§9.4

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

Let P be the population size and let t be the time variable, measured in hours. The system is modelled by the differential equation

$$\frac{dP}{dt} = 0.7944P.$$

Solving the separable equation and expifying both sides,

$$P = Ae^{0.7944t}.$$

From our initial condition $P(0) = 2$, $A = 2$. Therefore the population size after six days is $P(6) = 2e^{6(.7944)}$ (about 235).

3. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria.

(a) Find an expression for the number of bacteria after t hours.

Letting P be the number of bacteria and t be time measured in hours,

$$\frac{dP}{dt} = kP,$$

for some constant k, so

$$P = Ae^{kt}.$$

From $P(0) = 500$, $A = 500$, and from $P(3) = 8000$, $k = \frac{\ln 16}{3}$. Therefore

$$P(t) = 500\exp(\frac{\ln 16}{3}t) = 500 \cdot 16^{\frac{t}{3}}.$$

(b) Find the number of bacteria after 4 hours.

$$P(4) = 500 \cdot 16^{\frac{4}{3}}$$

(c) Find the rate of growth after 4 hours.

$$\frac{dP}{dt}(4) = kP(4) = \frac{\ln 16}{3} \cdot 500 \cdot 16^{\frac{4}{3}}.$$

(d) When will the population reach 30,000?

If $30,000 = P(t)$, then

$$t = \frac{3\ln 60}{\ln 16}$$

7. Let $P = N_2O_5$.

(a) Find an expression for the concentration P after t seconds if the initial concentration is C.

We solve the separable differential equation and use our initial condition, $P(0) = C$ to obtain

$$P(t) = C\exp(-0.0005t).$$

(b) How long will the reaction take to reduce the concentration P to 90% of its original value?

We would like to find a t such that $P(t) = .9C$. Using the equation obtained in part a,

$$t = -2000\ln(.9).$$

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after t years.

Let m be the mass of the sample in milligrams, and let t be time measured in years.
The system is modelled by the differential equation

$$\frac{dm}{dt} = km,$$

so $m(t) = Ae^{kt}$. From $m(0) = 100$, $A = 100$, and from $m(30) = .5 \times 100 = 50$, $k = \frac{-\ln 2}{30}$. Therefore

$$m(t) = 100\exp(\frac{-\ln 2}{30}t) = 100 \cdot 2^{\frac{-t}{30}}.$$

(b) How much of the sample remains after 100 years?

$$m(100) = 100 \cdot 2^{-100}30.$$

(c) After how long will only 1 mg remain?

We want to find t such that $m(t) = 1$, or equivalently so $\exp(\frac{-\ln 2}{30}t) = .01$. Therefore

$$t = \frac{30\ln 100}{\ln 2}$$

(since $-\ln .01 = \ln 100$).

§9.5

3. (a) By equation (4),

$$y(t) = \frac{8 \times 10^7}{1 + Ae^{-0.71t}}.$$

Using the initial condition $y(0) = 2 \times 10^7$, and equation (4) again,

$$A = \frac{6 \times 10^7}{2 \times 10^7} = 3.$$

Therefore $y(1) = \frac{8 \times 10^7}{1+3e^{-0.71}}$.

(b) We want to solve

$$\frac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7$$

for t. The solution is $t = \frac{\ln 3}{0.71}$.

7. (a) If the fraction of the population who has not heard the rumor is x, then $y + x = 1$, so $x = 1 - y$. The differential equation is

$$\frac{dy}{dt} = ky(1 - y).$$

(b) Using equation (4), the solution to the equation found in (a) is

$$y(t) = \frac{1}{1 + Ae^{-kt}}.$$

(c) We remember the y represents the fraction of the population that has heard the rumor. Let t represent time measured in hours after 8am. Then $y(0) = 0.08$, so (equation 4 again) $A = 11.5$. We are also told that $y(4) = .5$, so that we can find k. We have

$$k = .25 \ln(11.5).$$

Now we have

$$y(t) = \frac{1}{1 + 11.5e^{-.25 \ln(11.5)t}} = \frac{2}{2 + 23(\frac{2}{23})^{\frac{t}{4}}}$$

We are trying to solve the equation $y(t) = .9$ for t. This gives us

$$t = 4\frac{\ln(9 \times 11.5)}{\ln(11.5)} = 4\frac{\ln 9 - \ln \frac{2}{23}}{- \ln 223} = 4(1 - \frac{\ln 9}{\ln \frac{2}{23}}).$$