} h(w)\mathrm{d}S(w)\\ \tilde{u}(t,x,0) &= 0\end{aligned} \tag{8}$$

(The last equality was proved just before the proof of Darboux's Lemma.)

Motivated by our solution of the boundary value problem for 1D wave equation we take the odd extensions of $\tilde{g}$ and $\tilde{h}$ in ρ:

$$\tilde{u}(\rho) = \begin{cases} \tilde{g}(\rho) & \rho \geq 0\\ -\tilde{g}(-\rho) & \rho \leq 0\end{cases} \qquad \tilde{h}(\rho) = \begin{cases} \tilde{h}(\rho) & \rho \geq 0\\ -\tilde{h}(-\rho) & \rho \leq 0\end{cases}$$

This defines $\tilde{g}$ and $\tilde{h}$ on $\mathbb{R}$. Apply the 1-D formula.

$$\tilde{u}(t,x,\rho) = \frac{1}{2}\left(\tilde{g}(x,\rho+t) + \tilde{g}(x,\rho-t)\right) + \frac{1}{2}\int_{\rho-t}^{\rho+t}\tilde{h}(x,s)\mathrm{d}s$$

We take ρ small ($0 \leq \rho < t$). Since $\tilde{g}$ is an odd extension, we have

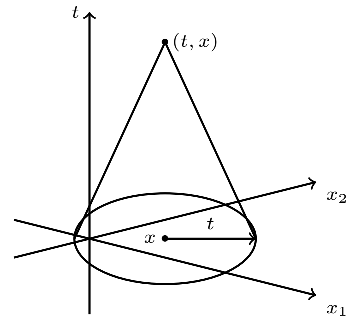
$$\begin{aligned}\tilde{u}(t,x,\rho) &= \frac{1}{2}\left(\tilde{g}(x,\rho+t) - \tilde{g}(x,t-\rho)\right) + \frac{1}{2}\int_{\rho-t}^{\rho+t}\tilde{h}(x,s)\mathrm{d}s\\ u(t,x) &= \lim_{\rho\to 0}\frac{1}{2\rho}\left[\left(\tilde{g}(x,t+\rho) - \tilde{g}(x,t-\rho)\right) + \left(\int_0^{t+\rho}\tilde{h}(x,s)\mathrm{d}s - \int_0^{t-\rho}\tilde{h}(x,s)\mathrm{d}s\right)\right]\\ &= \partial_t\tilde{g}(x,t) + \tilde{h}(x,t).\end{aligned}$$

(Here we used the fact that $\lim_{\rho\to 0}(F(t+\rho) - F(t-\rho)/(2\rho) = (d/dt)F(t)$.)

Substituting the definitions of $\tilde{g}(x,\rho)$ and $\tilde{h}(x,\rho)$ yields Kirchoff's Formula. □

The Light Cone

Although we can only plot in two dimensions we can describe the *domain of dependence* for the solutions as follows. (We cannot plot the additional x_3 axis). So, the circle on the x plane is actually the boundary $\partial B(t,x)$, which is a sphere. We see that *the solution $u(t,x)$ depends only on $g(y)$ and $\nabla g(y)$ for y the sphere $\partial B(x,t)$ and on $h(y)$ y near the sphere $\partial B(x,t)$.*



The claim about the dependence on h is clear from the formula. To see the claim about the dependence on g we note that using the change of variables $w = x + ty$,

$$\begin{aligned}
\partial_t \left(\frac{1}{4\pi t} \int_{\partial B(x,t)} g(w) dS(w) \right) &= \partial_t \left(\frac{t}{4\pi} \int_{\partial B(0,1)} g(x+ty) dS(y) \right) \\
&= \frac{1}{4\pi} \int_{\partial B(0,1)} g(x+ty) dS(y) + \int_{\partial B(0,1)} ty \cdot (\nabla g)(x+ty) dS(y) \\
&= \frac{1}{4\pi t^2} \int_{\partial B(x,t)} g(w) dS(w) + \frac{1}{4\pi t} \int_{\partial B(x,t)} (w-x) \cdot \nabla g(w) dS(w).
\end{aligned}$$

Uniqueness of the solution

The system

$$\begin{aligned}
\left(\partial_t^2 - \Delta_x\right) u &= 0 \\
u(0,x) &= g(x) \in C^2(\mathbb{R}^3) \\
\partial_t u(0,x) &= h(x) \in C^1(\mathbb{R}^3)
\end{aligned}$$

has a *unique solution* given by Kirchhoff's formula.

- **Proof:** Suppose there were two solutions u_1 and u_2 then u would solve

$$\left(\partial_t^2 - \Delta_x\right) u = 0, \quad u(0,x) = 0, \quad \partial_t u(0,x) = 0$$

We then define $\tilde{u}(t,x,\rho)$ by (4) and from Darboux's Lemma and (8) we see that for all x (treated as a parameter) and $\rho \geq 0$,

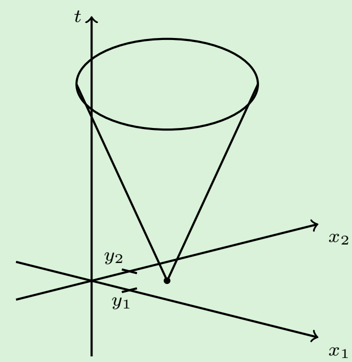
$$(\partial_t^2 - \partial_\rho^2)\tilde{u}(t,x,\rho) = 0, \quad \tilde{u}(0,x,\rho) = 0, \quad \partial_t \tilde{u}(0,x,\rho) = 0, \quad \tilde{u}(t,x,0) = 0.$$

But then uniqueness of the solution of 1D wave equation shows that $\tilde{u} \equiv 0$. Hence (6) shows that $u(t,x) \equiv 0$, that is $u_1 = u_2$.

Theorem (Huygens' Principle)

Suppose that $g(y) = 0$ and $h(y) = 0$ for $|y| \geq 0$. Then

$$|x| > R + t \implies u(t,x) = 0.$$



Proof: In the formula for $u(t,x)$, the values of $g(y)$, $\nabla g(y)$ and $h(y)$ are needed only when $y \in \partial B(x,t)$, that is when $|x-y| = t$. Now $|y| \geq |x| - |x-y|$ (by the triangle inequality ($|x-y| + |y| \geq |x|$). Hence if $|x| > R + t$ and $|x - y| = t$,

$$|y| \geq |x| - |x-y| = |x| - t > R.$$

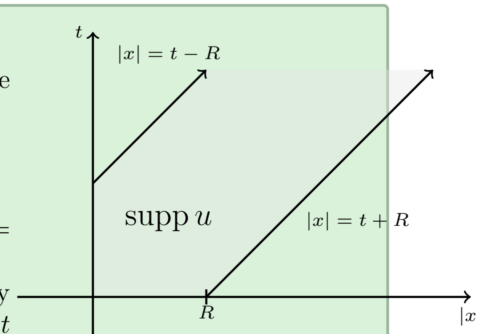
We conclude that $u(t,x) = 0$ as for such y, as then $g(y) = h(y) = 0$ and $\nabla g(y) = 0$. $\square$

The illustration of the forward light cone is once again in 2 dimensions. Again, there is a third axis x_3, but we can only illustrate x_1 and x_2. The vertical axis represents time and the horizontal plane represents space. It is important to remember that we are illustrating a 3-D space since the result for 2 dimensions is different.

Theorem (Strong Huygens' Principle)

Suppose that $g(y) = 0$ and $h(y) = 0$ for $|y| \geq 0$. Then the solution of the 3D wave equation satisfies

$$|x| < t - R \implies u(t,x) = 0.$$



Proof: In the formula for $u(t,x)$ we need $g(y)$, $\nabla g(y)$ and $h(y)$ for $|x-y| = t$ (that is for $y \in \partial B(x,t)$). Now $|y| \geq |x-y| - |x|$ (by the triangle inequality ($|x-y| \geq |x| + |y|$). Hence if $|x| < t - R$ and $|x-y| = t$, then $-|x| > R - t$ and

$$|y| \geq |x-y| - |x| = t + (R - t) = R.$$

We conclude that $u(t,x) = 0$ as for such y, $g(y) = h(y) = 0$ and $\nabla g(y) = 0$. $\square$

Two dimensional Wave Equation

The homogeneous second dimension wave equation for $c = 1$ is

$$\left(\partial_t^2 - \Delta\right) u = 0$$

where $u \in C^2$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$. The solution to the second dimension wave equation is given by Poisson's Formula.

Theorem: Poisson's Formula

The system

$$\begin{aligned} \left(\partial_t^2 - \Delta_x\right) u &= 0 \\ u(0,x) &= g(x) \in C^2(\mathbb{R}^2) \\ \partial_t u(0,x) &= h(x) \in C^1(\mathbb{R}^2) \end{aligned}$$

is solved by $u(t,x)$ where B is the ball in 2 dimensions:

$$u(t,x) = \partial_t \left[\frac{t}{2\pi} \int_{B(0,1)} \frac{g(x+tw)}{\sqrt{1-|w|^2}} \, \mathrm{d}^2 w \right] + \frac{t}{2\pi} \int_{B(0,1)} \frac{h(x+tw)}{\sqrt{1-|w|^2}} \, \mathrm{d}^2 w$$

- **Proof**: To solve this, we will use the method of descent using Kirchhoff's formula in 3-D to solve Poisson's formula in 2-D. Let

  $$\begin{aligned} g &= g(x_1, x_2) \in C^2(\mathbb{R}^3) \\ h &= h(x_1, x_2) \in C^1(\mathbb{R}^3) \end{aligned}$$

  meaning g and h are functions in $\mathbb{R}^3$, but they do not depend on the third variable. Solve this initial value problem *in* $\mathbb{R} \times \mathbb{R}^3$ using Kirchhoff's Formula and set $x_3 = 0$:

  $$u(t, x_1, x_2) = \partial_t \left[\frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} g(y_1, y_2) \mathrm{d}S(y) \right] + \frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1, y_2) \mathrm{d}S(y)$$

  We want to write this in 2 dimensions.

  Suppose that $y \in \partial B((x_1, x_2, 0), t)$, that is

  $$(y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^3 = t^2.$$

  We note that

  $$\frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1, y_2) \mathrm{d}S(y) = \frac{1}{2\pi t} \int_{\partial B_3((x_1,x_2,0),t) \cap \{y_3 \geq 0\}} h(y_1, y_2) \mathrm{d}S(y)$$

  We can parametrize the upper half sphere on the right hand side using $y_3 = \sqrt{t^2 - (y_1^2 + y_2^2)}$ where $(y_1, y_2) \in B_2((x_1, x_2), t)$. That means that the upper half sphere is the graph of $g(y) = \sqrt{t^2 - |y-x|^2}$, $y \in B_2(x, t)$. So, we have

  $$\begin{aligned} \nabla g &= \frac{-(y-x)}{\sqrt{t^2 - |y-x|^2}} \\ 1 + |\nabla g|^2 &= 1 + \frac{|y-x|^2}{t^2 - |y-x|^2} = \frac{t^2}{t^2 - |y-x|^2}, \\ \mathrm{d}S &= \frac{t \, \mathrm{d}^2 y}{\left(t^2 - |y-x|^2\right)^{1/2}}. \end{aligned}$$

Plugging this back into the original equation for $y_3 \geq 0$, we have

$$\frac{1}{2\pi t}\int_{\partial B_3((x_1,x_2,0),t)} h(y_1,y_2)\mathrm{d}S(y) = \frac{1}{2\pi t}\int_{B_2((x_1,x_2),t)} h(y)\frac{t}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}^2y$$

We can use the same process with the same result for the integral of $g(y_1,y_2)$. Plugging these back into the formula for u, we have

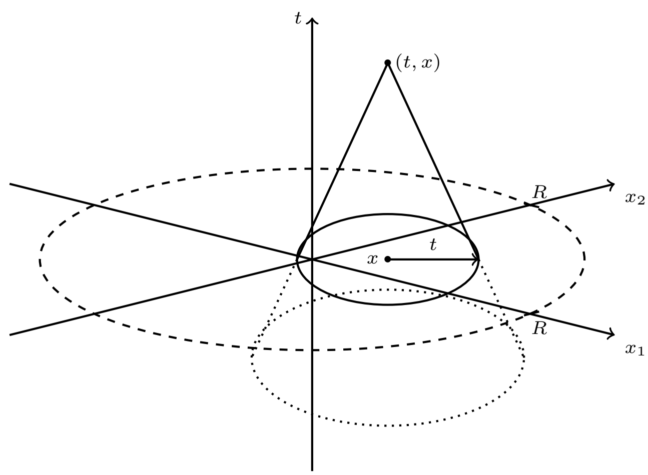
$$u(t,x_1,x_2) = \partial_t\left[\frac{1}{2\pi}\int_{B_2(x,t)}\frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}^2y\right] + \frac{1}{2\pi}\int_{B_2(x,t)}\frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}^2y$$

Using the change of variables $y = x + tw$ where $\mathrm{d}^2y = t^2\,\mathrm{d}^2w$, we have

$$u(t,x) = \partial_t\left[\frac{t}{2\pi}\int_{B_2(0,1)}\frac{g(x+tw)}{(1-|w|^2)^{1/2}}\,\mathrm{d}^2w\right] + \frac{t}{2\pi}\int_{B_2(0,1)}\frac{h(x+tw)}{(1-|w|^2)^{1/2}}\,\mathrm{d}^2w\,.$$

□

We can now consider the question of the speed of decay of solutions in 2D. Suppose $g(x) \equiv 0$ and $h(x) = 0$ for $|x| > R$ for some $R \in \mathbb{R}_{>0}$. Fix $x_0 \in \mathbb{R}^2$.



We want to describe $u(t,x_0)$ as $t \to \infty$. We have

$$u(t,x_0) = \frac{1}{2\pi t}\int_{B_2(x_0,t)} h(y)\left(1-\frac{|x_0-y|^2}{t^2}\right)^{-1/2}\mathrm{d}^2y$$

We take t large, so $t > |x_0| + R$. $B_2(0,R) \subset B_2(x_0,t)$ since $|y| \leq R$, so $|x_0 - y| \leq |x_0| + |y| \leq |x_0| + R < t$.

$$= \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\left(1-\frac{|x_0-y|^2}{t^2}\right)^{-1/2}\mathrm{d}^2y$$

We have $q = \frac{|x_0-y|^2}{t^2} \to 0$ as $t \to \infty$ since x_0 is fixed and w is bounded. We want to approximate $(1-q)^{-1/2}$.

$$(1+q)^a = 1 + aq + \frac{a(a-1)}{2}q^2 + \frac{a(a-1)(a-2)}{3!}q^3 + \cdots$$
$$(1-q)^{-1/2} = 1 + \frac{1}{2}q + \frac{3}{8}q^2 - \cdots$$

So, we have

$$u(t,x_0) = \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\left[1 + \frac{1}{2}\frac{|x_0-y|^2}{t^2} + \frac{3}{8}\frac{|x_0-y|^4}{t^4} + \cdots\right]\mathrm{d}y = \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\mathrm{d}y + E(t,x_0)$$

where $E(t,x_0)$ is the error term bounded by $1/t^3$. The wave decays at approximately $1/t$, which is far slower than the decay rate in 3-D where the solution eventually vanishes.

Summary of Wave Equations

Recall that the general form of the wave equation for $c = 1$ is

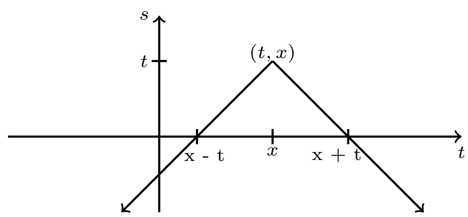
$$\left(\partial_t^2 - \Delta\right) u = 0$$

with initial conditions $u(0, x) = g(x) \in C^2(\mathbb{R}^n)$ and $\partial_t u(0, x) = h(x) \in C^1(\mathbb{R}^n)$ where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$.

- For $n = 1$, the wave equation is solved by

$$u(t, x) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\mathrm{d}s$$

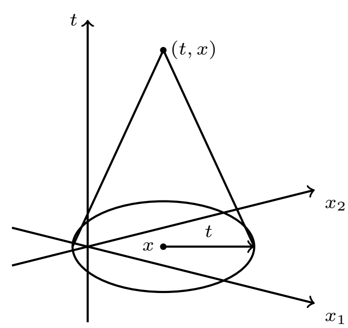
  The value of $u(t, x)$ is determined only by the values between $x - t$ and $x + t$.



- For $n = 2$, the wave equation is solved by

$$\begin{aligned} u(t, x) &= \partial_t \left[\frac{1}{2\pi}\int_{B(x,t)} \frac{g(w)}{\left(t^2 - |x - w|^2\right)^{1/2}}\,\mathrm{d}^2 w\right] + \frac{1}{2\pi}\int_{B(x,t)} \frac{h(w)}{\left(t^2 - |x - w|^2\right)^{1/2}}\,\mathrm{d}^2 w \\ &= \partial_t \left[\frac{t}{2\pi}\int_{B(0,1)} \frac{g(x + ty)}{\left(1 - |y|^2\right)^{1/2}}\,\mathrm{d}^2 y\right] + \frac{t}{2\pi}\int_{B(0,1)} \frac{h(x + ty)}{\left(1 - |y|^2\right)^{1/2}}\,\mathrm{d}^2 y \end{aligned}$$

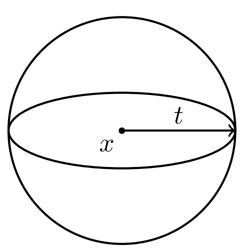
  The value of $u(t, x)$ is determined only by the values on the disk of radius t centered at x.

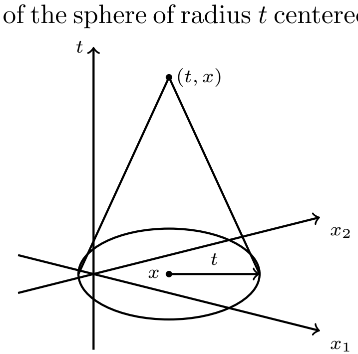


- For $n = 3$, the wave equation is solved by

$$u(t, x) = \partial_t \left[\frac{1}{4\pi t}\int_{\partial B(x,t)} g(y)\mathrm{d}S(y)\right] + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(y)\mathrm{d}S(y)$$

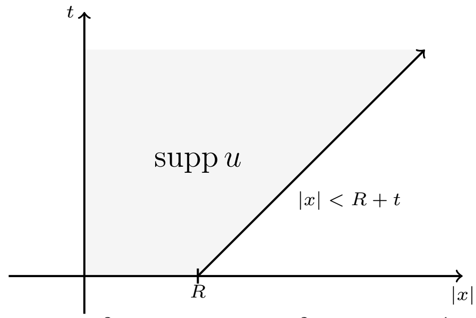
  The value of $u(t, x)$ is determined only by the values on the surface of the sphere of radius t centered at x.



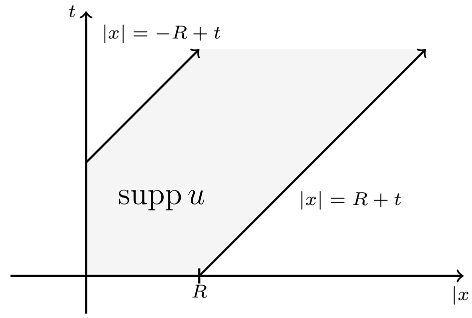


If $g(x) = 0$ and $h(x) = 0$ for $|x| > R$, then for $|x| > R + t$:

For $n = 1, 2$:



For $n = 3$:



To have $u \in C^2$, we need $g \in C^2$ and $h \in C^1$.

