Math 128A Spring 2003
Week 12 Solutions

Burden & Faires §5.9: 1b, 2b, 3, 5, 6, 7
Burden & Faires §5.10: 4, 5, 8
Burden & Faires §5.11: 1c, 2, 5, 6, 8

Burden & Faires §5.9. Higher-Order Equations and Systems of Differential Equations

1. *Use the Runge-Kutta method for systems to approximate the solution of the following system of first-order differential equations, and compare the result to the actual solution.*

   b.

$$u_1' = -4u_1 - 2u_2 + \cos t + 4\sin t, \quad 0 \le t \le 2, \quad u_1(0) = 0;$$
$$u_2' = 3u_1 + u_2 - 3\sin t, \quad 0 \le t \le 2, \quad u_2(0) = -1;$$
$$h = 0.1; \quad \textit{actual solutions } u_1(t) = 2e^{-t} - 2e^{-2t} + \sin t \quad \textit{and} \quad u_2(t) = -3e^{-t} + 2e^{-2t}.$$

Solution. b. Applying the Runge-Kutta method, we obtain the following results.

t_i	$w_{i,1}$	$u_1(t_i)$	$w_{i,2}$	$u_2(t_i)$
0.0	0.00000000	0.00000000	-1.00000000	-1.00000000
0.1	0.27204137	0.27204675	-1.07704549	-1.07705075
0.2	0.49548169	0.49549074	-1.11554333	-1.11555217
0.3	0.67952186	0.67953338	-1.12482019	-1.12483139
0.4	0.83138741	0.83140051	-1.11228951	-1.11230221
0.5	0.95671390	0.95672798	-1.08381950	-1.08383310
0.6	1.05986269	1.05987732	-1.04403240	-1.04404648
0.7	1.14417945	1.14419437	-0.99654769	-0.99656198
0.8	1.21220597	1.21222098	-0.94417953	-0.94419386
0.9	1.26585346	1.26586845	-0.88909695	-0.88911120
1.0	1.30654440	1.30655930	-0.83295364	-0.83296776
1.1	1.33532844	1.33534321	-0.77699300	-0.77700693
1.2	1.35297699	1.35299160	-0.72213299	-0.72214673
1.3	1.36006018	1.36007462	-0.66903470	-0.66904822
1.4	1.35700930	1.35702353	-0.61815747	-0.61817077
1.5	1.34416716	1.34418117	-0.56980329	-0.56981634
1.6	1.32182847	1.32184223	-0.52415236	-0.52416515
1.7	1.29027184	1.29028532	-0.48129154	-0.48130403
1.8	1.24978481	1.24979796	-0.44123705	-0.44124922
1.9	1.20068300	1.20069578	-0.40395251	-0.40396431
2.0	1.14332436	1.14333672	-0.36936318	-0.36937457

□

2. *Use the Runge-Kutta for Systems Algorithm to approximate the solution of the following higher-order differential equation, and compare the result to the actual solution.*

   b.

$$t^2y'' - 2ty' + 2y = t^3 \ln t, \quad 1 \le t \le 2, \quad y(1) = 1, \quad y'(1) = 0, \quad \textit{with } h = 0.1;$$
$$\textit{actual solution } y(t) = \frac{7}{4}t + \frac{1}{2}t^3 \ln t - \frac{3}{4}t^3.$$

Solution. b. Solving this higher-order differential equation is equivalent to solving the following system of differential equations.

$$u_1' = u_2, \quad 1 \le t \le 2, \quad u_1(1) = 1;$$

$$u_2' = \frac{2}{t}u_2 - \frac{2}{t^2}u_1 + t\ln t, \quad 1 \le t \le 2, \quad u_2(1) = 0;$$

where $y(t) = u_1(t)$ and $y'(t) = u_2(t)$. Applying Runge-Kutta to this system we get the following.

