MATH 1B—SOLUTION SET FOR CHAPTERS 11.12, 9.1, 9.2

Problem 11.12.25. *Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 10^{-5}.*

Solution. First of all, every derivative of e^x is e^x. Since e^x is increasing, the maximum of every derivative of e^x on $[0, 0.1]$ is $e^{0.1}$ itself.

Now, if we actually knew $e^{0.1}$, we wouldn't need to estimate it! Still, we know that it is well less than 2 (the square root of 3 is, and this is the 10^{th} root of something less than 3). Our error bound for truncating at the n^{th} term is thus at most

$$R_n \leq \frac{2 \cdot 10^{-n-1}}{(n+1)!}$$

We wish to ensure that this error is less than 10^{-5}. This will clearly be met for $n = 4$, and (with a bit more work) for $n = 3$. At $n = 2$ it won't be met. Thus, we have

$$\begin{aligned} e^{0.1} &\approx 1 + (0.1) + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 \\ &\approx 1.10517 \end{aligned}$$

Problem 11.12.26. *How many terms of the Maclaurin series for $\ln(1+x)$ do you need to use to estimate $\ln 1.4$ to within 0.001?*

Solution. First, $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$. We first must find the n^{th} derivative of $\ln(1+x)$. Well, $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$. It's easy to show that the n^{th} derivative of $\frac{1}{1+x}$ is $\frac{(-1)^n n!}{(1+x)^{n+1}}$: it's true for the 0^{th} derivative, and if it's true for the n^{th} then it's true for the next derivative as well. Thus, $\frac{d^n}{dx^n}\ln(1+x) = \frac{(-1)^{n-1}n-1}{(1+x)^n}$. These functions have strictly decreasing absolute value on $[0, 0.4]$, so we may take $M = 1$. Thus,

$$R_n(0.4) \leq \frac{0.4^{n+1}}{(n+1)!}$$

For simplicity, we can write this as $\frac{4^{n+1}10^{-n-1}}{(n+1)!}$. Obviously we needn't consider anything below $n = 2$. At $n = 2$, we have $\frac{64 \cdot 10^{-3}}{6}$, which is clearly greater than 10^{-3}. At $n = 3$ we have $\frac{64 \cdot 10^{-4}}{6}$, which is still greater than 10^{-3}. AT $n = 4$ we have $\frac{128 \cdot 10^{-5}}{3}$, which is less than 10^{-3}. We must therefore keep everything out to the $n = 4$ term, so we must keep 5 terms.

Problem 11.12.31. *An electric dipole consists of two electric charges of equal magnitude and opposite signs. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is*

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series of powers in $\frac{d}{D}$, show that E is approximately proportional to $\frac{1}{D^3}$ when P is far from the dipole.

Solution. First, let's write the expression in terms of $\frac{d}{D}$:

$$E = \frac{q}{D^2} + \frac{q}{D^2(1 + \frac{d}{D})^2}$$

Now, expand the second term on the right as a binomial series:

$$E = \frac{q}{D^2} + \frac{q}{D^2} \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{d}{D}\right)^n$$

At this point, we argue that since $d \ll D$, we can reasonably approximate E by truncating the series at two terms, leaving:

$$\begin{aligned} E &\approx \frac{q}{D^2}\left(1 + (-1) + 2\left(\frac{d}{D}\right)\right) \\ &= \frac{2qd}{D^3} \end{aligned}$$

Problem 9.1.1. *Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.*

Solution. Here we just plug in:

$$\begin{aligned} xy' + y &= x(1 + x^{-2}) + x - x^{-1} \\ &= x + x^{-1} + x - x^{-1} \\ &= 2x \end{aligned}$$

as desired.

Problem 9.1.2. *Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial-value problem*

$$y' + (\tan x)y = \cos^2 x; y(0) = -1$$

on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution. First, $y(0) = \sin(0)\cos(0) - \cos(0) = -1$, satisfying the initial condition. Next, plugging in, we have $\cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x$, as desired, so the solution is good as long as $\tan x$ is defined (which imposes the range).

Problem 9.1.3.

(a) For what nonzero values of k does the function $y = \sin kt$ satisfy the differential equation $y'' + 9y = 0$?

(b) For those values of k, verify that every member of the family of functions

$$y = A \sin kt + B \cos kt$$

is also a solution.

Proof.

(a) Pluggin in the putative solution, we get:

$$-k^2 \sin kt + 9 \sin kt = 0$$

whence $k^2 = 9$, or $k = \pm 3$.