Example

Let $n = 2$, $g(x) = 0$, and

$$h(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

So, we have

$$u(t, 0) = \begin{cases} \frac{1}{2\pi} \int_{B(0,t)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w & t \le 1 \\ \frac{1}{2\pi} \int_{B(0,1)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w & t > 1 \end{cases}$$

Let $w = t\,(r\cos(\theta), r\sin(\theta))$. Then, we have $|w|^2 = t^2 r^2$ meaning $r = \frac{|w|}{t}$ and $\mathrm{d}^2 w = t^2 r \mathrm{d}r\mathrm{d}\theta$. Solve for each condition of $u(t, 0)$. For $t \le 1$, we have $|w| \le t \le 1$ meaning $r \le 1$.

$$\begin{aligned} \frac{1}{2\pi} \int_{B(0,t)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{1}{(t^2 - t^2 r^2)^{1/2}} t^2 r \mathrm{d}r \mathrm{d}\theta \\ &= t \int_0^1 \frac{r}{(1 - r^2)^{1/2}} \mathrm{d}r \end{aligned}$$

Use u substitution: $s = r^2$, $\mathrm{d}s = 2r$.

$$\begin{aligned} &= \frac{t}{2} \int_0^1 \frac{1}{(1 - s)^{1/2}} \mathrm{d}s \\ &= t. \end{aligned}$$

Now, for $t > 1$, we have $|w| \ge t > 1$ meaning $0 \le r < \frac{1}{t}$.

$$\frac{1}{2\pi} \int_{B(0,1)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{1/t} \frac{t}{(1 - r^2)^{1/2}} r \mathrm{d}r \mathrm{d}\theta$$

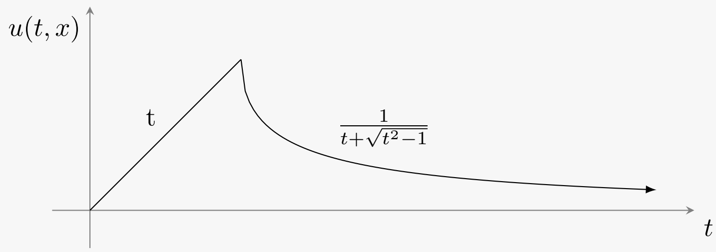
Use the same substitution:

$$\begin{aligned} &= \frac{t}{2} \int_0^{1/t^2} \frac{1}{(1 - s)^{1/2}} \mathrm{d}s \\ &= t\left(1 - \left(1 - \tfrac{1}{t^2}\right)^{1/2}\right) \\ &= t - \sqrt{t^2 - 1} \\ &= \frac{1}{t + \sqrt{t^2 - 1}} \\ &\approx \frac{1}{2t} \text{ as } t \to \infty \end{aligned}$$

So, we have our formula for $u(t,0)$:

$$u(t,0) = \begin{cases} t & t \leq 1 \\ \dfrac{1}{t+\sqrt{t^2-1}} & t > 0 \end{cases}$$

We can plot this solution:



Let us now consider a similar problem for $n = 3$: $g(x) = 0$ and

$$h = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

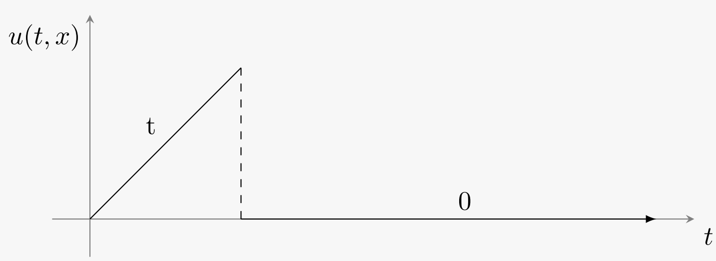
Similarly, we can find $u(t,0)$.

$$\begin{aligned} u(t,0) &= \begin{cases} \dfrac{1}{4\pi t}\displaystyle\int_{\partial B(x,t)} 1 \mathrm{d}S(y) & t \leq 1 \\ 0 & t > 1 \end{cases} \\ &= \begin{cases} \dfrac{1}{4\pi t} \cdot 4\pi t^2 & t \leq 1 \\ 0 & t > 1 \end{cases} \end{aligned}$$

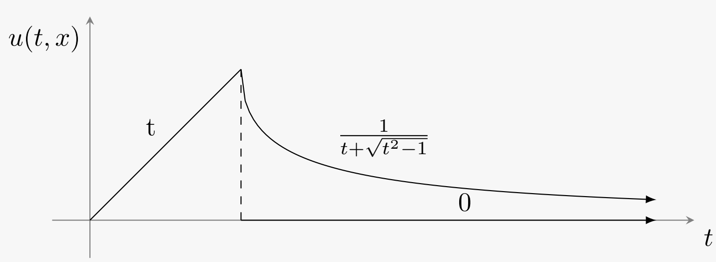
where $4\pi t^2$ is the surface area of the sphere of radius t

$$= \begin{cases} t & t \leq 1 \\ 0 & t > 1 \end{cases}$$

We can plot this solution as well.



Observe the difference between the two plots. In 2 dimensions, after passing a certain time, the wave slowly decays while in 3 dimensions, the wave immediately disappears.



Energy

Given the wave equation

$$\left(\partial_t^2 - \Delta\right) u = 0$$

with $u(0,x) = g \in C^2$ and $\partial_t u(0,x) = h \in C^1$, u exists with $u \in C^2$. For $g(x) = h(x) = 0$ for $|x| \geq R$ for some R, we have $u(t,x) = 0$ for $|x| \geq R + t$. (Strictly speaking we know it only for $n = 1, 2, 3$ but it is valid for all n.)

Then we define

$$E(u)[t] = \frac{1}{2}\int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2\right] \mathrm{d}^n x$$

to be the **energy** of a solution $u(t,x)$ at time t. This makes sense since $u = 0$ for $|x| > R + t$ and $u \in C^2$ so we are integrating a continuous function over a bounded set.

Theorem

Under the assumptions made above,

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2\right] \mathrm{d}^n x$$

is independent of t.

- **Proof**: it suffices to show that $\partial_t E(u) = 0$, meaning $E(u)$ is is constant as a function of t:

$$\begin{aligned}
\partial_t E(u) &= \partial_t \left[\frac{1}{2}\int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2\right] \mathrm{d}^n x\right] \\
&= \partial_t \left[\frac{1}{2}\int_{\mathbb{R}^2} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2\right] \mathrm{d}^n x\right]
\end{aligned}$$

Evaluate the components separately.

$$\begin{aligned}
\partial_t\left(\left(\frac{\partial u}{\partial t}\right)^2\right) &= 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} \\
\partial_t\left(\left(\frac{\partial u}{\partial x_j}\right)^2\right) &= 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}
\end{aligned}$$

So, we have

$$\partial_t E(u) = \int_{\mathbb{R}^n} \left[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1 \partial t} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial^2 u}{\partial x_n \partial t}\right] \mathrm{d}^n x$$

To evaluate $\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial u}{\partial t} \mathrm{d}x_1 \cdots \underline{\mathrm{d}x_j} \cdots \mathrm{d}x_n$, use integration by parts in the x_j variable. Integration by parts for functions for $v \in C^1$ and $w \in C^1$ which vanish for y large means

$$\int_{\mathbb{R}} \frac{\partial w}{\partial y} v \mathrm{d}y = -\int_{\mathbb{R}} w \frac{\partial v}{\partial y} \mathrm{d}y$$

(this follows from $\int \partial_y(uv)\mathrm{d}y = \int \left[\partial_y u \cdot v + u \cdot \partial_y v\right] \mathrm{d}y$ and the fact that $u, v = 0$ for $|y| > R$.)

So, we have

$$\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial u}{\partial t} \mathrm{d}x_1 \cdots \underline{\mathrm{d}x_j} \cdots \mathrm{d}x_n = -\int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t}\, \mathrm{d}^n x$$

meaning

$$\partial_t E(u) = \int_{\mathbb{R}^n} \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}\right) \mathrm{d}^n x = 0$$

So, $\partial_t E(u) = 0$ meaning $E(u)$ is independent of t. □

Suppose for $\Omega \subset \mathbb{R}^n$ bounded and $\partial\Omega$ is piecewise C^1. And suppose that $u \in C^2(\mathbb{R}_+ \times \overline{\Omega})$ solves $\left(\partial_t^2 - \Delta\right) u = 0$. Suppose $u(t,x) = 0$ when $g \geq 0$ and $x \in \partial\Omega$.

Theorem

Define

$$E(u) = \frac{1}{2}\int_\Omega \left|\partial_t u(t,x)\right|^2 + \left|\nabla u(t,x)\right|^2 \mathrm{d}^n x\,.$$

Then, $E(u)$ is independent of t.

The proof of this is the same as that of the previous theorem except that instead of using integration by parts, we use the divergence theorem – see the lemma below.

Theorem

Suppose $u \in C^2(\mathbb{R}_+ \times \overline{\Omega})$ solves $\left(\partial_t^2 - \Delta\right) u = f(t,x)$ with $u(0,x) = g(x) \in C^2$, $\partial_t u(0,x) = h(x) \in C^1$, and $u(t,x) = 0$ for $t \geq 0$ and $x \in \partial\Omega$. Then, u is unique.

- **Proof**: suppose that $\tilde{u}$ is another solution that satisfies the conditions. Consider $U(t,x) = u(t,x) - \tilde{u}(t,x)$. Then, we have

$$\left(\partial_t^2 - \Delta\right) U = 0$$

with $U(0,x) = 0$ and $\partial_t U(0,x) = 0$. From the previous theorem,

$$\begin{aligned}\int \left|\partial_t U(t,x)\right|^2 + \left|\nabla U(t,x)\right|^2 \mathrm{d}^n x &= \int \underbrace{\left|\partial_t U(0,x)\right|^2}_{=0} + \underbrace{\left|\nabla U(0,x)\right|^2}_{=0} \mathrm{d}^n x \\ &= 0\end{aligned}$$

So, $U(t,x) = 0$ meaning $u(t,x) = \tilde{u}(t,x)$, or u is unique. □

Lemma: Green's Identity

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}^n x = -\int v\Delta u \mathrm{d}x + \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S$$

where ν is the outward pointing normal vector.

- **Proof**: let $F = v \cdot \nabla u$. Then, $\mathrm{div}(F) = \nabla u \cdot \nabla v + v\Delta u$. We can apply the divergence theorem in reverse from this.

$$\begin{aligned}\nabla \cdot (v \cdot \nabla u) &= \nabla v \cdot vu + v \cdot \nabla \cdot \nabla u \\ &= \nabla v \cdot \nabla u + v\Delta u\end{aligned}$$

So,

$$
\begin{aligned}
\int_\Omega (v\Delta u + \nabla v \cdot \nabla u)\,\mathrm{d}^n x &= \int_\Omega \nabla \cdot (v \cdot \nabla u)\,\mathrm{d}^n x \\
&= \int_{\partial\Omega} \partial_\nu \left(v \cdot \nabla u\right)\mathrm{d}S && \text{Divergence Theorem} \\
&= \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S \\
\int_\Omega v\Delta u \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla u\,\mathrm{d}^n x &= \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S \\
\int_\Omega \nabla v \cdot \nabla u \mathrm{d}^n x &= -\int v\Delta u \mathrm{d}x + \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S && \square
\end{aligned}
$$

Separation of Variables

Separation of variables is a technique used to reduce PDEs to ODEs or simpler PDEs.

Proposition

Suppose $u \in C^2([a,b] \times \Omega)$ solves the wave equation $\left(\partial_t^2 - \Delta\right) u = 0$ and

$$u(t,x) = v(t)\varphi(x)$$

Then, in any set where $u \neq 0$, v and φ satisfy

$$\begin{aligned} \partial_t^2 v(t) &= \kappa v(t) \\ \Delta\varphi(x) &= \kappa\varphi(x) \end{aligned}$$

for some constant κ.

- **Proof**: by plugging $u(t,x) = v(t)\varphi(x)$ into the wave equation, we have

$$\begin{aligned} \left(\partial_t^2 - \Delta\right)(v(t)\varphi(x)) &= 0 \\ \partial_t^2 v(t)\varphi(x) - v(t)\Delta\varphi(x) &= 0 \\ \partial_t^2 v(t)\varphi(x) &= v(t)\Delta\varphi(x) \end{aligned}$$

  $u \neq 0$, so $v(t), \varphi(x) \neq 0$.

$$\frac{\partial_t^2 v(t)}{v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)}$$

  The LHS is independent of x and the RHS is independent of t. So,

$$\frac{\partial_t^2 v(t)}{v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)} = \text{constant}$$

  Let this constant be κ.

$$\begin{aligned} \partial_t^2 v(t) &= \kappa v(t) \\ \Delta\varphi(x) &= \kappa\varphi(x) \end{aligned}$$

$\square$

This is useful since $\partial_t^2 v = \kappa v$ is easy to solve:

$$v(t) = Ae^{\sqrt{\kappa}t} + Be^{-\sqrt{\kappa}t}$$

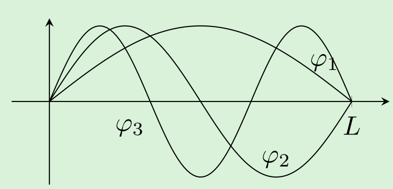
$\Delta\varphi = \kappa\varphi$ may be more difficult to solve depending on the dimension we are working in, that is n, $x \in \mathbb{R}^n$.

1-D Case

Theorem

The following two statements are equivalent (meaning the first is true if and only if the second is also true).

1. $\varphi \in C^2([0,L])$, $\varphi \neq 0$, $\varphi(0) = \varphi(L) = 0$, and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = \lambda\varphi(x)$
2. $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $\varphi = \varphi_n = A\sin\left(\sqrt{\lambda_n}x\right)$ for $n \in \mathbb{N}$



So, we have $v(t) = Ae^{i\sqrt{\lambda_n}t} + Be^{-i\sqrt{\lambda_n}t}$. This yields a single frequency solution

$$u_n(t,x) = \left(Ae^{i\sqrt{\lambda_n}t} + Be^{-i\sqrt{\lambda_n}t}\right)\sin\left(\sqrt{\lambda_n}x\right)$$

- **Proof**: in the reverse direction, we have

$$\begin{aligned}-\partial_x^2\varphi_n &= -\partial_x^2\left[A\sin\left(\sqrt{\lambda_n}x\right)\right] \\ &= -\partial_x\left(A\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right)\right) \\ &= A\lambda_n\sin\left(\sqrt{\lambda_n}x\right) \\ &= \lambda\varphi_n\end{aligned} \qquad \begin{aligned}\varphi(0) &= 0 \\ \varphi(L) &= A\lambda_n\sin\left(\sqrt{\lambda_n}L\right) \\ &= A\lambda_n\sin\left(\frac{n\pi}{L}\cdot L\right) \\ &= 0\end{aligned}$$

So, if 2. is true, then 1. is true.

Now, in the forward direction, we first want to show that if $-\varphi'' = \lambda\varphi$, $\varphi(0) = \varphi(L) = 0$ then $\lambda > 0$. For that we write

$$\begin{aligned}-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi\cdot\overline{\varphi} &= \lambda\varphi\overline{\varphi} \\ -\int_0^L\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi\right)\overline{\varphi}\mathrm{d}x &= \int_0^L\lambda\varphi\overline{\varphi}\mathrm{d}x\end{aligned}$$

Evaluate both sides separately: for the RHS,

$$\int_0^L\lambda\varphi(x)\overline{\varphi(x)}\mathrm{d}x = \lambda\int_0^L|\varphi(x)|^2\mathrm{d}x$$

For the LHS, use integration by parts. For any u and v, we have

$$-\int_0^L\frac{\mathrm{d}}{\mathrm{d}x}u\cdot v = u(0)v(0) - u(L)v(L) + \int_0^L u\cdot\frac{\mathrm{d}}{\mathrm{d}x}v$$

In our case, $u = \frac{\mathrm{d}}{\mathrm{d}x}\varphi$ and $v = \overline{\varphi}$. We know $v(0) = v(L) = 0$, so

$$-\int_0^L\left(\frac{d^2}{dx^2}\varphi\right)\overline{\varphi} = \int_0^L\left|\frac{d}{dx}\varphi(x)\right|^2\mathrm{d}x$$

So, we have

$$\underbrace{\int_0^L\left|\frac{d}{dx}\varphi(x)\right|^2\mathrm{d}x}_{\geq 0} = \lambda\underbrace{\int_0^L|\varphi(x)|^2\mathrm{d}x}_{\geq 0}$$

So, we know $\lambda \geq 0$. If $\lambda = 0$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}x} = 0$ and $\varphi(0) = 0$ meaning $\varphi = 0$ which is not allowed. So, $\lambda > 0$.

Now, we have $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi = \lambda\varphi$ where $\lambda > 0$. So, we have

$$\begin{aligned}\varphi(x) &= A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \\ \varphi(0) &= 0 = B \\ \varphi(L) &= 0 = A\sin\left(\sqrt{\lambda}L\right) \\ \sin\left(\sqrt{\lambda}L\right) &= 0\end{aligned}$$

$\sin(y) = 0$ when $y = n\pi$ for $n \in \mathbb{Z}$.

$$\begin{aligned}\sqrt{\lambda}L &= n\pi \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \qquad \text{for } n \in \mathbb{N}\end{aligned}$$

So, if 1. is true, then 2. is true. ☐

Theorem

The following two statements are equivalent (meaning the first is true if and only if the second is also true).

1. $\varphi \in C^2([0, L])$, $\varphi \neq 0$, $\varphi(0) = \varphi'(L) = 0$, and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = \lambda\varphi(x)$
2. $\lambda = \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ and $\varphi = \varphi_n = A\sin\left(\sqrt{\lambda_n}x\right)$ for $n \in \mathbb{N}$

- **Proof**: this has the same proof as the previous theorem except

$$\varphi'(L) = 0 = A\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)$$

$\cos(y) = 0$ when $y = \left(n + \frac{1}{2}\right)\pi$ for $n \in \mathbb{Z}$.

