Math 128A Spring 2003
Week 9 Solutions

Burden & Faires §5.1: 1a, 3a, 5, 7
Burden & Faires §5.2: 1ab, 2b, 5a, 7
Burden & Faires §5.3: 3a, 5ac, 7

Burden & Faires §5.1. The Elementary Theory of Initial-Value Problems

1. *Use Theorem 5.4 to show that the following initial-value problem has a unique solution, and find the solution.*

   a.
$$y' = y\cos t, \quad 0 \le t \le 1, \quad y(0) = 1.$$

Solution. a. In order to apply Theorem 5.4, we must show that $f(t,\, y) = y\cos t$ is continuous and satisfies a Lipschitz condition in the variable y on $\{(t,\, y)\,|\, 0 \le t \le 1,\, -\infty < y < \infty\}$. Clearly f is continuous and we have

$$\left|\frac{\partial f}{\partial y}(t,\, y)\right| = |\cos t| \le 1.$$

Thus f satisfies a Lipschitz condition with Lipschitz constant 1.

The solution to the equation is given by:

$$\begin{aligned}
\frac{dy}{dt} &= y\cos t \\
\int \frac{1}{y}\,dy &= \int \cos t\,dt \\
\ln|y| &= \sin t + C_0 \\
y &= C_1 e^{\sin t} \\
1 &= C_1 e^0 \\
y &= e^{\sin t}
\end{aligned}$$

□

3. *For the following initial-value problem, show that the given equation implicitly defines a solution. Approximate $y(2)$ using Newton's method.*

   a.
$$y' = -\frac{y^3 + y}{(3y^2+1)t}, \quad 1 \le t \le 2, \quad y(1) = 1; \quad y^3 t + yt = 2$$

Solution. a. First we must find the derivative of y implicitly.

$$\begin{aligned}
\frac{d}{dt}\left[y^3 t + yt\right] &= \frac{d}{dt}[2] \\
3y^2 ty' + y^3 + ty' + y &= 0 \\
(3y^2+1)ty' &= -(y^3 + y) \\
y' &= -\frac{y^3 + y}{(3y^2+1)t}.
\end{aligned}$$

Also, we have that $(1)^3(1) + (1)(1) = 2$. Thus the equation does define an implicit solution to the differential equation. Using Newton's method we can approximate $y(2)$ to be 0.6823278 by finding an approximate solution to $2y^3 + 2y = 2$.

$\square$

5. *Show that, for any constants a and b, the set $D = \{(t, y), \mid a \le t \le b, -\infty < y < \infty\}$ is convex.*

Proof. Suppose that (t_1, y_1) and (t_2, y_2) are points in D with $t_2 \ge t_1$. We need to show that for any $0 \le \lambda \le 1$, that the point $P = ((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D. Now, $a \le t_1 \le t_1 + \lambda(t_2 - t_1) = (1-\lambda)t_1 + \lambda t_2 = t_2 - (1-\lambda)(t_2 - t_1) \le t_2 \le b$, so P is in D and thus D is convex. $\square$

7. *Picard's method for solving the initial-value problem*

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is described as follows: Let $y_0(t) = \alpha$ for each t in $[a, b]$. Define a sequence $\{y_k(t)\}$ of functions by

$$y_k(t) = \alpha + \int_\alpha^t f(\tau, y_{k-1}(\tau))\, d\tau, \quad k = 1, 2, \ldots.$$

a. *Integrate $y' = f(t, y(t))$, and use the initial condition to derive Picard's method.*

b. *Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the intial-value problem*

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

c. *Compare the result in part (b) to the Maclaurin series of the actual solution $y(t) = t + e^{-t}$.*

Solution. a.

$$\begin{aligned}
\int_a^t y'(t)\, dt &= \int_a^t f(t, y)\, dt \\
y(t) - y(a) &= \int_a^t f(t, y)\, dt \\
y(t) - \alpha &= \int_a^t f(t, y)\, dt \\
y(t) &= \alpha + \int_a^t f(t, y)\, dt
\end{aligned}$$

b.