t_i	$w_{i,1}$	$y(t_i)$	$w_{i,2}$
1.0	1.00000000	1.00000000	0.00000000
1.1	0.99017818	0.99017892	-0.19451307
1.2	0.96152437	0.96152583	-0.37618726
1.3	0.91545502	0.91545714	-0.54240900
1.4	0.85363713	0.85363991	-0.69077449
1.5	0.77796897	0.77797237	-0.81905848
1.6	0.69056342	0.69056743	-0.92518958
1.7	0.59373369	0.59373830	-1.00723029
1.8	0.48998072	0.48998591	-1.06336076
1.9	0.38198213	0.38198790	-1.09186533
2.0	0.27258237	0.27258872	-1.09112119

□

3. *Change the Adams Fourth-Order Predictor-Corrector Algorithm to obtain approximate solutions to systems of first-order equations.*

Solution. The changed algorithm is as follows.

Adams Fourth-Order Predictor Corrector for Systems
To approximate the solution of the mth-order system of first-order initial-value problems

$$u_j' = f_j(t, u_1, u_2, \ldots, u_m) \quad a \le t \le b, \text{ with } u_j(a) = \alpha_j$$

for $j = 1, 2, \ldots, m$ at $(N+1)$ equally spaced numbers in the interval $[a, b]$:
INPUT endpoints a, b; number of equations m; integer N; initial conditions $\alpha_1, \ldots, \alpha_m$.
OUTPUT approximations of w_j to $u_j(t)$ at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t_0 = a$.

Step 2 For $j = 1, 2, \ldots, m$ set $w_j = \alpha_j$.

Step 3 **OUTPUT** $(t, w_1, w_2, \ldots, w_m)$.

Step 4 For $i = 1, 2, 3$ do steps 5–11.

Step 5 For $j = 1, 2, \ldots, m$ set
$k_{1,j} = hf_j(t, w_1, w_2, \ldots, w_m)$.

Step 6 For $j = 1, 2, \ldots, m$ set
$k_{2,j} = hf_j(t + \frac{h}{2}, w_1 + \frac{1}{2}k_{1,1}, w_2 + \frac{1}{2}k_{1,2}, \ldots, w_m + \frac{1}{2}k_{1,m})$.

Step 7 For $j = 1, 2, \ldots, m$ set
$k_{3,j} = hf_j(t + \frac{h}{2}, w_1 + \frac{1}{2}k_{2,1}, w_2 + \frac{1}{2}k_{2,2}, \ldots, w_m + \frac{1}{2}k_{2,m})$.

Step 8 For $j = 1, 2, \ldots, m$ set
$k_{4,j} = hf_j(t + h, w_1 + k_{3,1}, w_2 + k_{3,2}, \ldots, w_m + k_{3,m})$.

Step 9 For $j = 1, 2, \ldots, m$ set
$w_{i,j} = w_{i-1,j} + (k_{1,j} + 2k_{2,j} + 2k_{3,j} + k_{4,j})/6$.

Step 10 Set $t_i = a + ih$.

Step 11 **OUTPUT** $(t, w_1, w_2, \ldots, w_m)$.

Step 12 For $i = 4, \ldots, N$ do Steps 13–16.

Step 13 Set $t_i = a + ih$.

Step 14 For $j = 1, 2, \ldots, m$ set

$$
\begin{aligned}
w'_{i,j} \quad = \quad & w_{i-1,j} + h[55 f_j(t_{i-1}, w_{i-1,1}, w_{i-1,2}, \ldots, w_{i-1,m}) \\
& - 59 f_j(t_{i-2}, w_{i-2,1}, w_{i-2,2}, \ldots, w_{i-2,m}) \\
& + 37 f_j(t_{i-3}, w_{i-3,1}, w_{i-3,2}, \ldots, w_{i-3,m}) \\
& - 9 f_j(t_{i-4}, w_{i-4,1}, w_{i-4,2}, \ldots, w_{i-4,m})]/24
\end{aligned}
$$