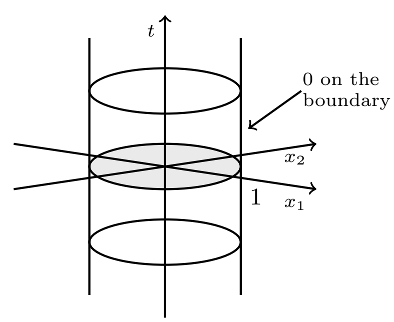
$$\lambda = \left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)^2 \qquad \square$$

λ_n are eigenvalues of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $[0, L]$ with

$$\begin{cases} \varphi(0) = \varphi(L) = 0 & \text{Dirichlet boundary conditions} \\ \varphi(0) = \varphi'(L) = 0 & \text{Mixed boundary conditions} \end{cases}$$

2-D Case

In the 2-D case, we have $\left(\partial_t^2 - \Delta\right)u(t, x) = 0$ for $t \in \mathbb{R}$ and we consider $x \in \Omega$ with the boundary condition $u(t, x) = 0$ if $x \in \Omega$ And example is given by $\Omega = B_2(0, 1)$. Then



Once again, we want to find solution which "separate": $u(t, x) = v(t)\varphi(x)$.

$$\begin{aligned} -\Delta\varphi(x) &= \lambda\varphi(x) && \text{for } |x| < 1 \\ \varphi(x) &= 0 && \text{for } |x| = 1 \\ -\partial_t^2 v(t) &= \lambda v(t) \\ v(t) &= Ae^{i\sqrt{\lambda}t} + Be^{-i\sqrt{\lambda}t} \end{aligned}$$

Lemma

Suppose Ω is bounded and $\partial\Omega$ is piecewise C^1. If $\varphi \in C^2(\overline{\Omega})$ and $-\Delta\varphi = \lambda\varphi$, if $\varphi|_{\partial\Omega} = 0$ where $\varphi \neq 0$, then $\lambda > 0$.

- **Proof**: we know $-\Delta\varphi \cdot \overline{\varphi} = \lambda\varphi\overline{\varphi}$, so $\int -\Delta\varphi \cdot \overline{\varphi} = \int \lambda\varphi\overline{\varphi}$. We apply the divergence theorem:

$$\operatorname{div}(\overline{\varphi} \cdot \nabla\varphi) = \nabla\overline{\varphi}\nabla\varphi + \overline{\varphi}\Delta\varphi$$

$$\begin{aligned} \int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x + \int_\Omega \overline{\varphi}\Delta\varphi &= \int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x + \int_{\partial\Omega} \underbrace{\overline{\varphi} \cdot \nu}_{=0} \cdot \nabla\varphi \\ &= \int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x. \end{aligned}$$

So, we have

$$0 \leq \int_\Omega |\nabla\varphi(x)|^2 = \lambda\int_\Omega |\varphi(x)|^2$$

Since $\varphi \neq 0$, the inequality is strict and we have $\lambda > 0$. $\square$

Example

Let $\Omega = [0, L_1] \times [0, L_2]$. $-\Delta\varphi = \lambda\varphi$, and we have Dirichlet boundary condition $\varphi|_{\partial\Omega} = 0$. We want to find solutions of the form $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$.

$$\begin{aligned} -\Delta\varphi = -\left(\partial_{x_1}^2\varphi_1\right)\varphi_2 - \left(\partial_{x_2}^2\varphi_2\right)\varphi_1 = \lambda\varphi_1\varphi_2 \\ -\frac{\partial_{x_1}^2\varphi_1(x_1)}{\varphi_1(x_1)} = \frac{\partial_{x_2}^2\varphi_2(x_2)}{\varphi_2(x_2)} + \lambda = \kappa \end{aligned}$$

So, we can solve for φ_1 and φ_2.

$$\begin{aligned} -\partial_{x_1}^2\varphi_1(x_1) &= \kappa\varphi_1(x_1) & \qquad -\partial_{x_2}^2\varphi_2(x_2) &= (\lambda-\kappa)\varphi_2(x_2) \\ \varphi_1(0) = \varphi_1(L_1) &= 0 & \varphi_2(0) = \varphi_2(L_2) &= 0 \end{aligned}$$

These are 1-D equations that we have already solved.

$$\begin{aligned} \kappa &= \left(\tfrac{n\pi}{L_1}\right)^2 & \qquad \lambda - \kappa & \left(\tfrac{m\pi}{L_2}\right)^2 \\ \varphi_1(x_1) &= \sin\left(\tfrac{n\pi}{L_1}x_1\right) & \varphi_2(x_2) &= \sin\left(\tfrac{m\pi}{L_2}x_2\right) \end{aligned}$$

So, we have

$$\lambda_{m,n} = \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2$$

for $m, n \in \mathbb{N}$.

$-\Delta\varphi = \lambda\varphi$ is the **eigennvalue problem** where $\lambda_{m,n}$ are the **eigenvalues** and φ is the eigenfunction or **eigenvector**.

Separation of Variables in Polar Coordinates

Consider $\Omega = D(0,1)$ which is the disk of radius 1 centered at $0 \in \mathbb{R}^2$. In polar coordinates, we have $x_1 = r\cos(\theta)$ and $x_2 = r\sin(\theta)$, and $\Omega = D(0,1)$ is given by $r < 1$. So, it is natural to look for solutions of the form

$$\varphi(r\cos(\theta), r\sin(\theta)) = f(r)g(\theta)$$

and we want $-\Delta\varphi = \lambda\varphi$ and $\varphi|_{\partial\Omega=0}$. We have the following conditions for $f(r)$ and $g(\theta)$:

$$f(1) = 0, \quad g(\theta) = g(\theta + 2\pi)$$

The first condition is for $\varphi|_\partial\Omega = 0$ as $\partial\Omega$ is given by $r = 1$ and the second one comes from the fact that θ and $\theta + 2\pi$ correspond to the same point. Recall the Laplacian for polar coordinates:

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

So, we have

$$\begin{aligned} -\Delta\left(f(r)g(\theta)\right) &= \lambda f(r)g(\theta) \\ \left[\left(-\partial_r^2 - \frac{1}{r}\partial_r\right)f(r)\right]g(\theta) - f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta) &= \lambda f(r)g(\theta) \\ \left(-\partial_r^2 - \frac{1}{r}\partial_r - \lambda\right)f(r)g(\theta) - f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta) &= 0 \\ \left(\partial_r^2 + \frac{1}{r}\partial_r + \lambda\right)f(r)g(\theta) &= -f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta) \\ \frac{\left(r^2\partial_r^2 + r\partial_r + \lambda r^2\right)f(r)}{f(r)} &= -\frac{\partial_\theta^2 g(\theta)}{g(\theta)} = \kappa \end{aligned}$$

To find g we note that it has to be periodic in θ, so

$$-\partial_\theta^2 g(\theta) = \kappa g(\theta), \quad g(\theta + 2\pi) = g(\theta)$$

since $g(\theta)$ is 2π-periodic. So, the general solution is

$$\begin{aligned} g(\theta) &= Ae^{i\sqrt{\kappa}\theta} + Be^{-i\sqrt{\kappa}\theta} \\ &= g(\theta + 2\pi) \\ &= Ae^{i\sqrt{\kappa}\theta + 2\pi i\sqrt{\kappa}} + Be^{-i\sqrt{\kappa}\theta - 2\pi i\sqrt{\kappa}} \implies \\ e^{i\theta\sqrt{\kappa} + 2\pi i\sqrt{\kappa}} &= e^{i\sqrt{\kappa}\theta} \implies e^{2\pi i\sqrt{\kappa}} = 1 \end{aligned}$$

So, $\sqrt{\kappa} \in \mathbb{Z}$ meaning $k = n^2$ for $n \in \mathbb{Z}$.
Now, solve for $f(r)$.

$$\begin{aligned} \left(r^2\partial_r^2 + r\partial_r + \lambda r^2 - n^2\right) f(r) &= 0 \\ f(1) &= 0 \end{aligned}$$

from the boundary conditions. Use change of variables $\lambda r^2 = z^2$, so $z = \sqrt{\lambda} r$.

$$\begin{aligned} \partial_r^2 &= \left(\frac{\partial z}{\partial r}\right)^2 \partial_z^2 = \lambda \partial_z^2 \\ r\partial_r^2 &= \left(\frac{z}{\lambda}\right)^2 \lambda\partial_z^2 = z^2\partial_z^2 \end{aligned}$$

Let $F(z) = f\left(\frac{z}{\sqrt{\lambda}}\right)$.

$$f(r) = F(\sqrt{\lambda} r)$$

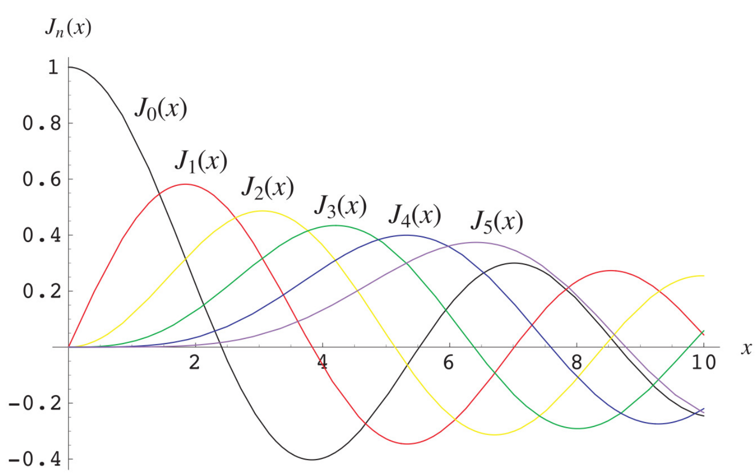
So, we have the following equation and the following boundary condition:

$$\begin{aligned} \left(z^2\partial_z^2 + z\partial_z + z^2 - n^2\right) F &= 0 \\ F(\sqrt{\lambda}) &= 0 \end{aligned}$$

The equation is called Bessel's Equation. It has 2 independent solutions $J_n(z)$ and $Y_n(z)$ which have the following properties:

$$\begin{aligned} J_n(z)\big|_{z\approx 0} &\approx z^n \\ Y_n(z)\big|_{z\approx 0} &\approx z^{-n} \end{aligned}$$

as $z \to 0$. The plots of J_n look as follows



We want $\lambda = \lambda_{n,k}$ such that $J_n\left(\sqrt{\lambda_{n,k}}\right) = 0$.

$$\varphi^{\pm}_{n,k}(r\cos(\theta), r\sin(\theta)) = e^{\pm in\theta} J_n\left(\sqrt{\lambda_{n,k}}r\right)$$

for $\lambda_{n,k} = j_{n,k}^2$ where $j_{n,1} < j_{n,2} < \cdots$ are the zeros of J_n.

This turns out to give us all the solutions which "separate":

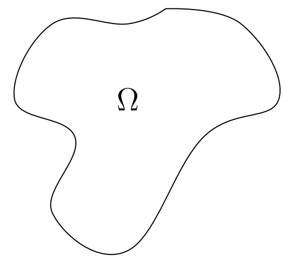
$$u(t,x) = u(r\cos(\theta), r\sin(\theta)) = \left(A\cos\left(\sqrt{\lambda_{n,k}}t\right) + B\sin\left(\sqrt{\lambda_{n,k}}\right)\right) J_k\left(\sqrt{\lambda_{n,k}}r\right)$$

Domains in 2-D

Suppose we have a domain Ω which is associated with the sequence of eigenvalues (frequencies)

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

where the first two inequalities are strict and the rest are not.



These are λ's for which $\exists\varphi \neq 0$ such that $-\Delta\varphi = \lambda\varphi$ and $\varphi|_{\partial\Omega} = 0$. We have found these λ's for two cases:

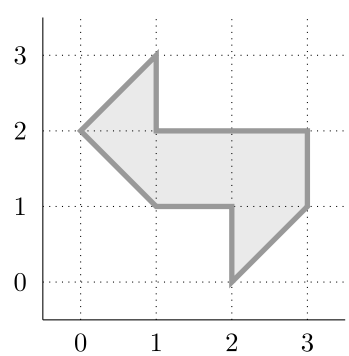
1. For $\Omega = [0, L_1] \times [0, L_2]$,
$$\lambda_{n,m} = \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2$$
2. For $\Omega = D(0,1)$,
$$\lambda_{n,k} = j_{n,k}^2 \text{ where } J_n(j_{n,k}) = 0$$

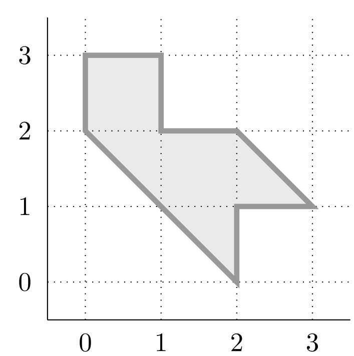
The λ's are nodes of oscillation over Ω. Each domain has a sequence of λ.

Question: if we have 2 different domains Ω, do we have 2 different sequences?

This question was popularized in the article "Can One Hear the Shape of a Drum?" written by Mark Kac in 1967. The idea is that if we know the λ_n sequences (all frequencies of possible oscillations) of a given domain, can we tell what the domain is? In other words, can we recover the shape of a domain from knowing the frequency of oscillations?

It turns out that multiple different domains can have the same sequence. It is still unknown if there are different smooth domains that have the same sequence. It is also unknown if there are different convex domains with the same sequence.





These two polygons have the same sequence. The reason for this is complex, they are both projected by the same shape in higher dimensions. Note that they are both not smooth and are both not convex.

3-D Case

In 3 dimensions, we have 2 main motivations.

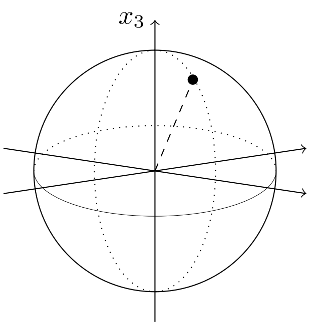
1. Solve the wave equation of a ball with Dirichlet boundary conditions given by
$$\left(\partial_t^2 - \Delta\right) u = 0$$
for $t \geq 0$ and $x \in B_3(0,1)$ where $u(t,x) = 0$ for $x \in \partial B_3(0,1)$. The pure mode mode solutions are
$$u(t,x) = \left(Ae^{it\sqrt{\lambda}} + Be^{-it\sqrt{\lambda}}\right)\varphi(x)$$
where $-\Delta\varphi = \lambda\varphi$ for $\varphi|_{\partial B(0,1)} = 0$.
2. Find the energies, λ, of the hydrogen atom given by the the Schrödinger equation
$$-\Delta - \frac{1}{|x|}$$
where $(-\Delta - 1/x) = \lambda\psi$ and $\psi \to 0$ as $|x| \to \infty$.

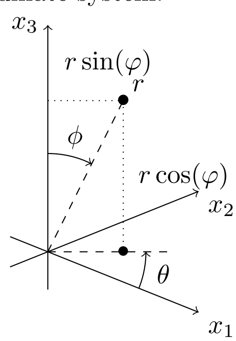
Separation of Variables in Spherical Coordinates

Observe the change of variables from Cartesian to spherical coordinates.

$$(x_1, x_2, x_3) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

We can illustrate the spherical coordinate system.





$$0 \leq \varphi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq r$$

The Laplacian in spherical coordinates is given by

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{r^2\sin^2(\varphi)}\partial_\theta^2$$

So, for function

$$w(r,\theta,\varphi) = u(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

we can write the Laplacian in spherical coordinates in terms of the Laplacian in Cartesian coordinates Δ_x:

$$\Delta w(r,\theta,\varphi) = (\Delta_x u)(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

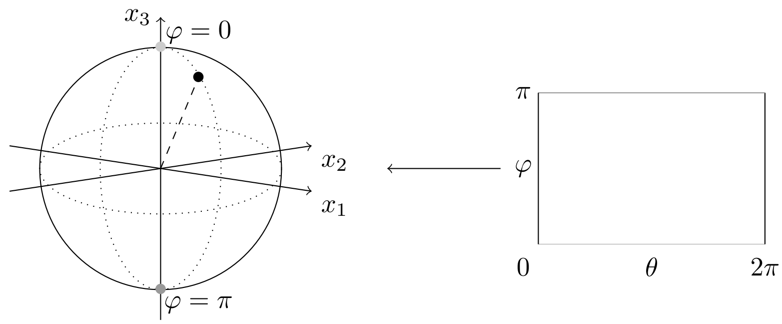
We can rewrite this as

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2} \tag{9}$$

where the spherical Laplacian, $\Delta_{\mathbb{S}^2}$, is given by

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{\sin^2(\varphi)}\partial_\theta^2$$

Observe the similarity, and differences, between 2 and 3 dimensions.



We know the following:

S^1	$-\partial_\theta^2 g = \lambda g,$	$g(\theta + 2\pi) = g(\theta)$
	$\lambda = n^2,$	$g(\theta) = e^{in\theta},\ n \in \mathbb{Z}$
S^2	$-\Delta_{S^2}\psi = \lambda\psi,$	φ is a "nice" function on S^2
	$\lambda = ?$	$\psi = ?$
S^3	?	

To find what happens on S^2, we want to solve

$$-\left[\frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{\sin^2(\varphi)}\partial_\theta^2\right]\psi = \lambda\psi$$

We write $\psi(\theta, \varphi) = h(\varphi)g(\theta)$ and $\lambda = \nu(\nu+1)$ which gives

$$-\left[\left(\frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right)\right)h(\varphi)\cdot g(\theta) + \frac{h(\varphi)}{\sin^2(\varphi)}\partial_\theta^2 g(\theta)\right] = \nu(\nu+1)g(\theta)h(\varphi)$$
$$\frac{\sin(\varphi)\partial_\varphi(\sin(\varphi)\partial_\varphi h(\varphi)) + \nu(\nu+1)\sin^2(\varphi)h(\varphi)}{h(\varphi)} = -\frac{\partial_\theta^2 g(\theta)}{g(\theta)} = \kappa$$

Note that the left fraction is independent of θ and the right is independent of φ. We want g be 2π periodic: $g(\theta + 2\pi) = g(\theta)$. Because of the equation for g and periodicity requirement we already know $\kappa = m^2$ and $g(\theta) = e^{im\theta}$ for $m \in \mathbb{Z}$ (see p.64). This gives us an equation for h:

$$\sin(\varphi)\partial_\varphi(\sin(\varphi)\partial_\varphi h(\varphi)) - \left(m^2 - \nu(\nu+1)\sin^2(\varphi)\right)h(\varphi) = 0$$

Use the change of variables $z = \cos(\varphi) \in [-1, 1]$. This means

$$\partial_\varphi = \frac{\partial z}{\partial\varphi}\partial_z = -\sin(\varphi)\partial_z,$$
$$\sin(\varphi)\partial_\varphi = -\sin^2(\varphi)\partial_z = -(1 - z^2)\partial_z$$

Plugging this back into the equation with $h(\varphi) = f(\cos(\varphi))$, we have

$$\left[(1-z^2)\partial_z(1-z^2)\partial_z - m^2 + \nu(\nu+1)(1-z^2)\right]f = 0$$
$$\left[(1-z^2)^2\partial_z^2 - (1-z^2)2z\partial_z - m^2 + \nu(\nu+1)(1-z^2)\right]f = 0$$
$$\left[(1-z^2)\partial_z^2 - 2z\partial_z + \nu(\nu+1) - \frac{m^2}{1-z^2}\right]f = 0$$

This is the **Associated Legendre Equation**. This has nice solutions that are smooth for $\nu = \ell \in \mathbb{Z}_{\geq 0}$ and $m \in \{-\ell, \dots, 0, \dots, \ell\} \in \mathbb{Z}$. Finding nice solutions to this allows us to find all possible λ's for the sphere and their associated eigenfunctions.

