Homework set #7 solutions, Math 128A

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Sec 4.4: 1a, 2a, 3a, 7abc, 17

1a. Compute by hand or use a program. Matlab code for the Composite Trapezoidal rule:

```
function integral = cmptrap(a,b,n,f)
h = (b-a)/n;
x = [a+h:h:b-h];
integral = h/2*(2*sum(feval(f,x))+feval(f,a)+feval(f,b));
```

Run with

```
cmptrap(1,2,4,'f')
```

where 'f' is the name of the function definition file

```
function y = f(t)
y = t.*log(t);  % pay attention to the dot
```

The result is ≈ 0.6399004.

2a. Matlab code for the Composite Simpson's rule

```
function integral = cmpsimp(a,b,n,f)
h = (b-a)/n;
xi0 = feval('f',a)+feval('f',b);
xi1 = 0;
xi2 = 0;
for i = 1:n-1
 x = a+i*h;
 if mod(i,2) == 0
 xi2 = xi2+feval('f',x);
 else
 xi1 = xi1+feval('f',x);
 end
end
xi = h*(xi0+2*xi2+4*xi1)/3;
xi
```

Result: 0.6363098.

3a. Approximation: $2\left(\frac{1}{6}\right)\left(\frac{7}{6}\ln\frac{7}{6}+\frac{9}{6}\ln\frac{9}{6}+\frac{11}{6}\ln\frac{11}{6}\right)=0.633096$.

7.

$$\begin{aligned} f(x) &= e^{2x}\sin 3x \\ f''(x) &= -5\,e^{2x}\sin(3x)+12\,e^{2x}\cos(3x) \\ f^{(4)}(x) &= -119\,e^{2x}\sin(3x)-120\;e^{2x}\cos(3x) \end{aligned}$$

For $\xi \in (0,2)$ we look for an upper bound for

$$\left|f''(\xi)\right| = \left|e^{2\xi}(-5\sin 3\xi + 12\cos 3\xi)\right| \leq e^4(5+12).$$

Or for a better upper bound

$$\left|f''(\xi)\right| = \left|e^{2\xi}(-5\sin 3\xi + 12\cos 3\xi)\right| \leq e^4\sqrt{5^2+12^2} = 13e^4.$$

We use the latter one(it's OK if you use the first one; then you will get a larger n/smaller h). Similarly

$$\begin{aligned}\left|f^{(4)}(\xi)\right| &= \left|-119\,e^{2x}\sin(3\xi x) - 120\,e^{2x}\cos(3\,\xi)\right| \\ &\leq e^4\sqrt{119^2+120^2} \leq 120\sqrt{2}e^4.\end{aligned}$$

Thus by the error for the Composite Trapezoidal rule

$$\begin{aligned}\left|-\frac{b-a}{12}h^2 f''(\xi)\right| \leq \frac{2}{12}h^2 13e^4 &< 10^{-4} \\ \implies h &< 9.1942\times 10^{-4} \\ \implies n &> 2175.3 \ (\textit{choose } 2176).\end{aligned}$$

Note: *it's quite possible that you got a different answer because you used a different upper bound. That's OK.*

Similarly for the Composite Simpson's rule

$$\begin{aligned}\left|-\frac{b-a}{180}h^4 f^{(4)}(\xi)\right| \leq \frac{2}{180}h^4 120\sqrt{2}e^4 &< 10^{-4} \\ \implies h &< 0.033557 \\ \implies n &> 57.6 \ (\textit{choose } 58).\end{aligned}$$

For the Composite Midpoint rule

$$\begin{aligned}\left|\frac{b-a}{6}h^2 f''(\xi)\right| \leq \frac{2}{6}h^2 13e^4 &< 10^{-4} \\ \implies h &< 6.5013\times 10^{-4} \\ \implies n &> 3076.3 \ (\textit{choose } 3077).\end{aligned}$$

17. Use the parametric equations for the ellipse

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}, \ t \in (-\pi, \pi].$$

Then by the arclength formula

$$\begin{aligned}L &= \int_{-\pi}^{\pi}\sqrt{[x'(t)]^2+[y'(t)]^2}dt = \int_{-\pi}^{\pi}\sqrt{9\sin^2 t + 4\cos^2 t}dt \\ &= \int_{-\pi}^{\pi}\sqrt{4+5\sin^2 t}dt \ (\textit{by } \sin^2 t + \cos^2 t = 1).\end{aligned}$$