Step 15 For $j = 1, 2, \ldots, m$ set

$$
\begin{aligned}
w_{i,j} \quad = \quad & w_{i-1,j} + h[9 f_j(t_i, w'_{i,1}, w'_{i,2}, \ldots, w'_{i,m}) \\
& + 19 f_j(t_{i-1}, w_{i-1,1}, w_{i-1,2}, \ldots, w_{i-1,m}) \\
& - 5 f_j(t_{i-2}, w_{i-2,1}, w_{i-2,2}, \ldots, w_{i-2,m}) \\
& + f_j(t_{i-3}, w_{i-3,1}, w_{i-3,2}, \ldots, w_{i-3,m})]/24
\end{aligned}
$$

Step 16 **OUTPUT** $(t, w_{i,1}, w_{i,2}, \ldots, w_{i,m})$.

Step 17 **STOP** □

5. *Repeat Exercise 2 using the algorithm developed in Exercise 3.*

Solution. b. Applying the algorithm, we arrive at the following results:

t_i	$w_{i,1}$	$w_{i,2}$	$y(t_i)$
1.00	1.00000000	0.00000000	1.00000000
1.10	0.99017818	-0.19451307	0.99017892
1.20	0.96152437	-0.37618726	0.96152583
1.30	0.91545502	-0.54240900	0.91545714
1.40	0.85363657	-0.69077432	0.85363991
1.50	0.77796798	-0.81905806	0.77797237
1.60	0.69056210	-0.92518887	0.69056743
1.70	0.59373213	-1.00722925	0.59373830
1.80	0.48997900	-1.06335939	0.48998591
1.90	0.38198032	-1.09186360	0.38198790
2.00	0.27258055	-1.09111909	0.27258872

□

6. *Suppose the swinging pendulum described in the lead example of this chapter is 2 ft long and that $g = 32.17$ ft/s^2. With $h = 0.1$ s, compare the angle θ obtained for the following two initial-value problems:*

a.
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0.$$

b.
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0,$$

at $t = 0$, 1, and 2 s.

Solution. a. The solution for this initial-value problem using the Runge-Kutta method for systems is as follows.

t_i	$w_{i,1}$	$w_{i,2}$
0.00	0.52359878	0.00000000
0.10	0.48385575	-0.78541857
0.20	0.37023789	-1.45935166
0.30	0.19927132	-1.91643290
0.40	-0.00300321	-2.07580771
0.50	-0.20479165	-1.90697790
0.60	-0.37439530	-1.44224266
0.70	-0.48601969	-0.76356628
0.80	-0.52345600	0.02331421
0.90	-0.48144588	0.80691343
1.00	-0.36590320	1.47584848
1.10	-0.19365646	1.92515511
1.20	0.00902565	2.07510419
1.30	0.21026965	1.89697050
1.40	0.37847931	1.42478516
1.50	0.48810413	0.74155126
1.60	0.52324540	-0.04666304
1.70	0.47896954	-0.82834582
1.80	0.36151324	-1.49218478
1.90	0.18800671	-1.93363505
2.00	-0.01505632	-2.07411834

b. The solution for this initial-value problem using the Runge-Kutta method for systems is as follows.

t_i	$w_{i,1}$	$w_{i,2}$
0.00	0.52359878	0.00000000
0.10	0.48205280	-0.81963042
0.20	0.36403762	-1.50919046
0.30	0.18827918	-1.95929754
0.40	-0.01733741	-2.09856146
0.50	-0.22019199	-1.90490736
0.60	-0.38810442	-1.40907486
0.70	-0.49443950	-0.68973862
0.80	-0.52233201	0.13897516
0.90	-0.46736160	0.94559536
1.00	-0.33825332	1.60216317
1.10	-0.15549276	2.00453096
1.20	0.05192440	2.08888257
1.30	0.25109267	1.84185471
1.40	0.41041699	1.30265398
1.50	0.50462485	0.55683428
1.60	0.51877504	-0.27727786
1.70	0.45062736	-1.06735616
1.80	0.31099724	-1.68806720
1.90	0.12203925	-2.04095266
2.00	-0.08626800	-2.07004691