The solutions to the Associated Legendre Equation are

$$P_\ell^m(z) := \frac{(-1)^m}{2^\ell \ell!}(1-z^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}z^{\ell+m}}\left(z^2-1\right)^\ell$$

Note that when m is even, P_ℓ^m is a polynomial. So, we have solutions to $-\Delta_{\mathbb{S}^2}\psi = \nu(\nu+1)\psi$ with $\nu = \ell \in \mathbb{Z}_{\geq 0}$ and $m \in \{-\ell, \dots, \ell\} \in \mathbb{Z}$.

$$\begin{aligned}\psi(\theta,\varphi) =& Y_\ell^m(\varphi,\theta)\\ :=& P_\ell^m(\cos(\varphi))e^{im\theta}\end{aligned}$$

These are **spherical harmonics.** The multiplicity $\ell(\ell+1)$ as en eigenvalue of $-\Delta_{\mathbb{S}^2}$ is the number of independent φ's it corresponds to, which is $2\ell+1$. You do not have to remember the formulas for P_ℓ^m but you should be aware of the general structure of spherical harmonics

We now consider

$$-\Delta F = \lambda F, \quad F|_{\partial B_3(0,1)} = 0, \quad F = G(r)\psi(\varphi,\theta),$$

that is (see (9) above)

$$-\Delta(G(r)\psi(\varphi,\theta)) = \frac{1}{r^2}(-\partial_r(r^2\partial_r G) + \ell(\ell+1)G)\psi(\varphi,\theta) = \lambda G(r)\psi(\varphi,\theta).$$

That gives us

$$r^2\partial_r^2 G + 2r\partial_r G + (\lambda r^2 - \ell(\ell+1))G = 0, \quad G(1) = 0.$$

We again make a change for variable (see the discussion of Bessel functions!)

$$x = \sqrt{\lambda}r, \quad g(x) = G(x/\sqrt{\lambda}), \quad G(r) = g(\sqrt{\lambda}r),$$

so that the equation becomes

$$x^2\partial_x^2 g + 2x\partial_x g + (x^2 - \ell(\ell+1))g = 0, \quad g(\sqrt{\lambda}) = 0.$$

This is almost like Bessel's equation. To bring it even close to that equation we put

$$f(x) = \sqrt{x}g(x) \implies x^2\partial_x^2 f + x\partial_x f + (x^2 - (\ell+\frac{1}{2})^2)f = 0.$$

that is $f(x) = J_{\ell+\frac{1}{2}}(x)$. We put

$$j_\ell(x) = x^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(x)$$

These functions are called *spherical Bessel functions.* Amazingly they have a simple representation:

$$j_\ell(x) = (-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\frac{\sin x}{x}.$$

Returning to the original problem we see that

$$F(r) = j_\ell(\sqrt{\lambda}r)Y_\ell^m(\varphi,\theta), \quad j_\ell(\sqrt{\lambda}) = 0.$$

This means that the *eigenvalues of the Dirichlet Laplacian* on the unit ball in $\mathbb{R}^3$ are given by

$$\lambda = k_{\ell,n}^2, \quad j_\ell(k_{\ell,n}) = 0.$$

Hydrogen Atom

Emission of light happens at very specific sequences of frequencies depending on the composition of a distant star.

This can be explained by the Schrödinger equation and the particular solvable case is the quantum Hamiltonian of the Hydrogen atom:

$$H = -\Delta - \frac{1}{|x|}.$$

Just as in the case of bounded domains we are interested in *eigenvalues* of H:

$$H\psi = \lambda\psi, \quad \psi(x) \to 0, \quad |x| \to \infty. \tag{10}$$

Why the condition on ψ? For those of you have studied quantum mechanics $|\psi(x)|^2$ should describe the probability density of finding the particle (the electron in this case) at x and hence we need $\int |\psi(x)|^2 dx = 1$.

To study (10) we can separate variables:

$$\begin{aligned}
\left[-\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{\Delta_{\mathbb{S}^2}}{r^2} - \frac{1}{r}\right] h(r)Y_\ell^m(\varphi,\theta) &= \lambda h(r)Y_\ell^m(\varphi,\theta) \\
\left[-\frac{1}{r^2}\partial_r(r^2\partial_r h(r)) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r}\right] h(r)Y_\ell^m(\varphi,\theta) &= \lambda h(r)Y_\ell^m(\varphi,\theta) \\
\left[-\frac{1}{r^2}\partial_r(r^2\partial_r h(r)) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r} - \lambda\right] h(r) &= 0
\end{aligned}$$

First, we want to understand the behaviour as $r \to 0$. We postulate that $h(r) \approx r^\alpha$ for some $\alpha > 0$.

$$\begin{aligned}
&\left[-\partial_r^2 - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r} - \lambda\right] r^\alpha \\
&= \underbrace{-\alpha(\alpha-1)r^{\alpha-2} - 2\alpha r^{\alpha-2} + \ell(\ell+1)r^{\alpha-2}}_{=0} + ar^{\alpha-1} + br^\alpha
\end{aligned}$$

for some constants a and b.

$$\begin{aligned}
\alpha(\alpha-1) + 2\alpha - \ell(\ell+1) &= 0 \\
\alpha(\alpha+1) - \ell(\ell+1) &= 0
\end{aligned}$$

We have $\alpha = \ell$ or $\alpha = -\ell - 1$. The case of $-\ell - 1 < 0$ which is not allowed (the function would blow up), so we have $\alpha = \ell$. So,

$$h(r) \sim r^\ell, \quad r \to 0,$$

Now, we consider the behaviour as $r \to +\infty$. We have $-\partial_r^2 h - \lambda h \approx 0$. Let $\lambda = -\sigma^2$. So, $h(r) \approx e^{-\sigma r} \to 0$ only when $\sigma > 0$ (otherwise h does not go to 0 as $r \to \infty$. Hence we can write h as

$$h(r) = r^\ell e^{-\sigma r} q(r)$$

where $e^{-\sigma r}q(r) \to 0$ as $r \to \infty$. We want to find q.

$$\begin{aligned}
-\partial_r^2\left[r^\ell e^{-\sigma r}q(r)\right] &= -\partial_r\left[\ell r^{\ell-1}e^{-\sigma r}q(r) - \sigma r^\ell e^{-\delta r} + r^\ell e^{-\sigma r}q'(r)\right] \\
&= -\ell(\ell-1)r^{\ell-2}e^{-\sigma r}q(r) + \sigma\ell r^{\ell-1}e^{-\sigma r}q(r) - \ell r^{\ell-1}e^{-\sigma r}q'(r) \\
&\quad + \sigma\ell r^{\ell-1}e^{-\sigma r}q(r) - \sigma^2 r^\ell e^{-\sigma r}q(r) + \sigma r^\ell e^{-\sigma r}q'(r) \\
&\quad - \ell r^{\ell-1}e^{-\sigma r}q'(r) + \sigma r^\ell e^{-\sigma r}q'(r) - r^\ell e^{-\sigma r}q''(r)
\end{aligned}$$

Plug this back into the equation.

$$rq''(r) + 2(\ell + 1 - r\sigma)q'(r) + (1 - 2\sigma(\ell+1))q(r) = 0$$

Let $q(r) = \sum_{k=0}^{\infty} a_k r^k$ with $a_0 = 1$.

$$\sum_{k=0}^{\infty} \left[k(k-1)a_k r^{k-1} + 2k(\ell+1-r\sigma)a_k r^{k-1} + (1-2\sigma(\ell+1))a_k r^k\right] = 0$$
$$\sum_{k=0}^{\infty} \left[k(k-1)a_k r^{k-1} + 2k(\ell+1)a_k r^{k-1} + (1-\sigma k-2\sigma(\ell+1))a_k r^k\right] = 0$$

Let $k-1=j$, meaning $k=j+1$.

$$\sum_{j=0}^{\infty} \left[((j+1)j + 2(j+1)(\ell+1))\, a_{j+1} r^j\right] + \sum_{k=0}^{\infty} \left[(-\sigma k - 2\sigma(\ell+1))\, a_k r^k\right] = 0$$
$$\sum_{j=0}^{\infty} \left[(j+1)(j+2(\ell+1))a_{j+1} + (1-\sigma j - 2\delta(\ell+1))\, a_j\right] r^j = 0$$

So, we have

$$a_{j+1} = \frac{2\sigma(j+\ell+1)-1}{(j+1)(j+2(\ell+1))} a_j. \tag{11}$$

This gives us an inductive way to compute a_j. Now, if the numerator never vanishes, then this relation show s that

$$a_j \sim \frac{(2\sigma)^j}{j!} \implies q \sim e^{2\sigma r}, \quad r \to \infty.$$

But then $h \sim e^{2\sigma r} e^{-\sigma r} = e^{\sigma r} \not\to 0$, $r \to \infty$. So, the only possibility is if the numerator vanishes for some j. But that happens *only* when

$$\sigma = \frac{1}{2n} \quad \text{for some } n \geq \ell+1.$$

In that case we have $a_k = 0$ for $k > n-\ell-1$.

Since $\lambda = -\sigma^2$ this gives us the eigenvalues

$$\lambda_n = -\frac{1}{4n^2}.$$

These were in fact the observed frequencies! (We did not prove that these are all the eigenvalues but it is in fact the case.)

The final comment is about multiplicities. We get our eigenfunction

$$H\psi = \lambda_m \psi$$

of the form

$$\psi(r,\theta,\varphi) = r^\ell q_{n,\ell}(r) e^{-r/2n} Y_\ell^m(\theta,\varphi).$$

Here $q_{n,\ell}$ is the polynomial determined by (11)

How many choices do we have? The ranges of ℓ and m are $\ell \in \{0,\cdots,n-1\}$ and $m \in \{-\ell,...,\ell\}$. So for each n we have $n-1$ choices of ℓ and $(2\ell+1)$ choices of m for each ℓ. Hence the multiplicity of the eigenvalue $\lambda_n = -1/4n^2$ is given by

$$\sum_{\ell=0}^{n-1}(2\ell+1) = n^2.$$

The ground state is given by the smallest eigenvalue which is $-\frac{1}{4}$. At the ground state,

$$\psi = r^\ell e^{-\frac{1}{2n}r} q(r)$$

The smallest possible ℓ and n is 0 and $-\frac{1}{4}$ respectively.

$$\begin{aligned}\psi &= e^{-\frac{1}{4}r}q(r)\\ &= e^{-r/2}\end{aligned}$$

where $q(r) = 1$ is the simplest possible $q(r)$.

$$\begin{aligned}\left(-\Delta - \frac{1}{|x|}\right)e^{-\frac{|x|}{2}} &= 0\\ \left(-\partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{2}\right)e^{-r/2} &= -\frac{1}{4}e^{-r/2} + \frac{1}{r}e^{-r/2} - \frac{1}{r}e^{-r/2}\\ &= -\frac{1}{4}re^{-r/2}\end{aligned}$$

So, the ground state (eigenfunction corresponding the smallest eigenvalue) of the hydrogen atom is $-\frac{1}{4}e^{-r/2}$.

The Heat Equation

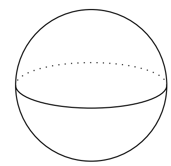
The heat equation takes the form

$$(\partial_t - \alpha\Delta)\, u = 0$$

where $\alpha > 0$ is the thermal diffusion constant and $u(t, x)$ is the temperature at time t and position x.

Let's consider a hot metal ball which we submerge submerge in freezing water. Then, we have

$$(\partial_t - \alpha\Delta u)\, u = 0$$



for $t \geq 0$ and $x \in B(0, 1) \subset \mathbb{R}^3$. The boundary conditions are given by $u(t, x) = 0$ for $x \in \partial B(0, 1)$ – that is because the temperature of the water is 0. What happens with the temperature distribution in the ball as time goes on?

Due to symmetries we should try spherical coordinates:

$$(x_1, x_2, x_3) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. Let us first assume that the initial temperature distribution depends only on the distance to the center of the ball. That suggests taking

$$u(t, r, \varphi, \theta) = v(t)h(r), \quad h(1) = 0.$$

We recall the Laplacian in spherical coordinates:

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{r^2\sin^2(\varphi)}\partial_\theta^2.$$

Hence

$$(\partial_t - \alpha\Delta)u = \partial_t v(t)h(r) - v(t)\alpha(\left(\partial_r^2 + \frac{2}{r}\partial_r\right) h(r),$$

which gives

$$\frac{\partial_t v(t)}{\alpha v(t)} = \frac{\partial_r^2 h(r) - (2/r)\partial_r h(r)}{h(r)} = -\lambda, \quad h(1) = 0,$$

or

$$\partial_t v + \alpha\lambda v(t) = 0, \quad -\partial_r^2 h(r) - \frac{2}{r}\partial_r h(r) = \lambda h(r), \quad h(1) = 0.$$

The equation for v is easy to solve:

$$v(t) = v(0)e^{-\lambda\alpha t}.$$

To consider the equation for $h(r)$ we notice that if we consider h as a function of x, $h(\sqrt{x_1^2 + x_2^2 + x_3^2})$, then $-\Delta h = \lambda h$, $h|_{\partial B(0,1)} = 0$, so $\lambda > 0$. Hence, as in the previous section, we can introduce $H(r) := \sqrt{r}h(r)$ and obtain

$$(r^2\partial_r^2 + r\partial_r + r^2\lambda - \tfrac{1}{4})H(r) = 0, \quad H(1) = 0.$$

With the substitution $z = \sqrt{\lambda}r$, and $f(\sqrt{\lambda}r) = H(r)$,

$$(r^2\partial_r^2 + r\partial_r + r^2 - \tfrac{1}{4})f(r) = 0, \quad f(\sqrt{\lambda}) = 0.$$

But this the Bessel equation and a solution which is nice at $r = 0$ is given by

$$f(z) = J_{\frac{1}{2}}(z) \implies h(r) = r^{-\frac{1}{2}}J_{\frac{1}{2}}(\sqrt{\lambda}r).$$

To satisfy $h(1) = 0$ we need $\lambda = j_{\frac{1}{2},k}$. Hence a solution to our problem is given by

$$u(t, x) = e^{-\alpha j_{\frac{1}{2},k}^2 t}J_{\frac{1}{2}}(j_{\frac{1}{2},k}r).$$

We can also consider solutions which involve higher spherical harmonics:

$$u(t, r, \varphi, \theta) = v(t)\,\underbrace{h(r)Y_\ell^m(\varphi, \theta)}_{\varphi(x)}$$

Recall

$$-\Delta_{\mathbb{S}^2} Y_\ell^m(\varphi, \theta) = \ell(\ell+1)Y_\ell^m(\varphi, \theta)$$

for $\ell = 0, 1, \ldots$ and $m = -\ell, \ldots, 0, \ldots, \ell$.

$$Y_\ell^m(\varphi, \theta) = P_\ell^m(\cos(\theta))e^{m\theta}$$

Plug this into the equation.

$$\begin{aligned}(\partial_t - \alpha\Delta)\,(v(t)\varphi(x)) = \partial_t v(t) \cdot \varphi(x) - v(t)\alpha\Delta\varphi(x) = 0 \\ \frac{\partial_t v(t)}{\alpha v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)} = -\lambda \\ \partial_t v + (\alpha\lambda)v = 0 \Longrightarrow v(t) = v(0)e^{-\lambda\alpha t}\end{aligned}$$

So, we want to find φ and λ such that $-\Delta\varphi = \lambda\varphi$ in $B(0,1)$ and $\varphi|_{\partial B(0,1)} = 0$. As shown before, $\lambda > 0$.

$$\begin{aligned}\varphi &= h(r)Y_\ell^m \\ -\Delta &= -\partial_r^2 - \frac{2}{r}\partial_r - \frac{\Delta_{\mathbb{S}}}{r^2} \\ -\Delta\varphi &= \left[\left(-\partial_r^2 - \frac{2}{r}\partial_r\right)h(r) + \frac{\ell(\ell+1)}{r^2}h(r)\right]Y_\ell^m \\ &= \lambda h(r)Y_\ell^m\end{aligned}$$

So, we have

$$\begin{aligned}\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2} + \lambda\right)h = 0 \\ \left(r^2\partial_r^2 + 2r\partial_r + \lambda r^2 - \ell(\ell+1)\right)h(r) = 0\end{aligned}$$

This is almost Bessel's Equation. Let $H(r) := \sqrt{r}h(r)$, which satisfies

$$\left(r^2\partial_r^2 + r\partial_r + r^2\lambda - (\ell + \tfrac{1}{2})^2\right)H(r) = 0$$

This is Bessel's Equation. Use the substitution $z = \sqrt{\lambda}r$ and $f(\sqrt{\lambda}r) = H(r)$.

$$\left(z^2\partial_z^2 + z\partial_z + z^2 - (\ell + \tfrac{1}{2})^2\right)f = 0$$

This has solutions $J_{\ell+\frac{1}{2}}(z)|_{z\approx 0} \approx z^{\ell+\frac{1}{2}}$ and $Y_{\ell+\frac{1}{2}}(z)|_{z\approx 0} \approx z^{-\ell-\frac{1}{2}}$. So, the physical solution is given by taking $J_{\ell+\frac{1}{2}}$. To satisfy the boundary condition we can take $J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r) = 0$ at $r = 1$.

$$\begin{aligned}h(r) &= r^{-1/2}J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r) \\ \sqrt{\lambda} &= j_{\ell+\frac{1}{2},k} \\ \lambda &= j_{\ell+\frac{1}{2},k}^2\end{aligned}$$

where $J_{\ell+\frac{1}{2}}\left(j_{\ell+\frac{1}{2},k}\right) = 0$. So, our final solution is

$$u(t, x) = e^{-\alpha j_{\ell+\frac{1}{2},k}^2 t}r^{-1/2}J_{\ell+\frac{1}{2}}\left(j_{\ell+\frac{1}{2},k}r\right)Y_\ell^m(\varphi, \theta).$$

1-D Case

The Scale-Invariant Solution

Consider the initial value problem

$$(\partial_t - \Delta)\, u(t,x) = 0, \quad u(0,x) = \varphi(x)$$

for $t > 0$ and $x \in \mathbb{R}^n$. When $n = 1$, we have

$$\left(\partial_t - \partial_x^2\right) u(t,x) = 0$$

The units of $\partial_t = \partial_x^2 = \frac{1}{\mathrm{m}^2}$ (meters). So, the variable $y = \frac{x}{\sqrt{t}}$ is dimensionless. It is then natural to look for solutions of the form

$$U(t,x) = q\left(\tfrac{x}{\sqrt{t}}\right).$$

We calculate:

$$\begin{aligned}\partial_t U &= \partial_t q\left(\tfrac{x}{\sqrt{t}}\right) \\ &= -\frac{1}{2}\frac{x}{t}\frac{1}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right)\end{aligned}$$

$$\begin{aligned}\partial_x^2 U &= \partial_x\left(\partial_x\left(q\left(\tfrac{x}{\sqrt{t}}\right)\right)\right) \\ &= \partial_x\left(\tfrac{1}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right)\right) \\ &= \frac{1}{t} q''\left(\tfrac{x}{\sqrt{t}}\right)\end{aligned}$$

$\left(\partial_t - \partial_x^2\right) u = 0$, so we have

$$\begin{aligned}-\frac{1}{2}\frac{x}{\sqrt{t}}\frac{1}{t} q'\left(\tfrac{x}{\sqrt{t}}\right) - \frac{1}{t} q''\left(\tfrac{x}{\sqrt{t}}\right) &= 0 \\ -\frac{1}{2}\frac{x}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right) - q''\left(\tfrac{x}{\sqrt{t}}\right) &= 0 \\ q''(y) &= -\frac{1}{2} y q'(y)\end{aligned}$$

where $y = \frac{x}{\sqrt{t}}$. Let $v(y) = q'(y)$.

$$\begin{aligned} v'(y) &= -\frac{1}{2} y v(y) \implies v(y) = v(0) e^{-y^2/4} \implies \\ q(y) &= \underbrace{q(0)}_{C_2} + \underbrace{v(0)}_{C_1} \int_0^y e^{-w^2/4} \mathrm{d}w \end{aligned}$$

As $t \to 0^+$, we have

$$\lim_{t \to 0+} U(t,x) = \begin{cases} C_2 + C_1 \int_0^{\infty} e^{-w^2/4} \mathrm{d}w & x > 0 \\ C_2 & x = 0 \\ C_2 + C_1 \int_0^{-\infty} e^{-w^2/4} \mathrm{d}w & x < 0 \end{cases}$$

Evaluate these integrals.