(b) If $y = A \sin \pm 3t + B \cos \pm 3t$, we have $y' = \pm 3A \sin \pm 3t \pm 3B \cos \pm 3t$, and $y'' = -9A \sin \pm 3t - 9B \cos \pm 3t$. Thus certainly $y'' + 9y = 0$, and these functions also satisfy the differential equation.

Problem 9.1.10. *A function y(t) satisfies the differential equation*

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$$

(a) What are the constant solutions of the equation?
(b) For what values of y is y increasing?
(c) For what values of y is y decreasing?

Solution.
(a) For the solution to be constant, we must have $\frac{dy}{dt} = 0$. This requires

$$\begin{aligned} y^4 - 6y^3 + 5y^2 &= 0 \\ y^2(y^2 - 6y + 5) &= 0 \\ y^2(y-5)(y-1) &= 0 \end{aligned}$$

Thus the constant solutions are $y = 0$, $y = 1$, and $y = 5$

(b,c) The function y is increasing where $\frac{dy}{dt} > 0$, and decreasing where $\frac{dy}{dt} < 0$ The factor y^2 is greater than zero everywhere except at 0, where it is zero. The factor $y - 1$ is negative on $y < 1$, positive on $y > 1$, and zero at 1. The factor $y - 5$ is negative on $y < 5$, positive on $y > 5$, and zero at 5. Just keeping track of the signs, we see that y is increasing on $(-\infty, 0) \cup (0, 1) \cup (5, \infty)$, decreasing on $(1, 5)$, and constant on $\{0, 1, 5\}$. As an aside, we note that this means that the equilibrium solution at 1 is stable, the equilibrium solution at 5 is unstable, and the equilibrium solution at 0 is unstable under positive perturbations but stable under negative perturbations (for most practical purposes, this simply means 'unstable," as you usually can't count on all perturbations being in a favorable direction.

Problem 9.1.11. *Explain why the function with the given graphs (not reproduced here, see p.592 of Stewart) can't be solutions of the differential equation*

$$\frac{dy}{dt} = e^t(y-1)^2$$

Solution.
(a) can't work, because it has negative slope on portions of its solutions. The slope, $e^t(y-1)^2$, is a nonnegative function. (b) can't work, because it has positive slope at $y = 1$, where the slope must be flat.

Note. On problems from 9.2, I won't actually draw the direction fields; instead I'll merely refer to them.

Problem 9.2.1. *A direction field for the differential equation* $y' = y(1 - \frac{1}{4}y^2)$ *is shown (p. 599)*
(a) Sketch the graphs of the solutions that satisfy the given initial conditions:
(i) $y(0) = 1$
(ii) $y(0) = -1$
(iii) $y(0) = -3$
(iv) $y(0) = 3$
(b) Find all the equilibrium solutions.

Solution.

(i) The solution asymptotically approaches $y = 2$ as you go in the $+x$ direction, and asymptotically approaches $y = 0$ as you go in the $-x$ direction

(ii) The solution asymptotically approaches $y = -2$ as you go in the $+x$ direction, and asymptotically approaches $y = 0$ as you go in the $-x$ direction.

(iii) The solution asymptotically approaches $y = -2$ as you go in the $+x$ direction, and diverges towards $-\infty$ as you go in the $-x$ direction.

(iv) The solution asymptotically approaches $y = 2$ as you go in the $+x$ direction, and diverges towards ∞ as you go in the $-x$ direction.

(b) There are stable equilibrium solutions at $y = 2$ and $y = -2$, and an unstable equilibrium solution at $y = 0$.

9.2.3-6. *These questions require you to match direction fields with differential equations:*

9.2.3: $y' = y - 1$
9.2.4: $y' = y^2 - x^2$
9.2.5 $y' = y - x$
9.2.6 $y' = y^3 - x^3$

Solutions.

9.2.3: This is the only of the differential solutions to have an equilibrium solution at $y = 1$. It thus corresponds to IV.

9.2.4: This should have zero slope where $y = x$, but not where $y = -x$ (excluding III). Moreover, the slope should change linearly with increasing x–it shouldn't linger near zero, then suddenly jump to high slopes (excluding I). This must correspond to II.

9.2.5: This should have zero slope both where $y = x$ and where $y = -x$. It must correspond to III.,

9.2.6: By elimination, this must be I. It has the same points of zero slope, but lingers near zero slope when x and y are both between -1 and 1, but takes off sharply outside this region (to see why, consider the graph of $y = x^3$.

9.2.10. *Sketch a direction field for the differential equation* $y' = x^2 - y^2$. *Then sketch three solution curves.*

Solution. This has nearly already been sketched for you: 9.2.4 above is the same function with x and y reversed!

My artistic skills are too limited for the solution curves—I'll leave them to your imagination!