Homework set #5 solutions, Math 128A

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Sec 3.4: 1, 7, 15, 17, 22*, 25

1. Let the free cubic spline

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3, & 0 \le x \le 1 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3, & 1 \le x \le 2 \end{cases}.$$

Then

$$\begin{aligned} &S_0(0) = f(0),\ S_0(1) = f(1),\ S_1(1) = f(1),\ S_1(2) = f(2), \\ &S_0'(1) = S_1'(1),\ S_0''(1) = S_1''(1), \\ &S_0''(0) = 0,\ S_1''(2) = 0. \end{aligned}$$

We then get a linear system of equations in the same order as the equations above

$$\begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & 1 & 1 & & & & \\ & & & & 1 & & & \\ & & & & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & & -1 & & \\ & & 2 & 6 & & & -2 & \\ & & 2 & & & & & \\ & & & & & & 2 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We get

$$a_0 = 0,\ b_0 = 1,\ c_0 = 0,\ d_0 = 0,\ a_1 = 1,\ b_1 = 1,\ c_1 = 0,\ d_1 = 0,$$

i.e.

$$S(x) \equiv x.$$

7. Use

$$S_0'(1) = S_1'(1),\ S_0''(1) = S_1''(1),\ S_1''(2) = 0$$

to get three equations with three unknowns. Solve them. $b = -1,\ c = -3,\ d = 1.$

15. Let $f(x) = a + bx + cx^2 + dx^3$. For any point x, f interpolates itself. It's easy to verify that conditions (a-e), and (ii) of (f) in definition 3.10 hold. Thus f is its own clamped cubic spline.

Now assume f is a natural cubic spline, then $f''(x) = 2c + 6dx = 0$. This can only hold at one single point $x = -\frac{c}{3d}$, instead of two. Thus (i) of (f) cannot be satisfied and f cannot be a natural cubic spline.

17. The linear interpolatng function through two points $(0, f(0)$ and $(0.05, f(0.05)$ is

$$S_0(x) = f(0)\frac{x - 0.05}{0 - 0.05} + f(0.05)\frac{x - 0}{0.05 - 0} = -20x + 1 + e^{0.1}20x,\ x \in [0, 0.05].$$

Similarly the linear interpolatng function through two points $(0.05, f(0.05)$ and $(0.1, f(0.1))$ is

$$S_1(x) = 20(e^{0.2} - e^{0.1})x + 2e^{0.1} - e^{0.2}, \ x \in (0.05, 0.1].$$

The piecewise linear approximation $F(x)$ to f is given by $S_0(x)$ and $S_1(x)$.

Thus

$$\int_0^{0.1} F(x)dx = 0.1107936.$$

The actual integral

$$\int_0^{0.1} f(x)dx = 0.1107014.$$

22. Contitions (i) and (ii) lead to five equations with six variables

$$\begin{aligned}
a_0 &= f(x_0) \\
a_1 &= f(x_1) \\
a_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 &= f(x_2) \\
a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 - a_1 &= 0 \ (\Longleftarrow S_0(x_1) = S_1(x_1)) \\
b_0 + 2c_0(x_1 - x_0) - b_1 &= 0 \ (\Longleftarrow S_0'(x_1) = S_1'(x_1)).
\end{aligned}$$

So we need an additional condition to make the solution unique. Considering the condition $S \in C^2[x_0, x_2]$, we get an extra condition

$$c_0 - c_1 = 0 \ (\Longleftarrow S_0''(x_1) = S_1''(x_1)).$$

Eliminate $a_0, a_1, c_1 (= c_0)$ and write the rest equations in matrix form

$$\begin{pmatrix} & (x_2 - x_1)^2 & x_2 - x_1 \\ x_1 - x_0 & (x_1 - x_0)^2 & \\ 1 & 2(x_1 - x_0) & -1 \end{pmatrix} \begin{pmatrix} b_0 \\ c_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} f(x_2) - f(x_1) \\ -f(x_0) + f(x_1) \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is $(x_2 - x_1)(x_1 - x_0)(x_2 - x_0) \neq 0$ because the three points are distinct. Thus the coefficient matrix is invertible. There is always a unique solution for the above linear system. And the problem has a meaningful solution then.

25. a. Program the clampled cubic spline. Or do it in matlab. Suppose the spline is

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \ x \in [x_i, x_{i+1}].$$

Run

```
x = [0 3 5 8 13]; y = [0 225 383 623 993]; cs = spline(x,[75 y 72])
```

This will give the output of the infomation about the cubic spline.

```
cs =
 form:  'pp'
 breaks:  [0 3 5 8 13]
 coefs:  [4x4 double]
 pieces:  4
```

```
order:  4
dim:  1
```

We can print out the coefficients of the pieciwise polynomials with

```
cs.coefs(:,4:-1:1)
```

The results

i	x_i	a_i	b_i	c_i	d_i
0	0	0	75.0000	-0.6593	0.2198
1	3	225.0000	76.9779	1.3186	-0.1538
2	5	383.0000	80.4071	0.3960	-0.1772
3	8	623.0000	77.9978	-1.1991	0.0799

And we can predict the position and the speed of the car at $t = 10$s respectively by

```
spline(x,[75 y 72],10)
```

which is 774.8384, and

```
cs.coefs(4,3)+2*cs.coefs(4,2)*(10-8)+3*cs.coefs(4,1)*(10-8)^2
```

which is 74.1603. Here we used the derivative $S'_3(10)$.

b,c. Compute the derivative of the spline with the derivatives of each S_i. For each S'_i, find the points where derivative $S''_i(x) = 0$. This can be done by Newton's method. Find the maximum value at those points and the endpoints. This gives the maximum speed $S'(x_m) = 80.7ft/s = 55.02mi/h > 55m/h$, where $x_m = 5.7448$.

