Selberg, Ihara and Berkovich

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Abstract

We use the Selberg zeta function to study the limit behavior of resonances in a degenerating family of Kleinian Schottky groups. We prove that, after a suitable rescaling, the Selberg zeta functions converge to the Ihara zeta function of a limiting finite graph associated to the relevant non-Archimedean Schottky group acting on the Berkovich projective line.

Moreover, we show that these techniques can be used to get an exponential error term in a result of McMullen (recently extended by Dang and Mehmeti) about the asymptotics for the vanishing rate of the Hausdorff dimension of limit sets of certain degenerating Schottky groups generating symmetric three-funnel surfaces. Here, one key idea is to introduce an intermediate zeta function capturing *both* non-Archimedean and Archimedean information (while the traditional Selberg, resp. Ihara zeta functions concern only Archimedean, resp. non-Archimedean properties).

Contents

1 Introduction

A *Schottky group* of rank g *over* $\mathbb{C}$ (also called Kleinian Schottky group) is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ which is purely loxodromic and isomorphic to the free group of rank g. These groups were introduced by Schottky in 1877 and they were classically studied in connection with the famous uniformization theorem for Riemann surfaces: indeed, Koebe proved in 1910 that any closed Riemann surface is uniformized by some Kleinian Schottky group.

Partly motivated by the great success of the study of degenerating families of Riemann surfaces (ultimately leading to the celebrated Deligne–Mumford compactification of moduli spaces and its numerous applications), we shall investigate in this paper the analytical and geometrical properties of *degenerating* families of Schottky groups over $\mathbb{C}$.

More concretely, given a family Γ_z of Schottky groups over $\mathbb{C}$ generated by 2×2 matrices whose entries are meromorphic functions of $z \in \mathbb{D}^*$, it is sometimes possible to introduce a Schottky group Γ^{na} over the non-Archimedean field $\mathbb{C}(\!(t)\!)$ (of Laurent series on t) whose features can be used to extract asymptotic information on Γ_z as $|z| \to 0$. In fact, this philosophy of studying degenerating families of complex objects via non-Archimedean limiting objects is nowadays widely spread in the literature (cf. [Kiw15], [BJ17], [DF19], [Fav20] and [Luo22] for some recent works) and, in the context of Schottky groups, this idea was exploited by Dang and Mehmeti [DM24, Theorem 3] to prove that certain degenerating families Γ_z of Schottky groups over $\mathbb{C}$ possess *limit sets* $L(\Gamma_z) \subset \mathbb{P}^1_{\mathbb{C}}$ whose Hausdorff dimensions satisfy

$$\dim(L(\Gamma_z)) \sim \frac{d_0}{\log(1/|z|)} \tag{1.1}$$

as $|z| \to 0$, where d_0 is the Hausdorff dimension of the limit set of a non-Archimedean Schottky group Γ^{na} acting on the *Berkovich projective line* $\mathbb{P}^{1,an}_{\mathbb{C}(\!(t)\!)}$. (We also recommend to the reader the very recent paper [CG24] of Courtois and Guilloux containing a partial extension of the work of Dang and Mehmeti to *higher* dimensions via an *alternative method*, namely, the study of the infinite dimensional hyperbolic space.)

In the present paper, we pursue this kind of philosophy in order to describe the Selberg zeta functions of certain degenerating families of Schottky groups over $\mathbb{C}$ in terms of Ihara zeta functions of finite graphs naturally attached to non-Archimedean Schottky groups: see Subsection 1.2 below for precise statements. As a consequence, we are able to significantly refine the result of Dang and Mehmeti mentioned above. In particular, we improve upon the work of McMullen [McM98] on the Hausdorff dimension of limit sets of Schottky reflection groups associated to symmetric three-funnel hyperbolic surfaces and the theoretical and numerical works of Weich [Wei15], Borthwick [Bor16] and Pollicott–Vytnova [PV19] on the convergence of rescaled Selberg zeta functions of symmetric three-funnel hyperbolic surfaces: see Subsection 1.1 below for concrete statements.

1.1 Example: symmetric three-funnel surfaces

Let $X(\ell)$ be the symmetric three-funnel hyperbolic surface with parameter ℓ, that is, the non-compact hyperbolic surface of genus zero that has three funnels of widths $\ell > 0$. This gives an example of a family of degenerating surfaces when ℓ tends to infinity. Denote by $\Gamma(\ell)$ the Schottky group uniformizing $X(\ell)$, and let $L(\Gamma(\ell))$ be the limit set of $\Gamma(\ell)$ on the visual boundary $\partial\mathbb{H}^2$. McMullen [McM98] studied the asymptotics of the Hausdorff dimension of $L(\Gamma(\ell))$ as $\ell \to \infty$ and he obtained $\dim(L(\Gamma(\ell))) \approx \frac{\log 4}{\ell}$.

For this particular example, as it will be explained in Example 6.7 below, the main results of this paper imply an exponential error term in McMullen's asymptotic formula above:

Theorem 1.1. *For the case of symmetric three-funnel hyperbolic surfaces, we have for any $\epsilon > 0$,*

$$\dim(L(\Gamma(\ell))) = \frac{\log 4}{\ell} + O_\epsilon(e^{-(1-\epsilon)\ell/4}).$$

Moreover, for the Selberg zeta function (see Eq. (1.2) for definition), we have

$$|Z(\Gamma(\ell), s/\ell) - (1 - 6e^{-s} + 9e^{-2s} - 4e^{-3s})| = O_{K,\epsilon}(e^{-(1-\epsilon)\ell/4}),$$

for any s in a fixed compact set K and any $\epsilon > 0$.

The convergence of the Selberg zeta function without speed is due to Weich [Wei15]. Pollicott–Vytnova [PV19] proved a polynomial error term $O(1/\sqrt{\ell})$ in the convergence of the Selberg zeta functions and $O(1/\sqrt{\ell^3})$ in the convergence of Hausdorff dimensions (and, in fact, Example 6.8 below shows how to recover their results from our techniques). The exponential error term above also explains that the numerics of resonances in Borthwick [Bor16] behave well even for small ℓ such as $\ell = 10$: see Example 6.5 below for more details.

Our method of proof is different from previous works on symmetric three-funnel surfaces. Indeed, this family is just an example of a degenerating Schottky family and, as it is explained below, we actually study the limit behavior of all such families.

1.2 Main results: degenerating Schottky families

The study of the degenerations at the origin $z = 0$ of meromorphic families Γ_z of Kleinian Schottky (and, more generally, non-elementary) groups depending on a complex parameter $z \in \mathbb{D}^*$ in the unit punctured disc of $\mathbb{C}$ is a topic of intense investigations with a vast literature around it. For instance, such families are useful to build compactifications of representation variety, character variety or moduli spaces (see, e.g., [CS83] [Ota15] and [PT21]), their Lyapunov exponents for the associated random walks satisfy fascinating asymptotics (see, e.g., [ACS83] and [DF19]), and, as we already mentioned above, Hausdorff dimensions of their limit sets attracted the attention of many authors.

In the present paper, the central objects of discussion will be the Selberg zeta functions. Recall that the Selberg zeta function for a Schottky group Γ of $\mathrm{SL}_2(\mathbb{C})$ is defined as follows. Let $\mathcal{P}$ be the set of oriented primitive periodic geodesics on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$.

$$Z(\Gamma, s) = \prod_{\gamma\in\mathcal{P}} \prod_{k_1,k_2\geqslant 0} (1 - e^{-(s+k_1+k_2)\ell(\gamma)} e^{-i\theta_\gamma(k_1-k_2)}). \tag{1.2}$$

Here, by a slight abuse of notation, we also denote by γ an oriented closed geodesic representing a loxodromic element $\gamma \in \mathrm{SL}_2(\mathbb{C})$, so that $\ell(\gamma)$ is the length of the periodic geodesic γ in $\Gamma\backslash\mathbb{H}^3$, and $e^{i\theta_\gamma}$ is the holonomy given by $\gamma'(\gamma_+) = e^{-\ell(\gamma)-i\theta_\gamma}$ with γ_+ being the attracting fixed point of γ. Similarly, for $\Gamma < \mathrm{SL}_2(\mathbb{R})$, the classical Selberg zeta function is defined as

$$Z(\Gamma, s) = \prod_{\gamma\in\mathcal{P}} \prod_{k\geqslant 0} (1 - e^{-(s+k)\ell(\gamma)}).$$

Note that the definitions for $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ do not coincide, and, for this reason, we will deal with the two cases separately.

The function $Z(\Gamma, s)$ is convergent and analytic in the region $\Re s > \delta_\Gamma$, where δ_Γ is the so-called critical exponent. It admits an analytic continuation to the whole complex plane $\mathbb{C}$, and hence we obtain an entire function $Z(\Gamma, s)$. Due to [PP01], we know that the poles of the resolvent of the Laplacian on $\Gamma\backslash\mathbb{H}^3$ (also called resonances) correspond to a subset of the zeros of the Selberg zeta function $Z(\Gamma, s)$.

For a degenerating family of Schottky groups, we define the *intermediate zeta function*. We will prove in Proposition 4.2 and Corollary 5.8 that it is well-defined and has a holomorphic extension to $s \in \mathbb{C}$.

Definition 1.2 (Intermediate zeta function)**.** Let $\gamma \in \mathrm{SL}_2(\mathbb{C}(\!(t)\!))$ and $M \in \mathbb{Z}_{\geqslant 0}$, we define the approximate length function

$$\ell_M(\gamma, z) = \ell^{na}(\gamma) + \Re \sum_{j=0}^{M} a_j(\gamma) z^j / \log(1/|z|)$$

where $a_j(\gamma)$'s are the coefficients in the expansion

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + \Re \sum_{j=0}^{\infty} a_j(\gamma) z^j / \log(1/|z|).$$

The intermediate M-zeta function is defined by

$$Z_M(\Gamma, z, s) = \prod_{[\gamma]\in\mathcal{P}} (1 - e^{-s\ell_M(\gamma,z)})$$

for $\Re s$ sufficiently large. [1]

Here is our main result:

Theorem 1.3. *Let Γ_z be a degenerating Schottky family satisfying condition (★) in Subsection 2.4, and denote by $\Gamma < \mathrm{SL}_2(\mathbb{C}((t)))$ be the corresponding non-Archimedean Schottky group. Then for any $s \in \mathbb{C}$, as $|z| \to 0$, we have*

$$Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s),$$

where $Z_I(\Gamma, s)$ is the Ihara zeta function associated to Γ (defined in Proposition 2.19) and the convergence on any compact set is uniform.

Moreover, for any compact set K and $\epsilon > 0$, we have for all $0 < |z| < 1/e$ and for any $s \in K$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_0(\Gamma, z, s)| \lesssim_{K,\epsilon} |z|^{1-\epsilon}.$$

Furthermore, for any $C, \epsilon > 0$ and $M \in \mathbb{Z}_{\geqslant 0}$, we have for $s \in [-C, C] + i[-C|z|^{-M}, C|z|^{-M}]$ and $0 < |z| < 1/e$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon}.$$

As a direct corollary, we obtain

Corollary 1.4. *Let Γ_z be a degenerating Schottky family satisfying the condition (★) in Subsection 2.4. Let $P(|z|)$ be the first zero of the intermediate zeta function $Z_0(\Gamma, z, s)$ of the variable s. We have for any $\epsilon > 0$ and $|z| < 1/e$*

$$\dim(L(\Gamma_z)) = P(|z|) + O_\epsilon(|z|^{1-\epsilon}).$$

For special examples, like symmetric three-funnel surfaces and Example 6.4, the main term has the form $P(|z|) = d_0/\log(1/|z|)$. For general examples, the main term $P(|z|)$ is a zero of a polynomial of exponential functions of the variable s like $\sum a_j e^{-s(b_j + c_j/\log(1/|z|))}$. It would be interesting to study the Taylor expansion of $P(|z|)$ on $1/\log(1/|z|)$.

Remark 1.5. Due to the definition of Z_0, we know $P(|z|)$ and the coefficients of the expansion only depend on the coefficients of $\mathrm{tr}(\gamma)$ with $\gamma \in \Gamma$. For example, if we write $Z_0(\Gamma, z, s) = Z_0(s, u)$ with $u = 1/\log(1/|z|)$, then by the implicit function theorem, we obtain

$$P(|z|) = \frac{d_0}{\log(1/|z|)} - \frac{\partial_2 Z_0}{\partial_1 Z_0}(d_0, 0)\frac{1}{(\log(1/|z|))^2} + O\left(\frac{1}{(\log(1/|z|))^3}\right).$$

Can we find other interpretations of the term $\frac{\partial_2 Z_0}{\partial_1 Z_0}(d_0, 0)$, such as the variations of Hausdoff dimension [McM08] or the expansion of Lyapunov exponent [ACS83]?

We also obtain a corollary on the zeros of zeta functions and, consequently, on resonances of the Laplacian.

Corollary 1.6. *The zeros of $Z_I(\Gamma, s)$ have the structure $\mathrm{Res}_I(\Gamma) = \{\mu_j\} + 2\pi i\mathbb{Z}$, where $\{\mu_j\}$ is a finite set. Outside $s = 0$, the resonances of Γ_z converge to $\mathrm{Res}_I(\Gamma)$ (counted with multiplicity) after rescaling, i.e.*

$$\log(1/|z|)\,\mathrm{Res}(\Gamma_z) \setminus \{0\} \to \mathrm{Res}_I(\Gamma) \setminus \{0\}, \quad |z| \to 0 \tag{1.3}$$

on compact sets. In particular, the spectral gap of Γ_z converges to 0 after rescaling by $\log(1/|z|)$.

Proof. It follows from Theorem 1.3 that the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ converges to the zeros of $Z_I(\Gamma, s)$ in any compact set. See Corollary 4.5 for a more detailed version. The structure of $\mathrm{Res}_I(\Gamma)$ follows from the fact that $Z_I(\Gamma, s)$ is a polynomial of e^{-s} (see Proposition 5.1). The convergence Eq. (1.3) follows from the fact that the resonances of the Laplacian coincides with the zeros of $Z(\Gamma_z, s)$ outside $s \in \mathbb{Z}_{\leqslant 0}$ (see [Bor16, Chapter 10] for the two dimensional case and [PP01] for the general case). □

[1] See Example 6.5 for one explicit but nontrivial intermediate zeta function for hyperbolic funneled torus.

Remark 1.7. 1. By Proposition 2.19, the Ihara zeta function $Z_I(\Gamma, s)$ associated to Γ is the Ihara zeta function of a finite graph which comes from some Berkovich space quotient by Γ.

2. It is classic that the Hausdorff dimension of the limit set equals the first zero of the Selberg zeta function. For the non-Archimedean case, see Section 7.1 for the equality between Hausdorff dimension and the first zero of the Ihara zeta function. We recover the convergence theorem of the Hausdorff dimension of the limit set of [DM24] Eq. (1.1) from Corollary 1.6.

3. The work [OW16] proves that the family of congruence Schottky surfaces has a uniform spectral gap (in both low and high frequencies). The works [MN20], [CMN24] study the optimal size of the low frequency spectral gap under random covers. Our work shows that the gap becomes small under degenerate situations. It is interesting to ask if the gap also gets smaller in high frequency, i.e. when $\Im s$ is big. The conjecture of Jakobson–Naud [JN12] suggests that this does not happen.

4. When the corresponding Mumford curve Σ_X of Γ is a q-regular graph with each edge length 1 (see Section 2.5 for definition), the Ihara zeta function is given by $Z_I(\Gamma, s) = (1-e^{-2s})^{r-1}\det(I - A_X e^{-s} + (q-1)e^{-2s})$ where r is the number of generators of Γ and A_X is the adjacency matrix of Σ_X (see for example [HST06]). Let $\lambda_n \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda_1 = q$ be the eigenvalues of A_X. Therefore the set of resonances $\mathrm{Res}_I(\Gamma)$ is determined by μ_j's, which satisfy $e^{\mu_j} = (\lambda_i \pm \sqrt{\lambda_i^2 - 4(q-1)})/2(q-1)$ or $e^{\mu_j} = \pm 1$. This is similar to the relation between $\mathrm{Res}(\Gamma_z)$ and eigenvalues/resonances of the Laplacian operator on the quotient manifold.

Moreover, if the graph Σ_X is a Ramanujan graph, that is $-2\sqrt{q-1} \leqslant \lambda_n \leqslant \cdots \leqslant \lambda_2 \leqslant 2\sqrt{q-1}$, then all μ_j's have real part $\log(q-1)/2$ except one μ_j equal to $\log(q-1)$ and some μ_j equal to 0 or πi.

5. The fractal Weyl law conjecture [Zwo17, Conjecture 5] states that the number of resonances in a strip grows according to the dimension of the limit set at $T \to \infty$:

$$\#\mathrm{Res}(\Gamma) \cap \{\Re s > -C,\, |\Im s| \leqslant T\} \sim T^{1+\delta}, \quad \delta = \dim(L(\Gamma)).$$

Moreover, it is also conjectured in [Zwo17, Conjecture 7] that the resonances concentrate near the axis $\Re s = \delta/2$. The upper bound is known by [GLZ04], but the sharp lower bound is not known in any nontrivial example of Schottky groups. Our work provides a perspective on this conjecture via the intermediate zeta functions, which we conjecture to have a similar growth pattern as the fractal Weyl law, see the discussion after Example 6.8.

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1.3 Notations

Throughout the paper, we adopt the following conventions.

- We write $A \lesssim B$ if there exists a positive constant $C > 0$ (that may differ from expression to expression) such that $A \leqslant CB$. We specify the dependencies of C by a subscript, e.g., $A \lesssim_\Gamma B$. We use $A \ll B$ to mean that A is much smaller than B.
- $D(a, r) := \{w \in \mathbb{C} : |w - a| < r\}$ is a disc of radius r centered at $a \in \mathbb{C}$.
- $\mathbb{D} = D(0, 1)$ is the unit disc and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Similarly, $\mathbb{D}_r = D(0, r)$ and $\mathbb{D}_r^* = \mathbb{D}_r \setminus \{0\}$.
- The variable t usually means the formal variable in the field of Laurent series $\mathbb{C}((t))$. This field is equipped with its usual t-adic norm $|f|_{na} = \exp(-\mathrm{ord}_{t=0}(f)) = e^{-n}$ for $f = t^n \sum_{m=0} f_m t^m \in \mathbb{C}((t))$ with $f_0 \neq 0$. The variable z usually means a complex number that parametrizes a degenerating family of Schottky groups over $\mathbb{C}$ or $\mathbb{R}$.

- The Banach ring $A_r \subset \mathbb{C}(\!(t)\!)$ is defined by $f = \sum a_n t^n \in \mathbb{C}(\!(t)\!)$ such that the hybrid norm
$$\|f\|_{A_r} = \sum \max\{|a_n|_\infty, |a_n|_0\} r^n$$
is finite ($|\cdot|_0$ is the trivial norm). We will usually take $r = 1/e$ and define $\|\cdot\|_{\mathrm{hyb}} := \|\cdot\|_{A_{1/e}}$.

- Let $M \in \mathbb{Z}_{\geqslant 0}$ be a nonnegative integer and $a = t^m \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}(\!(t)\!)$ with $a_0 \neq 0$. We define the leading M terms
$$\mathrm{lt}_M(a) := t^m \sum_{n=0}^{M} a_n t^n.$$

- We use Fraktur letters $\mathfrak{a}, \mathfrak{b}$ to mean an element of the (fixed) set of generators on Γ. We use bold letters $\mathbf{a}, \mathbf{b}$ to mean words of the generators.

- For an element $\gamma \in \mathrm{SL}(2, \mathbb{C}(\!(t)\!))$, we use γ_z to denote an element in $\mathrm{SL}(2, \mathbb{C})$ by evaluating at $t = z$ if all the coefficients of γ converge at z.