Next we can use certain integration rule, say, the Composite Trapezoidal rule to approximate the integral(Theorem 4.5)

$$\int_a^b f(t)dt = \frac{h}{2}[f(a) + 2\sum_{j=1}^{n-1} f(t_j) + f(b)] - \frac{b-a}{12}h^2 f''(\xi), \tag{1}$$

where $[a,b] = [-\pi, \pi]$, $h = (b-a)/n$, $f(t) = \sqrt{4 + 5\sin^2 t}$, $\xi \in (a,b)$. To decide h(also n) we look for an upper bound for the absolute value of the error. Now

$$f'(t) = \frac{5\sin t \cos t}{\sqrt{4 + 5\sin^2 t}} = \frac{5}{2}\frac{\sin 2t}{\sqrt{4 + 5\frac{1-\cos 2t}{2}}} = 5\frac{\sin 2t}{\sqrt{26 - 10\cos 2t}} \text{ (by trig identities)}$$

$$f''(t) = \frac{52\cos 2t - 10(1 + \cos^2 2t)}{(26 - 10\cos 2t)^{\frac{3}{2}}}$$

Thus

$$\left|f''(\xi)\right| = \left|\frac{52\cos 2t - 10(1 + \cos^2 2t)}{(26 - 10\cos 2t)^{\frac{3}{2}}}\right| \le \frac{52 + 10 \cdot 2}{(26 - 10)^{\frac{3}{2}}} = \frac{9}{8}$$

(max for the numerator, min for the denominator)

We want

$$\left|\frac{b-a}{12}h^2 f''(\xi)\right| = \left|\frac{2\pi}{12}\left(\frac{2\pi}{n}\right)^2 f''(\xi)\right| \le \frac{3\pi^3}{4n^2} < 10^{-6}$$

i.e.

$$n > \sqrt{\frac{3}{4}\pi^3 \times 10^6} = 4822.3.$$

So we can choose $n = 4823$. Use the Composite Trapezoidal rule (1). Try the matlab code in problem 1a. The result is 15.865439589, which is the approximation to the length of the ellipse.

You can also try the Composite Midpoint rule with code

```
function integral = cmpmid(a,b,n,f)
h = (b-a)/(n+2);
x = [a+h:2*h:b-h];
integral = 2*h*sum(feval(f,x));
```

The n will be slightly different.

Of course in maple you can use

```
f := sqrt(4+5*sin(t)^2);
evalf(int(f,t=-Pi..Pi));
```

to get 15.86543959.

It's possible that you get a larger upper bound and thus use a larger n.

And we can also use some iterative method, say, adaptive method, and use the tolerance 10^{-6}. This way we don't need to compute a specific n.

Sec 4.5: 1a, 3a, 5, 14

1a, 3a. Try by hand or use the Romberg code

```
 function R = romberg(f, a, b, max_k)
% function R = romberg(f, a, b, max_k)
% Computes the triangular extrapolation table for Romberg integration
% using the composite trapezoid rule, starting with h=b-a
% f:  function name (either a name in quotes, or an inline function)
% a, b:  lower and upper limits of integration
% max_k:  the number of extrapolation steps
% (= number of columns in R, plus one.)
% max_k=0 will do no extrapolation.
%
% Example:  R = romberg('f',1,1.5,8)
 h = b-a;
 R = zeros(max_k+1);
 R(1,1) = h/2*(feval('f',a)+feval('f',b));
 for k = 2:max_k+1
 R(k,1) = 0;
 for i = 1:2^(k-2)
 R(k,1) = R(k,1)+feval('f',a+(2*i-1)*h/2^(k-1));
 end
 R(k,1) = (R(k-1,1)+h/2^(k-2)*R(k,1))/2;
end
 for k=2:max_k+1
 for j=k:max_k+1
 R(j,k) = (4^(k-1)*R(j,k-1)-R(j-1,k-1))/(4^(k-1)-1);
 end
 end
```

Run in matlab

```
R = romberg('f',1,1.5,3)
```

we can get $R_{i,j},\ i,j = 1,2,3,4$

$i \setminus j$	1	2	3	4
1	0.22807412331084			
2	0.20120251138753	0.19224530741310		
3	0.19449447318109	0.19225846044561	0.19225933731444	
4	0.19281809429207	0.19225930132906	0.19225935738796	0.19225935770658

We then have $R_{3,3}$.

For problem 3a, to get the accuracy 10^{-6}, it's helpful to look at the details of the Richardson extrapolation and get familiar with the increasing of the error

orders. The error $O(h_k^{2k})$ is decreased in certain pattern(page 209). We want the error $\left|O(h_k^{2k})\right| < 10^{-6},\ h_k = \frac{1/2}{2^{k-1}}$. Thus approximately we want at least

$$\left(\frac{1}{2^k}\right)^{2k} < 10^{-6}.$$

When $k = 3, 4$ respectively, the left hand side $\approx 3.8 \times 10^{-6},\ 2.3 \times 10^{-10}$. Now as we need to consider the coefficient in the notation $O(h_k^{2k})$, we can choose for safety $k = 4$. $R_{4,4}$ is as above.

