

BOUNCING-BALL MODES IN ERGODIC BILLIARDS

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ABSTRACT. We show that for a horizontally stretched Bunimovich billiard, *for almost every stretching parameter*, there exists a sequence of eigenfunctions concentrating in the rectangular part. Thanks to the work of Markarian–Olfson Kamphorst–Pinto de Carvalho, we know that for an open set of parameters such billiards are ergodic.

The proof is a result of interaction with ChatGPT 6: we were curious if it could establish the existence of bouncing ball modes for Bunimovich or Sinai billiards. That was not successful until we suggested varying the wings. That produced an essentially complete argument for the existence of quasimodes. We then suggested using Hassell’s parameter-variation argument which led to the result about eigenfunctions. The argument was then significantly simplified and clarified by the authors.

1. INTRODUCTION

The celebrated theorem of Shnirelman [Sh74] states that classical ergodicity forces most high energy eigenfunctions to become uniformly distributed. For billiards, this means that an orthonormal basis of Dirichlet eigenfunctions contains a density-one subsequence whose probability densities converge to normalized area measure. That case was proved by Gérard–Leichtnam [GL93] for domains including Bunimovich billiards and by Zelditch–Zworski [ZZ96] for manifolds with arbitrary piecewise smooth boundaries.

The theorem allows however the possibility of exceptional sequences concentrating on sets of zero phase space volume. That was established by Hassell [Ha10] who showed that for almost every length of the rectangle (t in Figure 1 with $a = 1$) such a sequence exists for the Bunimovich stadium, that is, it is not quantum unique ergodic (QUE).

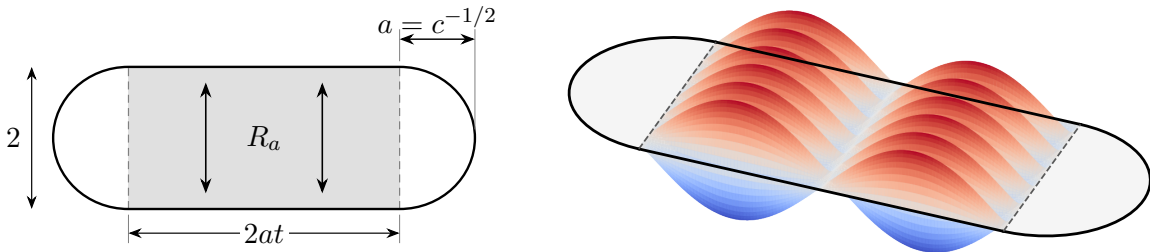


FIGURE 1. The elliptical stadium S_a , $a = c^{-1/2}$ and two bouncing-ball trajectories. Right: q_6 from (1.7), zero in the wings; Theorem 2 shows that it produces accurate quasimodes.

That sequence was not specified in [Ha10] and a natural candidate comes from the bouncing-ball trajectories perpendicular to the parallel sides of its rectangle. We prove the existence of such sequences for almost every member of the family with elliptical wings shown in Figure 1.

We state a rough version of the main result as follows:

Theorem 1. *Fix $t > 0$ and consider the billiard S_a shown in Figure 1:*

$$S_a := \left\{ (x, y) : |y| < 1, |x| < at + a\sqrt{1 - y^2} \right\}, \quad R_a := S_a \cap \{|x| < at\}.$$

There are integers $m_k \rightarrow \infty$ such that for almost every $a > 0$, there exist normalised Dirichlet eigenfunctions

$$-\Delta\psi_k = \lambda_k\psi_k, \quad \lambda_k = (m_k\pi)^2 + \pi^2/a^2t^2 + o(1), \quad \|\psi_k\|_{L^2(S_a)} = 1, \quad (1.1)$$

such that for every fixed $\varepsilon > 0$,

$$\int_{S_a \cap \{|x| \geq at + \varepsilon\}} |\psi_k|^2 dx dy \rightarrow 0. \quad (1.2)$$

In other words, the mass of ψ_k goes to 0 outside any neighbourhood of the rectangle R_a .

A more precise version justifying the bouncing ball title is presented in §7. The remark after Theorem 2 gives a statement valid for a family of fixed internal rectangles.

The limit (1.2) in Theorem 1 should be compared to a result of Burq–Zworski [BZ05] which states that for *any* sequence of normalised eigenfunctions and any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\liminf_{k \rightarrow \infty} \int_{S_a \cap \{|x| \geq at - \varepsilon\}} |\psi_k|^2 dx dy > \frac{1}{C_\varepsilon}. \quad (1.3)$$

Markarian–Olliffson Kamphorst–Pinto de Carvalho [MKC96, Theorem 2] proved ergodicity of the billiard flow for S_a under the condition that

$$1 < a < \sqrt{4 - 2\sqrt{2}}, \quad at > 2a^2\sqrt{a^2 - 1}. \quad (1.4)$$

The range of a 's and t 's is shown in Figure 2.

The theorem is a consequence of estimates needed to establish existence of good quasimodes obtained from bouncing ball modes. For that it is convenient to work in a fixed domain and consider a quarter billiard, which we will do from now on. Thus we fix $t > 0$ and define Ω to be the interior of $\overline{R} \cup \overline{C}$, where

$$R = (-t, 0) \times (0, 1), \quad \Gamma = \{0\} \times (0, 1), \quad C = \{x > 0, y > 0, x^2 + y^2 < 1\}. \quad (1.5)$$

For real $c > 0$, let H_c be the self-adjoint Dirichlet realization of

$$H_c = -c\partial_x^2 - \partial_y^2. \quad (1.6)$$

This is unitarily equivalent to the Euclidean Dirichlet Laplacian on the horizontally stretched quarter-stadium obtained from S_a by choosing $a = 1/\sqrt{c}$.

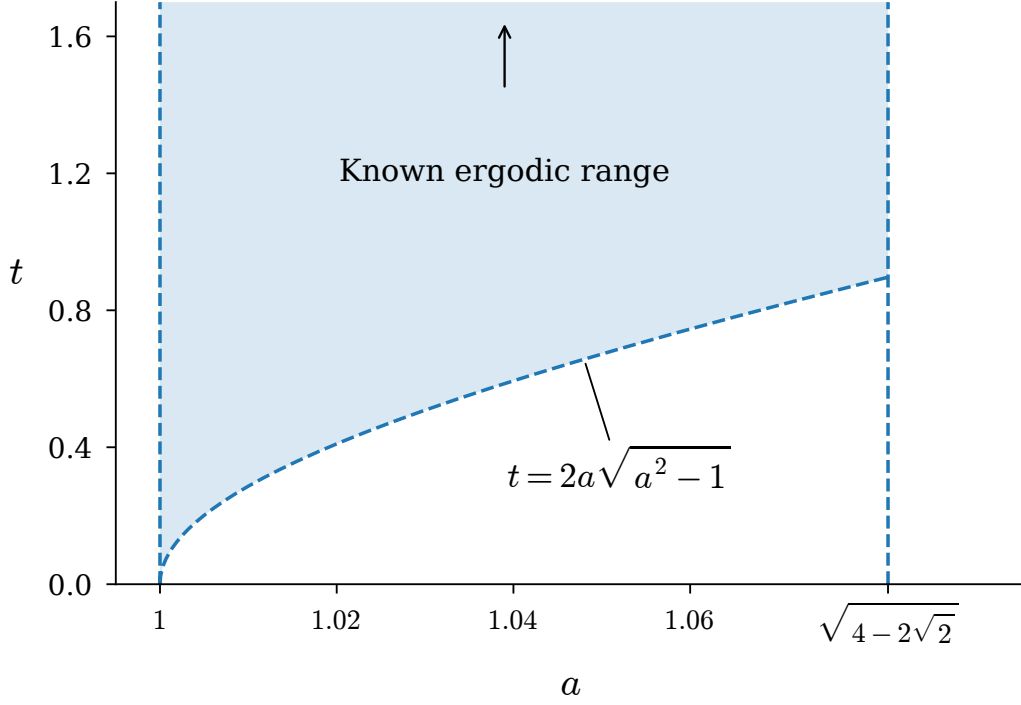


FIGURE 2. An illustration of the condition (1.4) from [MKC96] on (a, t) guaranteeing ergodicity of the billiard flow for S_a in Figure 1.