- $\ell(\gamma) \in \mathbb{R}_{\geqslant 0}$ means the length of a closed geodesic associated to $\gamma \in \Gamma$. $\ell^{na}(\gamma) \in \mathbb{Z}_{\geqslant 0}$ is the non-Archimedean length defined in Definition 2.16. $l(\gamma) \in \mathbb{Z}_{\geqslant 0}$ is the word length of γ for a fixed set of generators $\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}$ with $\mathfrak{a}_{i+g} = \mathfrak{a}_i^{-1}$, i.e. $l(\gamma) = n$ whenever $\gamma = \mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}$ with $\mathfrak{a}_{i_{j+1}} \neq \mathfrak{a}_{i_j}^{-1}$.

2 Preliminaries of Schottky groups

We recall some preliminaries on the Schottky groups.

2.1 Schottky groups of $\mathbf{PGL}_2(\mathbb{C})$

An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ is called *loxodromic* if it induces a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ on $\mathbb{C}$ which is conjugated to $z \mapsto \lambda z$ for a complex number λ with $0 < |\lambda| < 1$.

Definition 2.1. We say that a free subgroup $\Gamma = \langle M_1, \ldots, M_g \rangle \subset \mathrm{PGL}_2(\mathbb{C})$ with $g \geqslant 2$ generators is a *Schottky group of rank g* if

- every $\gamma \in \Gamma \setminus \{id\}$ is loxodromic;
- Γ admits a non-empty domain of discontinuity, i.e., it acts freely and properly on a non-empty connected invariant open subset of $\mathbb{P}^1(\mathbb{C})$.

Given a Jordan curve C in $\mathbb{P}^1(\mathbb{C})$ and a reference point $o \notin C$, we say that the exterior (resp. interior) of C is the connected component of $\mathbb{P}^1(\mathbb{C})\backslash C$ containing (resp. not containing) o. After Maskit [Mas67], a Schottky group has the form $\langle T_1, \ldots, T_g \rangle$, where T_j's induce Möbius transformations with the following property: there are a reference point o and Jordan curves $C_1, C_1', \ldots, C_g, C_g'$ with disjoint interiors such that each T_j maps the exterior of C_j to the interior of C_j'. If all the Jordan curves C_j, C_j' can be chosen to be circles, then the corresponding Schottky group is called a *classical Schottky group*.

2.2 Kleinian Schottky spaces

A loxodromic element $\gamma \in \mathrm{PGL}_2(\mathbb{C})$ is determined by three parameters: its attracting fixed point $\alpha \in \mathbb{P}^1(\mathbb{C})$, its repelling fixed point $\beta \in \mathbb{P}^1(\mathbb{C})$, and its contraction rate $\lambda \in \mathbb{D}^*$. In the sequel, we shall denote by $M(\alpha, \beta, \lambda)$ the loxodromic matrix parametrized by α, β and λ.

The *Schottky space $S_g(\mathbb{C})$ of rank $g \geqslant 2$* is the space of g elements $M_1, \ldots, M_g$ of $\mathrm{PGL}_2(\mathbb{C})$ that generate a Schottky group up to the natural equivalence given by simultaneous conjugation by Möbius transformations (cf. Bers [Ber75]). These elements are parametrized by $3g$ quantities $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \lambda_1, \ldots, \lambda_g$. Since the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is 3-point transitive, we can set $\alpha_1 = 0$, $\beta_1 = 2$, $\alpha_2 = 1$. In this way, we can regard $S_g(\mathbb{C})$ as a subset of $\mathbb{C}^{3g-3}$: each $p \in S_g(\mathbb{C})$ is parametrized by the so-called *Koebe coordinates*

$$(\alpha_3, \ldots, \alpha_g, \beta_2, \ldots, \beta_g, \lambda_1, \ldots, \lambda_g) = (\underline{\alpha}, \underline{\beta}, \underline{\lambda}) \in (\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, 2\})^{2g-3} \times (\mathbb{D}^*)^g.$$

As it was shown by Hejhal [Hej75] (see also [Ber75]), $S_g(\mathbb{C})$ is a connected complex manifold of dimension $3g - 3$.

2.3 Limit sets of Kleinian Schottky groups

Given a point $p \in S_g(\mathbb{C})$, let $L(\Gamma_p)$ be the limit set of the Schottky group $\Gamma = \Gamma_p$ associated to p. Sullivan [Sul79] showed that the Hausdorff dimension $\dim(L(\Gamma_p))$ coincides with the critical exponent δ_{Γ_p} of Γ which is given by

$$\delta_{\Gamma_p} = \inf\{s > 0 : \mathcal{P}(s) < \infty\}.$$

Here $\mathcal{P}(s) = \sum_{\gamma \in \Gamma_p} |\gamma'(o)|_\infty^s$ is the usual Poincaré series, o is a point outside $L(\Gamma_p)$ and $|.|_\infty$ is the norm with respect to the Euclidean metric on $\mathbb{C}$. Due to the work of Patterson [Pat76, Pat88, Pat89], we know that the critical exponent is also equal to the first zero of the Selberg zeta function.

As established by Anderson and Rocha [AR97], $\dim(L(\Gamma_p))$ is a real analytical function of $p \in S_g(\mathbb{C})$.

2.4 Degenerations of Kleinian Schottky groups

Similarly to the case of moduli space of complex algebraic curves, one can try to approach the boundary of $S_g(\mathbb{C})$ by studying degenerations of one-parameter families of Schottky groups. More concretely, we always consider a meromorphic function $p : \mathbb{D} \to \mathbb{C}^{2g-3} \times \mathbb{D}^g$, $z \mapsto p(z) = (\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))$ such that $p(z) \in S_g(\mathbb{C})$ for all $z \in \mathbb{D}^*$. From now on, when talking about Schottky family, we mean a family given by this meromorphic map $p(z)$.

Since it is not easy to handle *general* degenerations of Schottky groups, we shall focus on the degenerations $p(z)$ leading to nice (Schottky) non-Archimedean limits (in the sense of Poineau and Turchetti [PT22]) as $z \to 0$. For this sake, one follows Dang and Mehmeti [DM24] by considering the quantity

$$K_{\min} = \min_{\substack{i \neq j, i \neq k \\ u \in \{\alpha, \beta\}}} \mathrm{ord}_{z=0} \left(\lambda_i(z) \cdot [u_j(z) : u_k(z); \alpha_i(z) : \beta_i(z)]\right)$$

(based on the order of vanishing of certain holomorphic functions) measuring how close the attracting and repelling fixed points can get in terms of cross-ratios versus the contraction rates[2].

Definition 2.2. We say that a one-parameter family of Schottky groups satisfies condition (★) if

- $\lambda_j(z)$ vanishes at $z = 0$ for all $j = 1, \ldots, g$ and
- $K_{\min} > 0$.

For the explanation this condition using non-Archimedean norm on $\mathbb{C}((t))$, please see Remark 2.14.

In this setting, Dang and Mehmeti showed that

Theorem 2.3. *There exists $d_0 > 0$ with*

$$\delta_{\Gamma_z} = dim(L(\Gamma_z)) \sim d_0 / \log(1/|z|)$$

as $z \to 0$ whenever the condition (★) holds.

Remark 2.4. Similar results for Lyapunov exponents at the place of Hausdorff dimension were obtained by Favre [Fav20] (and, more recently, an error term was derived by Ingram et al. [IJM22] in arithmetic situations).

[2] For example, by taking $i = 1$, $j = k = 2$, one gets $\lambda_1(z) \cdot [1 : \beta_2(z); 0 : 2] = \lambda_1(z)(2 - \beta_2(z))/\beta_2(z)$.

Let $\mathbb{M}(\mathbb{D})$ is the ring of meromorphic function on $\mathbb{D}$ with possible pole at 0. From a Koebe coordinate $(\alpha(z), \beta(z), \lambda(z))$, we get a matrix $\gamma_z = \begin{pmatrix} \alpha(z) & \beta(z) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda(z) \end{pmatrix} \begin{pmatrix} \alpha(z) & \beta(z) \\ 1 & 1 \end{pmatrix}^{-1}$ in $\mathrm{GL}_2(\mathbb{M}(\mathbb{D}))$. In order to simplify later computations, we want a matrix in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. The only difficulty is that $\sqrt{\lambda(z)} = \sqrt{t^{-n} f(z)}$, for some $n \in \mathbb{Z}$ and holomorphic $f(z)$ nonzero at 0, may not be meromorphic. We can do a change of variable. Let $z = w^2$ for $w \in \mathbb{D}$, then due to $\mathbb{D}$ simply connected, the square root $\sqrt{\lambda(w^2)} = w^{-n}\sqrt{f(w^2)}$ is still meromorphic. Let $\beta_w = \begin{pmatrix} \alpha(w^2) & \beta(w^2) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w^n (f(w^2))^{-1/2} & 0 \\ 0 & w^{-n}(f(w^2))^{1/2} \end{pmatrix} \begin{pmatrix} \alpha(w^2) & \beta(w^2) \\ 1 & 1 \end{pmatrix}^{-1}$, which is in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. For $z = w^2$, from the definition we have $\ell(\gamma_z) = \ell(\beta_w)$ and $2\ell^{na}(\gamma) = \ell^{na}(\beta)$, where ℓ^{na} will be defined in Definition 2.16 and with respect to $\mathbb{C}(\!(t)\!)$ and $\mathbb{C}(\!(w)\!)$, respectively. Denote the Schottky group generated by β_w as Δ_w. Then the corresponding Selberg zeta functions and Ihara zeta functions satisfy

$$Z(\Gamma_z, s/\log(1/|z|)) = Z(\Delta_w, s/2\log(1/|w|)) \text{ and } Z_I(\Gamma, s) = Z_I(\Delta, s/2).$$

We can obtain the result of Γ_z from that of Δ_w. Hence, up to a possible change of variable $z = w^2$, the family of Schottky groups in $\mathrm{PGL}_2(\mathbb{M}(\mathbb{D}))$ can be replaced by a family in $\mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$. We don't give the detailed results in PGL_2. The readers are encouraged to write for themselves.

2.5 Non-Archimedean Schottky groups

Let $(k, |\cdot|_k)$ be a complete normed non-Archimedean field. See [PT22] for more details of this part.

Definition 2.5. For $n \in \mathbb{N}$, the *Berkovich analytification* $\mathbb{A}_k^{n,an}$ of the n-dimensional affine space $\mathbb{A}_k^n$ is the set of multiplicative seminorms on the polynomial ring $k[T_1, \cdots, T_n]$ which extend the norm on k. The topology on $\mathbb{A}_k^{n,an}$ is the coarsest topology such that for any polynomial P in $k[T_1, \cdots, T_n]$, the evaluation map from $\mathbb{A}_k^{n,an}$ to $\mathbb{R}$ given by $|\cdot|_x \mapsto |P|_x$ is continuous.

The *Berkovich analytification* $\mathbb{P}_k^{1,an}$ of the projective line $\mathbb{P}_k^1$ is constructed by gluing two copies of $\mathbb{A}_k^{1,an}$. More precisely, let $|\cdot|_x$ and $|\cdot|_y$ be two seminorms of two copies. They are equivalent if the map $T \to S^{-1}$ maps one seminorm on $k[T, T^{-1}]$ to the other on $k[S, S^{-1}]$.

Remark 2.6. When $k = \mathbb{C}$, $\mathbb{A}_{\mathbb{C}}^{n,an}$ is homeomorphic to the usual analytic $\mathbb{A}_{\mathbb{C}}^n$, and hence $\mathbb{P}_{\mathbb{C}}^{1,an}$ is homeomorphic to $\mathbb{P}_{\mathbb{C}}^1$. When k is non-Archimedean, $\mathbb{P}_k^{1,an}$ has the structure of a real tree.

Example 2.7. *Set $n = 1$. When k is non-Archimedean, for $a \in k$ and $r \in \mathbb{R}_{\geqslant 0}$, we can define the point $\eta_{a,r} \in \mathbb{A}_k^{1,an}$ by $\sum_n a_n(T-a)^n \mapsto \max_n |a_n|_k r^n$. In this setting, $a \in k$ corresponds to $\eta_{a,0} \in \mathbb{A}_k^{1,an}$ and the element $\eta_{0,1} \in \mathbb{A}_k^{1,an}$ is called the* Gauss point.

Definition 2.8. An open disk (resp. a closed disk) in $\mathbb{P}_k^{1,an}$ is an open subset (resp. a closed subset) isomorphic to a set

$$D^-(a, r) = \{x \in \mathbb{A}_k^{1,an} | |T-a|_x < r\}$$
$$(\text{resp. } D^+(a, r) = \{x \in \mathbb{A}_k^{1,an} | |T-a|_x \leqslant r\})$$

where $a \in k$ and $r \in \mathbb{R}_{>0}$, and $\mathbb{A}_k^{1,an}$ is an affine chart of $\mathbb{P}_k^{1,an}$.

The *Shilov boundaries* of both $D^-(a, r)$ and $D^+(a, r)$ are defined to be the singleton $\{\eta_{a,r}\}$.

The action of an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{PGL}_2(k)$ on $\mathbb{A}_k^{1,an} - \{-d/c\}$ is given by

$$|P(T)|_{\gamma(x)} = \left| P\left(\frac{aT+b}{cT+d}\right) \right|_x = \frac{|P_1(T)|_x}{|P_2(T)|_x}$$

for $P, P_1, P_2 \in k[T]$, and $P_1(T)/P_2(T) = P(\frac{aT+b}{cT+d})$. Moreover, it is possible to naturally extend this formula to obtain an action of γ on $\mathbb{P}_k^{1,an}$: see [PT21, II.1.3] for more details.

An element $\gamma \in \mathrm{PGL}_2(k)$ is called *loxodromic* of given a representative in $\mathrm{GL}_2(k)$, its eigenvalues in an algebraic closure k^{alg} of k have different absolute values (by [PT21, II.1.4], the eigenvalues are in k). Schottky groups of $\mathrm{PGL}_2(k)$ ([PT22, Definition 3.5.1]) are defined similarly to Schottky groups of $\mathrm{PGL}_2(\mathbb{C})$ (Definition 2.1) with the projective space $\mathbb{P}^1(\mathbb{C})$ replaced by $\mathbb{P}_k^{1,an}$.

Definition 2.9. For a loxodromic element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$ with $c \neq 0$ and $\lambda \in \mathbb{R}_{>0}$, let

$$D^-_{\gamma,\lambda} = \{x \in \mathbb{A}^{1,an}_k | \, |(cZ+d)(x)|^2 < \lambda |ad-bc|\},$$

$$D^+_{\gamma,\lambda} = \{x \in \mathbb{A}^{1,an}_k | \, |(cZ+d)(x)|^2 \leqslant \lambda |ad-bc|\},$$

where we only consider an affine chart $\mathbb{A}^{1,an}_k$ of $\mathbb{P}^{1,an}_k$ and Z is the variable of $k[Z]$. They are called open and closed *twisted Ford disks* respectively.

Remark 2.10. The original definition in [PT22, Definition 3.5.5] contains a typo. We need to define for all $x \in \mathbb{A}^{1,an}_k$ to have the desired property $\gamma D^-_{\gamma,\lambda} = \mathbb{P}^{1,an}_k - D^+_{\gamma^{-1},\lambda^{-1}}$. The definition relies on the choice of the affine chart $\mathbb{A}^{1,an}_k$. In all the argument, we fix the affine chart once for all, such that the ∞ is not in the limit set.

Definition 2.11 (Schottky figure)**.** Let Γ be a Schottky group in $\mathrm{PGL}_2(k)$. Let $\{\gamma_1, \cdots, \gamma_g\}$ be a set of free generators of Γ. Let $\mathcal{B} = \{D^+_{\gamma_i}, D^+_{\gamma_i^{-1}} : i = 1, \cdots g\}$ be a collection of disjoint closed disks in $\mathbb{P}^{1,an}_k$. Then $\mathcal{B}$ is called a *Schottky figure* adapted to $\{\gamma_1, \cdots, \gamma_g\}$ if each $i \in \{1, \cdots, g\}$ and $\epsilon \in \{1, -1\}$,

$$D^-_{\gamma_i^\epsilon} := \gamma_i^\epsilon(\mathbb{P}^{1,an}_k - D^+_{\gamma_i^{-\epsilon}})$$

is the maximal open disk inside $D^+_{\gamma_i^\epsilon}$.

The following theorem is due to Gerritzen and is stated in the notations of Poineau–Turchetti ([PT22, Theorem 3.5.9]).

Theorem 2.12 (Gerritzen)**.** *Let* Γ *be a Schottky group of* $\mathrm{PGL}_2(k)$*. Then there exists a set of free generators* $\{\gamma_1, \cdots, \gamma_g\}$ *of* Γ *and* $\lambda_i \in (0,1)$ *for* $i = 1, \cdots, g$ *such that the collection of twisted Ford disks*

$$\mathcal{B} = \{D^+_{\gamma_i^{-1},\lambda_i^{-1}}, D^+_{\gamma_i,\lambda_i} : i = 1, \cdots, g\}$$

is a Schottky figure adapted to $\{\gamma_1, \cdots, \gamma_g\}$*.*

Remark 2.13. There is no analogue of this theorem in the Archimedean setting: indeed, it is known that some Kleinian Schottky groups are not classical, i.e., they cannot be described in terms of Schottky figures involving only round discs.

Remark 2.14. A meromorphic family $(\Gamma_z)_{z \in \mathbb{D}^*}$ of Kleinian Schottky groups can be seen as a subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}(\!(t)\!))$. In this context, as it is shown in [PT22, Proposition 4.4.2], the condition (★) is *equivalent* to Γ being a Schottky subgroup of $\mathrm{PGL}_2(\mathbb{C}(\!(t)\!))$ (with $\mathbb{C}(\!(t)\!)$ equipped with the non-Archimedean norm $|\sum_{j \geqslant m} a_j t^j|_{na} = e^{-m}$ for $a_m \neq 0$).

Mumford curve and its skeleton For an analogue of periodic geodesics in non-Archimedean case, we need to use the description of the limit set in $\mathbb{P}^{1,an}_k$. The advantage is that $\mathbb{P}^{1,an}_k$ is a compact real tree, while $\mathbb{P}^1_k$ is discrete.

Let Γ be a Schottky group of rank g in $\mathrm{PGL}_2(k)$. Similarly to the special case where $k = \mathbb{C}$, a point $x \in \mathbb{P}^{1,an}_k$ is a *limit point* if there exist $x_o \in \mathbb{P}^{1,an}_k$ and a sequence $\{\gamma_n\}_n \subset \Gamma$ of distinct elements such that $\lim_{n\to\infty} \gamma_n x_0 = x$. Let $L(\Gamma)$ be the limit set of Γ on $\mathbb{P}^{1,an}_k$, that is the set of limit points. We have that $L(\Gamma) \subset \mathbb{P}^1_k$ [PT21, Corollary II.3.14].

Moreover, let $O = \mathbb{P}^{1,an}_k - L(\Gamma)$. It is shown that O is a domain of discontinuity of Γ, i.e., it is Γ-invariant and the action of Γ on O is free and proper. The quotient space $X = \Gamma \backslash O$ is a Mumford curve. (See [PT22, Theorem II.3.18].) Let $\Sigma = O \cap \cup_{x,y \in L}[x,y]$ where $\mathbb{P}^{1,an}_k$ has a structure of real tree and $[x,y]$ is the unique injective path from x to y, see [PT22, Section 2.5] [PT21, Proposition I.6.12].[3] Let $\Sigma_X = \Gamma \backslash \Sigma$, which is called the skeleton of the Mumford curve and is a finite graph of genus g. (See [PT22, Notation 4.2.2]) The skeleton Σ_X can also be obtained by identifying the Shilov boundary points p_i, q_i of $D^+_{\gamma_i}, D^+_{\gamma_i^{-1}}$, which is a Schottky figure adapted to $\{\gamma_1, \cdots, \gamma_g\}$.

