MATH 1B—SOLUTION SET FOR CHAPTERS 8.1, 8.2

Problem 8.1.1. *Use the arc length formula to find the length of the curve* $y = 2 - 3x, -2 \leq x \leq 1$. *Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.*

Solution. First, note:

$$\begin{aligned} y' &= -3 \\ \sqrt{1 + (y')^2} &= \sqrt{10} \end{aligned}$$

(Note that this is a constant, which is as it should be—the curve is a line, and a line should have the same amount of arc length per unit horizontal distance. In fact, it should be the secant of the angle the line makes with the x-axis!)

So, using the arc length formula, the length of the curve on $-2 \leq x \leq 1$ is

$$\begin{aligned} \int_{x=-2}^{x=1} ds &= \int_{-2}^{1} \sqrt{10}dx \\ &= \sqrt{10}\,[x]_{-2}^{1} \\ &= 3\sqrt{10} \end{aligned}$$

Of course, since this curve is a line, using the arc length formula is like using a flamethrower to kill ants. Since the line has endpoints at $(-2, 8)$ and $(1, -1)$, its length must be:

$$\sqrt{3^2 + (-9)^2} = \sqrt{90} = 3\sqrt{10}$$

as desired.

Problem 8.1.9. *Find the length of the curve given by* $x = \frac{1}{3}\sqrt{y}(y - 3), 1 \leq y \leq 9$.

Solution. In this case, we're probably (almost certainly) better off integrating up the y-axis. Taking the derivative, we have:

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{3}\left(\frac{y-3}{2\sqrt{y}} + \sqrt{y}\right) \\ &= \frac{1}{6\sqrt{y}}(3y - 3) \\ &= \frac{y-1}{2\sqrt{y}} \end{aligned}$$

Thus,

$$\begin{aligned} ds &= \sqrt{1 + \frac{(y-1)^2}{4y}} \\ &= \sqrt{\frac{(y+1)^2}{4y}} \end{aligned}$$

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On the range that we're interested in, $y+1$ is positive. Thus, the arc length is:

$$\int_{y=1}^{y=9} ds = \int_1^9 \frac{y+1}{2\sqrt{y}} dy$$

Substituting $u = \sqrt{y}$, so $du = \frac{1}{2\sqrt{y}}$, we now have

$$\begin{aligned} &= \int_1^3 (u^2+1) du \\ &= \left[\frac{u^3}{3} + u\right]_1^3 \\ &= \frac{32}{3} \end{aligned}$$

So the curve has arc length $\frac{32}{3}$.

Problem 8.1.13. *Find the arc length of the curve given by $y = \cosh x, 0 \leq x \leq 1$.*

Solution. As long as you remember how cosh is defined and what its derivative is, this one's easy. Recall:

$$y' = \sinh x$$

so

$$\begin{aligned} \sqrt{1+(y')^2} &= \sqrt{1+\sinh^2 x} \\ &= \sqrt{\cosh^2 x} \\ &= \cosh x \\ \int_{x=0}^{x=1} ds &= \int_0^1 \cosh x dx \\ &= [\sinh x]_0^1 \\ &= \frac{1}{2}e - \frac{1}{2e} \end{aligned}$$

Problem 8.1.30.

(a) Sketch the curve $y^3 = x^2$

(b) Set up two integrals for the arc length from $(0,0)$ to $(1,1)$, one along x and one along y.

(c) Find the length of the arc of this curve from $(-1,1)$ to (8.4).

Proof. (a) It's clear that this curve is single-valued, since $f(x) = x^3$ is invertible (so for any given x, there's only one value of y that satisfies the equation $y^3 = x^2$). Thus, the curve is the same as $y = x^{\frac{2}{3}}$. This function is even, and has first derivative $\frac{2}{3}x^{-\frac{1}{3}}$. This is positive on $x > 0$, negative on $x < 0$, and undefined at zero itself. The second derivative is $-\frac{2}{9}x^{-\frac{4}{3}}$, which is negative everywhere (except at 0, where it too is undefined). Thus the curve is concave down everywhere. Such a curve looks something like the plot of $\sqrt{|x|}$,