$$y_0(t) = 1$$

$$\begin{aligned}
y_1(t) &= 1 + \int_0^t f(\tau, y_0(\tau))\, d\tau \\
&= 1 + \int_0^t f(\tau, 1)\, d\tau \\
&= 1 + \int_0^t -1 + \tau + 1\, d\tau \\
&= 1 + \left[\frac{1}{2}\tau^2\right]_0^t \\
&= 1 + \frac{1}{2}t^2
\end{aligned}$$

$$
\begin{aligned}
y_2(t) &= 1 + \int_0^t f(\tau,\, y_1(\tau))\, d\tau \\
&= 1 + \int_0^t f(\tau,\, 1 + \frac{1}{2}\tau^2)\, d\tau \\
&= 1 + \int_0^t -(1 + \frac{1}{2}\tau^2) + \tau + 1\, d\tau \\
&= 1 + \int_0^t \tau - \frac{1}{2}\tau^2\, d\tau \\
&= 1 + \left[\frac{1}{2}\tau^2 - \frac{1}{6}\tau^3\right]_0^t \\
&= 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3
\end{aligned}
$$

$$
\begin{aligned}
y_3(t) &= 1 + \int_0^t f(\tau,\, y_2(\tau))\, d\tau \\
&= 1 + \int_0^t f(\tau,\, 1 + \frac{1}{2}\tau^2 - \frac{1}{6}\tau^3)\, d\tau \\
&= 1 + \int_0^t -(1 + \frac{1}{2}\tau^2 - \frac{1}{6}\tau^3) + \tau + 1\, d\tau \\
&= 1 + \int_0^t \tau - \frac{1}{2}\tau^2 + \frac{1}{6}\tau^3\, d\tau \\
&= 1 + \left[\frac{1}{2}\tau^2 - \frac{1}{6}\tau^3 + \frac{1}{24}\tau^4\right]_0^t \\
&= 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4
\end{aligned}
$$

c. The Maclaurin series is given by

$$
\begin{aligned}
t + e^{-t} &= t + (1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \dots) \\
&= 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \dots,
\end{aligned}
$$

so we see that $y_3(t)$ gives the first four terms of the Maclaurin series. □

Burden & Faires §5.2. Euler's Method

1. *Use Euler's method to approximate the solutions for each of the following initial-value problems.*

   a.
   $$y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0, \text{ with } h = 0.5$$

   b.
   $$y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1, \text{ with } h = 0.5$$

Solution. a.

$$\begin{aligned}
w_0 &= 0.0000000 \\
t_0 &= 0.000 \\
w_1 &= w_0 + hf(t_0, w_0) \\
&= 0 + 0.5[(0)e^{3(0)} - 2(0)] \\
&= 0.0000000 \\
t_1 &= 0.500 \\
w_2 &= 0 + 0.5[(0.5)e^{3(0.5)} - 2(0)] \\
&= 1.1204223 \\
t_2 &= 1.000
\end{aligned}$$

b.

$$\begin{aligned}
w_0 &= 1.0000000 \\
t_0 &= 2.000 \\
w_1 &= 1 + 0.5[1 + (2-1)^2] \\
&= 2.0000000 \\
t_1 &= 2.500 \\
w_2 &= 2 + 0.5[1 + (2.5-2)^2] \\
&= 2.6250000 \\
t_2 &= 3.000
\end{aligned}$$

□

2. *The actual solution to the initial-value problem in Exercise 1b is given here. Compare the actual error at each step to the error bound.*

   b.
$$y(t) = t + \frac{1}{1-t}$$

   Solution. The actual values of the function are as follows:

$$\begin{aligned}
y(2.000) &= 1.0000000 \\
y(2.500) &= 1.8333333 \\
y(3.000) &= 2.5000000.
\end{aligned}$$

   This gives actual errors:

$$\begin{aligned}
|w_0 - y(2.000)| &= 0.0000000 \\
|w_1 - y(2.500)| &= 0.1666667 \\
|w_2 - y(3.000)| &= 0.1250000.
\end{aligned}$$

   The differential equation does not satisfy a Lipschitz condition as required by Theorem 5.9, so we cannot obtain an error bound for it. □

5. *Given the initial-value problem*

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

   with exact solution $y(t) = t^2(e^t - e)$:

   a. *Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.*

Solution. a.