[3]The path is the union of two intervals given by $\{\eta_{x,r},\ r \in [0, |x-y|_k]\}$ and $\{\eta_{y,r},\ r \in [0, |x-y|_k]\}$ with $\eta_{x,|x-y|_k} = \eta_{y,|x-y|_k}$,

Remark 2.15. This skeleton Σ_X is the analogue of the convex core of a hyperbolic manifold for Archimedean case. This can be seen from the fact that $\mathbb{P}_k^{1,an}$ contains the Bruhat–Tits building (Bruhat–Tits tree in $\mathbb{P}_k^1$ case), which is the analogue of the simply connected hyperbolic manifold in Archimedean case.

Metric structure Recall from [PT21, Section I.8] [PT22, Section 2.5] the definition of multiplicative length. For $\alpha,\ \beta \in k$ and $0 < r \leqslant s$, we define the *additive* length of the elements $\eta_{\alpha,r}, \eta_{\beta,s}$ by:

- if $\max\{r, s\} \geqslant |\alpha - \beta|_k$, $d_a(\eta_{\alpha,r}, \eta_{\beta,s}) := \log(s/r)$;
- if $\max\{r, s\} < |\alpha - \beta|_k$, $d_a(\eta_{\alpha,r}, \eta_{\beta,s}) := \log(|\alpha - \beta|_k/r) + \log(|\alpha - \beta|_k/s)$.

This length is invariant under the action of Möbius transformations. The tree Σ consists of points of $\eta_{a,r}$ with $a \in k$ and $r \in \mathbb{R}_{>0}$.[4] Using this length, we can define length on Σ. The length of the edge of the graph $\Sigma_X = \Gamma\backslash\Sigma$ comes from the quotient of this additive length.

Definition 2.16. For a loxodromic element γ, we define the non-Archimedean length by

$$\ell^{na}(\gamma) = \log(|\mathrm{tr}\tilde{\gamma}|_k^2/|\det(\tilde{\gamma})|_k),$$

where $\tilde{\gamma}$ is a representative of γ in $\mathrm{GL}_2(k)$.

This definition also coincides with the logarithm of the attraction rate at the attracting fixed point of the loxordromic element. Hence a suitable analogue of length of periodic geodesics in the non-Archimedean case.

The following lemma connects the length of loops in the graph with non-Archimedean length of elements in Γ.

Lemma 2.17. *Let γ be a non-trivial loxodromic element in Γ and $\alpha, \alpha' \in \mathbb{P}_k^1$ be its two fixed points. Then for any $x \in [\alpha, \alpha'] - \{\alpha, \alpha'\}$, we have*

$$d_a(x, \gamma x) = \ell^{na}(\gamma),$$

where $[x, \gamma x] \subset [\alpha, \alpha'] - \{\alpha, \alpha'\} \subset \Sigma$.

For any $y = \eta_{\beta,s} \notin [\alpha, \alpha']$ with $0 < s < \infty$, we have

$$d_a(y, \gamma y) > \ell^{na}(\gamma).$$

Proof. Due to invariance under conjugation, up to conjugation, it is sufficient to consider $\gamma = \mathrm{diag}\{\lambda, 1\}$ with $|\lambda|_k > 1$. Then the line $[\alpha, \alpha'] = \{\eta_{0,r}, 0 \leqslant r \leqslant \infty\}$. We have $d_a(x, \gamma x) = d_a(\eta_{0,r}, \eta_{0,|\lambda|_k r}) = \log|\lambda|_k = \ell^{na}(\gamma)$.

The second statement follows from the fact that $\mathbb{P}_k^{1,an}$ is a tree and γ preserves the distance d_a. $\square$

Ihara zeta function The non-Archimedean case of Selberg zeta function is well studied. The analogue zeta function is called the Ihara zeta function for finite graphs. See [Iha66] and [HST06].

Suppose $G = (V, E)$ is a finite graph. Let $E = \{e_1, \cdots, e_{2J}\}$ be the set of oriented edges of G with e_j and e_{j+J} in opposite direction. Each edge has a positive length. A loop P in G is a finite sequence $P = (e_{i_1}, \cdots, e_{i_n})$ with the end point of e_{i_j} equals the starting point of $e_{i_{j+1}}$ (we use the convention $e_{i_{n+1}} = e_{i_1}$). The length $\ell(P)$ of a path P is the sum of the lengths of e_{i_j}. A path is non-backtracking if there are no consecutive edges that are inverse to each other, that is there is no j such that $e_{i_{j+1}} = e_{i_j+J}$. A loop is primitive if it cannot be written as a multiple of another loop. Two loops only differing by starting point are defined to be in the same equivalence class of loops, denoted by $[P]$. Let $\mathcal{P}$ be set of equivalence classes of primitive non-backtracking loops.

Theorem 2.18 (Ihara, Hashimoto, Bass)**.** *Let G be a finite graph. Define the Ihara zeta function by*

$$Z_I(G, s) = \prod_{[P]\in\mathcal{P}} (1 - e^{-s\ell(P)}).$$

Then the Ihara zeta function has an analytic extension to the whole complex plan.

[4]The points like $\eta_{x,0} = x$ are not in $\Sigma = (\mathbb{P}_k^{1,an} - L) \cap \cup_{x,y\in L}[x, y]$.

See Proposition 5.1 for a proof and a way to compute the Ihara zeta function.

Let Γ be Schottky group in $\mathrm{PGL}_2(k)$. Recall the Mumford curve $\Sigma_X = \Gamma\backslash\Sigma$ of Γ. Since Σ is a tree, which is simply connected, each non-backtracking loop P of Σ_X corresponds to a conjugacy class $[\gamma_P]$ in Γ. The length of each edge in Σ_X is given by the length of its lift in the Σ. Moreover, from Lemma 2.17, we obtain that for non-backtracking loop P, we have $\ell(P) = \ell^{na}(\gamma_P)$. Therefore, we obtain

Proposition 2.19. *Let $Z_I(\Gamma, s)$ be the Ihara zeta function for $\Gamma < \mathrm{PGL}_2(k)$*[5]*, that is*

$$Z_I(\Gamma, s) = \prod_{[\gamma]\in\mathcal{P}} (1 - e^{-s\ell^{na}(\gamma)}).$$

Then we have

$$Z_I(\Gamma, s) = Z_I(\Sigma_X, s). \tag{2.1}$$

Remark 2.20. If all the edges of G are of length 1, then

$$Z_I(s) = (1 - e^{-2s})^{r-1} \det(1 - A_G e^{-s} + Q_G e^{-2s})$$

where A is the adjacent matrix of the graph G, r is the rank of the fundamental group of G and Q_G is a diagonal matrix with i-th element equal to the degree minus 1 of the i-th vertex.

Therefore, if $|k^*| = r^{\mathbb{Q}}$ for some $r \in (0, 1)$ (in particular, the case of the algebraic closure of $\mathbb{C}(\!(t)\!)$), then we can use this formula to compute the Ihara zeta function.

2.6 Hybrid space

After Poineau–Turchetti [PT22], one can build a Schottky space S^{an}_{g,A_r} over a Banach ring A_r.

Set $r = 1/e$. The Banach ring A_r can be described as follows. We say that $f = \sum a_n t^n \in \mathbb{C}(\!(t)\!)$ belongs to A_r if its *hybrid norm*

$$\|f\|_{\mathrm{hyb}} := \sum \max\{|a_n|_\infty, |a_n|_0\}(1/e)^n \tag{2.2}$$

is finite. (Here, $|.|_0$ is the trivial norm.) Due to $\max\{|a_n|_\infty, |a_n|_0\} \geqslant \sum |a_n|_\infty$, the elements $f \in A_r$ can be realized as meromorphic functions on $\mathbb{D}_r$ with possible pole at 0.

Definition 2.21. The Berkovich affine space $\mathbb{A}^{3g-3,an}_{A_r}$ consists of elements x which is a bounded multiplicative seminorm $|\cdot|_x$ on the ring $R = A_r[\alpha_3, \dots, \alpha_g, \beta_2, \dots, \beta_g, \lambda_1, \dots, \lambda_g]$, i.e., $|\cdot|_x$ satisfies

- $|0|_x = 0, |1|_x = 1$;
- $|P + Q|_x \leqslant |P|_x + |Q|_x$ and $|P \cdot Q|_x = |P|_x |Q|_x$ for all $P, Q \in R$;
- $|f|_x \leqslant \|f\|_{\mathrm{hyb}}$ for all $f \in A_r$.

The topology is the weakest topology such that for all the element $P \in R$, the map from $\mathbb{A}^{3g-3,an}_{A_r}$ to $\mathbb{R}$ given by $|P|_x$ is continuous.

When $n = 0$, we denote by $\mathcal{M}(A_r)$ the space $\mathbb{A}^{0,an}_{A_r}$, and call it the *Berkovich spectrum* of A_r. It is known that $\mathcal{M}(A_r)$ is naturally homeomorphic to the closed disc $\overline{\mathbb{D}}_r$ and, by restricting the seminorms to the base ring, one obtains a continuous morphism $\Pi : \mathbb{A}^{3g-3,an}_{A_r} \to \mathcal{M}(A_r)$.

For each $x \in \mathbb{A}^{3g-3,an}_{A_r}$, we denote by $\mathcal{H}(x)$ the completion of the fraction field $R/\mathrm{Ker}|\cdot|_x$. We call x *Archimedean* if $\mathcal{H}(x)$ is Archimedean, *non-Archimedean* otherwise.

[5]This function is also called the Ruelle zeta function.

Continuous map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$

Let $p : \mathbb{D} \to \mathbb{C}^{3g-3}$, $z \mapsto p(z) = (\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))$, be a continuous function in $\overline{\mathbb{D}}^*_r$, holomorphic in $\mathbb{D}^*_r$, and meromorphic at 0. By abusing notation, we construct a map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$ as follows: for any $F = \sum f_n(t) P_n(\underline{\alpha}, \underline{\beta}, \underline{\lambda}) \in A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$, we set

$$|F|_{p(z)} = \begin{cases} \left|\sum f_n(z) P_n(\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z))\right|_\infty^{1/\log(1/|z|)}, & \text{if } z \neq 0 \\ \exp(-\mathrm{ord}_{z=0}(\sum f_n(z) P_n(\underline{\alpha}(z), \underline{\beta}(z), \underline{\lambda}(z)))), & \text{if } z = 0, \end{cases} \tag{2.3}$$

where $P_n(\underline{\alpha}, \underline{\beta}, \underline{\lambda})$ is a monomial of total degree n. The key advantage of this construction is that from [BJ17, Proposition A.4] the map $p : \overline{\mathbb{D}}_r \to \mathbb{A}^{3g-3,an}_{A_r}$ is continuous. In the beginning, we deal with meromorphic functions. Using this hybrid construction, we don't have a singularity at $z = 0$ and give a sense of the limiting behavior at $z = 0$.

For $z \in \overline{\mathbb{D}}^*_r$, we have $\mathcal{H}(|\cdot|_{p(z)}) = (\mathbb{C}, |\cdot|_\infty^{1/\log(1/|z|)})$, and therefore $p(z)$ is an Archimedean point. For $z = 0$, we have $\mathcal{H}(|\cdot|_{p(0)}) = (\mathbb{C}((t)), |\cdot|_{na})$, and therefore $p(0)$ is a non-Archimedean point.

Now, we build projective spaces where the Schottky groups attached to the points in Berkovich affine space act in a natural way. Given $x \in \mathbb{A}^{3g-3,an}_{A_r}$, let

$$\mathbb{P}^{1,an}_{\mathcal{H}(x)} \simeq \mathbb{P}^1(\mathbb{C})$$

if the residue field $\mathcal{H}(x)$ is Archimedean, and $\mathbb{P}^{1,an}_{\mathcal{H}(x)} \simeq$ a tree otherwise. In this context, a point $x \in \mathbb{A}^{3g-3,an}_{A_r}$ belongs to the Schottky space S^{an}_{g,A_r} if $0 < |\lambda_j|_x < 1$ for all $1 \leqslant j \leqslant g$, the points $0, 1, 2, \alpha_3, \ldots, \alpha_g, \beta_2, \ldots, \beta_g$ are mutually distinct, and the matrices $\Psi(x) \subset \mathrm{PGL}_2(\mathcal{H}(x))$ associated to x induce a Schottky group over $\mathbb{P}^{1,an}_{\mathcal{H}(x)}$. Interestingly enough, Poineau and Turchetti proved an analog of Hejhal's theorem by establishing that S^{an}_{g,A_r} is an open path connected subset of $\mathbb{A}^{3g-3,an}_{A_r}$.

For the family of Schottky groups satisfying condition ($\bigstar$), due to Remark 2.14, we know $p(0) \in S^{an}_{g,A_r}$. Due to openness of S^{an}_{g,A_r}, we recover [DM24, Proposition 6.1]

Proposition 2.22. *For the family of Schottky groups satisfying condition ($\bigstar$), there exists $0 < \eta \leqslant r$ such that the continuous map p restricted in $\mathbb{D}_\eta$ has image in S^{an}_{g,A_r}.*

Uniform Schottky basis For an element $\gamma \in \mathrm{PGL}_2(\mathbb{C}((t)))$, whose entries are meromorphic on $\mathbb{D}^*_r$ with a possible pole at 0, we define a family of twisted Ford disks:

$$D^+_{\gamma,\lambda}(z) = \begin{cases} \{x \in \mathbb{C} |\ |c(z)x + d(z)|_\infty^{2/\log(1/|z|)} \leqslant \lambda |(ad - bc)(z)|_\infty^{1/\log(1/|z|)}\} \text{ if } z \neq 0, \\ \{x \in \mathbb{A}^{1,an}_{\mathbb{C}((t))} |\ |(cZ + d)(x)|^2 \leqslant \lambda |(ad - bc)(x)|\} \text{ if } z = 0, \end{cases} \tag{2.4}$$

where $c(z)$ means the complex value of the meromorphic function c at t.

We state a result of uniform Schottky figures for all small parameters. This is a translation to this special case of [PT21, Theorem 4.3.2 and Corollary 4.3.3].

Lemma 2.23. *Let $\Gamma = \langle M_1, \cdots, M_g \rangle$ be a family of Schottky groups satisfying condition ($\bigstar$). There exist $s \in (0, r)$ and an automorphism $\tau \in Aut(F_g)$ such that the following holds. Let*

$$N_1 = \tau M_1, \cdots, N_g = \tau M_g \in \mathrm{PGL}_2(\mathbb{C}((t))).$$

There exist positive real numbers $\lambda_1, \cdots, \lambda_g$, such that the family of twisted Ford discs

$$(D^+_{N_1,\lambda_1}(z), \cdots, D^+_{N_g,\lambda_g}(z), D^+_{N_1^{-1},\lambda_1^{-1}}(z), \cdots, D^+_{N_g^{-1},\lambda_g^{-1}}(z))$$

is a Schottky figure adapted to the basis $(N_1, \cdots, N_g)$ for each $z \in \mathbb{C}$ with $|z| < s$.

Proof. We explain the language used in [PT22] and how to obtain the lemma.

Recall $S = S^{an}_{g,A_r}$ is the Schottky space. Let $\mathcal{O}(S)$ be the ring of analytic functions, which means locally the function can be uniformly approximated by rational functions.[6] Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}(S))$ be a loxodormic element with $c \neq 0$ and $0 < \lambda$. Let $\mathbb{A}^{1,an}_S$ be the affine line over S with variable Z. In [PT22], relative twisted Ford discs is defined by

$$D^+_{\gamma,\lambda} = \{x \in \mathbb{A}^{1,an}_S | \ |(cZ+d)(x)|^2 \leqslant \lambda |(ad-bc)(x)|\}.$$

We will prove that this gives the same family of twisted Ford disks as in Eq. (2.4).

The projection map π from $\mathbb{A}^{1,an}_S$ to S is the restriction of the seminorm to the ring $A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$. For any $y \in S$, the fiber $\pi^{-1}(y)$ is the affine line $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$.[7] Relative Ford discs are the union of twisted Ford discs on these fibers.

For the continuous map p, we know $p(0)$ is a non-Archimedean point in S. Then we can apply [PT21, Theorem 4.3.2 and Corollary 4.3.3] to find a relative twisted Ford disks on an open neighbourhood U of $p(0)$ in S, which is a Schottky figure adapted to a chosen basis. The new base is denoted by $N_1 = \tau M_1, \cdots, N_g = \tau M_g$ for some $\tau \in Aut(F_g)$ and with parameters $\lambda_1, \cdots, \lambda_g$. Moreover, for any $y \in U$, the restriction of relative twisted Ford discs to the fiber $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$ gives a Schottky figure on that fiber. Then we only need to prove this gives the twisted Ford disks in the statement of the Lemma.

For $\mathcal{H}(y)$ Archimedean, the norm $(\mathcal{H}(y), ||_y)$ is always given by $(\mathbb{C}, ||^{\epsilon(y)}_\infty)$ with some $\epsilon(y) \leqslant 1$. Due to our choice of $p(z)$, we know that $\epsilon(p(z)) = 1/\log(1/|z|)$ for $z \neq 0$. Over $(\mathbb{C}, ||^\epsilon_\infty)$, the analytification gives no new point. Hence a semi-norm $||_x \in \mathbb{A}^{1,an}_{\mathcal{H}(p(z))} = \mathbb{A}^{1,an}_{\mathbb{C},||^\epsilon_\infty}$ reads as

$$|P(Z)(x)| = \left|(\sum a_n Z^n)(x)\right| = \left|\sum a_n Z(x)^n\right|^{1/\log(1/|z|)}_\infty,$$

for some unique $Z(x) \in \mathbb{C}$ determined by x and for a polynomial $P \in \mathbb{C}[Z]$. Due to the openness of U and the continuity of the map p, we can find $0 < s < r$ such that $p(\mathbb{D}_s)$ is contained in U. Therefore, the twisted Ford discs at the fibre over $p(z)$ with $z \neq 0$ and $|z| < s$ reads as

$$\begin{aligned} D^+_{\gamma,\lambda}(z) &= \{x \in \mathbb{A}^{1,an}_{\mathcal{H}(p(z))} | \ |(cZ+d)(x)|^2 \leqslant \lambda|(ad-bc)(x)|\} \\ &= \{x \in \mathbb{C} | \ |c(z)x + d(z)|^{2/\log(1/|z|)}_\infty \leqslant \lambda |(ad-bc)(z)|^{1/\log(1/|z|)}_\infty\}. \end{aligned}$$

The proof is complete. □

We define the discs $D_{\mathfrak{a},z}$ to be the Ford discs

$$D_{\mathfrak{a},z} := D^-_{\mathfrak{a}^{-1},\lambda_{\mathfrak{a}^{-1}}}(z) \tag{2.5}$$

for any generator $\mathfrak{a}$ in the generator set $\{N_1, \cdots, N_g, N_1^{-1}, \cdots, N_g^{-1}\}$, where the inverse is because we want that the attracting fixed point $\mathfrak{a}^+$ of $\mathfrak{a}$ satisfies $\mathfrak{a}^+ \in D_\mathfrak{a}$. For any element $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \Gamma$, we define

$$D_{\gamma,z} = \mathfrak{a}_{i_1,z} \cdots \mathfrak{a}_{i_{n-1},z} D_{\mathfrak{a}_{i_n},z}.$$

We can further suppose the Schottky basis is uniformly bounded for $|z| \ll 1$. Indeed, the limit set and the Schottky figure of Γ are in the unit ball $D(0,1)$ of the Berkovich affine line $\mathbb{A}^{1,an}_k$, which can be realized by suitably conjugating the Schottky group. Then, for small $|z|$, the Schottky figure are in a uniform compact set in $\mathbb{C}$ due to the centers of Schottky disks being meromorphic functions and their radii being functions of the form $\lambda^{\log(1/|z|)}/|c(z)|$, where λ and $c(z)$ comes from the disc $D^-_{\mathfrak{a}^{-1},\lambda_{\mathfrak{a}^{-1}}}(z)$ for example $\mathfrak{a}_z = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$ and $\lambda_{\mathfrak{a}^{-1}} = \lambda$.