(b) Solving for y, we have $y = x^{\frac{2}{3}}$. Then $y' = \frac{2}{3}x^{-\frac{1}{3}}$, and so

$$\int ds = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}} dx$$

Because the integrand is undefined at $x = 0$, this integral is improper. We thus write:

$$\begin{aligned}
&= \lim_{s\to 0^+} \int_s^1 \sqrt{1 + \frac{4}{9}x^{-\frac{2}{3}}}dx \\
&= \lim_{s\to 0^+} \int_s^1 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + \frac{4}{9}}dx \\
&= \frac{3}{2}\lim_{s\to 0^+} \int_{s^{\frac{2}{3}}+\frac{4}{9}}^{\frac{13}{9}} \sqrt{u}du \\
&= \frac{3}{2}\lim_{s\to 0^+} \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{s^{\frac{2}{2}}+\frac{4}{9}}^{\frac{13}{9}} \\
&= \left(\frac{13}{9}\right)^{\frac{3}{2}} - \left(\frac{4}{9}\right)^{\frac{3}{2}} \\
&= \frac{13\sqrt{13}-8}{27}
\end{aligned}$$

We could instead have solved for x (on $0 \le x \le 1$, the curve is single-valued in either x or y). In this case, we have $x = y^{\frac{3}{2}}$, so

$$\frac{dx}{dy} = \frac{3}{2}y^{\frac{1}{2}}.$$

Our arc length is thus

$$\begin{aligned}
\int ds &= \int_0^1 \sqrt{1 + \frac{9}{4}y}dy \\
&= \frac{4}{9}\int_1^{\frac{13}{4}} \sqrt{u}du \\
&= \frac{8}{27}\left[u^{\frac{3}{2}}\right]_1^{\frac{13}{4}} \\
&= \frac{8}{27}\left[\frac{13\sqrt{13}-8}{8}\right] \\
&= \frac{13\sqrt{13}-8}{27}
\end{aligned}$$

In either case, we get the same answer, as we should—this is, after all, the arc length of a curve!

(c) Now we have to be careful. On the range $-1 \le x \le 8$, the curve is a function in y, but is not invertible. Probably the laziest (and therefore best) way to proceed is as follows: First, note that we already know the arc length between $(0, 0)$ and $(1, 1)$. Next, realize that since the function is odd, the length of the curve between $(-1, 1)$ and $(0, 0)$ must be the same as the length between $(0, 0)$ and $(1, 1)$. This leaves only the curve between $(1, 1)$ and $(8, 4)$. On this range, the curve is invertible, so we can just use the second method above, to get

$$\int dx = \int_1^4 \sqrt{1+\frac{9}{4}y}dy$$
$$= \frac{4}{9}\int_{\frac{13}{4}}^{10} \sqrt{u}du$$
$$= \frac{8}{27}\left[10\sqrt{10}-\frac{13\sqrt{13}}{8}\right]$$
$$= \frac{80\sqrt{10}-13\sqrt{13}}{27}$$

So, our total arc length is $2\frac{13\sqrt{13}-8}{27}+\frac{80\sqrt{10}-13\sqrt{13}}{27}$, or $\frac{80\sqrt{10}+13\sqrt{13}-8}{27}$.

Problem 8.1.31. *Find the arc length function for the curve $y = 2x^{\frac{3}{2}}$, starting with the point $P_0(1,2)$.*

Solution. The arc length function is defined by:

$$s(x) = \int_1^x \sqrt{1+(y')^2}dt$$

Since $y' = 3x^{\frac{1}{2}}$, this is

$$s(x) = \int_1^x \sqrt{1+9t}dt$$
$$= \frac{1}{9}\int_1 0^{1+9x}\sqrt{u}du$$
$$= \frac{2}{27}\left[(1+9x)^{\frac{3}{2}}-10\sqrt{10}\right]$$

So the arc length function is $s(x) = \frac{2}{27}\left[(1+9x)^{\frac{3}{2}}-10\sqrt{10}\right]$.

Problem 8.1.34. *A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80ft$ is given by*

$$y = 150-\frac{1}{40}(x-50)^2$$

Find the distance traveled by the kite.

Solution. It should be clear that the distance traveled by the kite is precisely the arc length of its path, as it travels along its parabolic path. (That the path above describes a downward-opening parabola isn't important to the problem, but is worth noting. It's always nice to see old friends like parabolae).