Thus we see that at 0, 1 and 2 s, we have 0.52359878, -0.36590320, and -0.01505632 radians respectively for the first initial-value problem and 0.52359878, -0.33825332, and -0.08626800 radians respectively for the second initial-value problem. □

7. *The study of mathematical models for predicting the population dynamics of competing species has its origin in independent works published in the early part of this century by A. J. Lotka and V. Volterra. Consider the problem of predicting the population of two species, one of which is*

a predator, whose population at time t is $x_2(t)$, feeding on the other, which is the prey, whose population is $x_1(t)$. We will assume that the prey always has an adequate food supply and that its birth rate at any time is proportional to the number of prey alive at that time; that is, birth rate (prey) is $k_1x_1(t)$. The death rate of the prey depends on both the number of prey and predators alive at that time. For simplicity, we assume death rate (prey) = $k_2x_1(t)x_2(t)$. The birth rate of the predator, on the other hand, depends on its food supply, $x_1(t)$, as well as on the number of predators available for reproductive purposes. For this reason, we assume that the birth (predator) is $k_3x_1(t)x_2(t)$. The death rate of the predator will be taken as simply proportional to the number of predators alive at the time; that is, death rate (predator) = $k_4x_2(t)$.

Since $x_1'(t)$ and $x_2'(t)$ represent the change in the prey and predator populations, respectively, with respect to time, the problem is expressed by the system of nonlinear differential equations

$$x_1'(t) = k_1x_1(t) - k_2x_1(t)x_2(t) \quad \text{and} \quad x_2'(t) = k_3x_1(t)x_2(t) - k_4x_2(t).$$

Solve this system for $0 \le t \le 4$, assuming that the initial population of the prey is 1000 and of the predators is 500 and that the constants are $k_1 = 3$, $k_2 = 0.002$, $k_3 = 0.0006$, and $k_4 = 0.5$. Sketch a graph of the solutions to this problem, plotting both populations with time, and describe the physical phenomena represented. Is there a stable solution to this population model? If so, for what values x_1 and x_2 is the solution stable?

Solution. Running the Runge-Kutta for systems algorithm, we get that at time $t = 4$, the number of prey is approximately $25.39254675 \approx 25$ and the number of predators is approximately $1257.67355448 \approx 1258$. There are two stable solutions to the population model occurring when x_1' and x_2' are both 0. One occurs when both x_1 and x_2 are 0. The other occurs when both $k_1 - k_2x_2(t) = 0$ and $k_3x_1(t) - k_4 = 0$ implying that

$$x_2(t) = \frac{k_1}{k_2} \quad \text{and} \quad x_1(t) = \frac{k_4}{k_3}.$$

With the current values of k_1, k_2, k_3, and k_4, this gives $x_1(t) = 833.33333$ and $x_2(t) = 1500.0000$ which cannot happen since we cannot have one third of one prey. □

Burden & Faires §5.10. Stability

4. *Consider the differential equation*

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

a. *Show that*

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3}y'''(\xi_1),$$

for some ξ, where $t_i < \xi_i < t_{i+2}$.

b. *Part (a) suggests the difference method*

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i), \quad \text{for } i = 0, 1, \ldots, N-2.$$

use this method to solve

$$y' = 1 - y, \quad 0 \le t \le 1, \quad y(0) = 0,$$

with $h = 0.1$. Use the starting values $w_0 = 0$ and $w_1 = y(t_1) = 1 - e^{-0.1}$.

c. *Repeat part (b) with $h = 0.01$ and $w_1 = 1 - e^{-0.01}$.*

d. *Analyze this method for consistency, stability, and convergence.*

Solution. a. Applying the three point method for approximating the derivative at t_i given the values at t_i, $t_{i+1} = t_i + h$, and $t_{i+2} = t_i + 2h$, we get the following equation:

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3}y'''(\xi_i),$$

for some ξ_i, where $t_i < \xi_i < t_{i+2}$.

b. Applying this method with $h = 0.1$ we obtain the following results.

t_i	w_i
0.00	0.00000000
0.10	0.09516258
0.20	0.18065033
0.30	0.25614608
0.40	0.31876341
0.50	0.35784461
0.60	0.33884089
0.70	0.15339866
0.80	-0.53515987
0.90	-2.77015572
1.00	-9.78217524

c. Applying this method with $h = 0.01$ we obtain the following results.