Finally run until $|R_{k-1,k-1} - R_{k,k}| < 10^{-6}$, or $k = 10$, we can get the approximation 0.1922593577.

5. We can use Romberg integration to get hight accuracy. The f values can be used in the formula (4.32) for Romberg integration approximation

$$R_{k,1} = \frac{1}{2}[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)].$$

Here the f values are used for the following 3 values

$$\begin{array}{ll} R_{1,1} = \frac{h_1}{2}[f(a) + f(b)], & h_1 = b - a \\ R_{2,1} = \frac{1}{2}[R_{1,1} + h_1 f(a + h_2)], & h_2 = \frac{b-a}{2} \\ R_{3,1} = \frac{1}{2}\{R_{1,1} + h_2[f(a + h_3) + f(a + 3h_3)]\}, & h_3 = \frac{b-a}{4} \end{array}$$

Thus If the given 5 f values are $f_1, \cdots, f_5$, then use them in

$$\begin{array}{ll} R_{1,1} = \frac{h_1}{2}[f_1 + f_5], & h_1 = b - a \\ R_{2,1} = \frac{1}{2}[R_{1,1} + h_1 f_3], & h_2 = \frac{b-a}{2} \\ R_{3,1} = \frac{1}{2}\{R_{2,1} + h_2[f_2 + f_4]\}, & h_3 = \frac{b-a}{4}. \end{array}$$

It's basically the Richardson extrapolation. This way $R_{3,3}$ can have the highest accuracy based on the given data. Now you can compute by hand, or by a code like

```
a = 1; b = 5;
h = b-a;
R = zeros(3);
f = [2.4142 2.6734 2.8974 3.0976 3.2804];
R(1,1) = h/2*(f(1)+f(5));
R(2,1) = (R(1,1)+h*f(3))/2;
R(3,1) = (R(2,1)+h/2*(f(2)+f(4)))/2;
for k=2:3
   for j=k:3
     R(j,k) = (4^(k-1)*R(j,k-1)-R(j-1,k-1))/(4^(k-1)-1);
  end
end
```

The corresponding $R_{i,j}(i,j=1,2,3)$ values are

$i \setminus j$	1	2	3
1	11.3892		
2	11.4894	11.5228	
3	11.5157	11.5244667	11.524577778

14. When we use Romberg integration, we want the error $\left|O(h_k^{2k})\right| < 10^{-7},\ h_k = \frac{1}{2^{k-1}}$. Thus approximately we want at least

$$\left(\frac{1}{2^{k-1}}\right)^{2k} < 10^{-7}.$$

When $k = 4, 5$ respectively, the left hand side $\approx 6 \times 10^{-8}$ and 9×10^{-13}. Now as we need to consider the coefficient in the notation $O(h_k^{2k})$, we can choose for safety $k = 5$. Then use the Romberg code in problem 1a

```
2/sqrt(pi)*romberg('f',0,1,5)
```

we can get the approximation $erf(1) \approx 0.84270079326867$(this has even higher accuracy than 10^{-7}). Of course in maple you can verify this:

```
evalf(erf(1.));
```

4.6: 1a, 3a, 5 (for sin(1/x)), 9

1a. Use a simple matlab code

```
function y = s(a,b)
h = (b-a)/2;
y = h/3*(f(a)+4*f(a+h)+f(b));
```

We have

$$\begin{aligned} S(a,b) &\approx 0.192245307 \\ S(a,(a+b)/2) &\approx 0.039372434 \\ S((a+b)/2,b) &\approx 0.152886026. \end{aligned}$$

The real value is

$$\begin{aligned} \int_1^{1.5} x^2 \ln x dx &= \frac{1}{3}x^3 \ln x|_1^{1.5} - \int_1^{1.5} \frac{1}{3}x^2\, dx \\ &= \frac{1.5^3}{3}\ln 1.5 - \frac{1}{9}(1.5^3 - 1) \\ &\approx 0.1922593577. \end{aligned}$$

