Homework for Fri 3/5

2. If $a_n = \frac{n-1}{n^2+n}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2-n}{n^2+n} = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$, so the series $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

5. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{-3\cdot 2^{3n}}{2^{3n}\cdot 2^3}\right| = \lim_{n\to\infty} \frac{3}{2^3} = \frac{3}{8} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{2^{k+1}}{(k+2)(k+3)} \cdot \frac{(k+1)(k+2)}{2^k}\right| = \lim_{k\to\infty} \left(2 \cdot \frac{k+1}{k+3}\right) = 2 > 1$, so the series diverges. *Or:* Use Test for Divergence.

13. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}\right| = \lim_{n\to\infty} \left[\frac{3(n+1)^2}{(n+1)n^2}\right] = 3 \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim_{n\to\infty} \sin n$ does not exist.

17. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$. By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

28. Since $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \leq e^{1/1} = e$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{e}{n^2}$ converges ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{1/n}-1}{1/n} = \lim_{x\to\infty} \frac{2^{1/x}-1}{1/x} =$(By l'Hospital Rule) $\lim_{n\to\infty} \frac{2^{1/x}\cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{n\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0$. So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$.