0.00	0
0.01	0.0099501663
0.02	0.019800665
0.03	0.029551165
0.04	0.039198677
0.05	0.048732236
0.06	0.058116888
0.07	0.067245488
0.08	0.075793627
0.09	0.082782952
0.10	0.0852668
0.11	0.074374002
0.12	0.023400946
0.13	-0.14803074
0.14	-0.68185779
0.15	-2.3062996
0.16	-7.213262
0.17	-22.000275
0.18	-66.52558
0.19	-200.5615
0.20	-604.01978
0.21	-1818.4258
0.22	-5473.7444
0.23	-16476.089
0.24	-49592.616
0.25	-149271.74
0.26	-449300.98
0.27	-1352374.2
0.28	-4070579.8
0.29	-12252244
0.30	-36878649
0.31	-1.1100291×10^{08}
0.32	-3.3411326×10^{08}
0.33	-1.0056644×10^{09}
0.34	-3.0269999×10^{09}
0.35	-9.1111200×10^{09}
0.36	-2.7424020×10^{10}
0.37	-8.2544943×10^{10}

0.38	-2.4845619×10^{11}
0.39	-7.4784083×10^{11}
0.40	-2.2509639×10^{12}
0.41	-6.7752899×10^{12}
0.42	-2.0393287×10^{13}
0.43	-6.1382785×10^{13}
0.44	-1.8475914×10^{14}
0.45	-5.5611587×10^{14}
0.46	-1.6738812×10^{15}
0.47	-5.0382997×10^{15}
0.48	-1.5165033×10^{16}
0.49	-4.5645997×10^{16}
0.50	-1.3739219×10^{17}
0.51	-4.1354370×10^{17}
0.52	-1.2447461×10^{18}
0.53	-3.7466240×10^{18}
0.54	-1.1277153×10^{19}
0.55	-3.3943672×10^{19}
0.56	-1.0216877×10^{20}
0.57	-3.0752294×10^{20}
0.58	-9.2562883×10^{20}
0.59	-2.7860970×10^{21}
0.60	-8.3860139×10^{21}
0.61	-2.5241487×10^{22}
0.62	-7.5975626×10^{22}
0.63	-2.2868287×10^{23}
0.64	-6.8832412×10^{23}
0.65	-2.0718215×10^{24}
0.66	-6.2360803×10^{24}
0.67	-1.8770293×10^{25}
0.68	-5.6497652×10^{25}
0.69	-1.7005514×10^{26}
0.70	-5.1185754×10^{26}
0.71	-1.5406659×10^{27}
0.72	-4.6373280×10^{27}
0.73	-1.3958128×10^{28}
0.74	-4.2013273×10^{28}
0.75	-1.2645787×10^{29}
0.76	-3.8063193×10^{29}
0.77	-1.1456833×10^{30}
0.78	-3.4484499×10^{30}
0.79	-1.0379664×10^{31}
0.80	-3.1242274×10^{31}
0.81	-9.4037697×10^{31}
0.82	-2.8304881×10^{32}
0.83	-8.5196291×10^{32}
0.84	-2.5643662×10^{33}
0.85	-7.7186153×10^{33}
0.86	-2.3232650×10^{34}
0.87	-6.9929126×10^{34}
0.88	-2.1048321×10^{35}
0.89	-6.3354403×10^{35}
0.90	-1.9069362×10^{36}