In this case, $y' = -\frac{1}{20}(x-50)$, so the arc length is:

$$\int ds = \int_0^{80} \sqrt{1+\frac{1}{400}(x-50)^2}dx$$
$$= \int_{-50}^{30} \sqrt{1+\frac{1}{400}u^2}du$$
$$= 20\int_{\arctan\left(-\frac{5}{2}\right)}^{\arctan\left(\frac{3}{2}\right)} \sec^3\theta d\theta$$

To find $\int \sec^3 \theta d\theta$, we use the usual trick:

$$\begin{aligned}
\int \sec^3 \theta d\theta &= \sec\theta\tan\theta - \int \sec\theta \tan^2\theta d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\
2\int \sec^3\theta &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \\
\int \sec^3\theta &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|
\end{aligned}$$

Thus, returning to our arc length problem, the distance traveled by the kite in feet is:

$$\begin{aligned}
d &= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_{\arctan\left(-\frac{5}{2}\right)}^{\arctan\left(\frac{3}{2}\right)} \\
&= \left[\frac{1}{2}\sqrt{1+\tan^2\theta}\tan\theta + \frac{1}{2}\ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right|\right]_{\arctan\left(-\frac{5}{2}\right)}^{\arctan\left(\frac{3}{2}\right)} \\
&= \left[\frac{1}{2}\left(\frac{3}{2}\right)\sqrt{\frac{13}{4}} - \frac{1}{2}\ln\left|\sqrt{\frac{13}{4}}+\frac{3}{2}\right| - \frac{1}{2}\sqrt{\frac{29}{4}}\left(\frac{-5}{2}\right) + \frac{1}{2}\ln\left|\sqrt{\frac{29}{4}}-\frac{5}{2}\right|\right] \\
&= \frac{3\sqrt{13}+5\sqrt{29}}{8} + \frac{1}{2}\ln\left|\frac{3+\sqrt{13}}{\sqrt{29}-5}\right|
\end{aligned}$$

Problem 8.2.1. *Set up, but do not evaluate, an integral for the area of the surface obtained by rotating*

$$y = \ln x, 1 \le x \le 3$$

about the x-axis.

Solution. This one's easy (since we don't have to evaluate the integral!): $y' = \frac{1}{x}$, so

$$A = \int_1^3 2\pi \ln x\sqrt{1+\frac{1}{x^2}}dx$$

Problem 8.2.3. *Set up, but do not evaluate, an integral for the area of the surface obtained by rotating*

$$y = \sec x, 0 \le x \le \pi/4$$

about the y-axis.

Solution. First, note that $y' = \sec x\tan x$. Thus,

$$A = \int_0^{\pi/4} 2\pi x\sqrt{1+\sec^2 x\tan^2 x}dx$$

Problem 8.2.7. *Find the area of the surface obtained by rotating the curve*

$$y = \sqrt{x}, 4 \leq x \leq 9$$

about the x-axis.

Solution. Since $y' = \frac{1}{2\sqrt{x}}$, we have

$$\begin{aligned}
A &= \int_4^9 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}dx \\
&= 2\pi\int_4^9 \sqrt{x+\frac{1}{4}}dx \\
&= 2\pi\int_{\frac{17}{4}}^{\frac{37}{4}} \sqrt{u}du \\
&= \frac{4\pi}{3}\left[u^{\frac{3}{2}}\right]_{\frac{17}{4}}^{\frac{37}{4}} \\
&= \frac{4\pi}{3}\left[\frac{37\sqrt{37}-17\sqrt{17}}{8}\right] \\
&= \frac{\pi(37\sqrt{37}-17\sqrt{17})}{6}
\end{aligned}$$

Problem 8.2.9. *Find the area of the surface obtained by rotating the curve*

$$y = \cosh x, 0 \leq x \leq 1$$

about the x-axis.

Proof. Since $y' = \sinh x$, we have

$$\begin{aligned}
A &= \int_0^1 2\pi\cosh x\sqrt{1+\sinh^2 x}dx \\
&= 2\pi\int_0^1 \cosh^2 x dx \\
&= 2\pi\int_0^1 \left(\frac{1}{4}e^{2x}+\frac{1}{2}+\frac{1}{4}e^{-2x}\right)dx \\
&= 2\pi\left[\frac{1}{8}e^{2x}+\frac{1}{2}x-\frac{1}{8}e^{-2x}\right]_0^1 \\
&= 2\pi\left[\frac{1}{8}e^2+\frac{1}{2}-\frac{1}{8}e^{-2}-\frac{1}{8}+\frac{1}{8}\right] \\
&= 2\pi\left[\frac{1}{4}\sinh 2+\frac{1}{2}\right] \\
&= \pi[1+\frac{1}{2}\sinh 2]
\end{aligned}$$