[6] We will only use meromorphic functions on $\mathbb{D}$, so we don't give detailed definition.

[7] The fiber $\pi^{-1}(y)$ is the set of seminorms on $R[Z]$ whose restriction to $R = A_r[\underline{\alpha}, \underline{\beta}, \underline{\lambda}]$ coincides with the seminorm y. Recall that $\mathcal{H}(y)$ is the completion of the fractional field of $R/\ker(y)$. Hence this set of seminorms, $\pi^{-1}(y)$, is isomorphic to the set of multiplicative seminorms on $\mathcal{H}(y)[Z]$ whose restriction to $\mathcal{H}(y)$ coincides with y, which is exactly the affine line $\mathbb{A}^{1,an}_{\mathcal{H}(y)}$.

3 Convergence of zeta function

Let $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ be a family of Schottky groups satisfying ($\bigstar$).

3.1 Convergence on half plane: Euler product

Theorem 3.1. *For $\Re s \gg 1$, we have that*

$$\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_I(\Gamma, s)} \to 1 \quad \text{as } z \to 0.$$

The following lemma gives the convergence of the length of each conjugacy class.

Lemma 3.2. *Let $[\gamma]$ be a non-trivial conjugacy class in Γ, then*

$$\lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{\ell^{na}(\gamma)} = 1.$$

Proof. Using the definition of non-Archimedean length (Definition 2.16) and the assumption that $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$, we have

$$|\operatorname{tr}(\gamma)|_{na} = e^{\ell^{na}(\gamma)/2}.$$

Every non-trivial element γ in Γ is loxodromic in $\mathrm{SL}_2(\mathbb{C}(\!(t)\!))$, hence $\ell^{na}(\gamma) > 0$. Since $\operatorname{tr}(\gamma_z)$ is meromorphic, we have $\lim_{z\to 0} |\operatorname{tr}(\gamma_z)|^{1/\log(1/|z|)} = |\operatorname{tr}(\gamma)|_{na}$. The assumption that Γ satisfies ($\bigstar$) yields $\ell(\gamma_z) \to \infty$ as $z \to 0$. We obtain

$$|\operatorname{tr}(\gamma_z)|/e^{\ell(\gamma_z)/2} = (e^{\ell(\gamma_z)/2} + e^{-\ell(\gamma_z)/2})/e^{\ell(\gamma_z)/2} \to 1.$$

We obtain

$$\lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{\ell^{na}(\gamma)} = \lim_{z\to 0} \frac{\ell(\gamma_z)/\log(1/|z|)}{2\log|\operatorname{tr}(\gamma_z)|/\log(1/|z|)} \frac{\log|\operatorname{tr}(\gamma_z)|/\log(1/|z|)}{\log|\operatorname{tr}(\gamma)|_{na}} \frac{2\log|\operatorname{tr}(\gamma)|_{na}}{\ell^{na}(\gamma)} = 1. \qquad \square$$

We also need a uniform bound independent of z for the lengths of geodesics. We first recall the uniform distortion estimates from Lemma 5.6 and Corollary 5.9 in [DM24]. Meanwhile, Proposition 2.22 and Lemma 2.23 state that there exists $\eta \in (0, r)$ such that the map $p : \mathbb{D}_\eta \to \mathcal{S}^{an}_{g,A_r}$ is continuous, and $\{D_{\mathfrak{a},z}\}$ is a uniform Schottky basis for any $z \in \mathbb{D}_\eta$. Hence, we can apply the results to obtain the following.

Lemma 3.3. *There exist $\eta_0 \in (0, r)$ and constants $R > 0$, $c \in (0, 1)$ for any $|z| \leqslant \eta_0$ and any non-trivial word γ, the radius $r_{\gamma,z}$ of $D_{\gamma,z}$ satisfies*[8]

$$r_{\gamma,z} \leqslant Rc^{l(\gamma)-1}. \tag{3.1}$$

Moreover, there exists a constant $N > 0$ such that for any $\gamma = \mathfrak{a}\mathbf{b}$ with $\mathfrak{a}\mathbf{b}$ reduced and $l(\mathfrak{a}) = 1$ and $l(\mathbf{b}) > N$,

$$\frac{\sup_{x\in D_{\mathbf{b},z}} |\mathfrak{a}'(x)|_z}{\inf_{x\in D_{\mathbf{b},z}} |\mathfrak{a}'(x)|_z} \leqslant \exp\left(\frac{1}{4} r_{\mathbf{b},z}\right). \tag{3.2}$$

Here, the radius and the derivative are computed with respect to the norm $|\cdot|_z$. More precisely, for $z \neq 0$, $|\cdot|_z$ is a norm on $\mathbb{C}$ given by $|\cdot|_z := |\cdot|_\infty^{1/\log(1/|z|)}$, and for $z = 0$, $|\cdot|_z$ is a norm on $\mathbb{C}(\!(t)\!)$ defined by $|\cdot|_z = |\cdot|_{na}$.

Lemma 3.4. *Let $\eta_0 \in (0, r)$, $R > 0$, $c \in (0, 1)$ be the constants given in Lemma 3.3. Then there exists $C > 0$ such that for any word $\gamma = \mathbf{a}\mathfrak{b}$ with $\mathbf{a}\mathfrak{b}$ reduced and $l(\mathfrak{b}) = 1$ and $\mathbf{a}$ of arbitrary length, we have*

$$\frac{\sup_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z}{\inf_{x\in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z} \leqslant C\exp(R/(1-c)) \quad \text{for any } z \in \mathbb{D}_{\eta_0}.$$

[8] Recall that $l(\gamma)$ stands for the word length: cf. Subsection 1.3.

Proof. Assume $l(\mathbf{a}) = n$ and write $\mathbf{a} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n}$. Recall that $\mathfrak{a}_{i_j} \cdots \mathfrak{a}_{i_n} D_{\mathfrak{b},z} = D_{\mathfrak{a}_{i_j} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}$ for any $z \in \mathbb{D}_{\eta_0}$. We estimate the distortion of $\mathbf{a}$:

$$\begin{aligned}\frac{\sup_{x \in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z}{\inf_{x \in D_{\mathfrak{b},z}} |\mathbf{a}'(x)|_z} &\leqslant \frac{\sup_{x \in D_{\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}} |\mathfrak{a}_{i_1}'(x)|_z}{\inf_{x \in D_{\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}} |\mathfrak{a}_{i_1}'(x)|_z} \cdot \frac{\sup_{x \in D_{\mathfrak{a}_{i_3} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}} |\mathfrak{a}_{i_2}'(x)|_z}{\inf_{x \in D_{\mathfrak{a}_{i_3} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}} |\mathfrak{a}_{i_2}'(x)|_z} \cdots \frac{\sup_{x \in D_{\mathfrak{b},z}} |\mathfrak{a}_{i_n}'(x)|_z}{\inf_{x \in D_{\mathfrak{b},z}} |\mathfrak{a}_{i_n}'(x)|_z} \\ &\lesssim \exp\left(\sum_{j<n-N} \frac{1}{4} r_{\mathfrak{a}_{i_{j+1}} \cdots \mathfrak{a}_{i_n} \mathfrak{b},z}\right) \lesssim \exp\left(\frac{1}{4} R/(1-c)\right)\end{aligned}$$

Here, to obtain the second and third inequalities, for j satisfying $n - j > N$, we estimate the distortion of $\mathfrak{a}_{i_j}$ by applying Eq. (3.2) and then Eq. (3.1), and for j satisfying $n - j \leqslant N$, we bound the distortion of $\mathfrak{a}_j$ by a constant only depending on Γ and N. □

Lemma 3.5. *There exists $c \in (0,1)$ depending on the Schottky family, such that for any conjugacy class $[\gamma]$ with $\ell^{na}(\gamma) > 1/c$ and $|z| < c$, we have*

$$\ell(\gamma_z) \geqslant c \log(1/|z|) \ell^{na}(\gamma).$$

Proof. Let $l(\gamma)$ be the word length of γ with the fixed generating set. Choose a γ in the conjugacy class that is cyclically reduced, which means that the first and last letters of the word γ are not inverse to each other.

Note that $\gamma D_{\mathfrak{a}_{i_1},z} = D_{\gamma \mathfrak{a}_{i_1},z}$ and the γ-attracting fixed point is inside $D_{\mathfrak{a}_{i_1},z}$, combined with Lemma 3.4, we obtain

$$-\log|\gamma'(\gamma\text{-attracting fixed point})|_z \geqslant \left|\log\left(\frac{r_{\gamma \mathfrak{a}_{i_1},z}}{r_{\mathfrak{a}_{i_1},z}}\right)\right| - \frac{R}{(1-c)}. \tag{3.3}$$

As we required γ to be cyclically reduced, we have $l(\gamma \mathfrak{a}_{i_1}) = l(\gamma) + 1$. Hence by Eq. (3.1),

$$\begin{aligned}\ell(\gamma_z) = -\log(1/|z|)\log|\gamma'(\gamma\text{-attracting fixed point})|_z &\geqslant \log(1/|z|)\left|\left|\log\left(\frac{r_{\gamma \mathfrak{a}_{i_1},z}}{r_{\mathfrak{a}_{i_1},z}}\right)\right| - \frac{R}{(1-c)}\right| \\ &\geqslant \log(1/|z|)\left(l(\gamma)|\log c| - |\log r_{\mathfrak{a}_{i_1},z}| - \log R - \frac{R}{(1-c)}\right) \gtrsim \log(1/|z|) l(\gamma),\end{aligned}$$

where the last inequality is true if $l(\gamma)$ is large enough compared to these constants. For the finitely many γ with $l(\gamma)$ small, we use Lemma 3.2 to get a uniform bound.

Lemma 3.6. *Let γ be a cyclically reduced word. For the word length $l(\gamma)$ and the non-Archimedean length $\ell^{na}(\gamma)$, they satisfy $\ell^{na}(\gamma) \lesssim l(\gamma)$.*

Proof. Since $\ell^{na}(\gamma)$ is the least translation length for the type of points $\eta_{\alpha,r}$ (Lemma 2.17), we have for the Gauss point $o = \eta_{0,1}$,

$$\begin{aligned}\ell^{na}(\gamma) &\leqslant d_a(o, \gamma o) \leqslant d_a(o, \mathfrak{a}_{i_1} o) + d_a(\mathfrak{a}_{i_1} o, \mathfrak{a}_{i_1}\mathfrak{a}_{i_2} o) + \cdots + d_a(\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{n-1}} o, \gamma o) \\ &\leqslant \sum_j d_a(o, \mathfrak{a}_{i_j} o) \lesssim l(\gamma).\end{aligned}$$

□

The proof is complete by combining the above lemma. □

Since Eq. (3.3) also works for non-Archimedean point $p(0)$, the same proof implies

Lemma 3.7. *Let Γ be a non-Archimedean Schottky group. For any cyclically reduced word γ, we have*

$$\ell^{na}(\gamma) \gtrsim l(\gamma).$$

Proof of Theorem 3.1. Suppose $s \in \mathbb{C}$ with $\Re s \gg 1$. We have

$$\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_I(\Gamma, s)} = \prod_{[\gamma] \in \mathcal{P}} \frac{1 - e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1 - e^{-s\ell^{na}(\gamma)}} \cdot \mathcal{R}(\gamma) \tag{3.4}$$

where

$$\mathcal{R}(\gamma)=\begin{cases}\prod_{k\geqslant 1}(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)}) & \text{if } \Gamma_z<\mathrm{SL}_2(\mathbb{R}),\\ \prod_{k_1+k_2\geqslant 1}(1-e^{-(s/\log(1/|z|)+k_1+k_2)\ell(\gamma_z)}e^{-i\theta_\gamma(k_1+k_2)}) & \text{if } \Gamma_z<\mathrm{SL}_2(\mathbb{C}).\end{cases}$$

We will use the inequality

$$1/(1-x)\leqslant \exp(2x) \ \text{ for } \ 0\leqslant x\leqslant 1/2. \tag{3.5}$$

We estimate $|\mathcal{R}(\gamma)|^{-1}$. Using Lemma 3.5, we have for $|z|$ small, if $\Gamma_z\subset \mathrm{SL}_2(\mathbb{R})$, then

$$|\mathcal{R}(\gamma)|^{-1}=\prod_{k\geqslant 1}|1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)}|^{-1}\leqslant \exp\left(2\sum_{k\geqslant 1}e^{-kc\log(1/|z|)\ell^{na}(\gamma)}\right)=\exp\left(\frac{2e^{-c\log(1/|z|)\ell^{na}(\gamma)}}{1-e^{-c\log(1/|z|)\ell^{na}(\gamma)}}\right).$$

If $\Gamma_z<\mathrm{SL}_2(\mathbb{C})$, then we have

$$\begin{aligned}|\mathcal{R}(\gamma)|^{-1}&\leqslant \prod_{k\geqslant 1}|1-e^{-(\Re s/\log(1/|z|)+k)\ell(\gamma_z)}|^{-(k+1)}\leqslant \exp\left(2\sum_{k\geqslant 1}(k+1)e^{-kc\log(1/|z|)\ell^{na}(\gamma)}\right)\\ &=\exp\left(\frac{2}{(1-e^{-c\log(1/|z|)\ell^{na}(\gamma)})^2}-2\right)\leqslant \exp\left(\frac{4e^{-c\log(1/|z|)\ell^{na}(\gamma)}}{(1-e^{-c\log(1/|z|)\ell^{na}(\gamma)})^2}\right).\end{aligned}$$

Combing these two cases together, we have for $|z|$ small,

$$\prod_{[\gamma]\in\mathcal{P}}|\mathcal{R}(\gamma)|^{-1}\leqslant \exp\left(8\sum_{[\gamma]\in\mathcal{P}}e^{-c\log(1/|z|)\ell^{na}(\gamma)}\right). \tag{3.6}$$

We first sum over $\{[\gamma]\in\mathcal{P}:\ \ell^{na}(\gamma)=n\}$ and then sum over $n\in\mathbb{N}$. Due to Lemma 3.7, there exists a constant C only depending on the family such that

$$\#\{[\gamma]\in\mathcal{P}:\ \ell^{na}(\gamma)=n\}\leqslant \#\{\gamma\,\text{cyclically reduced}:\ l(\gamma)\leqslant C'n\}\leqslant e^{Cn}. \tag{3.7}$$

Hence

$$Eq.\ (3.6)\leqslant \exp\left(\frac{8e^{-(c\log(1/|z|)-C)}}{1-e^{-(c\log(1/|z|)-C)}}\right),$$

which converges to 1 as $|z|$ tends to zero.

Now, we consider the other factor in Eq. (3.4). Take $N\gg 1$. We separate into two parts: $\ell^{na}(\gamma)\leqslant N$ and $\ell^{na}(\gamma)>N$. For the first part, which is a product of finite terms, due to Lemma 3.2,

$$\prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)\leqslant N}}\frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}}\to 1.$$

For the second part, due to Lemma 3.5, we have

$$\begin{aligned}\left|\log\prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}}\left|\frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}}\right|\right|&\leqslant 2\sum_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}}e^{-\Re s/\log(1/|z|)\ell(\gamma_z)}+e^{-\Re s\ell^{na}(\gamma)}\\ \leqslant 2\sum_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}}e^{-\Re sc\ell^{na}(\gamma)}+e^{-\Re s\ell^{na}(\gamma)}&\leqslant 2\sum_{n>N}e^{Cn}(e^{-n\Re sc}+e^{-\Re sn})=\frac{4e^{-N(\Re sc-C)}}{1-e^{-N(\Re sc-C)}},\end{aligned}$$

where the first inequality is due to Eq. (3.5) and the last inequality is due to Eq. (3.7). Therefore, given $\Re s\gg 1$, we have that for N sufficiently large, the product

$$\prod_{\substack{[\gamma]\in\mathcal{P}\\ \ell^{na}(\gamma)>N}}\frac{1-e^{-s/\log(1/|z|)\ell(\gamma_z)}}{1-e^{-s\ell^{na}(\gamma)}}$$

is close to 1. The proof is complete. □

3.2 Boundedness of zeta function through transfer operator

3.2.1 Preliminaries on trace class operators

We recall some preliminaries on trace class operators. See [DZ19, Appendix B] for more details.

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator between two Hilbert spaces. The singular values μ_ℓ of $\mathcal{A}$ are defined to be the eigenvalues λ_ℓ of $(\mathcal{A}^*\mathcal{A})^{1/2}$ in decreasing order, i.e.

$$\mu_\ell(\mathcal{A}) := \lambda_\ell((\mathcal{A}^*\mathcal{A})^{1/2}), \quad \ell = 0, 1, 2, \cdots .$$

We recall the minimax property of singular values:

$$\mu_\ell(\mathcal{A}) = \min_{\operatorname{codim}(V,\mathcal{H}_1)=\ell} \max_{v \in V\setminus\{0\}} \frac{\|\mathcal{A}v\|_{\mathcal{H}_2}}{\|v\|_{\mathcal{H}_1}}. \tag{3.8}$$

This follows directly from the minimax property for self-adjoint operators since $\|\mathcal{A}v\|^2_{\mathcal{H}_2} = (\mathcal{A}^*\mathcal{A}v, v)_{\mathcal{H}_2}$. As a corollary, for any compact operators $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{B} : \mathcal{H}_1 \to \mathcal{H}_2$, we have the inequality

$$\mu_\ell(\mathcal{A} + \mathcal{B}) \leqslant \mu_{\ell_1}(\mathcal{A}) + \mu_{\ell_2}(\mathcal{B}), \quad \forall\ \ell = \ell_1 + \ell_2. \tag{3.9}$$

Definition 3.8. A compact operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert spaces is a trace class operator if

$$\|\mathcal{A}\|_1 := \sum_\ell \mu_\ell(\mathcal{A}) < \infty.$$

$\|\cdot\|_1$ is called the trace norm.

For a trace class operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ on a single space, one can define the trace by

$$\operatorname{tr}(\mathcal{A}) := \sum_j \langle \mathcal{A}e_j, e_j\rangle \tag{3.10}$$

where $\{e_j\}$ is an orthonormal basis of $\mathcal{H}$. One can verify that the definition does not depend on the choice of $\{e_j\}$. By Lidskii's Theorem [DZ19, Proposition B.31]), we have

$$\operatorname{tr}(\mathcal{A}) = \sum_\ell \lambda_\ell(\mathcal{A}).$$

The Fredholm determinant is defined by

$$\det(1 - \mathcal{A}) := \prod_\ell (1 - \lambda_\ell(\mathcal{A})).$$

We have the estimates

$$|\operatorname{tr}(\mathcal{A})| \leqslant \sum_\ell |\lambda_\ell(\mathcal{A})| \leqslant \|\mathcal{A}\|_1, \quad |\det(1 - \mathcal{A})| \leqslant \prod_\ell (1 + \mu_\ell(\mathcal{A})) \leqslant e^{\|\mathcal{A}\|_1}. \tag{3.11}$$

For a trace class operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and a bounded operator $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$, it follows from the minimax property that $\mu_\ell(\mathcal{B}\mathcal{A}) \leqslant \|\mathcal{B}\|_{\mathcal{H}_2 \to \mathcal{H}_3} \mu_\ell(\mathcal{A})$ and thus

$$\|\mathcal{B}\mathcal{A}\|_1 \leqslant \|\mathcal{B}\|_{\mathcal{H}_2 \to \mathcal{H}_3} \|\mathcal{A}\|_1. \tag{3.12}$$

Moreover, it is shown in [DZ19, Eq (B.4.9)] that for a trace class operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and and a bounded operator $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_1$,

$$\operatorname{tr}(\mathcal{A}\mathcal{B}) = \operatorname{tr}(\mathcal{B}\mathcal{A}). \tag{3.13}$$

3.2.2 Transfer operator: $\mathrm{SL}_2(\mathbb{R})$ case

To estimate the Selberg zeta function in the region where the infinite product expression does not converge, we employ the transfer operator $\mathcal{L}_s$. The observation is that the transfer operator $\mathcal{L}_s$ is a trace class operator acting on the space of holomorphic functions with bounded L^2 norm, and its Fredholm determinant gives the Selberg zeta function.