The bouncing ball modes in R extended by 0 to all of Ω is given by

$$q_m(x, y) := \sqrt{\frac{2}{t}} \sin \frac{\pi(x+t)}{t} e_m(y) \mathbf{1}_R, \quad e_m(y) := \sqrt{2} \sin(m\pi y), \quad \|q_m\| = 1. \quad (1.7)$$

Theorem 2. *There are integers $m_k \rightarrow \infty$ such that for almost every real $c > 0$ there exist $u_k \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying*

$$\|u_k - q_{m_k}\| \rightarrow 0, \quad \|(H_c - (m_k\pi)^2 - c\nu)u_k\| \rightarrow 0, \quad \nu := \pi^2/t^2. \quad (1.8)$$

We stress that in Theorem 1 (and Theorem 3) we do not assert $\|\varphi_k - q_{m_k}\| \rightarrow 0$ for individual eigenfunctions.

Remark. One can also keep the physical rectangle fixed. Put $L = at$ and write

$$S_{L,a} = \{(x, y) : |y| < 1, \quad |x| < L + a\sqrt{1 - y^2}\}.$$

For almost every $L > 0$, the existence and concentration conclusions of Theorems 1 and 2 (and of Theorem 3) hold on $S_{L,a}$ for almost every $a > 0$. Thus the rectangle $(-L, L) \times (-1, 1)$ is fixed and only the elliptical wings vary. The spectral centres are now $E_m = (m\pi)^2 + \pi^2/L^2$, independent of a , and the eigenvalues in Theorems 1 and 3 satisfy $\lambda_k = E_{m_k} + o(1)$. The subsequences may depend on a . Indeed, for every fixed $t > 0$ the exceptional set of $c > 0$ has measure zero. After identifying the domains, the

norm and spectral conditions in the proofs are measurable in (c, t) , so Fubini's theorem gives a null exceptional set of pairs.

1.1. Outline of the proof. The functions q_m solve the eigenvalue equation except for a jump of normal derivatives at Γ . To remove it, we solve

$$(H_c - (m\pi)^2 - z)U_m = e_m \delta_\Gamma, \quad g_m(c, z) := \langle U_m|_\Gamma, e_m \rangle_{L^2(\Gamma)}.$$

For real $c > 0$ and $z = c\nu + i$, the energy identity gives $\text{Im } g_m = \|U_m\|^2$. The correction $v_m = q_m + c\gamma_t U_m(c, c\nu + i)$ with $\gamma_t := \sqrt{2}\pi t^{-3/2}$ then satisfies $(H_c - (m\pi)^2 - c\nu)v_m = i(v_m - q_m)$. Thus the smallness of g_m is the key to obtaining a quasimode from q_m . The bounded errors sufficient for [Ha10] would not give the concentration obtained here – making them tend to zero is the main additional difficulty.

To achieving this smallness we complexify c . At $c = 1 - i$ and $z = i$, the energy identity also controls the unscaled horizontal derivative $\partial_x U_m$. With this additional control, a semiclassical propagation argument, in the spirit of Burq–Zworski [BZ05, BZ04] used to obtain (1.3), forces the mass in the cap to vanish. A trace estimate then gives $g_m(1 - i, i) \rightarrow 0$. Positivity and “harmonic measure magic” transfer this decay to

$$\int_I \text{Im } g_m(c, c\nu + i) dc \rightarrow 0, \quad I \Subset (0, \infty).$$

Related uses of harmonic measure in spectral theory go back to Pearson [Pe94]. This averaged estimate gives the required quasimodes after selecting a subsequence, for almost every c .

Finally, Hassell's parameter-variation argument [Ha10], applied in shrinking spectral windows, selects exact eigenfunctions satisfying $\lambda_k^{-1} \|\partial_x \varphi_k\|^2 \rightarrow 0$. Propagation results for semiclassical defect measures from Burq [Bu97] then show the concentration on the closed bouncing-ball family.

We remark that the argument applies to more general partially rectangular domains as in Hillairet–Hassell [Ha10, Appendix]. To keep the presentation focused we restrict ourselves to the case of stretched Bunimovich billiards.

1.2. Review of the literature. Here we will review some physics and maths papers, which in addition to the already cited [Ha10], [BZ05], addressed the bouncing ball problem. We do not claim completeness, especially when it comes to the review of the physics literature.

1.2.1. Physics. Early work of Shapiro–Taylor–Brumer [STB84] used an adiabatic approximation to describe regular low energy eigenstates of the stadium. Bai et al. [B*85] developed a Born–Oppenheimer construction of bouncing ball modes, which were also observed in the numerical study of McDonald–Kaufman [MK88]. O'Connor–Heller [OH88] gave semiclassical arguments for the persistence of bouncing ball localization at arbitrarily high energies. These modes are associated with the neutral family of trajectories between the parallel sides. This differs from the scarring along unstable

periodic orbits introduced by Heller [He84]. Sieber et al. [S*93] derived semiclassically the oscillatory contribution of bouncing ball trajectories to the spectral counting function and its effect on spectral statistics, accounting for features of a measured stadium spectrum.

Bäcker–Schubert–Stifter [BSS97] and Tanner [Ta97] predicted that the number of bouncing ball modes below energy E in the stadium grows as $E^{3/4}$, using semiclassical constructions supported by numerical calculations. The effect of these exceptional modes on the rate of quantum ergodicity was studied in [BSS98]. Bäcker–Schubert [BS99] studied numerically the localization of bouncing ball modes in position and momentum space. Löck–Bäcker–Ketzmerick [LBK12] analysed their coupling to chaotic states and obtained counting laws from the resulting decay rates. More recently, Dietz–Nguyen–Čadež [DNC26] numerically compared bouncing ball modes in Schrödinger, neutrino and graphene stadium billiards.

On the experimental side, Stein–Stöckmann [SS92] visualized bouncing ball modes in a microwave stadium cavity (see their Figure 2a). Chinnery–Humphrey [CH96] visualized bouncing ball modes in an acoustic stadium, while Gräf et al. [G*92] and Alt et al. [A*99] studied the spectral contributions of bouncing ball trajectories in microwave stadium billiards. The work of Heller–Crommie–Lutz–Eigler [H*94] on standing electron waves in quantum corrals provides a related physical realization of the stadium geometry. Ge et al. [G*24] reported bouncing ball scars in stadium-shaped graphene quantum dots with electrostatic confinement – see in particular the numerically computed states in their Figure 4f, compared with the experimental pattern in Figure 2b.