0.91	-5.7397835×10^{36}
0.92	-1.7276464×10^{37}
0.93	-5.2001302×10^{37}
0.94	-1.5652134×10^{38}
0.95	-4.7112150×10^{38}
0.96	-1.4180524×10^{39}
0.97	-4.2682675×10^{39}
0.98	-1.2847274×10^{40}
0.99	-3.8669658×10^{40}
1.00	-1.1639376×10^{41}

d. From part (a), we get the following:

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - 4y(t_i) + 3y(t_{i-1})}{h} + 2f(t_{i-1}, y(t_{i-1})) = \frac{2h^2}{3}y'''(\xi_{i-1}).$$

Thus this method is consistent. The method has characteristic equation given by:

$$P(\lambda) = \lambda^2 - 4\lambda + 3.$$

This equation has roots 1 and 3, but since $|3| > 1$, the method is unstable. Thus the method is not convergent.

□

5. *Given the multistep method*

$$w_{i+1} = -\frac{3}{2}w_i + 3w_{i-1} - \frac{1}{2}w_{i-2} + 3hf(t_i, w_i), \quad \textit{for } i = 2, \ldots, N-1,$$

with starting values w_0, w_1, w_2*:*

a. *Find the local truncation error.*

b. *Comment on consistency, stability, and convergence.*

Solution. a. Using the polynomial approximation agreeing with $y(t)$ at t_{i+1}, t_i, t_{i-1}, and t_{i-2}, we have:

$$y(t) = P(t) + \frac{y^{(4)}(\xi(t))}{24}(t - t_{i+1})(t - t_i)(t - t_{i-1})(t - t_{i-2}).$$

Plugging $P(t)$ into

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) + \frac{3}{2}y(t_i) - 3y(t_{i-1}) + \frac{1}{2}y(t_{i-2})}{h} - 3f(t_i, y(t_i)),$$

we get 0 and plugging in the error term we get:

$$\frac{1}{4}h^3 y^{(4)}(\xi(t_i)).$$

b. Part (a) shows that the method is consistent. The characteristic equation is given by

$$P(\lambda) = \lambda^3 + \frac{3}{2}\lambda^2 - 3\lambda + \frac{1}{2} = 0.$$

Factoring, we get

$$(\lambda - 1)\left(\lambda^2 + \frac{5}{2}\lambda - \frac{1}{2}\right) = 0.$$

The two roots other than 1 are given by

$$\frac{-5 \pm \sqrt{33}}{4},$$

but

$$\left| \frac{-5 - \sqrt{33}}{4} \right| > 1,$$

which implies that the method is unstable and hence not convergent.

□

8. *Consider the problem $y' = 0$, $0 \le t \le 10$, $y(0) = 0$, which has the solution $y \cong 0$. If the difference method of Exercise 4 is applied to the problem, then*

$$w_{i+1} = 4w_i - 3w_{i-1}, \quad \text{for } i = 1, 2, \ldots, N-1,$$

$$w_0 = 0, \quad \text{and} \quad w_1 = \alpha_1.$$

Suppose $w_1 = \alpha_1 = \epsilon$, where ϵ is a small rounding error. Compute w_i exactly for $i = 2, 3, 4, 5, 6$ to find how the error ϵ is propagated.

Solution. Applying the method, we have

$$\begin{aligned} w_0 &= 0 \\ w_1 &= \varepsilon \\ w_2 &= 4w_1 - 3w_0 = 4\varepsilon \\ w_3 &= 13\varepsilon \\ w_4 &= 40\varepsilon \\ w_5 &= 121\varepsilon \\ w_6 &= 364\varepsilon \end{aligned}$$

□

Burden & Faires §5.11. Stiff Differential Equations

1. *Solve the following stiff initial-value problem using Euler's method, and compare the result with the actual solution.*

c.

$$y' = -20y + 20\sin t + \cos t, \quad 0 \le t \le 2, \quad y(0) = 1, \text{ with } h = 0.25;$$
$$\text{actual solution } y(t) = \sin t + e^{-20t}.$$