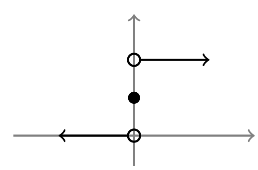
$$\begin{aligned}
\int_0^\infty e^{-w^2}\mathrm{d}w &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-x_1^2}\mathrm{d}x_1 \cdot \int_{-\infty}^{\infty} e^{-x_2^2}\mathrm{d}x_2\right)^{1/2} \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x_1^2}e^{-x_2^2}\mathrm{d}x_1\mathrm{d}x_2\right)^{1/2} \\
&= \frac{1}{2}\left(\int_{\mathbb{R}^2} e^{-|x|^2}\,\mathrm{d}^2x\right)^{1/2} \\
&= \frac{1}{2}\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty re^{-r^2}\mathrm{d}r\right)^{1/2} \\
&= \frac{1}{2}\left(\frac{2\pi}{2}\int_0^\infty (2r)e^{-r^2}\mathrm{d}r\right)^{1/2} \\
&= \frac{1}{2}\left(\pi\int_0^\infty e^{-s}\mathrm{d}s\right)^{1/2} \\
&= \frac{1}{2}(\pi)^{1/2} \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

So, $\int_0^\infty e^{-w^2/4}\mathrm{d}w = \frac{1}{2}\left(4\pi\right)^{1/2} = \sqrt{\pi}$.

$$\lim +_{t\to 0+}U(t,x) = \begin{cases} C_2 + C_1\sqrt{\pi} & x > 0 \\ C_2 & x = 0 \\ C_2 - C_1\sqrt{\pi} & x < 0 \end{cases}$$

We want to express this using the Heaviside Function which is defined as follows:

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$



For this to be true, we have $C_2 = \frac{1}{2}$ and $C_1 = \frac{1}{2\sqrt{\pi}}$.

This yields

$$U(t,x) := \frac{1}{2} + \frac{1}{\sqrt{4\pi}}\int_0^{x/\sqrt{t}} e^{-y^2/4}\mathrm{d}y$$

where $\left(\partial_t - \partial_x^2\right)U = 0$ and $U(t,x) \to H(x)$ as $t \to 0^+$.

Why do we use the Heaviside Function? For $\varphi \in C_c^1(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} \varphi'(z)\underbrace{H(x-z)}_{\substack{0 \text{ at } x-z<0 \\ 1 \text{ at } x-z>0}}\mathrm{d}z &= \int_{-\infty}^{x}\varphi'(z)\mathrm{d}z \\
&= \varphi(x) - \varphi(-\infty) \\
&= \varphi(x)
\end{aligned}$$

Consider

$$u(t,x) := \int_{\mathbb{R}} \varphi'(z)U(t,x-z)\mathrm{d}z$$

So, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \varphi'(x) H(x-z)\mathrm{d}z &= \varphi(x) && \text{as } t \to 0^+ \\
\left(\partial_t - \partial_x^2\right) u(t,x) &= \int_{\mathbb{R}} \varphi'(z) \underbrace{\left(\partial_t - \partial_x^2\right) U(t, x-z)}_{=0} \mathrm{d}z \\
&= 0 && \text{for } t > 0
\end{aligned}
$$

The Integral Solution Formula

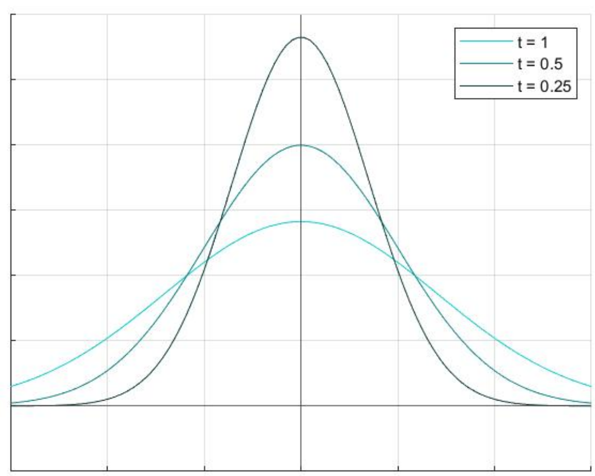
A better way to write this solution is

$$
\int_{\mathbb{R}} \varphi'(z) \left(\frac{1}{2} + \frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \mathrm{d}y \right) \mathrm{d}z.
$$

So,

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}} \varphi'(z)\mathrm{d}z &= 0 \\
\int_{\mathbb{R}} \varphi'(z) \left(\frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \mathrm{d}y \right) \mathrm{d}z &= -\int_{\mathbb{R}} \varphi(z) \partial_z \left[\frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \right] \mathrm{d}z \\
&= \int_{\mathbb{R}} \varphi(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z
\end{aligned}
$$

We have $\left(\partial_t - \partial_x^2\right) u(t,x) = 0$ where $u(t,x) \to \varphi(x) \in C_c^1(\mathbb{R})$ as $t \to 0^+$. For now, $\varphi \in C^0(\mathbb{R}^n)$ (bounded and continuous) is enough.



$$
u(t,x) = \int_{\mathbb{R}} \varphi(z) H_t(x-z)\mathrm{d}z =: H_t * \varphi(x),
$$

where

$$
H_t(y) := \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}
$$

is the heat kernel and $*$ is the convolution.

$$
\int_{\mathbb{R}} H_t(y)\mathrm{d}y = 1
$$

since by change of variables, we have

$$
\int_{-\infty}^{\infty} e^{-y^2/4}\mathrm{d}y = 2\sqrt{\pi}
$$

Compare this to the Dirac δ_0 "function" (for those of you with experience in quantum mechanics):

$$
\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \qquad \int \delta_0(x)\mathrm{d}x = 1
$$

In some sense, $H_0 = \delta_0$.

We summarize all this as follows: our final solution for the 1-D case $\left(\partial_t - \partial_x^2\right) u(t,x) = 0$ for $t > 0$, $u(t,x) \to \varphi(x) \in C_c^1(\mathbb{R})$ as $t \to 0^+$ is

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \varphi(z) e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z \\ &= H_t * \varphi(x) \end{aligned}$$

where the heat kernel is

$$H_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and the convolution of two functions φ and ψ is

$$\begin{aligned} \psi * \varphi(x) &= \int \psi(z)\varphi(x-z)\mathrm{d}z \\ &= \int \psi(x-z')\varphi(z')\mathrm{d}z' \end{aligned}$$

Example

Let $\varphi(x) = e^{-ax^2}$.

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-az^2} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z$$

Use completing the square to write $az^2 + \frac{(x-z)^2}{4t}$ as a square of a function of z plus a correction:

$$\begin{aligned} az^2 + \frac{1}{4t}(x-z)^2 &= \left(a + \frac{1}{4t}\right) z^2 - \frac{2}{4t} xz + \frac{1}{4t} x^2 \\ &= \left(a + \frac{1}{4t}\right)\left(z^2 - 2\frac{x}{4ta+1} z + \frac{x^2}{(4ta+1)^2}\right) + \frac{1}{4t}x^2 - \frac{x^2}{4ta+1} \cdot \frac{1}{4t} \\ &= \frac{1+4at}{4t}\left(z - \tilde{x}\right)^2 + \frac{a}{4ta+1} x^2 \end{aligned}$$

where $\tilde{x} = \frac{x}{4at+1}$. Let $z - \tilde{x} = s$.

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-ax^2} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\left(ax^2 + \frac{(x-z)^2}{4t}\right)} \mathrm{d}z \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4ta+1}x^2} \int_{\mathbb{R}} e^{-\left(\frac{1+4at}{4t}\right)s^2} \mathrm{d}s \end{aligned}$$

use $r = \left(\frac{1+4at}{4t}\right)^{1/2} s$, where $\mathrm{d}s = \left(\frac{4t}{1+4at}\right)^{1/2} \mathrm{d}w$.

$$\begin{aligned} &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4ta+1}x^2} \left(\frac{\sqrt{4t}}{(1+4at)^{1/2}}\right) \int_{\mathbb{R}} e^{-r^2} \mathrm{d}w \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4at+1}x^2} \left(\frac{\sqrt{4t}}{(1+4at)^{1/2}}\right) \cdot \sqrt{\pi} \\ &= \frac{e^{-\frac{a}{4at+1}x^2}}{(1+4at)^{1/2}} \end{aligned}$$

The plot of $u(t,x)$ with varying t is similar to the plot of the heat kernel.

Nonhomegenous problem

The nonhomegenous heat equation in one dimension is given by

$$\left(\partial_t - \partial_x^2\right) u(t,x) = f(t,x)$$

where $f(t,x) \in C_c^2([0,\infty) \times \mathbb{R})$ and $u(0,x) = 0$.

To find a solution to the nonhomegeous problem, take $\eta_s(t,x)$ which solves $\left(\partial_t - \partial_x^2\right)\eta_s(t,x) = 0$ and $\eta_s(s,x) = f(s,x)$:

$$\eta_s(t,x) = \int H_{t-s}(y) f(s, x-y)\mathrm{d}y$$

We now postulate

$$\begin{aligned} u(t,x) &= \int_0^t \eta_s(t,x)\mathrm{d}s = \int_0^t \int_{\mathbb{R}} H_{t-s}(y) f(s,x-y)\mathrm{d}y\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}} H_s(y) f(t-s, x-y)\mathrm{d}y\mathrm{d}s \end{aligned}$$

We check that this $u(t,x)$ solve the equation. It is more complicated than in the case of the wave equation because of the singularity of H_t as $t \to 0+$. Here we go:

$$(\partial_t - \Delta)\, u(t,x) = \int_{\mathbb{R}} H_t(y) f(0,x-y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} H_s(y)\left(\partial_t - \Delta_x\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s$$

We want to use integration by parts. Let $\varepsilon > 0$. Since $|f(t-s,x-y)|$ is bounded by some M and since f is in C^2,

$$\begin{aligned} \left|\int_0^\varepsilon \int_{\mathbb{R}} H_s(y)\left(\partial_t - \Delta_x\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s\right| &\leq M \int_0^\varepsilon \underbrace{\int_{\mathbb{R}} H_s(y)\mathrm{d}y}_{=1}\mathrm{d}s \\ &= M\varepsilon \to 0 \text{ as } \varepsilon \to 0 \end{aligned}$$

Note that $\partial_t f(t-s,x-y) = -\partial_s f(t-s,x-y)$ and $\Delta_x f(t-s,x-y) = \Delta_y f(t-s,x-y)$. So, we have

$$(\partial_t - \Delta)\, u(t,x) = \int_{\mathbb{R}} H_t(y) f(0,x-y)\mathrm{d}y + \lim_{\varepsilon\to 0}\int_\varepsilon^t \int_{\mathbb{R}} H_s(y)\left(-\partial_s - \Delta_y\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s \tag{12}$$

We use integration by parts to evaluate the second integral and we start with the term involving ∂_s:

$$\begin{aligned} \int_\varepsilon^t H_s(y)\left(-\partial_s f(t-s,x-y)\right)\mathrm{d}s &= -H_s(y) f(t-s,x-y)\Big|_\varepsilon^t + \int_\varepsilon^t \partial_s H_s(y) f(t-s,x-y)\mathrm{d}s \\ &= H_\varepsilon(y) f(t-\varepsilon,x-y) - H_t(y) f(0,x-y) + \int_\varepsilon^t \partial_s H_s(y) f(t-s,x-y)\mathrm{d}s \end{aligned}$$

Note that the second term cancels the first term on the right hand side of (12). We also have, by integration by parts in y using the fact that f is zero for large values of y,

$$\int_{\mathbb{R}} H_t(y)(-\Delta_x f(t-s,x-y))dy = \int_{\mathbb{R}} H_t(y)(-\Delta_y f(t-s,x-y))dy = \int_{\mathbb{R}} (-\Delta_y H_t(y)) f(t-s,x-y)dy.$$

Putting it all together (see (12)) we get

$$\begin{aligned} &(\partial_t - \Delta_x) u(t,x) = \\ &= \lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}} H_\varepsilon(y) f(t-\varepsilon,x-y)\mathrm{d}y + \int_\varepsilon^t \int_{\mathbb{R}} \underbrace{\left(\partial_s H_s(y) - \Delta_y H_s(y)\right)}_{=0} f(t-s,x-y)\mathrm{d}y\mathrm{d}s\right] \end{aligned}$$

since H_s solves the heat equation for $s > 0$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} H_\varepsilon(y) f(t-\varepsilon,x-y)\mathrm{d}y$$

We want to show that this equals $f(t,x)$. Use the change of variables $y = \varepsilon^{1/2}w$, meaning $\mathrm{d}y = \varepsilon^{1/2}\mathrm{d}w$.

$$H_\varepsilon(y) = \frac{1}{\sqrt{4\pi\varepsilon}}e^{-y^2/4\varepsilon}\mathrm{d}y = \frac{1}{\sqrt{4\pi}}e^{-w^2/4}\mathrm{d}w = H_1(y).$$

We want to show that as $\varepsilon \to 0$, $\int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon, x-y)\mathrm{d}y - f(t,x) \to 0$.

$$\begin{aligned}\int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon, x-y)\mathrm{d}y - f(t,x) &= \int_{\mathbb{R}} H_1(w)f(t-\varepsilon, x-\varepsilon^{1/2}w)\mathrm{d}w - f(t,x) \\ &= \int_{\mathbb{R}} H_1(w)\left[f(t-\varepsilon, x-\varepsilon^{1/2}w) - f(t,x)\right]\mathrm{d}w\end{aligned}$$

since $\int H_1(w) = 1$.

By the Mean Value Theorem and the fact that $f \in C_c^2$,

$$\left|f(t-\varepsilon, x-\varepsilon^{1/2}w) - f(t,x)\right| \le C(\varepsilon + \varepsilon^{1/2}|w|)$$

where $C = \max\limits_{s,y}\left(|\partial_s f(s,y)| + |\partial_y f(s,y)|\right)$. Hence,

$$\begin{aligned}\left|\int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon, x-y)\mathrm{d}y - f(t,x)\right| &\le \int_{\mathbb{R}} H_1(w)C(\varepsilon + \varepsilon^{1/2}|w|)\mathrm{d}w \\ &= C\varepsilon + \varepsilon^{1/2}\frac{1}{\sqrt{4\pi}}\underbrace{\int e^{-w^2/4}|w|\mathrm{d}w}_{\text{bounded}} \\ &\to 0 \text{ as } \varepsilon \to 0\end{aligned}$$

So, the solution to the nonhomegenous heat equation is

$$u(t,x) = \int_0^t \int_{\mathbb{R}^2} \frac{1}{\sqrt{4\pi s}}e^{-y^2/4s}f(t-s, x-y)\mathrm{d}y\mathrm{d}s$$

Higher Dimensions

We can perform dimensional analysis using the knowledge acquire in the 1-D case. Suppose our unit of distance is meters m. Then the unit of time should be m^2 (I know it sounds weird) since ∂_t and ∂_x^2 have to have the same dimension. Then u has no dimension but in the above formula f has to have dimension m^{-2}:

$$\int_0^t \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{4\pi s}}}_{\mathrm{m}^{-1}} \underbrace{e^{-y^2/4s}}_{\text{dimensionless}} \underbrace{f(t-s, x-y)}_{\mathrm{m}^{-2}} \underbrace{\mathrm{d}y}_{\mathrm{m}^1} \underbrace{\mathrm{d}s}_{\mathrm{m}^2}$$

We want $\mathrm{m}^{-1} \to \mathrm{m}^{-n}$ since $\mathrm{m}^1 \to \mathrm{m}^n$, so $\frac{1}{\sqrt{s}} \to \frac{1}{s^{n/2}}$. Our guess is $H_s(y) \in [0,\infty) \times \mathbb{R}^n$ is

$$H_s(y) = \frac{1}{(4\pi s)^{n/2}}e^{-|y|^2/4s}$$

Check that $(\partial_s - \Delta_y)\,H_s(y) = 0$ for $s > 0$. This follows from a calculation involving chain rule.

Theorem

Suppose $g \in C^0(\mathbb{R}^n)$ and g is bounded. Then,

$$u(t,x) = \int_{\mathbb{R}^n} H_t(x-y)g(y)\,\mathrm{d}^n y$$

solves $(\partial_t - \Delta)\,u(t,x) = 0$ for $t > 0$ and $u(t,x) \to g(x)$ as $t \to 0^+$.

- **Proof**: for $u(t,x) \to g(x)$, use change of variables.

$$\begin{aligned} |u(t,x) - g(x)| &= \left| \int_{\mathbb{R}^n} H_1(w) \left(g(x - t^{1/2} w) - g(x) \right) \mathrm{d}w \right| \\ &\le \int_{|w| \ge R} H_1(w) \left| g(x - t^{1/2} w) - g(x) \right| \mathrm{d}w + \int_{|w| \le R} H_1(w) \left| g(x) - t^{1/2} w - g(x) \right| \mathrm{d}w \end{aligned}$$

Let ε_1 be a small number and choose R so that

$$\int_{|w| \ge R} H_1(w) \mathrm{d}w < \varepsilon_1$$

Choose δ so that for $t < \delta$ small enough that that for $|w| \le R$,

$$\left| g(x - t^{1/2} w) - g(x) \right| < \varepsilon_1$$

g is bounded, so $|g(y)| \le M$ for some M. So, we have

$$|u(t,x) - g(x)| \le 2M\varepsilon_1 + \varepsilon_1 \to 0 \text{ for } 0 < t < \delta$$

But ε_1 was as small as we like (provided we make δ small). So, $u(t,x) \to g(x)$. $\square$

Example

Let $g(x) = \sin(kx)$ in $n = 1$.

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-z^2/4t} \sin(k(x-z)) \mathrm{d}z \\ &= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \int_{\mathbb{R}} e^{-z^2/4t} e^{ik(x-z)} \mathrm{d}z \end{aligned}$$

Use completing the squares to put the exponent of e into some form of z^2.

$$\begin{aligned} \frac{1}{4t} z^2 - ik(x-z) &= -ikx + \tfrac{1}{4t} \left(z^2 + 2(i2tk)z + (2itk)^2 \right) + \frac{1}{4t} \cdot 4t^2 k^2 \\ &= -ikx + \tfrac{1}{4t} (z + 2itk)^2 + tk^2 \end{aligned}$$

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \int_{\mathbb{R}} e^{-\frac{(z+2it)^2}{4t}} \mathrm{d}z \right] = \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \int_{\mathbb{R}} e^{-z^2/4t} \mathrm{d}z \right] \\ &= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \cdot \sqrt{4\pi t} \right] = \operatorname{Im} \left(e^{ikt} e^{-k^2 t} \right) = e^{-k^2 t} \sin(kx) \end{aligned}$$

Now, suppose we want to solve the 1-D heat equation for $x \in [0, \pi]$ and $u(t,0) = u(t,\pi) = 0$ with $u(0,x) = g(x)$. We want to answer the question of what $g(x)$ we can solve for $u(t,x)$ with.