$$\begin{aligned} w_0 &= 0.000000 \\ t_0 &= 1.0 \\ y(t_0) &= 0.000000 \end{aligned}$$

$$\begin{aligned} w_1 &= 0.000000 + 0.1\left[\frac{2}{1.0}0.000000 + (1.0)^2 e^{(1.0)}\right] \\ &= 0.271828 \\ t_1 &= 1.1 \\ y(t_1) &= 0.345920 \end{aligned}$$

$$\begin{aligned} w_2 &= 0.271828 + 0.1\left[\frac{2}{1.1}0.271828 + (1.1)^2 e^{(1.1)}\right] \\ &= 0.684756 \\ t_2 &= 1.2 \\ y(t_2) &= 0.866643 \end{aligned}$$

$$\begin{aligned} w_3 &= 0.684756 + 0.1\left[\frac{2}{1.2}0.684756 + (1.2)^2 e^{(1.2)}\right] \\ &= 1.27698 \\ t_3 &= 1.3 \\ y(t_3) &= 1.60722 \end{aligned}$$

$$\begin{aligned} w_4 &= 1.27698 + 0.1\left[\frac{2}{1.3}1.27698 + (1.3)^2 e^{(1.3)}\right] \\ &= 2.09355 \\ t_4 &= 1.4 \\ y(t_4) &= 2.62036 \end{aligned}$$

$$\begin{aligned} w_5 &= 2.09355 + 0.1\left[\frac{2}{1.4}2.09355 + (1.4)^2 e^{(1.4)}\right] \\ &= 3.18745 \\ t_5 &= 1.5 \\ y(t_5) &= 3.96767 \end{aligned}$$

$$\begin{aligned} w_6 &= 3.18745 + 0.1\left[\frac{2}{1.5}3.18745 + (1.5)^2 e^{(1.5)}\right] \\ &= 4.62082 \\ t_6 &= 1.6 \\ y(t_6) &= 5.72096 \end{aligned}$$

$$\begin{aligned} w_7 &= 4.62082 + 0.1\left[\frac{2}{1.6}4.62082 + (1.6)^2 e^{(1.6)}\right] \\ &= 6.46640 \\ t_7 &= 1.7 \\ y(t_7) &= 7.96387 \end{aligned}$$

$$
\begin{aligned}
w_8 &= 6.46640 + 0.1\left[\frac{2}{1.7}6.46640 + (1.7)^2 e^{(1.7)}\right] \\
&= 8.80912 \\
t_8 &= 1.8 \\
y(t_8) &= 10.7936
\end{aligned}
$$

$$
\begin{aligned}
w_9 &= 8.80912 + 0.1\left[\frac{2}{1.8}8.80912 + (1.8)^2 e^{(1.8)}\right] \\
&= 11.7480 \\
t_9 &= 1.9 \\
y(t_9) &= 14.3231
\end{aligned}
$$

$$
\begin{aligned}
w_{10} &= 11.7480 + 0.1\left[\frac{2}{1.9}11.7480 + (1.9)^2 e^{(1.9)}\right] \\
&= 15.3982 \\
t_{10} &= 2.0 \\
y(t_1 0) &= 18.6831
\end{aligned}
$$

□

7. *Given the initial-value problem*

$$y' = -y + t + 1, \quad 0 \le t \le 5, \quad y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$:

a. *Approximate* $y(5)$ *using Euler's method with* $h = 0.2$, $h = 0.1$, *and* $h = 0.05$.

b. *Determine the optimal value of* h *to use in computing* $y(5)$, *assuming* $\delta = 10^{-6}$ *and that Eq. (5.14) is valid.*

Solution. a. The exact value of $y(5)$ is 5.00674. Using algorithm 5.1, with $h = 0.2$, we get $y(5) \approx 5.00378$, with $h = 0.1$, we get $y(5) \approx 5.00515$, and with $h = 0.005$, we get $y(5) \approx 5.00592$.

b. On $0 \le t \le 5$, we have $y''(t) = \frac{d^2}{dt^2}[e^{-t} + t] = \frac{d}{dt}[-e^{-t} + 1] = e^{-t} \le 1$. Thus the optimum value is

$$h = \sqrt{\frac{2\delta}{M}} = \sqrt{\frac{2 \times 10^{-6}}{1}} = 0.0014142.$$

□

Burden & Faires §5.3: 3a, 5ac, 7

Burden & Faires §5.3. Higher-Order Taylor Methods

3. *Use Taylor's method of order two and four to approximate the solution for the following initial-value problem.*

a.

$$y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1, \quad \textit{with } h = 0.1$$

Solution. a. For the second order Taylor's method we have the difference equation:

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + h\left(\frac{w_i}{t_i} - \frac{w_i^2}{t_i^2}\right) + \frac{h^2}{2}\left(-\frac{w_i^2}{t_i^3} + 2\frac{w_i^3}{t_i^4}\right).
\end{aligned}
$$

Thus, $w_0 = 1.0000000$, $w_1 = 1.0050000$, and $w_2 = 1.0160294$. For the fourth order Taylor's method we have the difference equation:

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\left(\frac{w_i}{t_i} - \frac{w_i^2}{t_i^2}\right) + \frac{h^2}{2}\left(-\frac{w_i^2}{t_i^3} + 2\frac{w_i^3}{t_i^4}\right) + \frac{h^3}{6}\left(\frac{w_i^2}{t_i^4} - 6\frac{w_i^4}{t_i^6}\right) \\ &\quad + \frac{h^4}{24}\left(-2\frac{w_i^2}{t_i^5} - 2\frac{w_i^3}{t_i^6} + 12\frac{w_i^4}{t_i^7} + 24\frac{w_i^5}{t_i^8}\right). \end{aligned}$$

Thus, $w_0 = 1.0000000$, $w_1 = 1.0043000$, and $w_2 = 1.0149771$. The actual solution is approximately $y(t_0) = 1.0000000$, $y(t_1) = 1.0042790$, and $y(t_2) = 1.0149522$. □

5. *Given the initial-value problem*

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with the exact solution $y(t) = t^2(e^t - e)$*:*

a. *Use Taylor's method of order two with* $h = 0.1$ *to approximate the solution, and compare it with the actual values of* y.

b. *Use the answers generated in part (a) and linear interpolation to approximate* y *at the following values, and compare them to the actual values of* y.

i. *y(1.04)*

ii. *y(1.55)*

iii. *y(1.97)*

Solution. a. Taylor's method of order two is given by the difference equation:

$$\begin{aligned} w_0 &= 0.0 \\ w_{i+1} &= w_i + h\left(\frac{2}{t}y + t^2 e^t\right) + \frac{h^2}{2}\left(\frac{2}{t^2}y + 4te^t + t^2 e^t\right). \end{aligned}$$

Thus, the solution is approximated by w_i with exact value $y(t_i)$:

i	t_i	w_i	$y(t_i)$
0	1.00	0.0000000	0.0000000
1	1.10	0.3397852	0.3459199
2	1.20	0.8521434	0.8666425
3	1.30	1.581770	1.607215
4	1.40	2.580997	2.620360
5	1.50	3.910985	3.967666
6	1.60	5.643081	5.720962
7	1.70	7.860382	7.963874
8	1.80	10.65951	10.79362
9	1.90	14.15268	14.32308
10	2.00	18.46999	18.68310

b. Using linear interpolation, we have

$$\begin{aligned} y(1.04) &\approx 0.6y(1.00) + 0.4y(1.10) \\ &\approx 0.6(0.0000000) + 0.4(0.3397852) \\ &= 0.1359141 \\ y(1.55) &\approx 0.5y(1.50) + 0.5y(1.60) \\ &\approx 0.5(3.910985) + 0.5(5.643081) \\ &= 4.777033 \\ y(1.97) &\approx 0.3y(1.90) + 0.7y(2.00) \\ &\approx 0.3(14.15268) + 0.7(18.46999) \\ &= 17.17480. \end{aligned}$$

The actual values are given by

$$\begin{aligned} y(1.04) &= 0.1199875 \\ y(1.55) &= 4.788635 \\ y(1.97) &= 17.27930. \end{aligned}$$

□

7. *A projectile of mass $m = 0.11$kg shot vertically upward with initial velocity $v(0) = 8$m/s is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8\text{m/s}^2$ and $k = 0.002$kg/m. The differential equation for the velocity v is given by*

$$mv' = -mg - kv|v|.$$

   a. *Find the velocity after 0.1, 0.2, ..., 1.0s.*

   b. *To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.*

Solution. a. Using Taylor's method of order two, we get the following.

i	t_i	w_i
0	0.0	8.0000000
1	0.1	6.9108826
2	0.2	5.8501200
3	0.3	4.8129058
4	0.4	3.7948239
5	0.5	2.7917670
6	0.6	1.7998655
7	0.7	0.8154250
8	0.8	−0.1651307
9	0.9	−1.1449490
10	1.0	−2.1216520

b. Thus the projectile reaches its maximum height and begins falling at 0.8s to the nearest tenth of a second. □