Solution. c. Applying the Euler's method we get the following:

t_i	w_i	$y(t_i)$
0.00	1.00000000	1.00000000
0.25	-3.75000000	0.25414191
0.50	16.47924790	0.47947094
0.75	-63.30046827	0.68163907
1.00	256.79298911	0.84147099
1.25	-1022.82952594	0.94898462
1.50	4096.14185745	0.99749499
1.75	-16379.56227058	0.98398595
2.00	65523.12445056	0.90929743

□

2. *Repeat Exercise 1 using the Runge-Kutta fourth-order method.*

Solution. c. Applying the Runge-Kutta method to the previous differential equation, we get the following:

t_i	w_i	$y(t_i)$
0.00	1.00000000	1.00000000
0.25	13.95099966	0.25414191
0.50	188.30821458	0.47947094
0.75	2575.45822267	0.68163907
1.00	35296.67835654	0.84147099
1.25	483847.97544071	0.94898462
1.50	6632737.23998763	0.99749499
1.75	90923760.22686818	0.98398595
2.00	1246413200.45146275	0.90929743

□

5. *Solve the following stiff initial-value problem using the Runge-Kutta fourth-order method with (a) $h = 0.1$ and (b) $h = 0.025$.*

$$\begin{aligned} u_1' &= 32u_1 + 66u_2 + \frac{2}{3}t + \frac{2}{3}, \quad 0 \le t \le 0.5, \quad u_1(0) = \frac{1}{3}; \\ u_2' &= -66u_1 - 133u_2 - \frac{1}{3}t - \frac{1}{3}, \quad 0 \le t \le 0.5, \quad u_2(0) = \frac{1}{3}. \end{aligned}$$

Compare the results to the actual solution,

$$u_1(t) = \frac{2}{3}t + \frac{2}{3}e^{-t} - \frac{1}{3}e^{-100t} \quad \text{and} \quad u_2(t) = -\frac{1}{3}t - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-100t}.$$

Solution. a. When $h = 0.1$, the Runge-Kutta fourth-order method gives:

t_i	$w_{1,\,i}$	$u_1(t_i)$	$w_{2,\,i}$	$u_2(t_i)$
0.00	0.33333333	0.33333333	0.33333333	0.33333333
0.10	-96.33010833	0.66987648	193.66505417	-0.33491554
0.20	-28226.32084607	0.67915383	56453.66042303	-0.33957692
0.30	-8214056.30612105	0.69387881	16428113.65306053	-0.34693941
0.40	-2390290586.28645468	0.71354670	4780581173.64322948	-0.35677335
0.50	-695574560816.26318359	0.73768711	1391149121633.63305664	-0.36884355

b. When $h = 0.025$, the Runge-Kutta fourth-order method gives:

t_i	$w_{1,i}$	$u_1(t_i)$	$w_{2,i}$	$u_2(t_i)$
0.000	0.33333333	0.33333333	0.33333333	0.33333333
0.025	0.45072744	0.63951161	0.09885503	-0.27871330
0.050	0.52732922	0.66524030	-0.05342901	-0.32925118
0.075	0.57761256	0.66831130	-0.15248164	-0.33387911
0.100	0.61095960	0.66987648	-0.21708179	-0.33491554
0.125	0.63345088	0.67166336	-0.25940485	-0.33582982
0.150	0.64902611	0.67380522	-0.28734424	-0.33690245
0.175	0.66023691	0.67630467	-0.30601680	-0.33815232
0.200	0.66873489	0.67915383	-0.31873903	-0.33957692
0.225	0.67558811	0.68234415	-0.32766000	-0.34117207
0.250	0.68148632	0.68586719	-0.33417186	-0.34293359
0.275	0.68687403	0.68971475	-0.33917594	-0.34485737
0.300	0.69203679	0.69387881	-0.34325535	-0.34693941
0.325	0.69715713	0.69835157	-0.34678690	-0.34917578
0.350	0.70235087	0.70312539	-0.35001366	-0.35156270
0.375	0.70769063	0.70819285	-0.35309197	-0.35409643
0.400	0.71322103	0.71354670	-0.35612202	-0.35677335
0.425	0.71896869	0.71917986	-0.35916758	-0.35958993
0.450	0.72494850	0.72508543	-0.36226885	-0.36254272
0.475	0.73116791	0.73125670	-0.36545077	-0.36562835
0.500	0.73762953	0.73768711	-0.36872840	-0.36884355