1.2.2. Mathematics. Zelditch [Ze04] provided a conceptual precursor of Hassell’s argument by showing that the failure of QUE follows from the existence of bouncing ball modes within an approximate span of a uniformly bounded number of actual modes. In [BZ05], (1.3) was established, showing that modes cannot concentrate in subsets of the rectangle. That was refined by Burq–Hassell–Wunsch [BHW07] who gave lower bounds (going to zero with energy) on the leakage into the wings. Papers by Hillairet–Marzuola [HM12] and Christianson [Ch15] provided additional quantitative restrictions on bouncing-ball quasimodes.

The breakthrough by Hassell [Ha10] showed the failure of QUE for almost every Bunimovich stadium. (Comments on that and earlier results can be found on Tao’s blog [Ta07, Ta08].) Mangoubi–Weller Weiser [MW24] clarify some boundary regularity issues in Hassell’s argument at the junction of the straight and curved boundary pieces. Most recently, Li–Liu [LL26] used Hassell’s argument and bouncing ball modes to show physical-space non-equidistribution for almost every parameter, without however showing concentration in the rectangle.

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2. AN AUXILIARY BOUNDARY VALUE PROBLEM

If we consider q_m in (1.7) as an element of $H_0^1(\Omega)$, the mismatch of normal derivatives at Γ gives

$$(H_c - (m\pi)^2 - c\nu)q_m = -c\gamma_t e_m \delta_\Gamma, \quad \gamma_t := \sqrt{2}\pi t^{-3/2}, \quad (2.1)$$

where $e_m \delta_\Gamma$ is the boundary functional $V \mapsto \langle e_m, V|_\Gamma \rangle_{L^2(\Gamma)}$. To remove this term, we first solve the problem with the source given by $e_m \delta_\Gamma$, a complex spectral parameter z , and complexified c : Thus, for

$$\operatorname{Re} c > 0, \quad \operatorname{Im} c \leq 0, \quad z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad (2.2)$$

we let $U = U_m(c, z) \in H_0^1(\Omega)$ satisfy

$$(H_c - (m\pi)^2 - z)U = e_m \delta_\Gamma \quad \text{in } \Omega. \quad (2.3)$$

The restrictions $U_{m,C} = U_m|_C$ and $U_{m,R} = U_m|_R$ solve the homogeneous equation in their interiors, vanish on the exterior boundary, and satisfy transmission conditions

$$U_{m,R}|_\Gamma = U_{m,C}|_\Gamma, \quad c\partial_x U_{m,R}|_\Gamma - c\partial_x U_{m,C}|_\Gamma = e_m. \quad (2.4)$$

The conormal derivatives are understood weakly. We define the following crucial object which is the e_m component of the restriction to Γ :

$$g_m(c, z) = \langle U_m|_\Gamma, e_m \rangle_{L^2(\Gamma)}. \quad (2.5)$$

To show that this transmission problem has a solution, we rephrase it weakly using a quadratic form:

$$c\langle \partial_x U, \partial_x V \rangle + \langle \partial_y U, \partial_y V \rangle - ((m\pi)^2 + z)\langle U, V \rangle = \langle e_m, V|_\Gamma \rangle_{L^2(\Gamma)}, \quad (2.6)$$

for all $V \in H_0^1(\Omega)$. Inner products are linear in the first factor and the norms and inner products without a specified domain are over Ω .

Lemma 1. *For the parameters in (2.2), problem (2.6) has a unique solution. The functions $U_m(c, z)$ and $g_m(c, z)$ are holomorphic on*

$$\{(c, z) \in \mathbb{C}^2 : \operatorname{Re} c > 0, \operatorname{Im} c < 0, \operatorname{Im} z > 0\},$$

and extend holomorphically to a neighborhood of each (c_0, z_0) with $c_0 > 0$ real and $z_0 \in \mathbb{C}_+$. Moreover, for g_m defined in (2.5),

$$\operatorname{Im} g_m(c, z) = (-\operatorname{Im} c)\|\partial_x U_m(c, z)\|^2 + (\operatorname{Im} z)\|U_m(c, z)\|^2 \geq 0. \quad (2.7)$$

In particular, for fixed real $c > 0$, the function $z \mapsto g_m(c, z)$ is holomorphic on $\mathbb{C}_+$ and has nonnegative imaginary part there.

Remarks. 1. We stress that no uniformity in m is asserted in the conclusions of the lemma. It is an abstract functional analysis argument.

2. The energy identity (2.7) explains the significance g_m defined in (2.5). For $c > 0$ (the physically interesting case of billiards) and $z = c\nu + i$, the identity $\text{Im } g_m(c, z) = \|U_m(c, z)\|^2$ shows that $\text{Im } g_m \ll 1$ implies $\|U_m(c, z)\| \ll 1$. That means that the correction to (2.1) $r_m := c\gamma_t U_m(c, z)$ would be small and $q_m + r_m$ would produce a quasimode.

Proof of Lemma 1. The right hand side of (2.6), $V \mapsto \langle e_m, V|_\Gamma \rangle_{L^2(\Gamma)}$, is a continuous antilinear functional on $H_0^1(\Omega)$.

If the left hand side of (2.6) is denoted by $B(U, V)$, then for a sufficiently large positive M , depending on m, c, z ,

$$\begin{aligned} \text{Re}((1 + iM)B(U, U)) &= (\text{Re } c - M \text{Im } c)\|\partial_x U\|^2 + \|\partial_y U\|^2 \\ &\quad + (M \text{Im } z - (m\pi)^2 - \text{Re } z)\|U\|^2 \end{aligned}$$

is coercive on $H_0^1(\Omega)$. Consequently the Lax–Milgram theorem (see for instance [Ev10, §6.2.1, Theorem 1]) gives existence and uniqueness. The form B defines a bounded operator $\mathcal{B}(c, z) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by $(\mathcal{B}(c, z)U)(V) = B(U, V)$, where $H^{-1}(\Omega)$ is the continuous antilinear dual of $H_0^1(\Omega)$. It depends holomorphically, in operator norm, on c, z and so does the inverse guaranteed by the Lax–Milgram argument. Writing

$$U_m(c, z) = \mathcal{B}(c, z)^{-1}(e_m \delta_\Gamma),$$

we see that $(c, z) \mapsto U_m(c, z)$ is holomorphic with values in $H_0^1(\Omega)$. Applying the continuous linear functional $U \mapsto \langle U|_\Gamma, e_m \rangle_{L^2(\Gamma)}$ gives the same conclusion for g_m . The invertibility holds for c, z satisfying (2.2) and hence the holomorphy holds in a neighbourhood of any such point.