We know that if

$$g(x) = \sum_{k=1}^{n} a_n \sin(kx),$$

we have

$$u(t,x) = \sum_{k=1}^{n} e^{-tk^2} a_k \sin(kx).$$

So, we can ask what functions can be written as this $g(x)$. The answer to this requires the use of function spaces which is our next topic.

Function Spaces

Let V be a **complex vector space**. V must satisfy two properties:

1. For $v, w \in V$, $v + w \in V$
2. For $c \in \mathbb{C}$ and $v \in V$, $cv \in V$

Example

$$V = \{g \in C^0(\mathbb{R}^n) \mid \exists M \text{ such that } |g(x)| \le M\}$$

is the set of bounded continuous functions in $\mathbb{R}^n$. This V is a complex vector space since for any $g, h \in V$, $g + h \in V$ and $cg \in V$ for $c \in \mathbb{C}$.

The **norm** on V is a function $v \mapsto \|v\| \in [0, \infty)$ such that

1. $\|v\| \ge 0$ and if $\|v\| = 0$, then $v = 0$
2. $\|v + w\| \le \|v\| + \|w\|$
3. $\|cv\| = |c|\|v\|$

Example

$$\|g\| = \sup_{x \in \mathbb{R}^n} |g(x)|$$

for $g \in V$ as previously defined satisfies all properties.

The **inner product** (or scalar product) is a function $u, v \mapsto \langle u, v \rangle \in \mathbb{C}$ such that

1. $\langle v, v \rangle \ge 0$ and if $\langle v, v \rangle = 0$, then $v = 0$
2. $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3. $\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$ for $c_j \in \mathbb{C}$, $v_j \in V$. Note that by the second property, $\langle v, c_1 w_1 + c_2 w_2 \rangle = \bar{c}_1 \langle v, w_1 \rangle + \bar{c}_2 \langle v, w_2 \rangle$

Theorem: Cauchy-Schwarz Inequality

For a complex vector space V with $v, w \in V$,

$$|\langle v, w \rangle|^2 \le \langle v, v \rangle \langle w, w \rangle .$$

If the equality holds then v and w are parallel, that is $\alpha v + \beta w = 0$ for some $\alpha, \beta \in \mathbb{C}$.

- **Proof**: let $t \in \mathbb{R}$.

$$\begin{aligned} 0 &\le \langle v + t\langle v, w \rangle w, v + t\langle v, w \rangle w \rangle \\ &= \langle v, v \rangle + t \langle v, w \rangle \langle w, v \rangle + t\overline{\langle v, w \rangle} \langle v, w \rangle + t^2 \langle v, w \rangle \overline{\langle v, w \rangle} \langle w, w \rangle \\ &= \underbrace{\langle v, v \rangle}_{c} + \underbrace{2|\langle v, w \rangle|^2}_{b} t + \underbrace{|\langle v, w \rangle|^2 \langle w, w \rangle}_{a} t^2 \\ &= at^2 + bt + c \end{aligned}$$

This is ≥ 0 for every $t \in \mathbb{R}$. We know $a \ge 0$ and if $a = 0$, v or $w = 0$ and the equality holds. So, we can assume that $a > 0$. A quadratic polynomial is then non-negative if and only if $b^2 - 4ac \le 0$ (so that there are no real zero except possibly a double zero when $b^2 - 4ac = 0$). Hence:

$$\begin{aligned} 4|\langle v, w \rangle|^4 - 4\langle v, v \rangle \langle w, w \rangle |\langle v, w \rangle|^2 \le 0 \\ 4|\langle v, w \rangle|^4 \le 4\langle v, v \rangle \langle v, w \rangle |\langle v, w \rangle|^2 \\ |\langle v, w \rangle|^4 \le \langle v, v \rangle \langle v, w \rangle |\langle v, w \rangle|^2 \end{aligned}$$

If $\langle v, w \rangle = 0$, we have nothing to prove. Otherwise,

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle$$

If $|\langle v, w \rangle|^2 = \langle v, v \rangle \langle w, w \rangle$ then going back to the beginning,

$$\langle v + t \langle v, w \rangle w, v + t \langle v, w \rangle w \rangle = 0$$

for some t (as $b^2 - 4ac = 0$). But that means that $v + t \langle v, w \rangle w = 0$, that is v and w are parallel. $\square$

Example

Let $V = C^0([0, 1])$. We define $\langle g, h \rangle := \int_0^1 g(x)\overline{h(x)}\mathrm{d}x$. By the Cauchy-Schwarz Inequality,

$$\left| \int_0^1 g(x)\overline{h(x)}\mathrm{d}x \right|^2 \leq \int_0^1 |g(x)|^2 \mathrm{d}x \cdot \int_0^1 |h(x)|^2 \mathrm{d}x$$

Corollary

An inner product $\langle \bullet, \bullet \rangle$ defines a norm:

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

- We want to show that all properties of the norm are satisfied. The first property is automatic and the third follows from the second, so we will just prove the second.

$$\begin{aligned}
\|v + w\|^2 &= \langle v + w, v + w \rangle \\
&= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
&= \|v\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} + \|w\|^2 \\
&= \|v\|^2 + 2\,\mathrm{Re}\,\langle v, w \rangle + \|w\|^2 \\
&\leq \|v\|^2 + 2|\langle v, w \rangle| + \|w\|^2 \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 && \text{Cauchy-Schwarz} \\
&= (\|v\| + \|w\|)^2 \\
&= \|v + w\|^2
\end{aligned}$$ $\square$

We want to know if every norm comes from an inner product. To test this, use the parallelogram identity.

Lemma: Parallelogram Identity

Let $u, v \in V$ with norm $\| \bullet \|$. Then,

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

- **Proof**: for $u, v \in V$, from the previous corollary, we have

$$\begin{aligned}
\|u + v\|^2 + \|u - v\|^2 &= \langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\
&= (\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle) + (\langle u, u \rangle - \langle v, u \rangle - \langle u, v \rangle + \langle v, v \rangle) \\
&= 2\langle u, u \rangle + 2\langle v, v \rangle = 2\|u\|^2 + 2\|v\|^2
\end{aligned}$$ $\square$

> **Example**
> Let $V = C^0([0,1])$ where $\|g\| = \max_{x\in[0,1]} |g(x)|$ for $g \in V$. Is there an inner product behind it?
>
> Let $g(x) = x$ and $h(x) = 1-x$. $h+g = 1$ and $h-g = 1-2x$. So,
>
> $$\begin{aligned} \|g\| &= 1 \\ \|h\| &= 1 \\ \|g+h\| &= 1 \\ \|h-g\| &= 1 \end{aligned}$$
>
> But,
> $$\|h+g\|^2 + \|h-g\|^2 = 1+1 \neq 2+2 = 2\|h\|^2 + 2\|g\|^2$$
> So, there is no inner product associated to the norm of V.

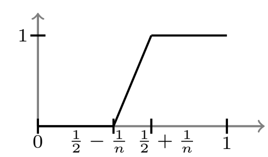
L^p Spaces

For sequence $v_n \in V$, we say v_n **converges** to v ($v_n \to v$) in the norm of V if $\|v - v_n\| \to 0$ as $n \to \infty$.
Take $V = C^0([0,1])$ where the norm $\| \bullet \|_2$ is defined as

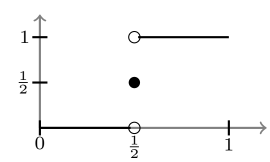
$$\|g\|_2 = \left(\int_0^1 |g(x)|^2 \mathrm{d}x \right)^{1/2}$$

This comes from the inner product $\langle g, h \rangle = \int_0^1 g(x)\overline{h(x)}\mathrm{d}x$. Take $g_n \in C^0([0,1])$ where

$$g_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right) & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$



$$g(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x = 1 \end{cases}$$



We have $|g(x) - g_n(x)| \le |g(x)| + |g_n(x)| \le 2$. So,

$$\|g - g_n\| = \left(\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} |g(x) - g_n(x)| \mathrm{d}x \right)^{1/2} \le \left(\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} 2^2 \right)^{1/2} = \frac{2}{\sqrt{2n}} = \frac{\sqrt{2}}{\sqrt{n}} \to 0 \text{ as } n \to \infty$$

So, we have

$$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{C} \;\middle|\; \int_0^1 |f(x)|^2 \mathrm{d}x < \infty \right\}$$

where $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x$ is well defined.

$$\|f\| = \left(\int_0^1 |f|^2 \mathrm{d}x \right)^{1/2}$$

We have $C^0([0,1]) \subsetneq L^2([0,1])$. This holds for $C^k([0,1])$ for any k.
An important property is that if $f \in L^2([0,1])$, then $\exists f_n \in C^k([0,1])$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$.

For any open connected $\Omega \subseteq \mathbb{R}^n$, we have

$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(x)|^2 \, \mathrm{d}^n x < \infty \right\} \qquad L^1(\Omega) := \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(x)| \, \mathrm{d}^n x < \infty \right\}$$

$$\langle f, g \rangle := \int_\Omega f(x) \overline{g(x)} \, \mathrm{d}^n x$$

$$\|f\|_{L^2(\Omega)} := \left(\int_\Omega |f(x)|^2 \, \mathrm{d}^n x \right)^{1/2} \qquad \|f\|_{L^1(\Omega)} := \int_\Omega |f(x)| \, \mathrm{d}^n x$$

Proposition

Suppose $\Omega \subset B(0, R)$ with $R < \infty$. Then, $L^2(\Omega) \subset L^1(\Omega)$.

- **Proof**: suppose $f \in L^2(\Omega)$. Note that $1 \in L^2(\Omega)$ since $\int_\Omega 1^2 \leq \int_{B(0,R)} 1 < \infty$.

$$\begin{aligned} \int_\Omega |f(x)| \mathrm{d}x &= \int_\Omega |f(x)| \cdot 1 \mathrm{d}x \\ &\leq \left(\int_\Omega |f(x)|^2 \right)^{1/2} \left(\int_\Omega 1 \right) \leq c \|f\|_2 \end{aligned}$$

by the Cauchy-Schwarz inequality. So, $\|f\|_{L^1(\Omega)} \leq c \|f\|_{L^2(\Omega)}$ for some constant c. $\square$

Measure of Sets

Recall for open set $\Omega \subseteq \mathbb{R}^n$, we have

$$\begin{aligned} L^2(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(x)|^2 \, \mathrm{d}^n x \right\} \\ L^1(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(x)| \, \mathrm{d}^n x \right\} \end{aligned}$$

For $\Omega \subseteq \mathbb{R}^n$, the **measure** of Ω is

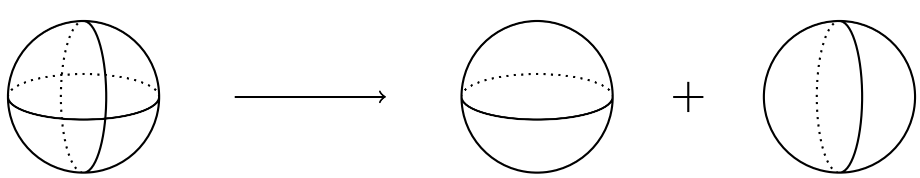
$$m(\Omega) := \int_\Omega 1 \mathrm{d}x.$$

For example, we have $m(\mathbb{R}^n) = \infty$, $m(B(x_0, r)) = c_n r^n$, $m(\{x_0\}) = 0$, and $m(\{(t,t) \mid t \in \mathbb{R}\}) = 0$. But we have to note that not all sets are measurable.

This can be seen in the Banach-Tarski paradox. Take a ball $B(0,1) \subseteq \mathbb{R}^3$. There exist sets $B_k \subset B(0,1)$, $k = 1, \cdots 5$, such that

$$B(0,1) = \bigcup_{k=1}^{5} B_k$$

and $B_k \cap B_j = \emptyset$ for $k \neq j$. But in addition (somewhat amazingly) $B_1 \cup B_2 = B(0,1)$ and $B_3 \cup B_4 \cup B_5 = B(0,1)$. This means that the pieces can be rearranged to make two balls.

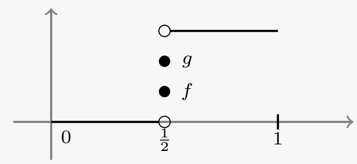


This works because the balls B_k are not measurable. Needless to say you should not worry about this in Math126!

Example

Let $f \in L^1([0,1])$.

$$\int_0^1 |f - g| \mathrm{d}x = 0$$



We say $f \equiv g$ if $m(\{x \mid f(x) \neq g(x)\}) = 0$. For $f : \Omega \to \mathbb{C}$, we have

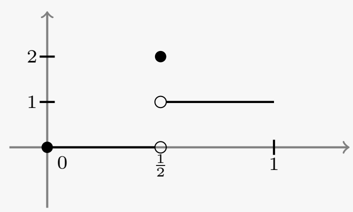
$$\|f\|_\infty = \inf \{a \in \mathbb{R} \mid m(\{x \mid f(x) > a\}) = 0\}$$

So, we have

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \mid \|f\|_\infty < \infty\}$$

Example

Now, for $\Omega = (0,1)$, we have $\|f\|_\infty = 1$.



Banach and Hilbert Spaces

Recall that for a vector space V, we have normed spaces and inner product spaces. We From the Cauchy-Schwarz Inequality, all inner product spaces are normed spaces, but the converse is not true.

For normed vector space V, $v_n \to v$ if and only if $\|v - v_n\|$ converges to 0. We say V is **complete** if and only if "all the limits lie in V".

- A complete normed space is a **Banach space**
- A complete inner product space is a **Hilbert space**

All Hilbert spaces are Banach spaces, but the converse is not true.

Example

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)| \mathrm{d}x \text{ is a Banach space and } \|f\|_{L^2(\Omega)} = \int_\Omega |f(x)|^2 \mathrm{d}x \text{ is a Hilbert space}$$

since

$$\langle f, g \rangle := \int_\Omega f(x)\overline{g(x)} \mathrm{d}x$$

Application to the Heat Equation

Let's apply this back to the heat equation. We know the solution to

$$(\partial_t - \Delta)\, u = 0$$

with $u(0,x) = g(x)$ and $t > 0$ is

$$u(t,x) = \int H_t(z) g(x-z)\, \mathrm{d}^n z$$

where

$$H_t(z) = \frac{1}{(4\pi t)^{n/2}} e^{-|z|^2/4t}$$

and $\int H_t(z)\mathrm{d}z = 1$.

Our goal is to understand estimates of u at fixed t in terms of g:

$$\|u(t,\bullet)\|_{L^?} \leq ?\|g\|_{L^?}$$

Let's list all L^p spaces we have defined.

$$\begin{aligned}
L^1(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)| \,\mathrm{d}^n x < \infty \right\} \\
\|f\|_{L^1(\Omega)} &:= \int_\Omega |f(x)| \,\mathrm{d}^n x \\
\\
L^2(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^2 \,\mathrm{d}^n x < \infty \right\} \\
\|f\|_{L^2(\Omega)} &:= \left(\int_\Omega |f(x)|^2 \,\mathrm{d}^n x \right)^{1/2} \\
\\
L^\infty(\Omega) &:= \{ f : \Omega \to \mathbb{C} \mid \|f\|_\infty < \infty \} \\
\|f\|_{L^\infty(\Omega)} &:= \inf\{ a \in \mathbb{R} \mid m(\{x \mid f(x) > a\}) = 0 \} \\
&\text{“}=\text{”} \sup |f(x)|
\end{aligned}$$

The “ $=$ ” is not a true equality but it holds in our case.

1. Find the bound in the form $\|u(t,\bullet)\|_{L^\infty} \leq ?$.

$$\begin{aligned}
|u(t,x)| &= \left| \int H_t(z) g(x-z) \mathrm{d}z \right| \\
&\leq \int H_t(z) |g(x-z)| \mathrm{d}z \\
&\leq \int H_t(z) \|g\|_\infty \mathrm{d}z \\
&= \|g\|_\infty
\end{aligned}$$

   So,

$$\|u(t,\bullet)\|_{L^\infty} \leq \|g\|_\infty.$$

   This is optimal if g is constant.

2. Find the bound in the form $\|u(t,\bullet)\|_{L^\infty} \leq ?\ \|g\|_{L^1}$.

$$\begin{aligned}
|u(t,x)| &= \left| \int H_t(z) g(x-z) \mathrm{d}z \right| \\
&\leq \int H_t(z) |g(x-z)| \mathrm{d}z \\
&\leq \sup_{z \in \mathbb{R}^n} (H_t(z)) \cdot \int |g(x-z)| \mathrm{d}z \\
&= \frac{1}{(4\pi t)^{n/2}} \|g\|_{L^1}
\end{aligned}$$

   So,

$$\|u(t,\bullet)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}} \|g\|_{L^1}.$$

Example
Let $g(x) = H_s(x)$ for $s > 0$. H solves the heat equation, so

$$\begin{aligned}
u(t,x) &= \int H_t(z) H_s(x-z)\mathrm{d}z \\
&= H_{t+s}(x) \\
&= \frac{e^{-|z|^2/4(t+s)}}{(4\pi(t+s))^{n/2}} \\
\|u(t,\bullet)\|_\infty &\le (4\pi(t+s))^{-n/2}, \quad \|g\|_1 = 1 \\
\|u(t,\bullet)\|_{L^\infty} &\le \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}
\end{aligned}$$

$$\|u(t,\bullet)\|_{L^\infty} \le \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}$$

is nearly optimal at $g = H_s$ for $s > 0$.

3. Find the bound in the form $\|u(t,\bullet)\|_{L^1} \le ?$.

$$\begin{aligned}
\|u(t,x)\|_{L^1} &= \int |u(t,x)|\mathrm{d}x \\
&= \int \left| \int H_t(z) g(x-z)\mathrm{d}z \right| \mathrm{d}z \\
&\le \int\int H_t(z)|g(x-z)|\mathrm{d}z\mathrm{d}x \\
&= \int H_t(z) \left(\int |g(x-z)|\mathrm{d}x \right) \mathrm{d}z \\
&= \int H_t(z) \left(\int |g(y)|\mathrm{d}y \right) \mathrm{d}z && y = x-z \\
&= \|g\|_{L^1}
\end{aligned}$$

So,
$$\|u(t,\bullet)\|_{L^1} \le \|g\|_{L^1}.$$
This is optimal if $g \ge 0$ or $g \le 0$ for every x.