Solve $S'_i(x) - 80.67 = 0,\ i = 0, 1, 2, 3$ in the corresponding intervals to get the smallest solution $x = 5.5$, which is the first time the car exceeds the speed $80.67ft/s = 55mi/h$.

Sec 3.5: 1a, 2a, 4

1a, 2a. $(x_0, y_0) = (0, 0),\ (x_1, y_1) = (5, 2),\ (x_0 + \alpha_0, y_0 + \beta_0) = (1, 1),\ (x_1 - \alpha_1, y_1 - \beta_1) = (6, 1)$. Thus $(\alpha_0, \beta_0) = (1, 1),\ (\alpha_1, \beta_1) = (-1, 1)$. Use formulas (3.22), (3.23) to get the cubic Hermite approximations

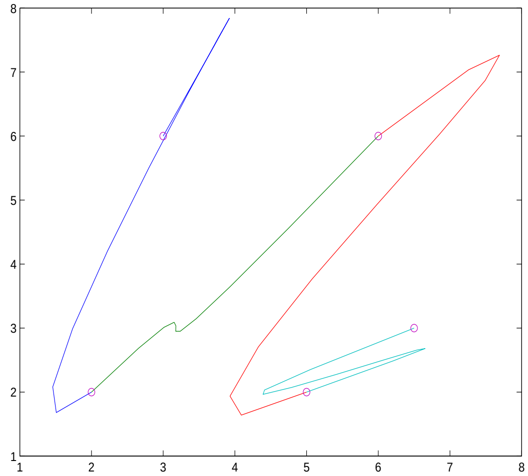
$$x(t) = -10t^3 + 14r^2 + t,\ y(t) = -2t^3 + 3t^2 + t.$$

Use formula (3.24), (3.25) to get the cubic Bezier polynomials

$$x(t) = -10t^3 + 12r^2 + 3t,\ y(t) = 2t^3 - 3t^2 + 3t.$$

4. Note here $(\alpha_i, \beta_i), (\alpha'_i, \beta'_i)$ correspond to $(\alpha_0, \beta_0), (\alpha_1, \beta_1)$ in the formulas. For each pair of points $(x_i, y_i), (x_{i+1}, y_{i+1})$, the left guide point is $(x_i, y_i) + (\alpha_i, \beta_i)$, and the right guide point is $(x_{i+1}, y_{i+1}) + (\alpha'_i, \beta'_i)$. Be careful with the correspondence of the values. Now you can write a matlab program:

```
x=[3, 2, 6, 5, 6.5];    y=[6, 2, 6, 2, 3];
al=[3.3, 2.8, 5.8, 5.5, 0];    bl=[6.5, 3.0, 5.0, 2.2, 0];
ar=[0, 2.5, 5.0, 4.5, 6.4];    br=[0, 2.5, 5.8, 2.5, 2.8];
xgpl=x+al;   ygpl=y+bl;   xgpr=x-ar;   ygpr=y-br;
N=length(x);
for nn=1:N-1
   a0(nn)=x(nn);
   b0(nn)=y(nn);
```




```
  a1(nn)=3*(xgpl(nn)-x(nn));
  b1(nn)=3*(ygpl(nn)-y(nn));
  a2(nn)=3*(x(nn)+xgpr(nn+1)-2*xgpl(nn));
  b2(nn)=3*(y(nn)+ygpr(nn+1)-2*ygpl(nn));
  a3(nn)=x(nn+1)-x(nn)+3*xgpl(nn)-3*xgpr(nn+1);
  b3(nn)=y(nn+1)-y(nn)+3*ygpl(nn)-3*ygpr(nn+1);
end
syms t
for i = 1:4
    ['x(i):'  a0(i)+a1(i)*t+a2(i)*t.^2+a3(i)*t.^3 ...
  'y(i):'  b0(i)+b1(i)*t+b2(i)*t.^2+b3(i)*t.^3]
end
```

This will output the Bezier polynomial. The polynomials are(the original output is in fractional form)

i	$x(i)$	$y(i)$
0	$3+9.9t-30.3t^2+19.4t^3$	$6+19.5t-58.5t^2+35t^3$
1	$2+8.4t-19.8t^2+15.4t^3$	$2+9t-23.4t^2+18.4t^3$
2	$6+17.4t-51.3t^2+32.9t^3$	$6+15t-49.5t^2+30.5t^3$
3	$5+16.5t-47.7t^2+32.7t^3$	$2+6.6t-18.6t^2+13t^3$

Use the following command to plot the curve(I'm keeping the format of output of the previous code for the polynomials). Note for each piece, the interval for t is always $[0,1]$. The curve is as above.

```
t = 0:0.1:1;
plot(3+99/10*t-303/10*t.^2+97/5*t.^3, 6+39/2*t-117/2*t.^2+35*t.^3, ...
2+42/5*t-99/5*t.^2+77/5*t.^3, 2+9*t-117/5*t.^2+92/5*t.^3, ...
6+87/5*t-513/10*t.^2+329/10*t.^3, 6+15*t-99/2*t.^2+61/2*t.^3, ...
5+33/2*t-477/10*t.^2+327/10*t.^3, 2+33/5*t-93/5*t.^2+13*t.^3, ...
x(1:5),y(1:5),'O')
```