Taking $V = U$ in (2.6) gives $B(U, U) = \bar{g}_m$, from which (2.7) follows. $\square$

3. THE COMPLEX-PARAMETER ESTIMATE

We now prove that the boundary coefficient (2.5) tends to zero at one fixed pair of complex parameters:

$$g_m(1 - i, i) \rightarrow 0, \quad m \rightarrow \infty. \quad (3.1)$$

This decay and “harmonic measure magic” will be used to obtain an average decay of $\text{Im } g_m(c, c\nu + i)$ for the relevant positive values of c . The particular values $1 - i$ and i in (3.1) are inessential: any fixed pair with $\text{Re } c > 0$, $\text{Im } c < 0$, $\text{Im } z > 0$ works, with constants depending on that pair.

3.1. Uniform bounds without a high-frequency loss. For $u \in H^1(C)$ vanishing on the circular arc, $x = \sqrt{1 - y^2}$, integration in x over $0 < x < \sqrt{1 - y^2}$ gives

$$\|u|_\Gamma\|_{L^2(\Gamma)}^2 = -2 \text{Re} \int_C \partial_x u \bar{u} dx dy \leq 2\|u\|_{L^2(C)} \|\partial_x u\|_{L^2(C)}. \quad (3.2)$$

In this section we write $U_m = U_m(1 - i, i)$ and $g_m = g_m(1 - i, i)$. Using the definition (2.5), formula (2.7), and the estimate (3.2) gives

$$\begin{aligned} |g_m|^2 &\leq \|U_{m,C}|_\Gamma\|_{L^2(\Gamma)}^2 \leq \|U_{m,C}\|_{L^2(C)}^2 + \|\partial_x U_{m,C}\|_{L^2(C)}^2 \\ &\leq \|U_m\|_{L^2(\Omega)}^2 + \|\partial_x U_m\|_{L^2(\Omega)}^2 = \operatorname{Im} g_m \leq |g_m|, \end{aligned} \quad (3.3)$$

We conclude that

$$|g_m| \leq 1, \quad \|U_m\|_{L^2(\Omega)}^2 + \|\partial_x U_m\|_{L^2(\Omega)}^2 \leq 1. \quad (3.4)$$

Taking the real part of (2.6), with $V = U_m$, also gives

$$\begin{aligned} (m\pi)^{-2} (\|\partial_y U_{m,C}\|_{L^2(C)}^2 + \|\partial_y U_{m,R}\|_{L^2(R)}^2) \\ = \|U_m\|_{L^2(\Omega)}^2 + (m\pi)^{-2} \operatorname{Re} g_m - (m\pi)^{-2} \|\partial_x U_m\|_{L^2(\Omega)}^2 \leq 2, \end{aligned} \quad (3.5)$$

where we used (3.4) to get the estimate. We note that the estimates on $\partial_x U_m$, unlike those on $\partial_y U_m$, do not involve any loss when $m \rightarrow \infty$. That is what we mean by the title of this subsection.

3.2. Cap compactness: a semiclassical argument. We use two integrations by parts and weak limits of $|u_m|^2 dx dy$ to pass from the estimates in §3.1 to decay in the cap C . This gives the following lemma.

Lemma 2. *Suppose $u_m \in H^1(C)$ vanishes on the arc and on $y = 0$, satisfies*

$$(-(1-i)\partial_x^2 - \partial_y^2 - (m\pi)^2 - i)u_m = 0 \quad \text{in } C, \quad (3.6)$$

and, for a constant K independent of m ,

$$\|u_m\|_{L^2(C)} + \|\partial_x u_m\|_{L^2(C)} + \|(m\pi)^{-1} \partial_y u_m\|_{L^2(C)} \leq K, \quad \|u_m|_\Gamma\|_{L^2(\Gamma)} \leq K.$$

Then

$$\|u_m\|_{L^2(C)} \rightarrow 0.$$

Proof. Odd reflection of u_m across $y = 0$ extends it to $H^1(D)$, where

$$D = \{(x, y) : x > 0, x^2 + y^2 < 1\}. \quad (3.7)$$

The extension solves (3.6) in D and vanishes on the circular arc. We put $h = (m\pi)^{-1}$ and $u = u(h) = u_m$ for the extension. The norms in this proof are over D and the restrictions to $x = 0$ are measured in $L^2((-1, 1)_y)$. The equation is

$$-h^2 u_{yy} - u = (1-i)h^2 u_{xx} + ih^2 u, \quad \|u\| + \|u_x\| + \|hu_y\| \leq C. \quad (3.8)$$

We first show that every weak limit of $|u(h)|^2 dx dy$ is constant along vertical segments. This uses two integrations by parts, in the spirit of the control argument of [BZ05] – see also [BZ04, §6.1].

We will use the local bound $\|hu_{xy}\|_{L^2(K)} \leq C_K$, $K \Subset D$. This follows from the interior elliptic regularity applied to u_x , which satisfies the same equation as u – see for instance [Zw12, Theorem 7.1].

For real $\psi \in C_c^\infty(D)$, multiplying (3.8) by $\psi \bar{u}$, integrating by parts, and taking real parts gives

$$\begin{aligned} \int_D \psi (h^2 |u_y|^2 - |u|^2) dx dy &= \frac{1}{2} h^2 \int_D \psi_{yy} |u|^2 dx dy - h^2 \int_D \psi |u_x|^2 dx dy \\ &\quad - h^2 \operatorname{Re} \int_D (1-i) \psi_x u_x \bar{u} dx dy = O_\psi(h^2). \end{aligned} \quad (3.9)$$

For real $\varphi \in C_c^\infty(D)$, we next multiply (3.8) by $2\varphi \bar{u}_y$, so that integration in y on the left and in x on the right yields

$$\begin{aligned} \int_D \varphi_y (h^2 |u_y|^2 + |u|^2) dx dy &= -2h^2 \operatorname{Re} \int_D (1-i) (\varphi_x u_x \bar{u}_y + \varphi u_x \bar{u}_{xy}) dx dy \\ &\quad + 2h^2 \operatorname{Re} \int_D i \varphi u \bar{u}_y dx dy = O_\varphi(h). \end{aligned} \quad (3.10)$$

The last estimate uses (3.8) and the local bound on hu_{xy} . Subtracting (3.9), with $\psi = \varphi_y$, from (3.10) gives

$$\int_D \varphi_y |u(h)|^2 dx dy = O_\varphi(h).$$

After passage to a subsequence, the bounded measures $|u(h)|^2 dx dy$ converge weakly on compact subsets of D to a nonnegative measure ρ . The preceding estimate gives $\partial_y \rho = 0$, in the sense of distributions. Thus

$$d\rho(x, y) = d\sigma(x) dy, \quad -L(x) < y < L(x), \quad L(x) = \sqrt{1-x^2}, \quad (3.11)$$

where σ is a nonnegative, locally finite measure on $(0, 1)$. Indeed, $\partial_y \rho = 0$ gives this product form on interior rectangles – see [Hö90, Theorem 3.1.4]

We next exclude mass in a shrinking boundary layer. Set $s(y) = \sqrt{1-y^2}$ and denote the horizontal collar of the arc by

$$\mathcal{A}_\varepsilon = \{(x, y) \in D : 0 < s(y) - x < \varepsilon\}.$$

For fixed y , apply the fundamental theorem of calculus to u_m itself, using $u(s(y), y) = 0$, and then Cauchy–Schwarz:

$$|u(x, y)|^2 = \left| \int_x^{s(y)} \partial_x u(r, y) dr \right|^2 \leq (s(y) - x) \int_x^{s(y)} |\partial_x u(r, y)|^2 dr.$$