We first consider the case for a Schottky group $\Gamma < \mathrm{SL}_2(\mathbb{R})$ of rank $g \geqslant 2$. Write $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{2g} \rangle$ with $\mathfrak{a}_i \mathfrak{a}_{i+g} = id$ for $i = 1, 2, \cdots, g$. By conjugating Γ if necessary, there exist open discs $D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \cdots, D_{\mathfrak{a}_{2g}}$ in $\mathbb{C}$ centered in $\mathbb{R}$ such that their closures are disjoint, and for each $\mathfrak{a} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$, we have $\mathfrak{a}(\hat{\mathbb{C}} - \overline{D_{\mathfrak{a}^{-1}}}) = D_{\mathfrak{a}}$ and $\mathfrak{a}(\hat{\mathbb{R}} - I_{\mathfrak{a}^{-1}}) = I_{\mathfrak{a}}^{\circ}$, where $I_{\mathfrak{a}} := \overline{D_{\mathfrak{a}}} \cap \mathbb{R}$.

We define the Hilbert space $\mathcal{H} := \oplus_{i=1}^{2g} \mathcal{H}(D_{\mathfrak{a}_i})$, where $\mathcal{H}(D_{\mathfrak{a}}) = \{u \in L^2(D_{\mathfrak{a}}) : u \text{ is holomorphic}\}$ is equipped with the norm $\|u\|^2_{\mathcal{H}(D_{\mathfrak{a}})} := |D_{\mathfrak{a}}|^{-1} \|u\|^2_{L^2(D_{\mathfrak{a}})}$. The transfer operator $\mathcal{L}_s : \mathcal{H} \to \mathcal{H}$ is defined as

$$\mathcal{L}_s u(w) := \sum_{\mathfrak{a}_i \neq \mathfrak{b}^{-1}} \mathfrak{a}_i'(w)^s u(\mathfrak{a}_i(w)) \quad \text{for} \quad w \in D_{\mathfrak{b}} \quad \text{with} \quad \mathfrak{b} \in \{\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}\}. \tag{3.14}$$

Here note that the derivative satisfies $\mathfrak{a}_i'(w) > 0$ for $w \in D_{\mathfrak{b}} \cap \mathbb{R}$, and hence we define $\mathfrak{a}_i'(w)^s := e^{s \log \mathfrak{a}_i'(w)}$ where we choose the branch of complex logarithm so that if $s \in \mathbb{R}$, then $\mathfrak{a}_i'(w)^s$ is the unique holomorphic function on $D_{\mathfrak{b}}$ satisfying $\mathfrak{a}_i'(w)^s > 0$ for $w \in D_{\mathfrak{b}} \cap \mathbb{R}$.

We introduce a few more notations. We use $\mathcal{W}^n$ to denote words of length n with respect to the generators $\mathfrak{a}_1, \cdots, \mathfrak{a}_{2g}$, i.e. words of the form $\mathfrak{a}_{i_1} \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_n}$ with $\mathfrak{a}_{i_{j+1}} \neq \mathfrak{a}_{i_j}^{-1}$. For $\mathbf{a} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \mathcal{W}^n$, we denote

$$\mathbf{a}(w) = \mathfrak{a}_{i_1}(\mathfrak{a}_{i_2}(\cdots \mathfrak{a}_{i_n}(w)))$$

and $\mathbf{a}' = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{n-1}}$.

For a disc D of radius R, and a real number $\rho > 0$, we use ρD to denote the disc with the same center and of radius ρR.

The following lemma shows that $\mathcal{L}_s$ is a trace class operator (see also [Bor16, Lemma 15.7]).

Lemma 3.9. *Suppose $\gamma : D_{\mathfrak{b}} \to \rho D_{\mathfrak{a}}$ is a holomorphic function between two discs with $0 < \rho < 1$ and $\mathfrak{a}, \mathfrak{b} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$, then the pullback operator*

$$\gamma^* u(w) = u(\gamma(w)) : \mathcal{H}(D_{\mathfrak{a}}) \to \mathcal{H}(D_{\mathfrak{b}})$$

is a trace class operator satisfying

$$\mu_\ell(\gamma^*) \leqslant C(1-\rho)^{-2} \rho^\ell \sqrt{\ell + 1}.$$

Consequently, for any $s \in \mathbb{C}$, $\mathcal{L}_s : \mathcal{H} \to \mathcal{H}$ is a trace class operator.

Proof. We may assume $D_{\mathfrak{a}}$ is centered at 0 and take an orthonormal basis $\phi_n(w) := \sqrt{n+1} \left(\frac{w}{R}\right)^n$ of $\mathcal{H}(D_{\mathfrak{a}})$ where R is the radius of $D_{\mathfrak{a}}$. Then

$$\|\gamma^* \phi_n\|^2_{\mathcal{H}(D_{\mathfrak{b}})} = |D_{\mathfrak{b}}|^{-1} \int_{D_{\mathfrak{b}}} (n+1) \frac{|\gamma(w)|^{2n}}{R^{2n}} dm(w) \leqslant (n+1) \rho^{2n}.$$

By the minimax property Eq. (3.8), we conclude

$$\mu_\ell(\gamma^*) \leqslant \sum_{n \geqslant \ell} \|\gamma^* \phi_n\|_{\mathcal{H}(D_{\mathfrak{b}})} \leqslant \sum_{n \geqslant \ell} \sqrt{n+1} \rho^n \leqslant C(1-\rho)^{-2} \rho^\ell \sqrt{\ell+1}.$$

Thus

$$\sum_\ell \mu_\ell(\gamma^*) \leqslant C(1-\rho)^{-2} \sum_\ell \rho^\ell \sqrt{\ell+1} < \infty$$

which implies that γ^* is a trace class operator. Since $\mathcal{L}_s$ is a finite direct sum of such pullback operators composed with multiplication by holomorphic functions, we conclude $\mathcal{L}_s$ is also a trace class operator using Eq. (3.9) and Eq. (3.12). ☐

Since $\mathcal{L}_s$ is a trace class operator, we can define its Fredholm determinant. We claim $\det(1-\mathcal{L}_s)$ is exactly the Selberg zeta function, see [Bor16, Lemma 15.10]. Here we include a proof for completeness but we will omit the proof later for similar cases. Note since $\mathcal{L}_s$ is in trace class and depends holomorphically on $s \in \mathbb{C}$, $\det(1-\mathcal{L}_s)$ is an entire function. So this gives an alternative proof of holomorphic extension of the Selberg zeta function.

Lemma 3.10. *For $s \in \mathbb{C}$, we have*

$$\det(1-\mathcal{L}_s) = Z(\Gamma, s). \tag{3.15}$$

Proof. Since both sides are entire functions on $\mathbb{C}$, it suffices to check the inequality for $\Re s \gg 1$. For $\Re s \gg 1$, we have

$$\det(1-\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(\mathcal{L}_s^n)\right).$$

On the other hand,

$$Z(\Gamma, s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{[\gamma],k} e^{-n(s+k)\ell(\gamma)}\right).$$

We again consider the orthonormal basis $\phi_k^{\mathfrak{a}}(w) = \sqrt{k+1}\left(\frac{w-w_{\mathfrak{a}}}{R_{\mathfrak{a}}}\right)^k$ where $w_{\mathfrak{a}}$ is the center of $D_{\mathfrak{a}}$ and $R_{\mathfrak{a}}$ is the radius of $D_{\mathfrak{a}}$. We write

$$\mathcal{L}_s = \sum_{\mathfrak{a} \neq \mathfrak{b}^{-1} \in \mathcal{W}} \mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} \quad \text{with} \quad \mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} : \mathcal{H}(D_{\mathfrak{a}}) \to \mathcal{H}(D_{\mathfrak{b}}) \quad \text{given by} \quad \mathcal{L}_{s,\mathfrak{a}\mathfrak{b}} u(w) = \mathfrak{a}'(w)^s u(\mathfrak{a}(w)) \quad \text{for } w \in D_{\mathfrak{b}}.$$

We have

$$\mathcal{L}_s^n = \sum_{\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}} \in \mathcal{W}^{n+1}} \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}} \quad \text{with} \quad \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}} := \mathcal{L}_{s,\mathfrak{a}_{i_n}\mathfrak{a}_{i_{n+1}}} \cdots \mathcal{L}_{s,\mathfrak{a}_{i_2}\mathfrak{a}_{i_3}} \mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}}.$$

The Schwartz kernel of $\mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}}$ is given by

$$\mathcal{L}_{s,\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_{n+1}}}(w,w') = \sum_k (\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})'(w)^s \phi_k^{\mathfrak{a}_{i_1}}(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}(w))\overline{\phi_k^{\mathfrak{a}_{i_1}}(w')}, \quad w \in D_{\mathfrak{a}_{i_{n+1}}},\ w' \in D_{\mathfrak{a}_{i_1}}.$$

Its restriction to the diagonal is nonzero if and only if $\mathfrak{a}_{i_{n+1}} = \mathfrak{a}_{i_1}$. Set $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ and $\mathcal{L}_{s,\gamma} := \mathcal{L}_{s,\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n+1}}}$. So $\mathcal{L}_{s,\gamma}u(w) = \gamma'(w)^s u(\gamma(w))$. A change of variables $w = gw'$ gives the operator $(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* : \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_1}}) \to \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_{n+1}}})$ given by

$$(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* u(w') = \gamma'(gw')^s u(g^{-1}\gamma g w').$$

Let $\gamma_1 = g^{-1}\gamma g$, and $\mathcal{L}_{s,\gamma_1} : \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_1}}) \to \mathcal{H}(g^{-1}D_{\mathfrak{a}_{i_{n+1}}})$ given by $\mathcal{L}_{s,\gamma_1}u(w') = \gamma_1'(w)^s u(\gamma_1 w')$. Then

$$(g)^* \circ \mathcal{L}_{s,\gamma} \circ (g^{-1})^* u(w') = \mathcal{B}^{-1}\mathcal{L}_{s,\gamma_1}\mathcal{B} \quad \text{with} \quad \mathcal{B}u(w') = (g'(w'))^s u(w').$$

Therefore, by Eq. (3.13), $\operatorname{tr}\mathcal{L}_{s,\gamma} = \operatorname{tr}\mathcal{L}_{s,\gamma_1}$. So we may choose the coordinate so that $\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}(w) = e^{-\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}w$, then

$$(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})'(w)^s = e^{-s\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}.$$

We may assume the center of $D_{\mathfrak{a}_{i_1}}$ is 0. Applying the definition of trace Eq. (3.10) to the basis $\phi_k^{\mathfrak{a}_{i_1}}(w)$, we obtain

$$\operatorname{tr}\mathcal{L}_{s,\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{n+1}}} = \frac{e^{-s\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}}{\pi R_{\mathfrak{a}_{i_1}}^2} \sum_k (k+1) \int_{|w| \leqslant R_{\mathfrak{a}_{i_1}}} \left(\frac{e^{-\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}w}{R_{\mathfrak{a}_{i_1}}}\right)^k \left(\frac{\bar{w}}{R_{\mathfrak{a}_{i_1}}}\right)^k dm(w) = \sum_k e^{-(s+k)\ell(\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n})}.$$

Suppose the closed loop $\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n} = \mathbf{a}^m$ where $\mathbf{a}$ is primitive. Then

$$\sum_n \frac{1}{n}\operatorname{tr}(\mathcal{L}_s^n) = \sum_m \frac{1}{m} \sum_{[\mathbf{a}] \in \mathcal{P}, k} e^{-m(s+k)\ell(\mathbf{a})}.$$

The proof is complete. □

For further applications, we also introduce modified transfer operators where we only use discs with word length $\geqslant N$. For $\mathbf{a} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n} \in \mathcal{W}^n$, let

$$D_{\mathbf{a}} := \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{n-1}}(D_{\mathfrak{a}_{i_n}}).$$

We define the modified transfer operator on $\mathcal{H} = \mathcal{H}_N := \oplus_{\mathbf{b} \in \mathcal{W}^N} \mathcal{H}(D_{\mathbf{b}})$:

$$\mathcal{L}_{s,N} u(w) := \sum_{\mathfrak{a} \neq \mathfrak{a}_{i_1}^{-1}} \mathfrak{a}'(w)^s u(\mathfrak{a}(w)) \quad \text{for} \quad w \in D_{\mathbf{b}} \quad \text{with} \quad \mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}. \tag{3.16}$$

When $N = 1$, it agrees with the previous definition Eq. (3.14). When $N \geqslant 2$, the modified transfer operator acts on a space of functions with a larger domain. We can still use the proof of Lemma 3.10 to show that for any $N \geqslant 1$,

$$\det(1 - \mathcal{L}_{s,N}) = Z(\Gamma, s).$$

In the computation, we need to use fixed points. By taking larger N, the fixed point of one loxodromic element is still contained in only one disc. The computations are the same.

Now we can state a much more precise estimate for the transfer operator.

Proposition 3.11. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is a Schottky group with the notations above. Suppose for some $A, B > 0$, $N \in \mathbb{Z}_{>0}$ and $\varphi \in [0, \pi]$, we have for any $\mathfrak{a} \in \mathcal{W}$ and $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$,*

- *$e^{-A} \leqslant |\mathfrak{a}'(w)| \leqslant e^A$ and $|\arg \mathfrak{a}'(w)| \leqslant \varphi \leqslant \pi$ for any $w \in D_{\mathbf{b}}$;*
- *$\mathfrak{a}(D_{\mathbf{b}}) \subset e^{-B} D_{\mathbf{a}}$ with $\mathbf{a} = \mathfrak{a}\mathbf{b}'$.*

Then the singular values of $\mathcal{L}_{s,N}$ satisfy

$$\mu_\ell(\mathcal{L}_{s,N}) \leqslant C(g, N, B) e^{(A|\Re s| + \varphi|\Im s| - B\ell)/C(g,N)} \sqrt{\ell + 1}. \tag{3.17}$$

Proof. The transfer operator $\mathcal{L}_{s,N}$ is the direct sum of operators

$$\mathcal{L}_{s,\mathbf{ab}} : \mathcal{H}(D_{\mathbf{a}}) \to \mathcal{H}(D_{\mathbf{b}}) \quad \text{given by} \quad \mathcal{L}_{s,\mathbf{ab}} u(w) = \mathfrak{a}'(w)^s u(\mathfrak{a}(w)),$$

where $\mathfrak{a} \in \mathcal{W}$ and $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$ and $\mathbf{a} = \mathfrak{a}\mathbf{b}'$. We have

- $|\mathfrak{a}'(w)^s| \leqslant \max(|\mathfrak{a}'(w)|^{\Re s}, |\mathfrak{a}'(w)|^{-\Re s}) |\exp(|\arg \mathfrak{a}'(w)||\Im s|) \leqslant e^{A|\Re s| + \varphi|\Im s|}$;
- By Lemma 3.9, the singular values of the pullback operator $\mathfrak{a}^* : \mathcal{H}(D_{\mathbf{a}}) \to \mathcal{H}(D_{\mathbf{b}})$ satisfy

$$\mu_\ell(\mathfrak{a}^*) \leqslant C(1 - e^{-B})^{-2} e^{-B\ell} \sqrt{\ell + 1}.$$

Then the proposition follows from Eq. (3.9) and Eq. (3.12). □

Recall from Lemma 2.23, for a family of Schottky groups satisfying condition (★), we have a uniform Schottky basis.

Proposition 3.12. *Suppose we have a degenerating family of Schottky groups $\Gamma_z < \mathrm{SL}_2(\mathbb{R})$ satisfying (★) with their Schottky discs uniformly bounded away from ∞. For any $N \geqslant 1$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a} \neq \mathfrak{a}_{i_1}^{-1}$, and sufficiently small $0 < |z| \ll_N 1$, we have for any $w \in D_{\mathbf{b},z}$, and the disc $D_{\mathbf{a},z}$ with $\mathbf{a} = \mathfrak{a}\mathbf{b}'$,*

$$\begin{aligned} |\log |\mathfrak{a}'(w)|| &\lesssim_N \log(1/|z|), \\ \mathfrak{a}(D_{\mathbf{b},z}) &\subset \frac{1}{10} D_{\mathbf{a},z} \end{aligned} \tag{3.18}$$

in terms of the absolute norm. Moreover, for any $M > 0$, there exists $N(M) \geqslant 1$ such that if we require $\mathbf{b} \in \mathcal{W}^{N(M)}$ additionally, then

$$|\arg \mathfrak{a}'(w)| \lesssim_M |z|^M.$$

Consequently, given any $C, M > 0$, the family of zeta functions $Z(\Gamma_z, s/\log(1/|z|))$ with $0 < |z| < 1/e$ is uniformly bounded (depending on C and M) in the region

$$|\Re s| \leqslant C, \quad |\Im s| \leqslant C|z|^{-M} \log(1/|z|).$$

Proof. By assumption, $\mathfrak{a} \in \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$, so we can write $\mathfrak{a} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{M}(\mathbb{D})$. Fix any $0 < |z| \ll 1$. The uniform separation $\mathfrak{a}(D_{\mathbf{b},z}) \subset \frac{1}{10} D_{\mathbf{a},z}$ follows from Corollary 3.23.

For any $w \in D_{\mathfrak{b},z}$, we have $\mathfrak{a}(w) = \frac{a_z w + b_z}{c_z w + d_z}$, where the lower subscript means to evaluate the functions at z. Then $\mathfrak{a}'(w) = 1/(c_z w + d_z)^2$. The estimate $|z|^A \leqslant |\mathfrak{a}'(w)|$ follows from the fact that the Schottky discs are uniformly bounded away from ∞. The other estimate $|\mathfrak{a}'(w)| \leqslant |z|^{-A}$ follows from that the definition of uniform Schottky discs Eq. (2.4) and the convention Eq. (2.5) give

$$\inf_{w \in D_{\mathbf{b},z}} |w + d_z/c_z| \geqslant d(\partial D_{\mathfrak{a}^{-1},z}, \mathfrak{a}^{-1}\infty) \geqslant \mathrm{diam}(D_{\mathfrak{a}^{-1},z})/2.$$

Finally, it follow from Lemma 3.3 that by taking N sufficiently large, we can make sure $\mathrm{diam}(D_{\mathbf{b},z}) = O(|z|^{cN})$. The estimate $|\arg \mathfrak{a}'(w)| \leqslant |z|^M$ for $w \in D_{\mathbf{b},z}$ follows from the observation that

$$\mathfrak{a}'(w) = (c_z w + d_z)^{-2} = c_z^{-2}((\Re w + d_z/c_z) + i\Im w)^{-2}$$

and $D_{b,z} \cap \mathbb{R} \neq \emptyset$.

For the boundedness of $Z(\Gamma_z, s/\log(1/|z|))$, it suffices to assume $0 < |z| \ll_N 1$ since otherwise, the statement follows from the continuity of the zeta function. Then the hypothesis of Proposition 3.11 is satisfied with $A = A_0 \log(1/|z|)$ where A_0 is fixed and $\varphi = O(|z|^M)$. We can apply the conclusion of Proposition 3.11 and Eq. (3.11) to obtain the boundedness of $Z(\Gamma_z, s/\log(1/|z|)$. ☐

3.2.3 Transfer operator: $\mathrm{SL}_2(\mathbb{C})$ case

Let Γ be a classical Schottky group in $\mathrm{SL}_2(\mathbb{C})$ of rank $g \geqslant 2$. The discussion is similar but has several complications, which we detail below.