□

6. *Show that the fourth-order Runge-Kutta method,*

$$\begin{aligned} k_1 &= hf(t_i,\, w_i), \\ k_2 &= hf(t_i + h/2,\, w_i + k_1/2), \\ k_3 &= hf(t_i + h/2,\, w_i + k_2/2), \\ k_4 &= hf(t_i + h,\, w_i + k_3), \\ w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned}$$

when applied to the differential equation $y' = \lambda y$, can be written in the form

$$w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right) w_i.$$

Solution. We calculate:

$$\begin{aligned}
k_1 &= h\lambda w_i \\
k_2 &= h\lambda\left(w_i + \frac{k_1}{2}\right) \\
&= \left(h\lambda + \frac{1}{2}(h\lambda)^2\right) w_i \\
k_3 &= h\lambda\left(w_i + \frac{1}{2}k_2\right) \\
&= \left(h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{4}(h\lambda)^3\right) w_i \\
k_4 &= h\lambda(w_i + k_3) \\
&= \left(h\lambda + (h\lambda)^2 + \frac{1}{2}(h\lambda)^3 + \frac{1}{4}(h\lambda)^4\right) w_i \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= w_i + \frac{1}{6}\Bigg(h\lambda + 2h\lambda + (h\lambda)^2 + 2h\lambda + (h\lambda)^2 + \frac{1}{2}(h\lambda)^3 \\
&\quad + h\lambda + (h\lambda)^2 + \frac{1}{2}(h\lambda)^3 + \frac{1}{4}(h\lambda)^4\Bigg) w_i \\
&= \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right) w_i
\end{aligned}$$

□

8. *The Backward Euler one-step method is defined by*

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}), \quad \textit{for } i = 0, \ldots, N-1.$$

   a. *Show that* $Q(h\lambda) = 1/(1 - h\lambda)$ *for the Backward Euler method.*
   b. *Apply the Backward Euler method to the differential equation given in Exercise 1.*

Solution. a. Applying the Backward Euler method to the differential equation $y' = \lambda y$, we get:

$$\begin{aligned}
w_{i+1} &= w_i + hf(t_{i+1}, w_{i+1}) \\
&= w_i + h\lambda w_{i+1} \\
(1 - h\lambda)w_{i+1} &= w_i \\
w_{i+1} &= \frac{w_i}{1 - h\lambda} \\
&= Q(h\lambda)w_i \\
Q(h\lambda) &= \frac{1}{1 - h\lambda}
\end{aligned}$$

b. Applying the Backward Euler method to the differential equation from Exercise 1, we get:

$$\begin{aligned}
w_{i+1} &= w_i + hf(t_{i+1}, w_{i+1}) \\
&= w_i + h(-20w_{i+1} + 20\sin(t_{i+1}) + \cos(t_{i+1})) \\
(1 + 20h)w_{i+1} &= w_i + 20h\sin(t_{i+1}) + h\cos(t_{i+1}) \\
w_{i+1} &= \frac{w_i + 20h\sin(t_{i+1}) + h\cos(t_{i+1})}{1 + 20h}.
\end{aligned}$$

Thus the results are:

t_i	w_i	$y(t_i)$
0.00	1.00000000	1.00000000
0.25	0.41320798	0.25414191
0.50	0.50495522	0.47947094
0.75	0.68267854	0.68163907
1.00	0.83751817	0.84147099
1.25	0.94354531	0.94898462
1.50	0.99145076	0.99749499
1.75	0.97780316	0.98398595
2.00	0.90337560	0.90929743

□