HW Solutions: 11.9

April 11, 2004

1 Section 11.9

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a

power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \iff |x| < 1$, so $R = 1$

and $I = (-1, 1)$.

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1-x^3} = \sum\limits_{n=0}^{\infty} (x^3)^n = \sum\limits_{n=0}^{\infty} x^{3n}$. The series converges when $|x^3| < 1$

$\iff |x|^3 < 1 \iff |x| < \sqrt[3]{1} \iff |x| < 1$. Thus, $R = 1$ and $I = (-1, 1)$.

9. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$

$= \dfrac{x}{9}\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{9^n} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{9^{n+1}}$. The geometric series $\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when

$\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \iff \dfrac{|x^2|}{9} < 1 \iff |x|^2 < 9 \iff |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2},

we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable

by 2. This gives us $\dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

2

34. (a) $J_1(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1)\, x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and

$$J_1''(x) = \sum_{n=1}^{\infty} \dfrac{(-1)^n (2n+1)(2n)\, x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}.$$

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x)$$

$$= \sum_{n=1}^{\infty} \dfrac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

$$+ \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

$$= \sum_{n=1}^{\infty} \dfrac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \dfrac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

$$- \sum_{n=1}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \begin{bmatrix} \text{Replace } n \text{ with } n-1 \\ \text{in the third term} \end{bmatrix}$$

$$= \dfrac{x}{2} - \dfrac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\dfrac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!\,(n+1)!\,2^{2n+1}} \right] x^{2n+1} = 0$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^{2n}\,(n!)^2} \quad \Rightarrow$

$$J_0'(x) = \sum_{n=1}^{\infty} \dfrac{(-1)^n (2n) x^{2n-1}}{2^{2n}\,(n!)^2} = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}\, 2(n+1) x^{2n+1}}{2^{2n+2}\,[(n+1)!]^2} \qquad [\text{Replace } n \text{ with } n+1]$$

$$= -\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \qquad [\text{cancel } 2 \text{ and } n+1; \text{ take } -1 \text{ outside sum}] \quad = -J_1(x)$$

35. (a) $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \dfrac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = f(x)$

39. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have

$$\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}} \right) = \sum_{n=0}^{\infty} (-1)^n \dfrac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{3} \right)^n \dfrac{1}{\sqrt{3}} \dfrac{1}{2n+1}, \text{ so}$$

$$\pi = \dfrac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)3^n}.$$