Write $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{2g} \rangle$ with $\mathfrak{a}_i \mathfrak{a}_{i+g} = id$ for $i = 1, 2, \cdots, g$. By conjugating Γ if necessary, there exist open discs $D_{\mathfrak{a}_1}, D_{\mathfrak{a}_2}, \cdots, D_{\mathfrak{a}_{2g}} \subset \mathbb{C}$ such that their closures are disjoint, and for each $\mathfrak{a} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}$, we have $\mathfrak{a}(\hat{\mathbb{C}} - \overline{D_{\mathfrak{a}^{-1}}}) = D_{\mathfrak{a}}$.

To study the Selberg zeta function, we identify $\mathbb{C}$ with $\mathbb{R}^2$, and given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, complexify the linear fractional transformation $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\gamma(x, y) = \left(\Re \frac{a(x+iy)+b}{c(x+iy)+d}, \Im \frac{a(x+iy)+b}{c(x+iy)+d} \right), \quad (x, y) \in \mathbb{R}^2 \tag{3.19}$$

to

$$\widetilde{\gamma}(w_1, w_2) = \frac{1}{2} \left(\frac{a(w_1+iw_2)+b}{c(w_1+iw_2)+d} + \frac{\bar{a}(w_1-iw_2)+\bar{b}}{\bar{c}(w_1-iw_2)+\bar{d}}, -i\frac{a(w_1+iw_2)+b}{c(w_1+iw_2)+d} + i\frac{\bar{a}(w_1-iw_2)+\bar{b}}{\bar{c}(w_1-iw_2)+\bar{d}} \right), \quad (w_1, w_2) \in \mathbb{C}^2 \tag{3.20}$$

For each generator, suppose $D_{\mathfrak{a}} = \{|w - w_{\mathfrak{a}}| \leqslant R_{\mathfrak{a}}\}$. Let $\widetilde{D}_{\mathfrak{a}}$ be the polydisc defined by

$$\widetilde{D}_{\mathfrak{a}} = \{(w_1, w_2) \in \mathbb{C}^2 : \max(|w_1 - \Re w_{\mathfrak{a}}|, |w_2 - \Im w_{\mathfrak{a}}|) \leqslant R_{\mathfrak{a}}/3\},$$

and we also view $D_{\mathfrak{a}} \cap \widetilde{D}_{\mathfrak{a}}$ as a subset of $\widetilde{D}_{\mathfrak{a}}$.

We introduce the Hilbert space $\widetilde{\mathcal{H}} = \oplus_{i=1}^{2g} \mathcal{H}(\widetilde{D}_{\mathfrak{a}_i})$ where $\mathcal{H}(\widetilde{D}_{\mathfrak{a}_i}) = \{u \in L^2(\widetilde{D}_{\mathfrak{a}_i}) : u \text{ is holomorphic}\}$ with the norm $\|u\|^2_{\mathcal{H}(\widetilde{D}_{\mathfrak{a}_i})} := |\widetilde{D}_{\mathfrak{a}_i}|^{-1} \|u\|^2_{L^2(\widetilde{D}_{\mathfrak{a}_i})}$. Suppose $\mathfrak{a}_i$ maps $\widetilde{D}_{\mathfrak{b}}$ into a compact subset of $\widetilde{D}_{\mathfrak{a}_i}$ for any $\mathfrak{b} \neq \mathfrak{a}_i^{-1}$. The transfer operator $\mathcal{L}_s : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is defined as

$$\mathcal{L}_s u(w_1, w_2) := \sum_{\mathfrak{a}_i \neq \mathfrak{b}^{-1}} [\mathfrak{a}_i'(w_1, w_2)]^s u(\widetilde{\mathfrak{a}}_i(w_1, w_2)) \quad \text{for} \quad (w_1, w_2) \in \widetilde{D}_{\mathfrak{b}} \quad \text{with} \quad \mathfrak{b} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{2g}\}. \tag{3.21}$$

Here $[\mathfrak{a}_i'(w_1, w_2)]$ is a holomorphic function in $\widetilde{D}_{\mathfrak{b}}$ such that $[\mathfrak{a}_i'(w_1, w_2)] = |\mathfrak{a}_i'(x + iy)|$ for $(w_1, w_2) = (x, y) \in D_{\mathfrak{b}}$. More precisely, if $\mathfrak{a}_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, then

$$[\mathfrak{a}_i'(w_1, w_2)] = \frac{1}{(c(w_1 + iw_2) + d)(\bar{c}(w_1 - iw_2) + \bar{d})}. \tag{3.22}$$

Eq. (3.21) requires choosing a branch of complex logarithm such that $\log[\mathfrak{a}_i'(w_1,w_2)]$ is a holomorphic function on $\widetilde{D}_{\mathfrak{b}}$ satisfying $\log[\mathfrak{a}_i'(w_1,w_2)] = \log|\mathfrak{a}_i'(x+iy)|$ for $(w_1,w_2)=(x,y)\in D_{\mathfrak{b}}\cap\widetilde{D}_{\mathfrak{b}}$. This is achievable because the inequality $|w_{\mathfrak{b}}+d/c| > R_{\mathfrak{b}}$ gives $|w_1+iw_2+d/c| > R_{\mathfrak{b}}/3$ and hence $[\mathfrak{a}_i'(w_1,w_2)]\neq 0$ for $(w_1,w_2)\in\widetilde{D}_{\mathfrak{b}}$. Define $[\mathfrak{a}_i'(w_1,w_2)]^s := \exp(s\log[\mathfrak{a}_i'(w_1,w_2)])$ on $\widetilde{D}_{\mathfrak{b}}$.

We show that the pullback operator $\widetilde{\gamma}^*$ is again a trace class operator.

Lemma 3.13. *Suppose $\widetilde{\gamma}:\widetilde{D}_{\mathfrak{b}}\to\rho\widetilde{D}_{\mathfrak{a}}$ is a holomorphic function between polydiscs in $\mathbb{C}^2$ with $0<\rho<1$ and $\mathfrak{a},\mathfrak{b}\in\{\mathfrak{a}_1,\dots,\mathfrak{a}_{2g}\}$. Then the pullback operator*

$$\widetilde{\gamma}^*u(w_1,w_2) = u(\widetilde{\gamma}(w_1,w_2)) : \mathcal{H}(\widetilde{D}_{\mathfrak{a}})\to\mathcal{H}(\widetilde{D}_{\mathfrak{b}})$$

is a trace class operator satisfying

$$\mu_\ell(\widetilde{\gamma}^*) \leqslant C_\rho\rho^{\sqrt{\ell}}(\ell+1).$$

In the following, we will use the multi-index notation:

$$\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}^2_{\geqslant 0},\quad |\alpha|=\alpha_1+\alpha_2,\quad\text{and}\quad \alpha! = \alpha_1!\alpha_2!.$$

Proof. We may assume $\widetilde{D}_{\mathfrak{a}} := \{(w_1,w_2)\in\mathbb{C}^2 : |w_1|<R_1, |w|_2<R_2\}$. Consider the orthonormal basis $\phi_\alpha(w_1,w_2) := \sqrt{(\alpha_1+1)(\alpha_2+1)}(w_1/R_1)^{\alpha_1}(w_2/R_2)^{\alpha_2}$ of $\mathcal{H}(\widetilde{D}_{\mathfrak{a}})$. Then

$$\|\widetilde{\gamma}^*\phi_\alpha\|^2_{\mathcal{H}(\widetilde{D}_{\mathfrak{b}})} = |\widetilde{D}_{\mathfrak{b}}|^{-1}\int_{\widetilde{D}_{\mathfrak{b}}}(\alpha_1+1)(\alpha_2+1)\frac{|\pi_1(\gamma(w_1,w_2))|^{2\alpha_1}}{R_1^{2\alpha_1}}\frac{|\pi_2(\gamma(w_1,w_2))|^{2\alpha_2}}{R_2^{2\alpha_2}}dm(w_1,w_2)\leqslant(|\alpha|+1)^2\rho^{2|\alpha|},$$

where $\pi_i:\mathbb{C}^2\to\mathbb{C}$ is to project on the i-th coordinate. By the minimax property Eq. (3.8), we conclude

$$\mu_\ell(\widetilde{\gamma}^*)\leqslant\sum_{|\alpha|^2\geqslant\ell-10}\|\widetilde{\gamma}^*\phi_\alpha\|_{\mathcal{H}(\widetilde{D}_{\mathfrak{b}})}\leqslant\sum_{k^2\geqslant\ell-10}(k+1)^2\rho^k\leqslant C_\rho\rho^{\sqrt{\ell}}(\ell+1).$$

Thus

$$\sum_\ell\mu_\ell(\widetilde{\gamma}^*)\leqslant C_\rho\sum_\ell\rho^{\sqrt{\ell}}(\ell+1)<\infty,$$

which implies $\widetilde{\gamma}^*$ is a trace class operator. $\square$

Similar to the $\mathrm{SL}_2(\mathbb{R})$ case, we have

$$\det(1-\mathcal{L}_s) = Z(\Gamma,s)\quad\text{for any}\quad s\in\mathbb{C}.$$

We refer to [GLZ04] for more details.

As in the $\mathrm{SL}_2(\mathbb{R})$ case, we also consider the modified transfer operators to obtain further applications. The precise construction is as follows. For $\mathbf{a}=\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}\in\mathcal{W}^n$, define the polydisc $\widetilde{D}_{\mathbf{a}}$ in $\mathbb{C}^2$ associated to $D_{\mathbf{a}} = D(w_{\mathbf{a}},R_{\mathbf{a}})\subset\mathbb{C}$ by

$$\widetilde{D}_{\mathbf{a}} = \{(w_1,w_2) : \max(|w_1-\Re w_{\mathbf{a}}|, |w_2-\Im w_{\mathbf{a}}|) < R_{\mathbf{a}}/3\}.$$

For any $N\in\mathbb{N}$, the modified transfer operator $\mathcal{L}_{s,N}$ on $\mathcal{H}=\mathcal{H}_N := \oplus_{b\in\mathcal{W}^N}\mathcal{H}(\widetilde{D}_{\mathbf{b}})$ is defined by

$$\mathcal{L}_{s,N}u(w_1,w_2) := \sum_{\mathfrak{a}\neq\bar{\mathfrak{a}}_{i_1}}[\mathfrak{a}'(w_1,w_2)]^s u(\widetilde{\mathfrak{a}}(w_1,w_2))\quad\text{for}\quad(w_1,w_2)\in\widetilde{D}_{\mathbf{b}}\quad\text{with}\quad\mathbf{b}=\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}.$$

Again we have for any $N\in\mathbb{N}$,

$$\det(1-\mathcal{L}_{s,N}) = Z(\Gamma,s)\quad\text{for any}\quad s\in\mathbb{C}.$$

We state a proposition similar to Proposition 3.11 for $\mathrm{SL}_2(\mathbb{C})$.

Proposition 3.14. *Suppose $\Gamma<\mathrm{SL}_2(\mathbb{C})$ is a Schottky group with the notations above. Suppose for some $A,B>0$, $N\in\mathbb{Z}_{>0}$ and $\varphi\in[0,\pi]$, we have for any $\mathfrak{a}\in\mathcal{W}$ and $\mathbf{b}=\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}\in\mathcal{W}^N$ with $\mathfrak{a}_{i_1}\neq\mathfrak{a}^{-1}$,*

- $e^{-A} \leqslant |[\mathfrak{a}'(w_1, w_2)]| \leqslant e^A$ *and* $|\arg[\mathfrak{a}'(w_1, w_2)]| \leqslant \varphi \leqslant \pi$ *for any* $(w_1, w_2) \in \widetilde{D}_{\mathbf{b}}$;
- $\widetilde{\mathfrak{a}}(\widetilde{D}_{\mathbf{b}}) \subset e^{-B}\widetilde{D}_{\mathbf{a}}$ *where* $\mathbf{a} = \mathfrak{a}\mathbf{b}'$.

Then the singular values of $\mathcal{L}_s$ *satisfy*

$$\mu_\ell(\mathcal{L}_{s,N}) \leqslant C(g, N, B)e^{(A|\Re s|+\varphi|\Im s|-B\ell^{1/2})/C(g,N)}(\ell + 1).$$

Consequently, the operator $\mathcal{L}_{s,N}$ *is a trace class operator.*

Proof. The proof is similar to Proposition 3.11. The operator $\mathcal{L}_{s,N}$ is a direct sum of the following operators: Consider each component of $\mathcal{L}_{s,N}$:

$$\mathcal{L}_{s,\mathbf{ab}} : \mathcal{H}(\widetilde{D}_{\mathbf{a}}) \to \mathcal{H}(\widetilde{D}_{\mathbf{b}}), \quad \mathcal{L}_{s,\mathbf{ab}}u(w_1, w_2) := [\mathfrak{a}'(w_1, w_2)]^s u(\widetilde{\mathfrak{a}}(w_1, w_2)), \quad \mathbf{a} = \mathfrak{a}\mathbf{b}',$$

where $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}_{i_1} \neq \mathfrak{a}^{-1}$, and $\mathbf{a} = \mathfrak{a}\mathbf{b}'$.

We have

- $|[\mathfrak{a}'(w_1, w_2)]^s| \leqslant \max\left(|[\mathfrak{a}'(w_1, w_2)]|^{\Re s}, |[\mathfrak{a}'(w_1, w_2)]|^{-\Re s}\right) \exp(|\arg[\mathfrak{a}'(w_1, w_2)]||\Im s|) \leqslant e^{A|\Re s|+\varphi|\Im s|}$;
- By Lemma 3.13, we know
$$\mu_\ell(\widetilde{\mathfrak{a}}^*) \leqslant C(B)e^{-B\ell^{1/2}}(\ell + 1).$$

Then the theorem follows from Eq. (3.9) and Eq. (3.12). □

Similar to Proposition 3.12, we have the following.

Proposition 3.15. *Suppose we have a degenerating family of Schottky groups* $\Gamma_z < \mathrm{SL}_2(\mathbb{C})$ *satisfying* (★) *with their Schottky discs uniformly bounded away from* ∞. *For* $N \geqslant 1$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ *with* $\mathfrak{a} \neq \mathfrak{a}_{i_1}^{-1}$, *and sufficiently small* $0 < |z| \ll_N 1$, *we have for any* $(w_1, w_2) \in \widetilde{D}_{\mathbf{b},z}$, *and the disc* $\widetilde{D}_{\mathbf{a},z}$ *with* $\mathbf{a} = \mathfrak{a}\mathbf{b}'$,

$$|\log|[\mathfrak{a}'(w_1, w_2)]|| \lesssim \log(1/|z|)$$
$$\widetilde{\mathfrak{a}}(\widetilde{D}_{\mathbf{b},z}) \subset \frac{1}{10}\widetilde{D}_{\mathbf{a},z} \tag{3.23}$$

in terms of the absolute norm. Moreover, for any $M > 0$, *there exists* $N(M) \geqslant 1$ *such that if we require* $\mathbf{b} \in \mathcal{W}^{N(M)}$ *additionally, then*

$$|\arg[\mathfrak{a}'(w_1, w_2)]| \lesssim |z|^M.$$

Consequently, given any $C, M > 0$, *the family of zeta functions* $Z(\Gamma_z, s/\log(1/|z|))$ *with* $0 < |z| < 1/e$ *is uniformly bounded (depending on* C *and* M*) in the region*

$$|\Re s| \leqslant C, \quad |\Im s| \leqslant C|z|^{-M}\log(1/|z|).$$

Proof. By Corollary 3.23, there exists $v > 0$ such that for any $N \in \mathbb{N}$, $\mathfrak{a} \in \mathcal{W}$, $\mathbf{b} = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N} \in \mathcal{W}^N$ with $\mathfrak{a}\mathfrak{a}_{i_1} \neq id$, and sufficiently small $0 < |z| \ll 1$, we have

$$\mathfrak{a}(D_{\mathbf{b},z}) \subset |z|^v D_{\mathbf{a},z}. \tag{3.24}$$

where $\mathbf{a} = \mathfrak{a}\mathbf{b}'$.

Fix $0 < |z| \ll 1$. We first show Eq. (3.23). Suppose $D_{\mathbf{a},z} = D(w_{\mathbf{a},z}, R_{\mathbf{a},z})$ and $D_{\mathbf{b},z} = D(w_{\mathbf{b},z}, R_{\mathbf{b},z})$. Then $|\mathfrak{a}(w) - w_{\mathfrak{a},z}| \leqslant |z|^v R_{\mathbf{a},z}$ for $w \in D_{\mathbf{b},z}$. Using the Cauchy integral formula, we estimate the derivatives of $\mathfrak{a}$ on $D_{\mathbf{b},z}$ at $w_{\mathbf{b},z}$: for any $k \in \mathbb{N}$,

$$|\frac{d^k}{dw^k}\mathfrak{a}(w_{\mathbf{b},z})| = k!\left|\int_{\partial D_{\mathbf{b},z}} \frac{\mathfrak{a}(w)}{(w - w_{\mathbf{b},z})^{k+1}}dw\right| = k!\left|\int_{\partial D_{\mathbf{b},z}} \frac{\mathfrak{a}(w) - w_{\mathbf{a},z}}{(w - w_{\mathbf{b},z})^{k+1}}dw\right| \leqslant 2\pi\frac{k!}{R_{\mathbf{b},z}^k}|z|^v R_{\mathbf{a},z}.$$

We can then form the analytic continuation

$$\widetilde{\mathfrak{a}}(w_1,w_2)=\mathfrak{a}(w_{\mathbf{b},z})+ \sum_{\substack{\alpha\in\mathbb{Z}^2_{\geqslant 0}\\ \text{with } |\alpha|>0}} \frac{1}{\alpha!}(w_1-\Re w_{\mathbf{b},z})^{\alpha_1}(w_2-\Im w_{\mathbf{b},z})^{\alpha_2}\left(\partial_x^{\alpha_1}\partial_y^{\alpha_2}\Re\mathfrak{a}(x+iy),\partial_x^{\alpha_1}\partial_y^{\alpha_2}\Im\mathfrak{a}(x+iy)\right)|_{x+iy=w_{\mathbf{b},z}}\in\mathbb{C}^2.$$

Combining the above two equations, we obtain that for any $(w_1,w_2)\in\widetilde{D}_{\mathbf{b},z}$,

$$\begin{aligned}|\widetilde{\mathfrak{a}}(w_1,w_2)-w_{\mathbf{a},z}|&\leqslant|z|^vR_{\mathbf{a},z}+4\pi\sum_\alpha\frac{|\alpha|!}{\alpha!R_{\mathbf{b}}^{|\alpha|}}|z|^vR_{\mathbf{a},z}(R_{\mathbf{b},z}/3)^{|\alpha|}\\ &=|z|^vR_{\mathbf{a},z}+4\pi\sum_k\left(\frac{2}{3}\right)^k|z|^vR_{\mathbf{a},z}\leqslant(8\pi+1)|z|^vR_{\mathbf{a},z}.\end{aligned}$$

Thus, we have Eq. (3.23).