Problem 8.2.25. *If the region $\mathcal{R} = \{(x, y) \mid x \geq 1, 0 \leq y \leq \frac{1}{x}\}$ is rotated about the x-axis, the resulting surface has infinite area.*

Proof. We are interested in the surface $y = \frac{1}{x}$, which has derivative $y' = -\frac{1}{x^2}$. Thus, the area is

$$\begin{aligned} A &= \int_1^\infty \frac{2\pi}{x}\sqrt{1 + \frac{1}{x^4}}dx \\ &= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + x^{-4}}dx \end{aligned}$$

At this point, the integrand is positive and is everywhere on our domain greater than $\frac{1}{x}$. Since $\int_1^\infty \frac{dx}{x}$ diverges to infinity, so does A, by the comparison test. □

Problem 8.2.27. *(a) If $a > 0$, find the area of the surface generated by rotating the loop of the curve $3ay^2 = x(a - x)^2$ about the x-axis.*

(b) Find the surface area if the loop is rotated about the y-axis.

Solution.

(a) The first step here is to work out what this "loop" is that's mentioned in the problem. Looking at the equation that defines the curve, first note that the left-hand side is necessarily nonnegative, while the right hand side is negative for all $x < 0$. Thus, no points with $x < 0$ can satisfy the equation. Now, if we solve for y, we see

$$y = \pm\frac{\sqrt{x}|a - x|}{\sqrt{3a}}$$

, so the curve will be double-valued whenever the right-hand size is nonzero. The zeros occur at 0 and and a, so the curve between 0 and a will indeed form a loop of sorts. We don't care about the curve beyond a. On $0 \leq x \leq a$, we know the sign of $(a - x)$. Since we're only interested in the top half of the loop (we're rotating about the x-axis, so the "loop" generates the same surface as its top half), we can consider the function $y = \frac{\sqrt{x}(a-x)}{\sqrt{3a}}$.

Now, $y' = \frac{1}{\sqrt{3a}}\frac{a}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$, so the area of the surface rotated about the x-axis is

$$\begin{aligned} A &= \int_0^a 2\pi\frac{\sqrt{x}(a - x)}{\sqrt{3a}}\sqrt{1 + \frac{1}{3a}\left[\frac{a^2}{4x} - \frac{6a}{4} + \frac{9x}{4}\right]}dx \\ &= \frac{2\pi}{\sqrt{3a}}\int_0^a \sqrt{x}(a - x)\sqrt{1 + \frac{1}{3a}\left[\frac{a^2 - 6ax + 9x^2}{4x}\right]}dx \\ &= \frac{2\pi}{\sqrt{3a}}\int_0^a \sqrt{x}(a - x)\sqrt{\frac{1}{3a}\left[\frac{a^2 + 6ax + 9x^2}{4x}\right]}dx \\ &= \frac{\pi}{3a}\int_0^a (a - x)(a + 3x)dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{3a}\int_0^a (a^2 + 2ax - 3x^2)dx \\
&= \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a \\
&= \frac{\pi}{3a}\left[a^3 + a^3 - a^3\right] \\
&= \frac{a^2\pi}{3}
\end{aligned}
$$

(b) If the loop is rotated about the y-axis, things become more unpleasant. First, we have to take both the upper and lower portions of the loop into account. Since they're symmetrical with respect to the x axis and give the same contribution to surface area, this is best handled by multiplying by 2. Then, we simply have

$$
\begin{aligned}
A &= 2\int_0^a 2\pi x\sqrt{\frac{1}{3a}\left[\frac{a^2 + 6ax + 9x^2}{4x}\right]} \\
&= \frac{2\pi}{\sqrt{3a}}\int_0^a \left(ax^{\frac{1}{2}} + 3x^{\frac{3}{2}}\right)dx \\
&= \frac{2\pi}{\sqrt{3a}}\left[\frac{2a}{3}x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}}\right]_0^a \\
&= \frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}a^{\frac{5}{2}} + \frac{6}{5}a^{\frac{5}{2}}\right] \\
&= \frac{56\pi\sqrt{3}a^2}{45}
\end{aligned}
$$