4. Find the bound in the form $\|u(t,\bullet)\|_{L^2} \le ?$.

$$\begin{aligned}
\|u(t,x)\|_{L^2}^2 &= \int |u(t,x)|^2\mathrm{d}x \\
&= \int \left| \int H_t(z) g(x-z)\mathrm{d}z \right|^2 \mathrm{d}x \\
&\le \int \left| \int H_t(z)|g(x-z)|\mathrm{d}z \right|^2 \mathrm{d}x
\end{aligned}$$

Apply the Cauchy-Schwarz Inequality: $\left|\int f(z)\overline{h(z)}\mathrm{d}z\right|^2 \le \int |f(z)|^2\mathrm{d}z \int |h(z)|^2\mathrm{d}z$. By applying this directly, we get

$$\le \int \left(\int H_t(z)^2\mathrm{d}z \right) \left(\int |g(x-z)|^2\mathrm{d}z \right) \mathrm{d}x$$

but this does not help us. Instead, use

$$\begin{aligned}
\|u(t,x)\|_{L^2}^2 &\le \int \left| \int H_t(z)|g(x-z)|\mathrm{d}z \right|^2 \mathrm{d}x \\
&= \int \left| \int H_t(z)^{1/2} \left(H_t(z)^{1/2}|g(x-z)| \right) \mathrm{d}z \right|^2 \mathrm{d}x
\end{aligned}$$

We have $f(z) = H_t(z)^{1/2}$ and $h(z) = H_t(z)^{1/2}|g(x-z)|$. So, by applying the Cauchy-Schwarz Inequality, we have

$$\le \int \underbrace{\left(\int H_t(z)\mathrm{d}z \right)}_{=1} \left(\int H_t(z)|g(x-z)|^2 \mathrm{d}z \right) \mathrm{d}x$$

By Fubini's Theorem, we can switch the order of integration.

$$\begin{aligned}
&= \int H_t(z) \left(\int |g(x-z)|^2 \mathrm{d}x \right) \mathrm{d}z \\
&= \int H_t(z)\|g\|_{L^2}^2 \mathrm{d}z \\
&= \|g\|_{L^2}^2
\end{aligned}$$

So,

$$\|u(t,\bullet)\|_{L^2} \le \|g\|_{L_2}.$$

This is nearly optimal.

In summary, we have the following:

1. $$\|u(t,\bullet)\|_{L^\infty} \le \|g\|_\infty$$

   (optimal at g constant)

2. $$\|u(t,\bullet)\|_{L^\infty} \le \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}$$

   (nearly optimal at $g = H_s$ for $s > 0$)

3. $$\|u(t,\bullet)\|_{L^1} \le \|g\|_{L^1}$$

   (optimal if $g \le 0$ or $g \ge 0$ for any x)

4. $$\|u(t,\bullet)\|_{L^2} \le \|g\|_{L^2}$$

   (nearly optimal)

Hilbert Spaces

Let H be a Hilbert with inner product $\langle \bullet, \bullet \rangle$. $L^2(\Omega)$ for open $\Omega \subseteq \mathbb{R}^n$ is an example of a Hilbert space. Take $u, v \in H$. u and v are **orthogonal** if $\langle u, v \rangle = 0$.

- A set $\{v_j\}_{j \in J}$ is **orthogonal** if $\langle v_i, v_j \rangle = 0$ for every $i \neq j$.
- A set $\{e_j\}_{j \in J}$ is **orthonormal** if $\langle e_i, e_j \rangle = 0$ for every $i \neq j$ and $\langle e_i, e_i \rangle = 1$ for every i.

The set of indices J could be finite or it could be $J = \mathbb{Z}$ or $J = \mathbb{N}$.

Suppose $\{e_j\}_{j \in J}$ is an orthonormal set. Then, if $v = \sum_{i=1}^{N} a_j e_j$, then $a_k = \langle v, e_k \rangle$ since

$$\langle v, e_k \rangle = \left\langle \sum a_j e_j, e_k \right\rangle = \sum a_j \langle e_j, e_k \rangle = a_k \langle e_k, e_k \rangle = a_k$$

Define $c_k[v] := \langle v, e_k \rangle$ and suppose $\{e_j\}_{j=1}^{\infty}$ is an orthonormal set.

Theorem: Bessel's Inequality

For $v \in H$, let $S_n v := \sum_{j=1}^{n} c_j[v] e_j$. Then,

$$\sum_{k=1}^{\infty} |c_j[v]|^2 \leq \|v\|^2$$

The equality holds if and only if $S_n v$ converges to v in H, that is,

$$\|S_n v - v\| \to 0, \quad n \to \infty.$$

- **Proof**:

$$\begin{aligned} 0 &\leq \|v - S_n v\|^2 \\ &= \langle v - S_n v, v - S_n v \rangle \\ &= \langle v, v \rangle - \langle S_n v, v \rangle - \langle v, S_n v \rangle + \langle S_n v, S_n v \rangle \end{aligned}$$

Evaluate each of these terms separately.

$$\langle S_n v, v \rangle = \sum_{j=1}^{n} \langle c_j[v] e_j, v \rangle = \sum_{j=1}^{n} c_j[v] \langle e_j, v \rangle = \sum_{j=1}^{n} c_j[v] \overline{\langle v, e_j \rangle} = \sum_{j=1}^{n} |c_j[v]|^2$$

By the same argument, note that $\langle v, S_n v \rangle$ is equal to the same.

$$\begin{aligned} \langle S_n v, S_n v \rangle &= \left\langle \sum_{j=1}^{n} c_j[v] e_j, \sum_{\ell=1}^{n} c_\ell[v] e_\ell \right\rangle \\ &= \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_j[v] \overline{c_\ell[v]} \langle e_j, e_\ell \rangle \\ &= \sum_{j=1}^{n} |c_j[v]|^2 \end{aligned}$$

Entering this back into the equation of $\|v - S_n v\|^2$, we have

$$\|v - S_n v\|^2 = \|v\|^2 - \sum |c_j[v]|^2 - \sum |c_j[v]|^2 + \sum |c_j[v]|^2 = \|v\|^2 - \sum |c_j[v]|^2$$

Since $\|v - S_n v\|^2 \geq 0$,

$$\sum_{j=1}^{n} |c_j[v]|^2 = \|v\|^2 - \|v - S_n v\|^2 \leq \|v\|^2$$

As $n \to \infty$, $\sum_{j=1}^{n} |c_j[v]|^2$ is a non decreasing sequence. $\|v\|^2$ is an upper bound of $\sum_{j=1}^{n} |c_j[v]|^2$, so

$$\sum_{n=1}^{\infty} |c_j[v]|^2 \leq \|v\|^2. \qquad \square$$

The concept of a **basis** is complicated in infinite dimensions and we will restrict ourselves to *orthonormal bases.*

Definition:
The set $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis if it is orthonormal and for any $v \in H$ we have

$$\|S_n v - v\| \to 0, \quad n \to \infty, \quad S_n v := \sum_{j=1}^{\infty} c_j[v] e_j, \quad c_j[v] := \langle v, e_j \rangle.$$

Remark: The index set in the definition is $J = \{1, 2, \cdots\}$ but it could also be a finite set (if H is finite dimensional, that is, has a basis, which is a finite set, or, it could be $J = \mathbb{Z}$. In that case we should put

$$S_n v := \sum_{j=-n}^{n} c_j[v] e_j.$$

We record (without proof) the following important fact (used, for instance, in Problem 3, in HW9):

Theorem
An orthonormal set $\{e_j\}_{j \in J}$ is an orthonormal basis if

$$\forall\, j \in J \ \langle v, e_j \rangle = 0 \implies v = 0.$$

As in finite dimensions we have a way to generate orthonormal sets:

Gram-Schmidt Process: Suppose $\{v_j\}_{j=1}^{\infty}$ is a set of elements of H. Then the following set is orthonormal:

$$\begin{aligned}
&e_1 = v_1/\|v_1\| \\
&e_2 = (v_2 - \langle v_2, e_1 \rangle e_1)/\|v_2 - \langle v_2, e_1 \rangle e_1\| \\
&\vdots \\
&e_n = (v_n - \sum_{j=1}^{n-1} \langle v_n, e_j \rangle e_j)/\|v_n - \sum_{j=1}^{n-1} \langle v_n, e_j \rangle e_j\|
\end{aligned}$$

Example: Define the following inner product for functions on $\mathbb{R}$:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} e^{-x^2} dx, \quad H = \{f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |f(x)|^2 e^{-x^2} dx < \infty\}.$$

If we take $v_\ell := x^\ell$, $\ell = 0, 1, \cdots$, the Gramm Schmidt process gives us an orthonormal set:

$$e_0 = \pi^{-\frac{1}{4}}, \quad e_1 = \sqrt{2}\pi^{-\frac{1}{4}} x, \quad e_2 = \sqrt{2}\pi^{-\frac{1}{4}}(x^2 - \tfrac{1}{2}), \cdots$$

Interlude: Review of Linear Algebra

Let V be t finite dimensional vector space $V = \mathbb{C}^N$ where for $v \in V$, $v = (v_1 \cdots v_N)^\intercal$. For $v, w \in V$, we have

$$\langle v, w \rangle = \sum_{n=1}^{N} v_n \overline{w_n}$$

For $A : V \to V$ linear, $A(c_1 v_1 + c_2 v_2) = c_1 AV_1 + c_2 AV_2$. We say A is **self-adjoint** if

$$\langle Av, w \rangle = \langle v, Aw \rangle$$

i.e. for $A = (A_{ij})_{1 \le i,j \le N}$, $A_{ij} = \overline{A_{ji}}$.

If $A : V \to V$ is self-adjoint, then $\exists$ an orthonormal basis of V $\{\varphi_j\}_{j=1}^N$ such that $A\varphi_j = \lambda_j \varphi_j$. So, we have an orthonormal basis of eigenvectors of A.

We have two properties of the orthonormal basis.

1. All λ_j's are real.
   - **Proof**: We know $\langle A\varphi_j, \varphi_j \rangle = \lambda_j |\varphi_j|^2$.
   $$\begin{aligned} \langle A\varphi_j, \varphi_j \rangle &= \langle \varphi_j, A\varphi_j \rangle \\ &= \overline{\langle A\varphi_j, \varphi_j \rangle} \end{aligned}$$
   So, $\langle A\varphi_j, \varphi_j \rangle$ must be real. $|\varphi_j|^2$ is also real, so λ_j must be real. □
2. If $\lambda_i \neq \lambda_j$, then $\langle \varphi_i, \varphi_j \rangle = 0$.
   - **Proof**:
   $$\begin{aligned} \lambda_i \langle \varphi_i, \varphi_j \rangle &= \langle A\varphi_i, \varphi_j \rangle \\ &= \langle \varphi_i, A\varphi_j \rangle \\ &= \langle \varphi_i, \lambda_j \varphi_j \rangle \\ &= \lambda_j \langle \varphi_i, \varphi_j \rangle \end{aligned}$$
   $\lambda_i \neq \lambda_j$, so we must have $\langle \varphi_i, \varphi_j \rangle = 0$. □

Suppose $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = A(\varphi(t))$ with $\varphi(0) = \varphi$. φ_j's are a basis, so

$$\varphi = \sum_{j=1}^{N} a_j \varphi_j$$

$$\begin{aligned} \langle \varphi, \varphi_k \rangle &= \left\langle \sum a_j \varphi_j, \varphi_k \right\rangle \\ &= \sum a_j \langle \varphi_j, \varphi_k \rangle \\ &= a_k \end{aligned}$$

since this is an orthonormal basis. So, for any a_j, $a_j = \langle \varphi, \varphi_j \rangle$.

$$\varphi(t) = \sum a_j e^{\lambda_j t} \varphi_j$$

We can check this:

$$\begin{aligned} \varphi(0) &= \sum a_j \varphi_j \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \sum a_j e^{\lambda_j t} \lambda_j \varphi_j \\ &= \sum a_j e^{\varphi_j t} A\varphi_j \\ &= A(\varphi(t)) \end{aligned}$$

This gives us a way of solving the equation by decomposing an arbitrary element of the vector space in terms of an orthonormal basis of the self-adjoint matrix A. We will observe how this can be done in the case of differential operators.

Heat equation on $[0, L]$.

Theorem
The set $(2/L)^{\frac{1}{2}} \sin(\pi n x/L)$, $n = 1, 2, 3, \cdots$ is an orthonormal basis of $L^2([0, L])$.

Proof. We will only show that it is an orthonormal set. The property that it is a basis will be discussed in the next section. We need to show that (make a change of variables $y = \pi x/L$)

$$\frac{2}{L}\int_0^L \sin(\pi n x/L)\sin(\pi m x/L)dx = \frac{2}{\pi}\int_0^\pi \sin ny \sin my dy = \delta_{nm}.$$

Writing $\sin y = (e^{iy} - e^{-iy})/2i$ so that

$$\begin{aligned}\int_0^\pi \sin nx \sin mx dx &= -\tfrac{1}{4}(e^{inx} - e^{-inx})(e^{imx} - e^{-imx})dx \\ &= -\tfrac{1}{4}\int_0^\pi (e^{i(n+m)x} + e^{-i(n+m)x} - e^{i(m-n)x} - e^{-i(n-m)x} dx \\ &= -\tfrac{1}{2}\int_0^\pi (\cos(n+m)x - \cos(n-m)x)dx = \begin{cases} \pi/2 & m = m, \\ 0 & m \neq n. \end{cases}\end{aligned}$$

□

We now use this to solve

$$(\partial_t - \partial_x^2)u(t,x) = 0,\ t > 0,\ x \in [0, L],\quad u(t,0) = u(t,L) = 0,\ t > 0,\quad \lim_{t\to 0+} u(t,x) = g(x). \tag{13}$$

We see that any sine series (rescaled to the interval L)

$$F(x) = \sum_{k=1}^{\infty} s_k(F)(2/\pi)^{\frac{1}{2}} \sin(\pi k x/L), \quad s_k(F) := (2/L)^{\frac{1}{2}} \int_0^L F(x)\sin(\pi k x/L)dx.$$

will satisfy the boundary conditions. Hence it is natural to expand $g(x)$ into a sine series which gives

$$g(x) = \sum_{k=1}^{\infty} s_k(g)(2/L)^{\frac{1}{2}} \sin(\pi k x/L), \quad s_k(g) := (2/L)^{\frac{1}{2}} \int_0^L g(x)\sin(\pi k x/L)dx. \tag{14}$$

We can then postulate

$$u(t,x) = \sum_{k=1}^{\infty} e^{-(\pi k/L)^2 t} s_k(g)(2/\pi)^{\frac{1}{2}} \sin(\pi k x/L).$$

For $t > 0$ we have very rapid convergence and we can differentiate under the sum and that shows that

$$(\partial_t - \partial_x^2)u(t,x) = 0.$$

In view of (14) we have

$$u(t,x) \to g(x), \quad t \to 0+.$$

It is believable (and not too difficult) to justify this in the L^2 sense for any $g \in L^2([0, L])$:

$$\int_0^L |u(t,x) - g(x)|^2 dx \to 0, \quad t \to 0+.$$

If $g(x)$ has an odd extension which is $2L$-periodic and C^1 then we have nice pointwise convergence and can say that $u(0,x) = g(x)$.

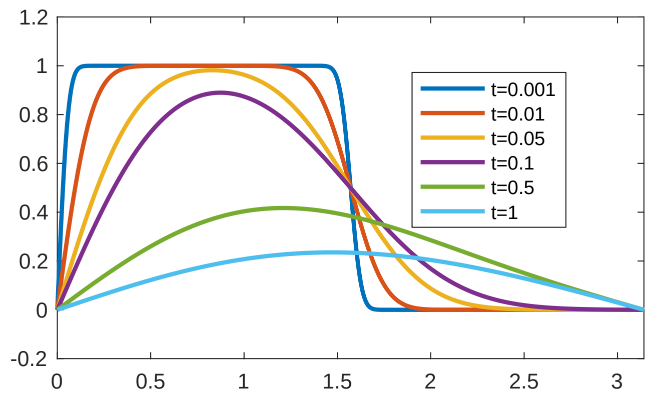


Figure 6: Solution of (13) with $L = \pi$, $g(x) = 1$ for $0 \le x \le \pi/2$, $g(x) = 0$ for $\pi/2 < x \le \pi$. We use the sine series of g obtained in the second example in the previous section. If you click on the figure you will see a movie of the solution and a comparison with the approximation by the first term in the Fourier expansion $e^{-t} 2\sin(x)/\pi$ – it is very accurate once time increases.

Fourier Series

We now consider $H = L^2([0, 2\pi])$,

$$\langle f, g\rangle := \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

and the following set

$$e_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k \in \mathbb{Z}.$$

Proposition

We have

$$\langle e_k, e_\ell\rangle = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

In other words, $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal set.

Proof. For $k = \ell$,

$$e_k(x)\overline{e_k(x)} = |e_k(x)|^2 = \frac{1}{2\pi}$$

and since we are integrating over $[0, 2\pi]$ the integral is 1. For $k \neq \ell$, we calculate using the fact that for $c \neq 0$, $(e^{cx}/c)' = e^{cx}$:

$$\int_0^{2\pi} e_k(x)\overline{e_\ell(x)}dx = \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}e^{-i\ell x}dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-\ell)x}dx = \frac{1}{2\pi i(k-\ell)} e^{i(k-\ell)x}|_0^{2\pi} = 0.$$

That's it! ◻

The important fact which we prove at the end of this section is

Theorem

The set $e_k(x) := e^{ikx}/\sqrt{2\pi}$, $k \in \mathbb{Z}$, is an orthonormal basis of $L^2([0, 2\pi])$.

This theorem implies that

$$S_n f(x) := \sum_{j=-n}^{n} c_j[f]e_j(x), \quad c_j[f] := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)e^{-ijx}dx,$$

satisfies

$$\int_0^{2\pi} |f(x) - S_n f(x)|^2 dx \to 0, \quad n \to \infty,$$

for every $f \in L^2([0, 2\pi])$.

We sometimes write

$$\widehat{f}(j) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} dx,$$

so that, with *convergence meant in* L^2,

$$f(x) = \sum_{-\infty}^{\infty} \widehat{f}(j) e^{ijx}. \tag{15}$$

In view of the Bessel's inequality (or rather *equality* since we have an orthonormal basis),

$$\int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{j \in \mathbb{Z}} |\widehat{f}(j)|^2$$

(To remember the constants check this for $f(x) \equiv 1$ so that $\widehat{f}(0) = 1$ and $\widehat{f}(j) = 0$, $j \neq 0$.)

Theorem

Suppose that $f \in C^k(\mathbb{R})$, $f(x + 2\pi) = f(x)$. Then considering f as a function on $[0, 2\pi]$ we have, for some constant, C,

$$|\widehat{f}(j)| \leq C j^{-k}, \quad \forall\, j \neq 0.$$

Remarks: 1. We see that if $k > 1$ and $f \in C^k(\mathbb{R})$, $f(x + 2\pi) = f(x)$, then the series in (15) converges absolutely for all $x \in \mathbb{R}$ and the limit is f. This remains true for $k = 1$ but the proof is trickier (see section 8.3 and 8.4 in the text). However, it is not true for $f \in C^0$

2. When $f \in C^\infty(\mathbb{R})$, that is $f \in C^k(\mathbb{R})$ for all k, then for every k there exists as constant k such that

$$|\widehat{f}(j)| \leq C_k j^{-k}, \quad \forall\, j \neq 0. \tag{16}$$

Proof of Theorem. We note that for $j \neq 0$,

$$e^{-ijx} = \left(-\frac{d}{dx}\right)^k \left(\frac{e^{-ijx}}{ij^k}\right).$$

Also, since $f \in C^k(\mathbb{R})$, $d^k/dx^k f(x + 2\pi) = d^k/dx^k f(x)$. We can then integrate by parts k times with boundary terms cancelling thanks to this periodicity:

$$\widehat{f}(j) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ijx} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) \left(-\frac{d}{dx}\right)^k \left(\frac{e^{-ijx}}{ij^k}\right) dx = j^{-k} \frac{1}{2\pi i^k} \int_0^{2\pi} \frac{d^k f(x)}{dx^k} e^{-ijx} dx.$$

Since

$$\left| \frac{1}{2\pi i^k} \int_0^{2\pi} \frac{d^k}{dx^k} f(x) e^{-ijx} dx \right| \leq \max |d^k f / dx^k|,$$

the theorem follows. $\square$

Why is e_j an othonormal basis of $L^2([0,\pi])$? We give some indications of this.