Integrating x from $\max(0, s(y) - \varepsilon)$ to $s(y)$ gives

$$\int_{\max(0, s(y) - \varepsilon)}^{s(y)} |u(x, y)|^2 dx \leq \frac{\varepsilon^2}{2} \int_0^{s(y)} |\partial_x u(r, y)|^2 dr.$$

Thus the factor ε^2 comes from the x integration, while y still ranges over $(-1, 1)$. Integrating in y proves

$$\int_{\mathcal{A}_\varepsilon} |u|^2 dx dy \leq \frac{\varepsilon^2}{2} \|\partial_x u\|_{L^2(D)}^2 \leq C\varepsilon^2. \quad (3.12)$$

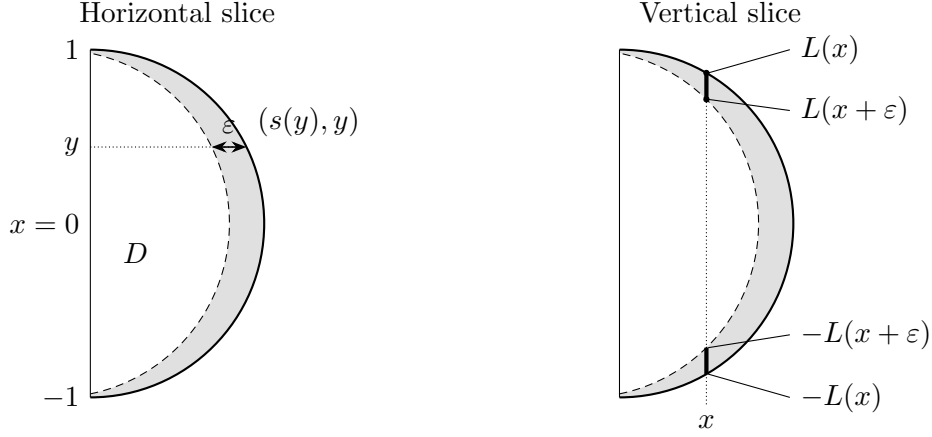


FIGURE 3. The same collar $\mathcal{A}_\varepsilon$ (shaded) in the half-disk D . Its horizontal width is at most ε . At a fixed x away from the endpoints, its vertical slice consists of the two thick segments, of total length $2(L(x) - L(x + \varepsilon))$.

Similarly, $u(x, y) = u(0, y) + \int_0^x \partial_x u(r, y) dr$ gives

$$\int_{\{0 < x < \varepsilon\} \cap D} |u|^2 dx dy \leq 2\varepsilon \|u|_{x=0}\|_{L^2(-1,1)}^2 + \varepsilon^2 \|\partial_x u\|_{L^2(D)}^2 \leq C\varepsilon. \quad (3.13)$$

Together these bounds exclude loss of L^2 mass at ∂D .

Fix $0 < \delta < 1/2$, put $K_\delta = [\delta, 1 - \delta]$, and take $0 < \varepsilon < \delta/2$. For $x \in K_\delta$, the condition $0 < s(y) - x < \varepsilon$ is equivalent to $L(x + \varepsilon) < |y| < L(x)$ (these are the upper and lower segments in Figure 3). Moreover,

$$L(x) - L(x + \varepsilon) = \int_x^{x+\varepsilon} \frac{r}{\sqrt{1-r^2}} dr \geq \delta\varepsilon.$$

To pass (3.12) to ρ , first test with compactly supported nonnegative functions in $\mathcal{A}_\varepsilon$ and then increase them to its indicator. Using (3.11), we obtain

$$\begin{aligned} C\varepsilon^2 &\geq \rho(\mathcal{A}_\varepsilon) \geq \int_{K_\delta} \left(\int_{-L(x)}^{-L(x+\varepsilon)} dy + \int_{L(x+\varepsilon)}^{L(x)} dy \right) d\sigma(x) \\ &= 2 \int_{K_\delta} (L(x) - L(x + \varepsilon)) d\sigma(x) \geq 2\delta\varepsilon \sigma(K_\delta). \end{aligned}$$

Letting $\varepsilon \rightarrow 0+$ gives $\sigma(K_\delta) = 0$. Since δ is arbitrary, $\rho = 0$. The bounds (3.12)–(3.13) then give $\|u(h)\|_{L^2(D)} \rightarrow 0$, $h \rightarrow 0$: outside these two collars one has a compact subset of D , where the limiting mass is zero. Every subsequence with a nonzero limiting norm would give the same contradiction. $\square$

Apply Lemma 2 to $U_{m,C}(1 - i, i)$. Its equation in the cap follows from (2.6), and its bounds from (3.3)–(3.5). Hence

$$\|U_{m,C}(1 - i, i)\|_{L^2(C)} \rightarrow 0.$$

The trace inequality (3.2) now gives

$$|g_m(1-i, i)|^2 \leq 2\|U_{m,C}(1-i, i)\|_{L^2(C)}\|\partial_x U_{m,C}(1-i, i)\|_{L^2(C)} \rightarrow 0.$$

Finally, (2.7) yields

$$\|U_m(1-i, i)\|_{L^2(\Omega)}^2 + \|\partial_x U_m(1-i, i)\|_{L^2(\Omega)}^2 = \operatorname{Im} g_m(1-i, i) \rightarrow 0.$$

This last identity includes both the cap and the rectangle components.

4. HARMONIC MEASURE AND REAL PARAMETERS

The positivity of $\operatorname{Im} g_m$ (see (2.7)) lets us convert $c \mapsto g_m(c, i)$ to a bounded holomorphic function using the map $g \mapsto (g-i)/(g+i)$ from the upper half-plane to the unit disk. Thus for $\operatorname{Re} c > 0$, $\operatorname{Im} c < 0$ we put

$$\begin{aligned} h_m(c) &:= 1 + \operatorname{Re} \frac{g_m(c, i) - i}{g_m(c, i) + i} = \frac{2(|g_m(c, i)|^2 + \operatorname{Im} g_m(c, i))}{|g_m(c, i) + i|^2}, \\ 0 &\leq h_m(c) \leq 2, \quad h_m(1-i) \rightarrow 0, \quad m \rightarrow \infty, \end{aligned} \tag{4.1}$$

where the last statement follows from (3.1).

The Poisson formula for the lower half-plane $\operatorname{Im} w < 0$ evaluated at $w = -2i$ is given by

$$F(-2i) = \int_{\mathbb{R}} \frac{2}{\pi(\xi^2 + 4)} F^*(\xi) d\xi$$

where F is a bounded harmonic function on $\operatorname{Im} w < 0$, and where F^* is its almost-everywhere defined boundary value – see for instance [HK76, Chapter 1]. The map $w = c^2$ sends $\operatorname{Re} c > 0$, $\operatorname{Im} c < 0$ onto $\operatorname{Im} w < 0$ and sends $1-i$ to $-2i$. Hence, the change of variables gives

$$h_m(1-i) = \int_0^\infty P(s) h_m(s) ds + \int_0^\infty P(s) h_m^*(-is) ds.$$

where

$$P(s) = \frac{2}{\pi(s^4 + 4)} 2s = \frac{4s}{\pi(s^4 + 4)}, \quad s > 0. \tag{4.2}$$

We should stress that the second term uses boundary limits, not solutions of (2.6) at $\operatorname{Re} c = 0$.