3a. Note: the matlab builtin function "quad" used adaptive Simpson's quadrature method just as our text. The following is one version of adaptive Simpson's quadrature code. It will be used for all the problems in this section. *If you get slightly different answers from your own code, it's OK.*

```
function [y,errest,iflg,nofun] = adpsim(a,b,tol,fun)
% an implementation of adaptive Simpson quadrature
% user must supply the integrand,fun.
% a,b :limits of integration,tol:absolute error tolerance
% errest:  estimated error
% iflg:  mode of return, gives number of subintervals where
% maximum number (levmax=10) of bisections is required and value is
% accepted by default.  The larger iflg, the less confidence one
% should have in the computed value, y.
% nofun:  number of fun evaluations
%initialization
y=0;iflg=0;jflg=0;errest=0;levmax=10;
fsave=zeros(levmax,3);xsave=zeros(levmax,3);simpr=zeros(levmax);
%protect against unreasonable tol
tol2=tol+10*eps;
tol1=tol2*15/(b-a);
x=a:(b-a)/4:b;
for j=1:5
    f(j)=feval(fun,x(j));
end
nofun=5;
level=1;
%level=0 means entire interval is covered, hence finished
while level>0
% save right half subinterval info
for k=1:3
    fsave(level,k)=f(k+2);
    xsave(level,k)=x(k+2);
end
h=(x(5)-x(1))/4;
simpr(level)=(h/3)*(f(3)+4*f(4)+f(5));
if jflg<=0
    simp1=2*(h/3)*(f(1)+4*f(3)+f(5));
end
simpl=(h/3)*(f(1)+4*f(2)+f(3));
simp2=simpl+simpr(level);
diff=abs(simp1-simp2);
if diff<=tol1*4*h
%accept approx on current subinterval
    level=level-1;
    jflg=0;
%we use the conservative strategy of accepting the two-panel composite

%Simpson rule.  The method described on the web page uses y=y+simp1.
    y=y+simp2;
```

```
        errest=errest+diff/15;
        if level<=0
            fprintf('predicted error bound= %g.  \n',errest)
            fprintf('number of times integrand was evaluated:  %g.  \n',
nofun)
            fprintf('the computed value of the integral is :  %g.  \n',y)
            return
        end
        for j=1:3
            jj=2*j-1;
            f(jj)=fsave(level,j);
            x(jj)=xsave(level,j);
        end

    else
        level=level+1;
        simp1=simpl;
        if level <= levmax
            jflg=1;
            f(5)=f(3);f(3)=f(2);
            x(5)=x(3);x(3)=x(2);
        else%levmax is reached, accept value computed and continue
            iflg=iflg+1;
            level=level-1;
            jflg=0;
            y=y+simp2;
            errest=errest+diff/15;
        end
    end
        for k=1:2
            kk=2*k;
            x(kk)=.5*(x(kk+1)+x(kk-1));
            f(kk)=feval(fun,x(kk));
        end
    nofun=nofun+2;
    end
```

Define the function in f.m. Run in matlab with:

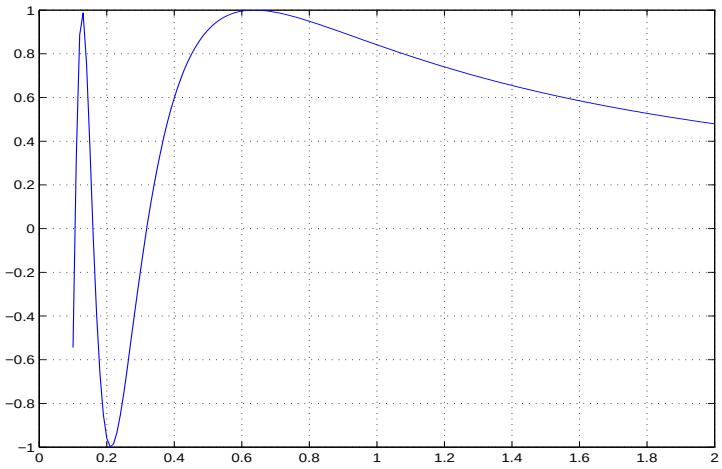
```
adpsim(1,3,1e-5,'f')
```

The result is ≈ 108.5552806.

5. Run the previous code for the function here.

```
adpsim(0.1,2,1e-3,'f')
```

The result is ≈ 1.1454993. It's possible that you may get slightly different answers. The graph is as follows.



9. Define the functions in c.m and s.m. Call the code in problem 1a.

```
fprintf(' t c(t) s(t)\n');
for t = 0.1:0.1:1
 fprintf('%16.12g %16.12f %16.12g\n', t, ...
       adpsim(0,t,1e-3,'c'), adpsim(0,t,1e-3,'s'));
end
```

Results(you may get slightly different ones):

t	$c(t)$	$s(t)$
0.1	0.099997526203	0.000523589386911
0.2	0.199920852789	0.00418758860931
0.3	0.299399445796	0.0141166481358
0.4	0.397474592626	0.0333568374407
0.5	0.492327199200	0.064720316156
0.6	0.581060998724	0.110498453372
0.7	0.659604949810	0.172021508022
0.8	0.722827448864	0.249078463858
0.9	0.764971727407	0.339270624009
1	0.780505925697	0.438216495327