Next, we estimate $|\log|[\mathfrak{a}'(w_1,w_2)]||$. We write $\mathfrak{a}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ with $a,b,c,d\in\mathbb{M}(\mathbb{D})$. For any $w\in D_{\mathfrak{b},z}$, we have $\mathfrak{a}(w)=\frac{a_zw+b_z}{c_zw+d_z}$ and $\mathfrak{a}'(w)=1/(c_zw+d_z)^2$, where the lower subscript means to evaluate the functions at z. By Eq. (3.22), we have for any $(w_1,w_2)\in\widetilde{D}_{\mathfrak{b},z}$,

$$\begin{aligned}[\mathfrak{a}'(w_1,w_2)]&=|c_z|^{-2}(w_1+iw_2-w_{\mathfrak{b},z}+w_{\mathfrak{b},z}+d_z/c_z)^{-1}(w_1-iw_2-\bar{w}_{\mathbf{b},z}+\bar{w}_{\mathbf{b},z}+\bar{d}_z/\bar{c}_z)^{-1}\\ &=|\mathfrak{a}'(w_{\mathbf{b},z})|^2\left(1+\frac{w_1+iw_2-w_{\mathbf{b},z}}{w_{\mathbf{b},z}+d_z/c_z}\right)^{-1}\left(1+\frac{w_1-iw_2-\bar{w}_{\mathbf{b},z}}{\bar{w}_{\mathbf{b},z}+\bar{d}_z/\bar{c}_z}\right)^{-1}\end{aligned}\tag{3.25}$$

Using $|w_{\mathfrak{b},z}+d_z/c_z|>R_{\mathfrak{b},z}$, and $\max(|w_1-\Re w_{\mathfrak{b}_z}|,|w_2-\Im w_{\mathfrak{b},z}|)\leqslant R_{\mathfrak{b},z}/3$, we have that both the absolute values of the second and third factors are bounded above by $5/3$ and below by $1/3$. Using the argument of $\mathrm{SL}_2(\mathbb{R})$ case (proof of Proposition 3.12), we have that there exists $A>0$, such that $|z|^A\leqslant|\mathfrak{a}'(w_{\mathbf{b},z})|\leqslant|z|^{-A}$. Hence, we have established the estimate for $|\log|[\mathfrak{a}'(w_1,w_2)]||$.

We estimate $|\arg[\mathfrak{a}'(w_1,w_2)]|$. It follows from Corollary 3.23 that $D_{\mathbf{b},z}\subset|z|^vD_{\mathfrak{a}_{i_1},z}$ and $R_{\mathbf{b},z}\leqslant|z|^{vN}R_{\mathfrak{a}_{i_1},z}$. Then, we have

$$\frac{|w_1+iw_2-w_{\mathbf{b},z}|}{|w_{\mathbf{b},z}+d_z/c_z|}\leqslant\frac{2}{3}|z|^v(1-|z|^v)^{-1},$$

which yields the estimate $|\arg[\mathfrak{a}'(w_1,w_2)]|\lesssim|z|^M$ using Eq. (3.25).

Finally, we use Proposition 3.14 to obtain the boundedness of the zeta functions. □

3.3 Convergence on the whole plane

Lemma 3.16. *Let Ω be a connected open set in $\mathbb{C}$ and $f(z)$, $f_n(z)$ be a family of holomorphic functions on Ω. Suppose*

- *$|f_n(z)|\leqslant C$ for any $z\in\Omega$ for some constant $C>0$;*
- *$f_n(z)\to f(z)$ pointwisely in some open set $\Omega'\subset\Omega$, as $n\to\infty$.*

Then $f_n(z)\to f(z)$ uniformly on any compact subset of Ω as $n\to\infty$.

Proof. Since $f_n(z)$ is uniformly bounded, by Montel's theorem, any subsequence of $\{f_n\}$ has a subsequence that converges uniformly on any compact subset. Suppose the lemma is false, then there exists a subsequence of f_n that converges locally uniformly but not to $f(z)$. This contradicts the second condition. □

Therefore, the convergence part of Theorem 1.3 follows from Lemma 3.16 by combining Theorem 3.1, Proposition 3.12 and Proposition 3.15.

3.4 Uniform separation

In this subsection, we verify the uniform separation property in Eq. (3.18) of Proposition 3.12 and Eq. (3.24) of Proposition 3.15 using the continuity of the hybrid model. The idea is to relate discs in Archimedean and non-Archimedean cases and then compute the discs under the action of Γ in both cases. The key input is Lemma 2.23.

We need [PT21, Lemma 3.2.2][9]

Lemma 3.17. *Let k be a complete valued field. If k is Archimedean, we suppose $(k, |\cdot|)$ is isometrically embedded into $(\mathbb{C}, |\cdot|_\infty^\epsilon)$ for some $\epsilon \in (0,1]$, where $|\cdot|_\infty$ is the usual absolute value on $\mathbb{C}$. Let $\rho > 0$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$. If $\gamma D^+(\alpha,\rho) \subset \mathbb{A}_k^{1,an}$, then $|\alpha + (d/c)| > \rho$ and*

$$\gamma D^+(\alpha,\rho) = \begin{cases} D^+\left(\left(\frac{a}{c} - \frac{ad-bc}{c^2} \frac{\overline{\alpha + d/c}}{|\alpha+d/c|^{2/\epsilon} - \rho^{2/\epsilon}}, \frac{|ad-bc|\rho}{|c|^2(|\alpha+d/c|^{2/\epsilon}-\rho^{2/\epsilon})^\epsilon}\right)\right), & \text{if } k \text{ is Archimedean;} \\ D^+\left(\frac{b+a\alpha}{c(\alpha+d/c)}, \frac{|ad-bc|\rho}{|c|^2|\alpha+d/c|^2}\right), & \text{otherwise.} \end{cases} \tag{3.26}$$

Recall that for the non-Archimedean case, an open disc $D^-(\beta,\lambda)$ is contained in a closed disc $D^+(\alpha,\rho)$ with $\alpha, \beta \in k$ and $\lambda, \rho > 0$ iff

$$\lambda < \rho \quad \text{and} \quad |\alpha - \beta| < \rho.$$

We recall the continuous map $p : \overline{\mathbb{D}}_r \to \mathbb{A}_{A_r}^{3g-3,an}$ in the hybrid model: for $0 < |z| < r$, the Archimedean norm $|\cdot|_z$ evaluates a function $f \in \mathbb{C}((t))$ which is holomorphic on $\mathbb{D}_r^*$ and meromorphic at 0 by the formula

$$|f|_z = |f(z)|_\infty^{1/\log(1/|z|)},$$

and we have as $|z| \to 0$

$$|f|_z \to |f|_{na}.$$

A ball with center α and radius ρ satisfies

$$D_z^\pm(\alpha,\rho) := D_\infty^\pm(\alpha, \rho^{\log(1/|z|)}) = D_\infty^\pm(\alpha, |z|^{\log(1/\rho)}).$$

Here, we use the lower subscript z (resp. na) to emphasize that the ball is measured using the norm $|\cdot|_z$ (resp. $|\cdot|_{na}$).

Lemma 3.18. *Let $\gamma \in \mathrm{PGL}_2(\mathbb{C}((t)))$, and $\alpha, \beta \in \mathbb{C}((t))$ which are holomorphic on $\mathbb{D}_r^*$ and meromorphic at 0. Suppose for some $\rho, \lambda > 0$, we have $\gamma D_{na}^-(\alpha,\rho) \subset D_{na}^+(\beta,\lambda)$. Then there exists $v > 0$ such that for any $0 < |z| \ll 1$, we have*

$$\gamma_z D_z^-(\alpha(z),\rho) \subset D_z^+(\beta(z), \lambda e^{-v})$$

and in terms of the absolute norm, we have

$$\gamma_z D_\infty^-(\alpha(z), |z|^{\log(1/\rho)}) \subset |z|^v D_\infty^+(\beta(z), |z|^{\log(1/\lambda)}).$$

Proof. Assume $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{C}((t))$. From Lemma 3.17 and $\gamma D_{na}^-(\alpha, r) \subset \mathbb{A}_k^{1,an}$ with $k = \mathbb{C}((t))$, we have

$$|\alpha + d/c|_{na} > \rho \quad \text{and} \quad \frac{|ad-bc|_{na}\rho}{|c|_{na}^2 |\alpha + d/c|_{na}^2} < \lambda. \tag{3.27}$$

For the Archimedean radius, let $\epsilon = 1/\log(1/|z|)$, then by Lemma 3.17 the radius of $\gamma_z D_z^-(\alpha(z),\rho)$ is equal to

$$\frac{|ad-bc|_z \rho}{|c|_z^2 (|\alpha+d/c|_z^{2/\epsilon} - \rho^{2/\epsilon})^\epsilon} = \frac{|(ad-bc)|_z\rho}{|c|_z^2|\alpha+d/c|_z^2} \frac{1}{(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon}.$$

Due to $\rho/|\alpha+d/c|_z \to \rho/|\alpha+d/c|_{na} < 1$ and $\epsilon \to 0$ as $|z| \to 0$, we obtain that

$$(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon \to 1, \tag{3.28}$$

[9] In [PT21, Lemma 3.2.2], they forget to take into account the ϵ in the Archimedean case.

as $|z| \to 0$. Hence the Archimedean radius converges to the non-Archimedean radius, and by continuity, we obtain that for $|z|$ small, the Archimedean radius also satisfies

$$\frac{|ad-bc|_z \rho}{|c|_z^2(|\alpha+d/c|_z^{2/\epsilon}-\rho^{2/\epsilon})^\epsilon} < \lambda.$$

In particular, there exists $v_0 > 0$ such that for $|z|$ small, we have a stronger inequality

$$\frac{|ad-bc|_z \rho}{|c|_z^2(|\alpha+d/c|_z^{2/\epsilon}-\rho^{2/\epsilon})^\epsilon} < \lambda e^{-v_0}.$$

In terms of the absolute norm, for $|z|$ sufficiently small, the Archimedean radius (the left-hand side of the following inequality) satisfies

$$\left(\frac{|ad-bc|_z r}{|c|_z^2(|\alpha+d/c|_z^{2/\epsilon}-(r)^{2/\epsilon})^\epsilon}\right)^{1/\epsilon} \leqslant |z|^{v_0}\lambda^{1/\epsilon}.$$

Next, we consider the center of the ball $\gamma_z D_z^+(\alpha, r)$ for $0 \leqslant |z| \ll 1$. For the non-Archimedean case (i.e., $z=0$), from the hypothesis, we have $|\frac{b+a\alpha}{c(\alpha+d/c)} - \beta|_{na} < \lambda$. Using the convergence $|\cdot|_z \to |\cdot|_{na}$ as $|z| \to 0$, we obtain a constant $v_1 > 0$ such that for $0 < |z| \ll 1$, we have

$$\left|\frac{b+a\alpha}{c(\alpha+d/c)} - \beta\right|_z < e^{-v_1}\lambda, \quad \text{i.e.,} \quad \left|\frac{b+a\alpha}{c(\alpha+d/c)} - \beta\right|_\infty < |z|^{v_1}\lambda^{1/\epsilon}.$$

Using Lemma 3.17, the distance between the Archimedean centre δ_a and the non-Archimedean center δ_{na} (view them as functions of z) is given by

$$\begin{aligned}\delta_{na} - \delta_a = \frac{b+a\alpha}{c(\alpha+d/c)} - \left(\frac{a}{c} - \frac{ad-bc}{c^2}\frac{\overline{\alpha+d/c}}{|\alpha+(d/c)|_z^{2/\epsilon}-\rho^{2/\epsilon}}\right) &= -\frac{ad-bc}{c^2(\alpha+d/c)}\left(1-\frac{1}{1-(\rho/|\alpha+d/c|_z)^{2/\epsilon}}\right)\\ &= \frac{ad-bc}{c^2(\alpha+d/c)}\frac{(\rho/|\alpha+d/c|_z)^{2/\epsilon}}{1-(\rho/|\alpha+d/c|_z)^{2/\epsilon}}.\end{aligned}$$

Its $|\cdot|_z$ norm equals (notice that for a real number $|x|_z = |x|^\epsilon$)

$$\begin{aligned}|\delta_{na}-\delta_a|_z &= \left|\frac{ad-bc}{c^2(\alpha+d/c)}\right|_z \left(\frac{\rho}{|(\alpha+d/c)|_z}\right)^2 \frac{1}{(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon}\\ &= \rho\left|\frac{ad-bc}{c^2(\alpha+d/c)^2}\right|_z \frac{\rho}{|(\alpha+d/c)|_z}\frac{1}{(1-(\rho/|\alpha+d/c|_z)^{2/\epsilon})^\epsilon} < \lambda,\end{aligned} \tag{3.29}$$

where the inequality is due to Eq. (3.27) and Eq. (3.28), and holds for $|z|$ small. As the estimate for the radius, we can replace λ by $e^{-v_2}\lambda$ for some $v_2 > 0$ in the previous inequality, which holds for $|z|$ small. Combining the estimate of radius and distance between centers, we obtain for $|z|$ sufficiently small and any $x \in \gamma_z D_z^-(\alpha(z), \rho)$,

$$\begin{aligned}|x-\beta|_z = |x-\beta|_\infty^\epsilon &\leqslant (|x-\delta_a|_\infty + |\delta_a - \delta_{na}|_\infty + |\delta_{na}-\beta|_\infty)^\epsilon\\ &\leqslant (|z|^{v_0}\lambda^{1/\epsilon} + |z|^{v_2}\lambda^{1/\epsilon} + |z|^{v_1}\lambda^{1/\epsilon})^\epsilon \leqslant 3^\epsilon \lambda e^{-v} < \lambda e^{-v/2},\end{aligned}$$

where $v = \min\{v_0, v_1, v_2\}$. This completes the proof. □

Remark 3.19. From Eq. (3.29), as $z \to 0$, the $|\cdot|_z$-norm of the difference of two centers does not go to zero.

We are ready to prove the main result of this part:

Proposition 3.20. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★). There exists $v > 0$ such that for any $0 < |z| \ll 1$, the corresponding Schottky group and the Schottky basis given by Lemma 2.23 satisfy*

$$\mathfrak{a}_z(D_{\mathfrak{b},z}) \subset |z|^v D_{\mathfrak{a},z},$$

in terms of the absolute norm, for any generators $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} \neq id$.

Proof. Recall that Lemma 2.23 gives a uniform Schottky basis for all $|z|$ small. In particular, for $z = 0$, the Schottky basis for $\Gamma < \mathrm{PGL}_2(k)$ with $k = \mathbb{C}((t))$ equipped with the norm $|\cdot|_{na}$ is given by

$$\mathcal{B} = \{D^+_{\delta_i,\lambda_i}, D^+_{\delta_i^{-1},\lambda_i^{-1}},\ i = 1, \cdots, g\},$$

Here the closed discs are given as in Eq. (2.4). Writing $\delta_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, we have

$$\begin{aligned} D^+_{\delta_i,\lambda_i} &:= D^+_{na}\left(\delta_i^{-1}\infty, \lambda_i^{1/2}\left|\frac{(a_id_i - b_ic_i)}{c_i^2}\right|^{1/2}_{na}\right), \\ \delta_i^{-1}D^-_{\delta_j,\lambda_j} &\subset D^+_{\delta_i,\lambda_i} \quad \text{for any} \quad i \neq j. \end{aligned} \tag{3.30}$$

Then we apply Lemma 3.18 to obtain the uniform separation. More precisely, take any $\delta_i = \mathfrak{a}^{-1}$ and $\delta_j = \mathfrak{b}^{-1}$ with $i \neq j$. For the non-Archimedean case $z = 0$, the Schottky discs are given by

$$D_{\mathfrak{a},0} = D^-_{\delta_i,\lambda_i} \quad \text{and} \quad D_{\mathfrak{b},0} = D^-_{\delta_j,\lambda_j}. \tag{3.31}$$

The Schottky discs for the Archimedean case are given by balls (see Eq. (2.4) and Eq. (2.5))

$$D_{\mathfrak{a},z} = D^-_z\left((\delta_i^{-1}\infty)(z), \lambda_i^{1/2}\left|\frac{(a_id_i - b_ic_i)}{c_i^2}\right|^{1/2}_z\right) \quad \text{and} \quad D_{\mathfrak{b},z} = D^-_z\left((\delta_j^{-1}\infty)(z), \lambda_j^{1/2}\left|\frac{(a_jd_j - b_jc_j)}{c_j^2}\right|^{1/2}_z\right).$$

Set

$$\lambda_{na} = \lambda_i^{1/2}\left|\frac{(a_id_i - b_ic_i)}{c_i^2}\right|^{1/2}_{na} \quad \text{and} \quad \rho_{na} = \lambda_j^{1/2}\left|\frac{(a_jd_j - b_jc_j)}{c_j^2}\right|^{1/2}_{na}.$$

Similarly, we introduce the notations $\lambda(z)$ and $\rho(z)$ by replacing $|\cdot|_{na}$ with $|\cdot|_z$.

We can apply Lemma 3.18 to the discs $D_{\mathfrak{b},0}$ and $D_{\mathfrak{a},0}$ due to the property Eq. (3.30) and the convention Eq. (3.31), and use the convergence of the radii $\rho(z) \to \rho_{na}$ and $\lambda(z) \to \lambda_{na}$ as $z \to 0$. This yields a constant $v' > 0$ such that for any $0 < |z| \ll 1$, we have

$$\mathfrak{a}_z D^-_z\left((\delta_j^{-1}\infty)(z), \rho(z)\right) = (\delta_i^{-1})_z D^-_z\left((\delta_j^{-1}\infty)(z), \rho(z)\right) \subset D^-_z\left((\delta_i^{-1}\infty)(z), e^{-v'}\lambda(z)\right), \tag{3.32}$$

and in terms of the absolute norm, we have

$$\mathfrak{a}_z D_{\mathfrak{b},z} \subset |z|^{v'} D_{\mathfrak{a},z}.$$

Repeating this argument for all possible pairs, we finish the proof. □

Remark 3.21. This proposition also implies the radius estimate in Lemma 3.3.

Remark 3.22. If the Schottky group Γ is indeed inside $\mathrm{SL}_2(\mathbb{R}((t)))$, then for $z \in \mathbb{R}$ the discs $D_{\mathfrak{a},z}$ are centered in $\mathbb{R}$ and the group Γ_z preserves $\mathbb{R} \cup \infty$. Therefore, the intersections $I_{\mathfrak{a},z} = D_{\mathfrak{a},z} \cap \mathbb{R}$ give a Schottky figure for Γ_z as a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Corollary 3.23. *Suppose $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★). There exists $v_1 > 0$ such that for any $0 < |z| \ll 1$ and for any $n \in \mathbb{N}$, the corresponding Schottky group and the Schottky basis given by Lemma 2.23 satisfy*

$$\mathfrak{a}_z(D_{\mathbf{b},z}) \subset |z|^{v_1} D_{\mathbf{a},z},$$

in terms of the absolute norm, where $\mathbf{a} = \mathfrak{a}\mathbf{b}'$ with $\mathfrak{a}$ a generator and $\mathbf{b} = \mathfrak{b}_{i_1} \cdots \mathfrak{b}_{i_n} \in \mathcal{W}^n$ satisfying $\mathfrak{a}\mathfrak{b}_{i_1} \neq id$.

Proof. Notice that $D_{\mathbf{a},z} = \mathfrak{a}\mathfrak{b}_{i_1} \cdots \mathfrak{b}_{i_{n-2}} D_{\mathfrak{b}_{i_{n-1}},z}$, $\mathfrak{a}_z(D_{\mathbf{b},z}) = \mathfrak{a}\mathfrak{b}_{i_1} \cdots \mathfrak{b}_{i_{n-2}} D_{\mathfrak{b}_{i_{n-1}}\mathfrak{b}_{i_n},z}$. The idea is to apply the distortion estimate of Lemma 3.4 to

$$\gamma := \mathfrak{a}\mathfrak{b}_{i_1} \cdots \mathfrak{b}_{i_{n-2}}, \quad D(\alpha(z), \lambda(z)) := D_{\mathfrak{b}_{i_{n-1}},z}, \quad D(\beta(z), \rho(z)) := D_{\mathfrak{b}_{i_{n-1}}\mathfrak{b}_{i_n},z},$$

and use the result that $D(\beta(z),\rho(z)) \subset D(\alpha(z), e^{-v}\lambda(z))$ for some $v>0$ due to Eq. (3.32). More precisely, let w_0 be the center of $D(\alpha(z),\lambda(z))$. For any points $w_1 \in \partial D(\alpha(z),\lambda(z))$ and $w_2 \in \partial D(\beta(z),\rho(z))$, we have

$$(1-e^{-v})\lambda(z) \leqslant |w_1 - w_2|_z \leqslant \int_0^1 \left|(\gamma^{-1} f(t))'\right|_z dt \leqslant |\gamma w_1 - \gamma w_2|_z \cdot |\gamma' w_0|_z^{-1} \cdot C\exp(R/(1-c)), \tag{3.33}$$

where $f:[0,1]\to\mathbb{C}$ is given by $f(t) = (1-t)\gamma w_2 + t\gamma w_1$, and the constants $C, R>0$ and $c\in(0,1)$ are given as in Lemma 3.4.