Since the boundary values, $h_m^*(s) = h_m(s)$ (g_m is defined for $\operatorname{Re} c > 0$, $\operatorname{Im} c = 0$ – see Lemma 1) and $h_m^*(-is)$, $s > 0$, are nonnegative, dropping the second integral gives

$$\int_0^\infty P(s) h_m(s) ds \leq h_m(1-i) \rightarrow 0. \tag{4.3}$$

For a compact interval $I \Subset (0, \infty)$, the continuous function P has a positive minimum on I . Hence

$$\int_I h_m(c) dc \rightarrow 0. \tag{4.4}$$

We upgrade this estimate slightly with a modified spectral parameter:

Lemma 3. *For every compact interval $I \Subset (0, \infty)$ there is a constant C_I , independent of m , such that for,*

$$\mathcal{H}(g) := 1 + \operatorname{Re} \frac{g - i}{g + i}, \quad \nu > 0, \quad (4.5)$$

$$\int_I \mathcal{H}(g_m(c, c\nu + i)) dc \leq C_I \int_I \mathcal{H}(g_m(c, i)) dc \leq C_I \mathcal{H}(g_m(1 - i, i)) \rightarrow 0. \quad (4.6)$$

Proof. For each real $c > 0$, the function $z \mapsto \mathcal{H}(g_m(c, z))$ is nonnegative and harmonic on $\mathbb{C}_+$. The points i and $c\nu + i$, for $c \in I$, lie in a fixed compact subset of $\mathbb{C}_+$. Harnack's inequality [HK76, Theorem 1.18], applied along a finite chain of disks in $\mathbb{C}_+$, gives

$$\mathcal{H}(g_m(c, c\nu + i)) \leq C_I \mathcal{H}(g_m(c, i)), \quad c \in I.$$

The constant is independent of m and c . Integrating in c proves the first inequality. The second one was already established in (4.3) using $\min_I P > 0$. $\square$

5. EXISTENCE OF STRONG QUASIMODES

5.1. Correction of the interface defect. For $c > 0$ we put $E_m(c) = (m\pi)^2 + c\nu$, and define the following candidate for a quasimode (see Remark 2 after Lemma 1):

$$v_m = q_m + c\gamma_t U_m(c, c\nu + i), \quad (H_c - E_m(c))v_m = i(v_m - q_m), \quad (5.1)$$

where the last identity follows from (2.1) and (2.3). Since $v_m \in H_0^1(\Omega)$ and $(H_c - E_m(c))v_m = i(v_m - q_m)$ belongs to $L^2(\Omega)$, $v_m \in D(H_c) = H^2(\Omega) \cap H_0^1(\Omega)$.

The quality of the quasimode is measured by

$$A_m(c) = \|v_m - q_m\|^2 = c^2 \gamma_t^2 \operatorname{Im} g_m(c, c\nu + i), \quad (5.2)$$

where the last equality comes from (2.7). We also note that (5.1) gives (using the resolvent identity $(H - \bar{z})^{-1} - (H - z)^{-1} = -2i \operatorname{Im} z (H - \bar{z})^{-1} (H - z)^{-1}$)

$$A_m(c) = 1 - \operatorname{Im} \langle (H_c - E_m(c) - i)^{-1} q_m, q_m \rangle, \quad (5.3)$$

and that

$$\|v_m\|^2 = 1 - A_m(c), \quad \operatorname{Re} \langle v_m, q_m \rangle = 1 - A_m(c). \quad (5.4)$$

In particular, $0 \leq A_m(c) < 1$, since (5.1) rules out $v_m = 0$.

5.2. The averaged estimate. We now fix $I \Subset (0, \infty)$, and consider write $g = g_m(c, c\nu + i)$, $c \in I$. For $\operatorname{Im} g \geq 0$, the definition (4.5) gives

$$\mathcal{H}(g) \geq \frac{2 \operatorname{Im} g}{1 + \operatorname{Im} g} = \frac{2A_m(c)}{c^2 \gamma_t^2 + A_m(c)}.$$

Since $A_m(c) < 1$, it follows that

$$A_m(c) \leq \frac{1}{2}(c^2 \gamma_t^2 + 1) \mathcal{H}(g) \leq C_I \mathcal{H}(g), \quad c \in I. \quad (5.5)$$

Integrating (5.5) and applying Lemma 3 gives

$$\begin{aligned} \int_I A_m(c) dc &\leq C_I \int_I \mathcal{H}(g_m(c, c\nu + i)) dc \\ &\leq C_I \mathcal{H}(g_m(1 - i, i)) \rightarrow 0. \end{aligned} \quad (5.6)$$

Thus the intermediate integral is precisely the one in (4.6).

5.3. Normalization and selection of a subsequence. Writing $A_m = A_m(c)$ and $u_m = v_m / \sqrt{1 - A_m}$, we obtain from (5.4)

$$\begin{aligned} \|(H_c - E_m(c))u_m\| &= \sqrt{\frac{A_m}{1 - A_m}}, \quad \|u_m\|_{L^2(C)}^2 \leq \frac{A_m}{1 - A_m}, \\ \|u_m - q_m\|^2 &= 2 - 2\sqrt{1 - A_m}. \end{aligned} \quad (5.7)$$

The cap estimate uses $q_m|_C = 0$ and $\|v_m - q_m\|^2 = A_m$. The last equality follows from $\operatorname{Re}\langle v_m, q_m \rangle = 1 - A_m$ and $\|u_m\| = \|q_m\| = 1$. Also $u_m \in D(H_c)$.

To choose a common subsequence, use (5.6) to select $m_k > m_{k-1}$, for $k \geq 2$, such that

$$\int_{1/k}^k A_{m_k}(c) dc \leq 2^{-k}.$$

For each fixed $j \geq 2$, Tonelli's theorem gives $\sum_{k \geq j} A_{m_k}(c) < \infty$ for almost every $c \in [1/j, j]$. Hence $A_{m_k}(c) \rightarrow 0$ for almost every $c > 0$. Equations (5.7) prove Theorem 2. The change of parameter $a = c^{-1/2}$ preserves null sets on compact positive intervals, giving the geometric formulation.

6. EXACT EIGENFUNCTIONS

Now we begin to work toward a more precise version of Theorem 1. We will show in Theorem 3 that there exists a sequence of eigenfunctions that localize to the bouncing-ball modes in the closed rectangle. To do so, we first construct a sequence of high-energy eigenfunctions whose L^2 mass of the semiclassical x -derivative tend to zero (see Lemma 5). This means that the sequence must concentrate on the vertical modes.

The strategy is a variation of Hassell's argument in [Ha10, §§2–4]. Using the averaged estimate (5.6), we find that for almost every $c > 0$, there exists nonempty spectral windows centered at

$$E_m(c) := (m\pi)^2 + c\nu$$

in which we have control over the eigenvalue derivative $\lambda'_j(c)$. This in turn controls the x derivative of the corresponding eigenfunction.

