HW Solutions: 11.11

April 11, 2004

1 Section 11.11

1. The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)x^n}{2^n \cdot n!} \text{ for } |x| < 1, \text{ so } R = 1\end{aligned}$$

2. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \sum\limits_{n=0}^{\infty} \dbinom{-4}{n} x^n$. The binomial coefficient is

$$\begin{aligned}\binom{-4}{n} &= \frac{(-4)(-5)(-6) \cdot \cdots \cdot (-4-n+1)}{n!} = \frac{(-4)(-5)(-6) \cdot \cdots \cdot [-(n+3)]}{n!} \\ &= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!} = \frac{(-1)^n (n+1)(n+2)(n+3)}{6}\end{aligned}$$

Thus, $\dfrac{1}{(1+x)^4} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n$ for $|x| < 1$, so $R = 1$.

7. We must write the binomial in the form (1+ expression), so we'll factor out a 4.

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1+\left(-\tfrac{1}{2}\right)\frac{x^2}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3+\cdots\right]$$

$$= \frac{x}{2}+\frac{x}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$$

$$= \frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4}<1 \iff \frac{|x|}{2}<1 \iff$$

$|x|<2$, so $R=2$.

13. (a) $\sqrt[3]{1+x} = (1+x)^{1/3} = \sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$

$$= 1+\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\cdots$$

$$= 1+\frac{x}{3}+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}x^n$$

(b) $\sqrt[3]{1+x} = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\cdots$. $\sqrt[3]{1.01} = \sqrt[3]{1+0.01}$, so let $x=0.01$. The sum of the first two terms is then $1+\frac{1}{3}(0.01)\approx 1.0033$. The third term is $\frac{1}{9}(0.01)^2\approx 0.000\,01$, which does not affect the fourth decimal place of the sum, so we have $\sqrt[3]{1.01}\approx 1.0033$.

17. (a) $\left(1+x^2\right)^{1/2} = 1+\left(\frac{1}{2}\right)x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3+\cdots$

$$= 1+\frac{x^2}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n=5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^4\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!} \Rightarrow f^{(10)}(0) = 10!\left(\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\right) = 99{,}225.$$

19. (a) $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \Rightarrow \quad g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1}$, so

$$(1+x)g'(x) = (1+x)\sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^n$$

$$= \sum_{n=0}^{\infty} \binom{k}{n+1} (n+1) x^n + \sum_{n=0}^{\infty} \binom{k}{n} n x^n \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n+1 \\ \text{in the first series}\end{array}\right]$$

$$= \sum_{n=0}^{\infty} (n+1) \frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!} x^n$$

$$+ \sum_{n=0}^{\infty} \left[(n) \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \right] x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!} \left[(k-n)+n\right] x^n$$

$$= k \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = kg(x)$$

Thus, $g'(x) = \dfrac{kg(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k} g(x) \quad \Rightarrow$

$$h'(x) = -k(1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x) \quad \text{[Product Rule]}$$

$$= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} \frac{kg(x)}{1+x} \quad \text{[from part (a)]}$$

$$= -k(1+x)^{-k-1} g(x) + k(1+x)^{-k-1} g(x) = 0$$

(c) From part (b) we see that $h(x)$ must be constant for $x \in (-1, 1)$, so $h(x) = h(0) = 1$ for $x \in (-1, 1)$.

Thus, $h(x) = 1 = (1+x)^{-k} g(x) \quad \Leftrightarrow \quad g(x) = (1+x)^k$ for $x \in (-1, 1)$.