Theorem
Suppose that $f \in L^2([0,2\pi])$ then there exists $f_t \in C^\infty(\mathbb{R})$, $f_t(x+2\pi) = f_t(x)$, $t > 0$, such that

$$\|f_t - f\|^2_{L^2([0,\pi])} = \int_0^{2\pi} |f_t(x) - f(x)|^2 dx \to 0, \quad t \to 0+ .$$

In other words any L^2 function on $[0,\pi]$ can be approximated by a smooth periodic function.

Proof. We can only give some indications of the proof. First extend f by periodicity to a function on R, $f(x+2\pi) = f(x)$. This function is *not* in $L^2(\mathbb{R})$ (unless it is 0) but you can check that the following definition makes sense:

$$f_t(x) := \int_{\mathbb{R}} H_t(x-y) f(y) dy,$$

where $H_t(y)$ is the heat kernel. We know that for $t > 0$, $x \mapsto H_t(x-y)$ is smooth function so $f_t \in C^\infty(\mathbb{R})$. Also, f_t is periodic:

$$\begin{aligned} f_t(x+2\pi) &= \int_{\mathbb{R}} H_t(x+2\pi-y) f(y) dy = \int_{\mathbb{R}} H_t(x-(y-2\pi)) f(y-2\pi+2\pi) dy \\ &= \int_{\mathbb{R}} H_t(x-z) f(z+2\pi) dz = \int_{\mathbb{R}} H_t(x-z) f(z) dz = f_t(x), \end{aligned}$$

where we used the fact that $f(x)$ is periodic (we extended the original f by periodicity. In view of the properties of H_t described in the Heat Equation section it is now easy to believe (though not so easy to prove) that

$$\|H_t * f - f\|_{L^2([0,2\pi])} \to 0, \quad t \to 0+ .$$

That gives the theorem. □

Remark
You should notice that in the process of proving the theorem we solved the following problem: suppose that $f \in C(\mathbb{R})$, $f(x+2\pi) = f(x)$. Then $u(t,x) := H_t * f(x)$ solves

$$(\partial_t - \partial_x^2) u(t,x) = 0, \quad u(0,x) = f(x), \quad u(t,x+2\pi) = u(t,x).$$

We now notice that the Fourier series converge in the case of smooth functions:

Theorem
Suppose that $g \in C^\infty(\mathbb{R})$, $g(x+2\pi) = g(x)$. Then for all x

$$g(x) = \sum_{n \in \mathbb{Z}} \widehat{g}(k) e^{ikx} = \lim_{N \to \infty} S_N g(x), \tag{17}$$

with absolute convergence (which is uniform in $x \in \mathbb{R}$).

Proof. It is enough to show (17) holds for $x = 0$ and for $g(0) = 0$, as otherwise we can consider $y \mapsto g_x(y) := g(y+x) - g(x)$ noticing that

$$\begin{aligned} \widehat{g}_x(k) &= \frac{1}{2\pi} \int_0^{2\pi} g_x(y) e^{-iky} dy - g(x)\delta_{k0} = \frac{1}{2\pi} \int_0^{2\pi} g(y+x) e^{-ik(y+x)+ikx} dy - g(x)\delta_{k0} \\ &= e^{ikx} \frac{1}{2\pi} \int_0^{2\pi} g(z) e^{-ikz} dy - g(x)\delta_{k0} = e^{ikx} \widehat{g}(k) - g(x)\delta_{k0}, \end{aligned}$$

and hence

$$g(x) = \sum_{n\in\mathbb{Z}} \widehat{g}(k)e^{ikx}$$

is the same as

$$0 = g_x(0) = \sum_{k\in\mathbb{Z}} \widehat{g}_x(k) = \sum_{k\in\mathbb{Z}} (e^{ikx}\widehat{g}(k) - g(x)\delta_{k0}) = -g(x) + \sum_{k\in\mathbb{Z}} e^{ikx}\widehat{g}(k).$$

Thus we assume that $g(0) = 0$ and prove (17) for $x = 0$. The key observation is that

$$G(x) := \frac{g(x)}{e^{ix} - 1} \in C^\infty(\mathbb{R}), \quad G(x + 2\pi) = G(x).$$

This follows from the fact that $x \mapsto e^{ix} - 1$ is 2π-periodic and has a simple zero at $x = 0$. (You can use L'Hôpital's rule if you must). It follows that

$$g(x) = (e^{ix} - 1)G(x) = e^{ix}G(x) - G(x),$$

and a simple computation shows that

$$\widehat{g}(k) = \widehat{G}(k - 1) - \widehat{G}(k).$$

From (16) we see that the series $\sum \widehat{G}(k)$ and $\sum \widehat{g}(k)$ converge absolutely and hence

$$\sum_{k\in\mathbb{Z}} \widehat{g}(k) = \sum_{k\in\mathbb{Z}} (\widehat{G}(k-1) - \widehat{G}(k)) = \sum_{k\in\mathbb{Z}} \widehat{G}(k-1) - \sum_{k\in\mathbb{Z}} \widehat{G}(k) = \sum_{k\in\mathbb{Z}} \widehat{G}(k) - \sum_{k\in\mathbb{Z}} \widehat{G}(k) = 0,$$

proving the claim. □

We now can prove

Theorem
Suppose that $f \in L^2([0, \pi])$. Then

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx, \tag{18}$$

and in view of the optimality of Bessel's inequality

$$\|f - S_N f\|_{L^2([0,2\pi]} \to 0, \quad N \to \infty.$$

In particular $e_k(x) := (2\pi)^{-\frac{1}{2}} e^{ikx}$, $k \in \mathbb{Z}$, form an orthnormal basis of $L^2([0, 2\pi]$.

Proof. We will use the notation

$$c_k[f] := \langle f, e_k \rangle.$$

If we define

$$S_\infty f(x) := \sum_k c_k[f]e_k(x),$$

then Bessel's inequality shows that $S_\infty \in L^2$ and $\|S_N f - S_\infty f\|_{L^2} \to 0$ as $N \to \infty$. We want to show that $S_\infty f = f$. In the notation of the approximation theorem above we know that $S_\infty f_t = f_t$ (since f_t is smooth and periodic – see (17)) and $\|f_t - f\| \to 0$ as $t \to +$. Hence, using Bessel's inequality, $\|S_\infty F\|_{L^2} = \sum |c_k[F]|^2 \le \|F\|_{L^2}$ with $F = f - f_t$, we get

$$\begin{aligned} \|S_\infty f - f\|_{L^2} &= \lim_{t\to 0+} \|S_\infty f - f_t\|_{L^2} = \lim_{t\to 0+} \|S_\infty f - S_\infty f_t\|_{L^2} \\ &\le \lim_{t\to 0+} \|f - f_t\| = 0. \end{aligned}$$

This means that $\|f - S_N f\|_{L^2} \to 0$ and and we have the equality (18). □

By rescaling $y = \pi(x+L)/L$ and applying the theory above in y variables we see that

The set
$$(2L)^{-\frac{1}{2}} e^{\pi inx/L}, \quad n \in \mathbb{Z},$$
is an orthormal basis of $L^2([-L,L])$. This means that
$$f(x) = \sum_{-\infty}^{\infty} \widehat{f}(n) e^{2\pi inx/L}, \quad \widehat{f}(n) := \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\pi nx/L} dx, \tag{19}$$
with the convergence in $L^2([0,L])$:
$$\int_0^L |f(x) - \sum_{-N}^{N} \widehat{f}(n) e^{\pi inx/L}|^2 dx \to 0, \quad N \to \infty$$
If $f \in C^k(\mathbb{R})$, $k \geq 1$, and $f(x+2L) = f(x)$, then the series in (19) converges absolutely to $f(x)$.

Also, note that
$$\int_{-L}^{L} |f(x)|^2 dx = 2L \sum_{j\in\mathbb{Z}} |\widehat{f}(j)|^2$$
(To remember the constants check this for $f(x) \equiv 1$ so that $\widehat{f}(0) = 1$ and $\widehat{f}(j) = 0$, $j \neq 0$.)

The orthonormal basis, $(2L)^{-\frac{1}{2}} e^{\pi inx/L}$, $n \in \mathbb{Z}$, is particularly well suited to the study of $2L$-periodic functions.

Example: Consider $f(x) = 1$ for $0 \leq x \leq \pi$ and $f(x) = 0$ for $\pi < x \leq 2\pi$. Then
$$\widehat{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_0^{\pi} e^{-inx} = \begin{cases} 2^{-1} & n = 0 \\ (\pi in)^{-1} & n = 2k+1 \\ 0 & n = 2k. \end{cases}$$
The partial sums of Fourier series are then given by
$$S_N f(x) := \tfrac{1}{2} + \sum_{k=0}^{N} \frac{e^{i(2k+1)x} - e^{-i(2k+1)x}}{\pi i(2k+1)} = \tfrac{1}{2} + \sum_{k=0}^{N} \frac{2\sin(2k+1)x)}{(2k+1)\pi}. \tag{20}$$
General theory tells us that
$$\int_0^{2\pi} |f(x) - S_N f(x)|^2 dx \to 0, \quad N \to \infty.$$
It is also true that
$$S_N(x) \to f(x), \quad x \in (0,\pi) \cup (\pi, 2\pi),$$
but
$$S_N(x) \to \tfrac{1}{2} \neq f(x), \quad x = 0, \pi, 2\pi.$$

In this example, if we take $x = \pi/2$, then, using
$$\sin\big(\pi(k+\tfrac{1}{2})\big) = (-1)^k,$$
we obtain
$$1 = \tfrac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi(2k+1)} \implies \pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

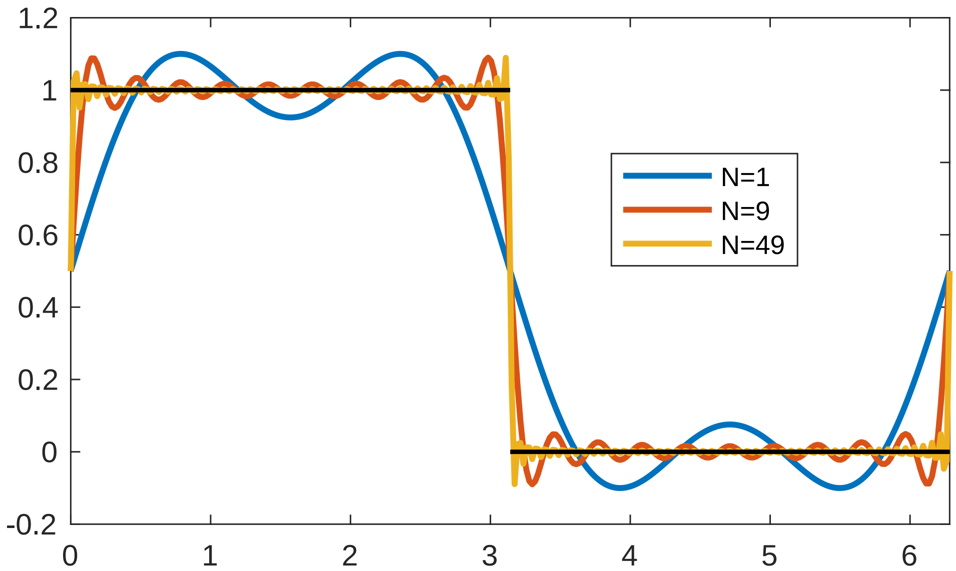


Figure 7: Plots of partial sums (20). Note that at point of the series gives 0.5 which is half of the jump. That is, in fact, a general phenomenon.

This is an example of many cool formulas we can obtain using Fourier series. This particular *conditionally* convergent series is not a very good approximation to π.

Sine and cosine series

Suppose $f \in L^2([-\pi,\pi])$ satisfies $f(x) = -f(-x)$, that is f is odd. We then have

$$\widehat{f}(k) = \int_{-\pi}^{\pi} f(x)e^{-kix}dx = \int_{-\pi}^{\pi} f(-x)e^{ikx}dx = -\int_{-\pi}^{\pi} f(x)e^{kix}dx = -\widehat{f}(-k).$$

Note that $\widehat{f}(0) = -\widehat{f}(0)$ so $\widehat{f}(0) = 0$.

This means that, with convergence of the series in L^2 (or pointwise convergence if $f(x+2\pi) = f(x)$ and $f \in C^k(\mathbb{R})$, $k \geq 1$),

$$f(x) = \sum_{k=1}^{\infty} \widehat{f}(k)(e^{ikx} - e^{-ikx}) = \sum_{k=1}^{\infty} 2i\widehat{f}(k)\sin kx. \tag{21}$$

We repeat an observation we made before: for $k, m \in \{1, 2, \cdots\}$,

$$\int_0^{\pi} \sin kx \sin mx dx = \begin{cases} \pi/2 & m = k, \\ 0 & m \neq k. \end{cases}$$

This means that

$$\left\{(2/\pi)^{\frac{1}{2}} \sin kx\right\}_{k=1}^{\infty} \subset L^2([0,\pi]) \text{ is an orthonormal set. .}$$

If $g \in L^2([0,\pi])$ we can take its odd extension to $[-\pi,\pi]$

$$f(x) = \begin{cases} g(x) & 0 \leq x \leq \pi \\ -g(-x) & -\pi \leq x \leq 0 \end{cases}$$

We then note that

$$
\begin{aligned}
s_k[g] &:= (2/\pi)^{\frac12}\int_0^\pi g(x)\sin kx dx == (2/\pi)^{\frac12}\frac{1}{2i}\int_0^\pi g(x)(e^{ikx}-e^{-ikx})dx \\
&= (2/\pi)^{\frac12}\frac{1}{2i}\int_0^\pi(-g(-x)e^{ikx}-g(x)e^{-ikx})dx = (2/\pi)^{\frac12}\frac{i}{2}\int_{-\pi}^\pi f(x)e^{-ikx}dx \\
&= (2/\pi)^{\frac12}\pi i\widehat{f}(k) = (\pi/2)^{\frac12}2i\widehat{f}(k).
\end{aligned}
$$

From (21) we conclude that, with convergence in $L^2([0,\pi])$,

$$
g(x) = \sum_{k=0}^\infty s_k[g](2/\pi)^{\frac12}\sin kx,
$$

that is,

$$
\left\{(2/\pi)^{\frac12}\sin kx\right\}_{k=1}^\infty \quad \text{is an orthonormal } \textit{basis} \text{ of } L^2([0,\pi]).
$$

We can do the same with even functions and even extensions of $g \in L^2([0,\pi])$. That shows that

$$
\{(1/\pi)^{\frac12}\cup\left\{(2/\pi)^{\frac12}\cos kx\right\}_{k=1}^\infty \quad \text{is an orthonormal } \textit{basis} \text{ of } L^2([0,\pi]).
$$

That means that, with convergence in $L^2([0,\pi])$,

$$
g(x) = c_0[g](1/\pi)^{\frac12} + \sum_{k=0}^\infty c_k[g](2/\pi)^{\frac12}\cos kx,
$$

$$
c_0[g] := (1/\pi)^{\frac12}\int_0^\pi g(x)dx, \quad c_k[g] := (2/\pi)^{\frac12}\int_0^\pi g(x)\cos kx dx, \quad k \neq 0.
$$

Example: Consider $f(x) = 1$ for $0 \leq x \leq \pi/2$ and $f(x) = 0$ for $\pi/2 < x \leq \pi$. This is essentially the same function as in the previous example but now we want to expand it into a sine series. This means that we need to compute calculate

$$
\begin{aligned}
s_k[f] &= (2/\pi)^{\frac12}\int_0^\pi f(x)\sin kx dx = (2/\pi)^{\frac12}\int_0^{\pi/2}\sin kx = (2/\pi)^{\frac12}\frac1k(-\cos kx)|_0^{\pi/2} \\
&= (2/\pi)^{\frac12}\frac1k(1-\cos(k\pi/2)) = \begin{cases} (2/\pi)^{\frac12}(2m+1)^{-1} & k = 2m+1 \\ (2/\pi)^{\frac12}(2m)^{-1}(1-(-1)^m) & k = 2m \end{cases}
\end{aligned}
$$

The partial sums of the sine series are then given by equality)

$$
S_N f(x) := \sum_{k=1}^N \frac{2(1-\cos(k\pi/2))\sin(kx)}{\pi k} \tag{22}
$$

General theory tells us that

$$
\int_0^\pi |f(x) - S_N f(x)|^2 dx \to 0, \quad N \to \infty.
$$

It is also true that

$$
S_N(x) \to f(x), \quad x \in (0,\pi)\cup(\pi,2\pi),
$$

but

$$
S_N(x) \to \tfrac12 \neq f(x), \quad x = \tfrac{\pi}{2},
$$

and

$$
S_N(x) \to 0 \neq f(x), \quad x = 0, \pi,
$$

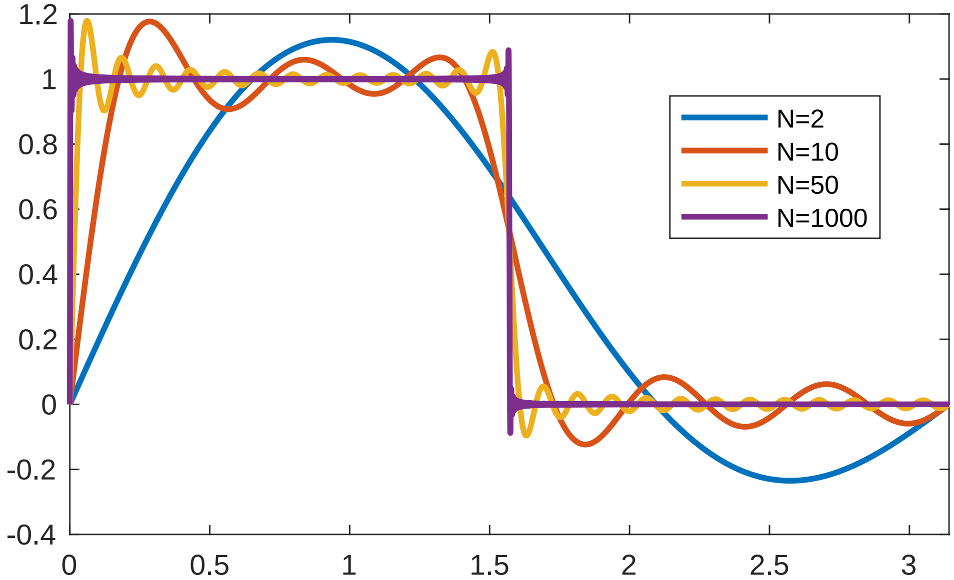


Figure 8: Plots of partial sums (22). Note the overshooting at the jumps – it does not go away as N gets larger.