Homework for Wed 3/3

4. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim\limits_{n\to\infty}\frac{2^n}{n^4}=\infty$, so $\lim\limits_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

7. $\lim\limits_{n\to\infty}|a_n|=\lim\limits_{n\to\infty}\frac{n}{5+n}=\lim\limits_{n\to\infty}\frac{1}{5/n+1}=1$, so $\lim\limits_{n\to\infty}a_n\neq 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\sum\limits_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series: $\lim\limits_{n\to\infty}\frac{n/(n^2+1)}{1/n}=\lim\limits_{n\to\infty}\frac{n^2}{n^2+1}=1$. But $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ converges by the Alternating Series Test: $\left\{\frac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1}\right)'=\frac{1-x^2}{(x^2+1)^2}\leq 0$ for $x\geq 1$, and $\lim\limits_{n\to\infty}\frac{n}{n^2+1}=0$. Thus $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ is conditionally convergent.

16. $n^{2/3}-2>0$ for $n\geq 3$, so $\frac{3-\cos n}{n^{2/3}-2}>\frac{1}{n^{2/3}-2}>\frac{1}{n^{2/3}}$ for $n\geq 3$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{n^{2/3}}$ diverges ($p=\frac{2}{3}\leq 1$), so does $\sum\limits_{n=1}^{\infty}\frac{3-\cos n}{n^{2/3}-2}$ by the Comparison Test.

23. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\frac{n^2+1}{2n^2+1}=\lim\limits_{n\to\infty}\frac{1+1/n^2}{2+1/n^2}=\frac{1}{2}<1$, so the series $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series $1-\frac{1\cdot 3}{3!}+\frac{1\cdot 3\cdot 5}{5!}-\frac{1\cdot 3\cdot 5\cdot 7}{7!}+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n-1)!}+\cdots=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n-1)!}$.

$\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdots(2n-1)}\right|=\lim\limits_{n\to\infty}\left|\frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!}\right|=\lim\limits_{n\to\infty}\frac{1}{2n}=0<1$, so the given series is absolutely convergent and therefore convergent.

27. $\sum\limits_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{n!}=\sum\limits_{n=1}^{\infty}\frac{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots(2\cdot n)}{n!}=\sum\limits_{n=1}^{\infty}\frac{2^n n!}{n!}=\sum\limits_{n=1}^{\infty}2^n$, which diverges by the Test for Divergence since $\lim\limits_{n\to\infty}2^n=\infty$