6.1. Eigenvalue derivatives and counting. Let $\lambda_j(c)$ denote the eigenvalues of H_c , repeated with multiplicity and arranged in increasing order. One can view H_c as a family of Laplacians with respect to an analytic family of metrics on Ω and find that off crossing parameters, $\lambda_j(c)$ can be followed analytically – compare [Ha10, §2]. For a normalized eigenfunction belonging to an analytic branch, differentiation of the form gives

$$\lambda'_j(c) = \|\partial_x \varphi_j(c)\|^2 \geq 0. \quad (6.1)$$

At a crossing one diagonalizes the form $\langle \partial_x u, \partial_x v \rangle$ in the eigenspace. Isolated crossings form a null set. If a multiple eigenvalue persists on an interval, this form restricted to the eigenspace is $\lambda'_j(c)$ times the identity. Consequently the estimates below apply to all eigenfunctions in a given eigenspace, outside a null set. No simplicity hypothesis is used.

Fix a compact parameter interval $I = [c_-, c_+] \Subset (0, \infty)$. Write

$$N_m(c, \eta) = \#\{j : |\lambda_j(c) - E_m(c)| \leq \eta\}, \quad 0 < \eta \leq 1.$$

The Weyl law reads

$$N_c(L) = \frac{|\Omega|}{4\pi\sqrt{c}}L + o(L), \quad N_c(L) := \#\{j : \lambda_j(c) \leq L\},$$

where the $o(L)$ remainder is uniform over $c \in I$. Therefore, it follows that

$$r_m := \frac{1}{(m\pi)^2} \int_I N_m(c, 1) dc \rightarrow 0. \quad (6.2)$$

We have the following lemma which controls the average number eigenvalues in a size η window of $E_m(c)$ with poorly behaved derivative in c .

Lemma 4. *For $0 < \eta \leq 1$ and $\beta > 0$, let*

$$F_m(c; \eta, \beta) = \#\{j : |\lambda_j(c) - E_m(c)| < \eta, \lambda'_j(c) - \nu > \beta(m\pi)^2\}.$$

There is C_I independent of m, η, β such that

$$\int_I F_m(c; \eta, \beta) dc \leq \frac{C_I \eta + \nu r_m}{\beta}. \quad (6.3)$$

Proof. Put $f_j(c) := \lambda_j(c) - E_m(c) = \lambda_j(c) - (m\pi)^2 - c\nu$. Note that $f'_j \geq -\nu$ where the derivative exists. Let

$$T_\eta(s) := -\eta \mathbf{1}_{(-\infty, -\eta)} + s \mathbf{1}_{[-\eta, \eta]} + \eta \mathbf{1}_{(\eta, \infty)}.$$

Then $\frac{d}{dc} T_\eta(f_j(c)) = \mathbf{1}_{\{|f_j| < \eta\}} f'_j(c)$, so

$$\int_I \mathbf{1}_{\{|f_j| < \eta\}} f'_j dc = T_\eta(f_j(c_+)) - T_\eta(f_j(c_-)) \leq 2\eta.$$

Using $f'_j \geq -\nu$, we therefore obtain

$$\int_I \mathbf{1}_{\{|f_j| < \eta\}} (f'_j)_+ dc \leq 2\eta + \nu \int_I \mathbf{1}_{\{|f_j| < \eta\}} dc.$$

Chebyshev's inequality then gives us control over the size of the set of parameters for which $f'_j = \lambda'_j(c) - \nu$ is poorly controlled:

$$|\{c \in I : |f_j(c)| < \eta, f'_j(c) > \beta(m\pi)^2\}| \leq \frac{1}{\beta(m\pi)^2} \left(2\eta + \nu \int_I \mathbf{1}_{\{|f_j| < \eta\}} dc \right).$$

By the Weyl law, there exists $C_I > 0$ depending only on I such that if $|\lambda_j(c) - E_m(c)| < \eta$ for some $c \in I$, then $j < C_I m^2$. Therefore, we can bound

$$\begin{aligned} \int_I F_m(c; \eta, \beta) dc &\leq \sum_{j < C_I m^2} |\{c \in I : |f_j(c)| < \eta, f'_j(c) > \beta(m\pi)^2\}| \\ &\leq \frac{C_I \eta + \nu r_m}{\beta}. \end{aligned} \tag{6.4}$$

Here, $C_I > 0$ may change from line to line but remains independent of m, η, β . This recovers (6.3). $\square$

6.2. Selection in shrinking windows. From (6.1), we see that controlling $\partial_x \varphi_j(c)$ amounts to controlling $\lambda'_j(c)$. Lemma 4 tells us that on average in the parameter c , we have good control in a spectral window as long as the size of the window η is small. In the following lemma, we will pick a good sequence of shrinking spectral windows so that we have the desired control with probability 1.

Lemma 5. *For almost every $c > 0$ there are integers $m_k \rightarrow \infty$ and normalized real eigenfunctions $\varphi_k \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$\begin{aligned} H_c \varphi_k &= \lambda_k \varphi_k, \quad \lambda_k = (m_k \pi)^2 + c\nu + o(1), \\ \lambda_k^{-1} \|\partial_x \varphi_k\|_{L^2(\Omega)}^2 &\rightarrow 0. \end{aligned} \tag{6.5}$$

On each compact parameter interval the integers m_k may be chosen independently of c , outside a null set.

Proof. Recall the quantity $A_m(c)$ from (5.3). Set $\alpha_m(I) = \int_I A_m(c) dc$, which tends to zero by (5.6). The spectral theorem gives

$$A_m(c) = \int_{\mathbb{R}} \frac{(s - E_m(c))^2}{1 + (s - E_m(c))^2} d\mu_{q_m, c}(s), \quad 0 \leq A_m(c) < 1. \tag{6.6}$$

Let $\Pi_m(c, \eta) = \mathbf{1}_{[E_m(c) - \eta, E_m(c) + \eta]}(H_c)$. Then

$$\|(1 - \Pi_m(c, \eta))q_m\|^2 \leq \frac{1 + \eta^2}{\eta^2} A_m(c). \tag{6.7}$$

In particular, it follows from Chebyshev's inequality that

$$\begin{aligned}\epsilon_m &:= |\{c \in I : \Pi_m(c, \eta)q_m = 0\}| \leq |\{c \in I : (1 + \eta^2)A_m(c) \geq \eta^2\}| \\ &\leq \frac{1 + \eta^2}{\eta^2} \alpha_m(I).\end{aligned}\tag{6.8}$$

In other words, the measure of parameters for which the spectral window is empty is at most $(1 + \eta^2)\alpha_m(I)/\eta^2$.

For large m , choose

$$\eta_m = (\alpha_m(I) + m^{-1})^{1/4}, \quad \beta_m = (\eta_m + r_m)^{1/2}.$$

Both tend to zero. Moreover, choosing ϵ_m with η_m in the spectral projector $\Pi_m(c, \eta_m)$, we from (6.8) that $\epsilon_m \rightarrow 0$. By Lemma 4, the measure of parameters with any eigenvalue in the window satisfying $\lambda'_j - \nu > \beta_m(m\pi)^2$ also tends to zero, that is

$$\rho_m := |\{c \in I : F_m(c; \eta_m, \beta_m) \geq 1\}| \rightarrow 0.$$

We can choose a sequence $m_k \rightarrow \infty$ so that $\sum_k (\epsilon_{m_k} + \rho_{m_k}) < \infty$. Then by the Borel–Cantelli lemma, we see that for almost every $c \in I$, all sufficiently large k then have a nonempty window and every eigenvalue in its interior satisfies

$$\lambda'_j(c) \leq \nu + \beta_{m_k}(m_k\pi)^2.\tag{6.9}$$

An endpoint equality causes no difficulty: unless the analytic function $\lambda_j(c) - E_m(c)$ is constant, it occurs on a null set; if it persists then $\lambda'_j = \nu$.

Choose any normalized real eigenfunction in the window. Equations (6.1) and (6.9) give

$$H_c \varphi_k = \lambda_k \varphi_k, \quad \lambda_k = (m_k\pi)^2 + c\nu + o(1), \quad \lambda_k^{-1} \|\partial_x \varphi_k\|^2 \rightarrow 0\tag{6.10}$$

as desired. $\square$

Observe that (6.7) also shows that the projections of q_{m_k} onto these windows approach q_{m_k} in norm for almost every c . This does *not* assert that a single eigenfunction approaches q_{m_k} . Finally take a countable cover of $(0, \infty)$ by compact intervals.

7. LOCALIZATION OF THE EIGENFUNCTIONS

We work on the full physical domain S_a to avoid the artificial corners of a quarter-domain. Let ψ_k be the normalized odd reflections of the quarter-domain eigenfunctions φ_k satisfying (6.5). Recall that such eigenfunctions were selected in Section 6. Set $h_k = \lambda_k^{-1/2}$. In particular, put

$$\psi_k(X, Y) = (2\sqrt{a})^{-1} \varphi_k(X/a - t, Y), \quad X, Y \geq 0$$

in the first quadrant and extend oddly across both axes. The change of variables gives

$$\|h_k \partial_X \psi_k\|_{L^2(S_a)}^2 = c \frac{\|\partial_x \varphi_k\|_{L^2(\Omega)}^2}{\lambda_k} \rightarrow 0.\tag{7.1}$$

We work in the full stadium coordinates from now on, and by a slight abuse of notation, we write (x, y) to mean the full stadium (X, Y) -coordinates from now on.

Extend ψ_k by zero to $\mathbb{R}^2$. The extensions belong to $H^1(\mathbb{R}^2)$ and satisfy $\|\psi_k\|_{L^2} = 1$ and $\|h_k \nabla \psi_k\|_{L^2} = 1$. The Dirichlet semiclassical measure construction for $C^{1,1}$ boundaries [GL93] gives, after passage to a subsequence, a probability measure μ with support on the unit cosphere bundle of $\overline{S_a}$. Moreover, by (7.1), we further see that

$$\text{supp } \mu \subset \{(x, y; \xi) : (x, y) \in \overline{S_a}, \xi = (0, \pm 1)\}. \quad (7.2)$$

The support of μ propagates along generalized billiard trajectories at smooth boundary points: see [Bu97, Théorème 15 and (iv'), pp. 180–181], applied to the time-harmonic wave solutions. A vertical trajectory in an open wing meets the elliptical arc transversally and its reflected direction is no longer vertical. At a tip the vertical trajectory is gliding, and the tangent to the ellipse immediately ceases to be vertical. Both possibilities contradict (7.2). Hence μ vanishes over the open wings. We have used propagation only at smooth points of the boundary, not at the joins.

Interior propagation now shows that the spatial marginal of μ is constant along vertical intervals in $|y| < 1$. It gives no mass to the straight boundary, including the joins.

Combining this with Lemma 5, we have the following strengthening of Theorem 1:

Theorem 3. *There are integers $m_k \rightarrow \infty$ such that for almost every $c > 0$ there exist normalized real eigenfunctions $\psi_k \in H^2(S_a) \cap H_0^1(S_a)$ such that*

$$-\Delta \psi_k = \lambda_k \psi_k, \quad \lambda_k = (m_k \pi)^2 + c\nu + o(1).$$

In addition, every semiclassical measure of this sequence has total mass one, gives zero mass to ∂S_a , and is supported on

$$\overline{B}_a = \{(x, y; 0, \pm 1) : |x| \leq at, |y| \leq 1\}.$$

In particular, (1.2) holds.

Proof. Interior propagation shows that the spatial marginal of μ is constant along the vertical lines $\{(x_0, y) : |y| < 1\}$ for all $|x_0| \leq at$. It remains to check that there is no mass on the straight boundary. Away from the joins $|y| = 1$ and $|x| = at$, the result follows from [Bu97] and at the joints we could use the $C^{1,1}$ analysis for [GL93]. Instead, we present an elementary argument based on the Rellich identities.

So, we write $u = \psi_k$, $h = h_k$, let n be the outward unit normal, and define $b(y) = y(1 - y^2)$. Using that $(-h^2 \Delta - 1)u = 0$ and the boundary condition $u|_{\partial S_a} = 0$, it follows from Green's formula that

$$\begin{aligned} \int_{\partial S_a} b n_y |h \partial_n u|^2 ds &= \text{Re} \int_{S_a} [-h^2 \Delta, b \partial_y] u \bar{u} dx dy \\ &= 2 \int_{S_a} b' |h \partial_y u|^2 dx dy - \frac{h^2}{2} \int_{S_a} b''' |u|^2 dx dy. \end{aligned}$$

Here we used $[-h^2\Delta, b\partial_y] = -2h^2b'\partial_y^2 - h^2b''\partial_y$ and integrated by parts. Testing the eigenfunction equation against $b'\bar{u}$ also gives

$$\int_{S_a} b'(|h\nabla u|^2 - |u|^2) dx dy = \frac{h^2}{2} \int_{S_a} b'''|u|^2 dx dy.$$

Combining these identities and using $\|u\| = 1$, we obtain

$$2 \int_{S_a} b'|u|^2 dx dy = \int_{\partial S_a} b n_y |h\partial_n u|^2 ds + O(\|h\partial_x u\|^2 + h^2). \quad (7.3)$$

On ∂S_a we have $0 \leq b n_y \leq x n_x$. The same Green identity with $x\partial_x$, for which $[-h^2\Delta, x\partial_x] = -2h^2\partial_x^2$, gives Rellich's identity:

$$\int_{\partial S_a} x n_x |h\partial_n u|^2 ds = -2h^2 \operatorname{Re} \int_{S_a} \partial_x^2 u \bar{u} dx dy = 2\|h\partial_x u\|^2 \rightarrow 0.$$

Therefore, the right-hand-side of (7.3) tends to zero as $h \rightarrow 0$. This means that $\int b'(y) d\rho = 0$, where ρ is the spatial marginal of the semiclassical measure μ . The semiclassical measure restricted to the interior of the rectangle has the form $d\sigma(x) dy$, so the interior contributes zero since $\int_{-1}^1 b'(y) dy = 0$. Since $b'(\pm 1) = -2$, the mass on the rectangular boundary $y = \pm 1$ is zero. $\square$

Note that for each fixed $\varepsilon > 0$, vanishing mass on $\{|x| \geq at + \varepsilon\}$ follows by a continuous cutoff and compactness, so Theorem 1 indeed follows from Theorem 3. Again, we stress that this does not imply that $\|\varphi_k - q_{m_k}\| \rightarrow 0$ for individual eigenfunctions.

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