Let $w_3 \in D(\alpha(z),\lambda(z))$ be such that γw_3 is the center of $\gamma D(\alpha(z),\lambda(z))$. By a similar argument, we have

$$|\gamma w_3 - \gamma w_1|_z \leqslant 2\lambda(z)\cdot|\gamma'(w_0)|_z \cdot C\exp(R/(1-c)). \tag{3.34}$$

Combining Eq. (3.33) and Eq. (3.34), we obtain

$$|\gamma w_1 - \gamma w_2|_z \geqslant \frac{1}{2}C^{-2}\exp(-2R/(1-c))\cdot(1-e^{-v})\cdot|\gamma w_3 - \gamma w_1|_z,$$

which implies there exists some $v_1>0$ such that for $0<|z|\ll 1$,

$$\mathfrak{a}_z(D_{\mathbf{b},z}) \subset |z|^{v_1} D_{\mathbf{a},z}$$

in terms of the absolute norm. □

4 Speed of convergence

Let $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ be a family of Schottky groups satisfying (★). In this section, we establish a logarithmic rate for the convergence $Z(\Gamma, s/\log(1/|z|)) \to Z_I(\Gamma, s)$.

Theorem 4.1. *Given any bounded region $K\subset\mathbb{C}$, we have for any $0<|z|<1/e$,*

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_I(\Gamma,s)| \lesssim_K \frac{1}{\log(1/|z|)} \quad \textit{for} \quad s\in K.$$

4.1 Expansion of lengths

The key to obtaining an effective convergence result such as Theorem 4.1 lies in establishing the following expansion of $\ell(\gamma_z)$ with the leading term $\ell^{na}(\gamma)\log(1/|z|)$ and exponential bounds for the coefficients $a_j(\gamma)$.

Proposition 4.2. *There exists a constant $C>0$ that depends only on Γ such that for any $\gamma\in\Gamma\backslash\{id\}$, we have for any $0<|z|<e^{-C\ell^{na}(\gamma)}$, the function $\ell(\gamma_z)$ in z has the expansion*

$$\ell(\gamma_z) = \ell^{na}(\gamma)\log(1/|z|) + \Re\left(\sum_{j\geqslant 0} a_j(\gamma) z^j\right), \tag{4.1}$$

where $a_j(\gamma)$'s are complex numbers satisfying

$$|a_0(\gamma)| \leqslant C\ell^{na}(\gamma) \quad \textit{and} \quad |a_j(\gamma)| \leqslant Ce^{C\ell^{na}(\gamma)(j+1)} \quad \textit{for} \quad j\in\mathbb{N}. \tag{4.2}$$

The proof of Proposition 4.2 is based on the following Laurent expansion of the trace.

Proposition 4.3. *There exists a constant $C>0$ that depends only on Γ such that for any $\gamma\in\Gamma\backslash\{id\}$, the function $\mathrm{tr}(\gamma_z)$ in z has the Laurent expansion*

$$\mathrm{tr}(\gamma_z) = z^{-\ell^{na}(\gamma)/2}\left(\sum_{j\geqslant 0} A_j(\gamma) z^j\right) \tag{4.3}$$

where $A_j(\gamma)$'s are complex numbers satisfying

$$|A_0(\gamma)| \geqslant e^{-C\ell^{na}(\gamma)}/C \quad \textit{and} \quad |A_j(\gamma)| \leqslant Ce^{j+C\ell^{na}(\gamma)} \quad \textit{for} \quad j\in\mathbb{Z}_{\geqslant 0}. \tag{4.4}$$

Proof. Since each entry of γ is an element of $\mathbb{C}((t))$, $\operatorname{tr}(\gamma)$ is also an element of $\mathbb{C}((t))$, and the first term on the right-hand side of Eq. (4.3) is obtained using the definition of $\ell^{na}(\gamma)$.

For the upper bound of $A_j(\gamma)$, recall $\{\mathfrak{a}_k,\ 1 \leqslant k \leqslant 2g\}$ be a finite symmetric generating set of Γ. Write $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_m}$ which we may assume the word on the right-hand side is cyclically reduced as $\operatorname{tr}(\gamma_z)$ is invariant under conjugation. It follows from Lemma 3.7 that $m \lesssim \ell^{na}(\gamma)$. Each entry $\gamma(i,l)$ of γ is a sum of at most 2^m terms, with each term a product of m entries of generators. By subadditivity and submultiplicativity of the hybrid norm, the entries of γ satisfy[10]

$$\sup_{1 \leqslant i,l \leqslant 2} \|\gamma(i,l)\|_{\text{hyb}} \leqslant 2^m \sup_{\substack{1 \leqslant k \leqslant 2g \\ 1 \leqslant i,l \leqslant 2}} \{\|\mathfrak{a}_k(i,l)\|_{\text{hyb}}\}^m \leqslant e^{C\ell^{na}(\gamma)}.$$

Each entry of γ has the form $\sum_{j \geqslant j_0} c_j z^j$. By the definition of hybrid norm, we have $|c_j| \leqslant \|\sum_{j \geqslant j_0} c_j z^j\|_{\text{hyb}} e^j$. Combining these two estimates, we obtain the upper bound of $A_j(\gamma)$.

Now we prove the lower bound. By Lemma 3.5, there exists $c > 0$ such that for any conjugacy class $[\gamma]$ with $\ell^{na}(\gamma) > 1/c$ and $0 < |z| < c$, we have

$$\ell(\gamma_z) \geqslant c \log(1/|z|)\ell^{na}(\gamma).$$

Let $\lambda_1(\gamma_z), \lambda_2(\gamma_z)$ be the two eigenvalues of γ_z with $|\lambda_1(\gamma_z)| > 1$. Notice that $2\log|\lambda_1(\gamma_z)| = \ell(\gamma_z) = -2\log|\lambda_2(\gamma_z)|$. For $0 < |z| < c$

$$|\operatorname{tr}(\gamma_z)| \geqslant e^{\ell(\gamma_z)/2} - e^{-\ell(\gamma_z)/2} \geqslant \frac{1}{2}(1/|z|)^{c\ell^{na}(\gamma)/2} \geqslant 1. \tag{4.5}$$

Recall

$$\operatorname{tr}(\gamma_z) = z^{-\ell^{na}(\gamma)/2}(A_0(\gamma) + A_1(\gamma)z + \cdots). \tag{4.6}$$

Let $f(z) = A_0(\gamma) + A_1(\gamma)z + \cdots$, which is an analytic function on $|z| < 1/e$. Using Eqs. (4.5) and (4.6) we obtain for $0 < |z| < c$,

$$\log|f(z)| \geqslant -\frac{1}{2}\ell^{na}(\gamma)\log(1/|z|).$$

Since γ_z is loxodromic for all $z \in \mathbb{D}^*$, the analytic function $f(z)$ has no zeros in $\mathbb{D}$. Applying the maximal principle to the harmonic function $\log|f(z)|$, we have

$$\log|A_0(\gamma)| = \log|f(0)| \geqslant \min_{|z|=c/2} \log|f(z)| \geqslant -\frac{1}{2}\ell^{na}(\gamma)\log(2/c).$$

Therefore, we obtain an exponential lower bound of $|A_0(\gamma)|$ in terms of $\ell^{na}(\gamma)$. □

Proof of Proposition 4.2. We use the same notations as in the proof of Proposition 4.3. We have $\lambda_1(\gamma_z) = \operatorname{tr}(\gamma_z)\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\operatorname{tr}(\gamma_z)^2}}\right)$ [11] is a meromorphic function at 0. This is because Equation (4.5) gives $\lambda_2(\gamma_z)/\lambda_1(\gamma_z) \to 0$ as $z \to 0$, which implies $\log(\frac{1}{4} - \frac{1}{\operatorname{tr}(\gamma_z)^2}) = \log\left(\frac{1-\lambda_2(\gamma_z)/\lambda_1(\gamma_z)}{2(1+\lambda_2(\gamma_z)/\lambda_1(\gamma_z))}\right)^2 = 2\log\left(\frac{1-\lambda_2(\gamma_z)/\lambda_1(\gamma_z)}{2(1+\lambda_2(\gamma_z)/\lambda_1(\gamma_z))}\right)$ is holomorphic in a neighborhood of 0. We continue to use Equation (4.3) to obtain

$$\begin{aligned}\lambda_1(\gamma_z) &= z^{-\ell^{na}(\gamma)/2}A_0(\gamma)\left(1 + \sum_{j \geqslant 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right)\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{z^{\ell^{na}(\gamma)}}{A_0(\gamma)^2}\left(1 + \sum_{j \geqslant 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right)^{-2}}\right) \\ &=: z^{-\ell^{na}(\gamma)/2}A_0(\gamma) \cdot f(z) \cdot g(z).\end{aligned}$$

By the previous estimate, we have $|A_j(\gamma)/A_0(\gamma)| \leqslant e^{C\ell^{na}(\gamma)+j}$ and $|A_0(\gamma)|^{-1} \leqslant e^{C\ell^{na}(\gamma)}$ for some $C > 0$. Thus for $|z| < e^{-10C\ell^{na}(\gamma)}/20$, we have the bounds

$$\left|\sum_{j \geqslant 1}\frac{A_j(\gamma)}{A_0(\gamma)}z^j\right| \leqslant 2e^{2C\ell^{na}(\gamma)}|z| \leqslant \frac{1}{10}, \quad \frac{|z|}{|A_0(\gamma)|^2} \leqslant e^{2C\ell^{na}(\gamma)}|z| \leqslant \frac{1}{10}.$$

[10] The finiteness of hybrid norm follows from that the entries are meromorphic function on the unit disc with possible pole at 0.

[11] On $\mathbb{C}\backslash(-\infty, 0]$, we choose the principal of $\log z$: $\log z = \log|z| + i\arg z$ with $|\arg z| < \pi$; set $\sqrt{z} = e^{\log z/2}$.

This implies that on the disc $|z| < e^{-10C\ell^{na}(\gamma)}/20$

$$\log(f(z)g(z)) = \log(f(z))\log(g(z)).$$

Moreover, we apply Cauchy's integral formula to obtain upper bounds of the form Eq. (4.2) for the coefficients of the power series expansion of $\log(f(z))$ and $\log(g(z))$. We complete the proof by using the equation

$$\begin{aligned}\ell(\gamma_z) &= 2\log|\lambda_1(\gamma_z)| = \ell^{na}(\gamma)\log(1/|z|) + 2\log|A_0(\gamma)| + 2\log|f(z)| + 2\log|g(z)| \\ &= \ell^{na}(\gamma)\log(1/|z|) + 2\log|A_0(\gamma)| + 2\Re\left(\sum_{j\geqslant 1} b_j z^j\right)\end{aligned} \tag{4.7}$$

where $\sum_{j\geqslant 1} b_j z^j$ is the power series expansion of $\log(f(z)) + \log(g(z))$ about 0. □

4.2 Convergence to intermediate zeta functions: statement of results

To obtain a better convergence rate, we introduce the following *intermediate zeta functions*: for each $M \in \mathbb{Z}_{\geqslant 0}$ and $z \in \mathbb{D}^*$,

$$Z_M(\Gamma, z, s) = \prod_{[\gamma]\in\mathcal{P}} (1 - e^{-s\ell_M(\gamma,z)}) \tag{4.8}$$

where $\ell_M(\gamma, z)$ is the *Mth-expansion of length* defined by

$$\ell_M(\gamma, z) = \ell^{na}(\gamma) + \Re(\sum_{j=0}^{M} a_j(\gamma)z^j / \log(1/|z|)) \tag{4.9}$$

with $a_j(\gamma)$'s the coefficients given in the expansion Eq. (4.1).

A priori, an intermediate zeta function is defined for $\Re s$ large. In Section 5, we will prove that it admits an analytic extension to $s \in \mathbb{C}$.

We establish the following polynomial rate for the convergence of the $\log(1/|z|)$-rescaled zeta functions to an intermediate zeta function.

Theorem 4.4. *For any $C > 0$, $M \in \mathbb{Z}_{\geqslant 0}$ and $\epsilon > 0$, we have for any $0 < |z| < 1/e$,*

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon} \quad \text{for} \quad |\Re s| \leqslant C,\ |\Im s| \leqslant C|z|^{-M}.$$

As a corollary, we obtain the convergence of zeros of zeta functions: in particular, the convergence of the first zeros gives the convergence of the Hausdorff dimensions of limit sets.

Corollary 4.5. *Let $R > 0$, $\epsilon > 0$ and $M \in \mathbb{Z}_{\geqslant 0}$. Suppose that no zero of $Z_I(\Gamma, s)$ lies on the boundary of $D_R := \{s : |s| < R\}$. Let*

- *$\rho_1, \cdots, \rho_{A_z}$ be the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ in D_R;*
- *$\rho_1^M, \cdots, \rho_{B_z}^M$ be the zeros of $Z_M(\Gamma, z, s)$ in D_R;*
- *$\rho_1^{na}, \cdots, \rho_N^{na}$ be the zeros of $Z_I(\Gamma, s)$ in D_R.* [12]

Then there exists $t_0 > 0$ such that for $0 < |z| < t_0$, we have $A_z = B_z = N$, and we can order the zeros such that for each $j \in \mathbb{N} \cap [1, N]$

$$|\rho_j - \rho_j^M| \leqslant |z|^{\frac{1}{m_j^{na}} - \epsilon},$$

where m_j^{na} is the multiplicity of ρ_j^{na} as a zero of $Z_I(\Gamma, s)$.

Corollary 4.6. *Let $C > 0$, $\epsilon > 0$ and $M \in \mathbb{Z}_{\geqslant 0}$. Let $D \subset [-C, C] + i[-C|z|^{-M}, C|z|^{-M}]$ be an open connected set with its boundary a simple closed curve and*

[12] Our convention is to list zeros according to their multiplicities.

- *$\rho_1, \cdots, \rho_{A_z}$ be the zeros of $Z(\Gamma_z, s/\log(1/|z|))$ in D;*
- *$\rho_1^M, \cdots, \rho_{B_z}^M$ be the zeros of $Z_M(\Gamma, z, s)$ in D.*

Suppose $|Z_M(\Gamma, z, s)| \geqslant |z|^{1-\epsilon}$ on ∂D for $0 < |z| < t_0$, then there exists $t_1 \in (0, t_0)$ such that for $0 < |z| < t_1$, we have $A_z = B_z$.

The difference between Corollary 4.5 and Corollary 4.6 lies in the regions we compare $Z(\Gamma_z, s/\log(1/|z|))$ and $Z_M(\Gamma, z, s)$. Corollary 4.6 will be used in Example 6.8 to recover [PV19] on the structure of the zeros with large imaginary parts of the Selberg zeta function for a three-funnel surface.

4.3 Convergence to intermediate zeta functions: proofs

In this section, we prove the effective convergence results Theorem 4.1, Theorem 4.4, Corollary 4.5 and Corollary 4.6.

First, we recall the Hadamard three-circle theorem.

Proposition 4.7. *Let $r_1 < r_2 < r_3 \in (0, \infty)$ and set $\alpha = \frac{\log(r_3/r_2)}{\log(r_3/r_1)}$. Let $f : \{z : r_1 \leqslant |z-a| \leqslant r_3\} \to \mathbb{C}$ be a holomorphic function with $a \in \mathbb{C}$. Then*

$$\max_{|z-a|=r_2} |f(z)| \leqslant \left(\max_{|z-a|=r_1} |f(z)|\right)^{\alpha} \left(\max_{|z-a|=r_3} |f(z)|\right)^{1-\alpha}.$$

Proof. This follows from the fact that $\log|f(z)|$ is a subharmonic function. □

The strategy of the proof of Theorem 4.4 is to prove convergence in the region $\Re s \gg 1$ and then use Proposition 4.7 to obtain convergence in a larger region.

We need a lower bound for the function $\ell_M(\gamma, z)$ given in Eq. (4.9), whose proof will be provided at the end of Section 5.2.

Lemma 4.8. *There exists $C > 1$ depending only on Γ such that for any $M \in \mathbb{Z}_{\geqslant 0}$ and for any $\gamma \in \Gamma\backslash\{id\}$, we have for any $0 < |z| \leqslant e^{-C(M+1)}$,*

$$\ell_M(\gamma, z) \geqslant \ell^{na}(\gamma)/2.$$

Proposition 4.9. *There exist $s_0 > 1$ and $C_\Gamma > 0$ depending on Γ, such that for any $M \in \mathbb{Z}_{\geqslant 0}$, we have for any $\log(1/|z|) > C_\Gamma(M+1)$ and any $C_0 > 0$,*

$$\left|\frac{Z(\Gamma_z, s/\log(1/|z|))}{Z_M(\Gamma, z, s)} - 1\right| \lesssim_{C_0} |z| \quad \text{for} \quad \Re s > (M+1)s_0, \ |\Im s| \leqslant C_0|z|^{-M}. \tag{4.10}$$

Proof. We establish the convergence of lengths of geodesics. We use C to refer to a constant depending only on Γ, which may vary from line to line.

We first deal with the case $\Re s < 10\log(1/|z|)$. Due to Proposition 4.2, for each $\gamma \in \Gamma\backslash\{id\}$ and $0 < |z| < e^{-C\ell^{na}(\gamma)}$,

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + \Re\left(\sum_{j \geqslant 0} a_j(\gamma) z^j\right) / \log(1/|z|) \tag{4.11}$$

where $a_j(\gamma)$'s are complex numbers satisfying

$$|a_0(\gamma)| \leqslant C\ell^{na}(\gamma), \quad |a_j(\gamma)| \leqslant Ce^{C\ell^{na}(\gamma)(j+1)}. \tag{4.12}$$

By Eq. (4.11) and Eq. (4.12), we have for $\log(1/|z|) \gg C$ and $\ell^{na}(\gamma) \ll C^{-1}\log(1/|z|)$,

$$\ell_M(\gamma, z) \geqslant \ell^{na}(\gamma)/2, \quad \ell(\gamma_z)/\log(1/|z|) \geqslant \ell^{na}(\gamma)/2, \tag{4.13}$$

and

$$|\ell_M(\gamma, z) - \ell(\gamma_z)/\log(1/|z|)| \leqslant C(e^{C\ell^{na}(\gamma)}|z|)^{M+1}/\log(1/|z|). \tag{4.14}$$

Using the following basic inequality: for $|z_1|, |z_2| < 1/2$,

$$|\log(1-z_1) - \log(1-z_2)| \leqslant 2|z_1 - z_2|, \tag{4.15}$$

we have for $1 \ll \Re s < 10\log(1/|z|)$ and $\ell^{na}(\gamma) \leqslant \epsilon_0 \log(1/|z|)$ with $\epsilon_0 > 0$ sufficiently small,

$$\begin{aligned}&\left|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1-e^{-s\ell_M(\gamma,z)})\right| \lesssim \left|e^{-s\ell(\gamma_z)/\log(1/|z|)} - e^{-s\ell_M(\gamma,z)}\right| \\ &= e^{-\Re s\ell_M(\gamma,z)}\left|1 - e^{-s\ell(\gamma_z)/\log(1/|z|)+s\ell_M(\gamma,z)}\right| \lesssim e^{-\Re s\ell^{na}(\gamma)/2}\frac{|s(e^{C\ell^{na}(\gamma)}|z|)^{M+1}|}{\log(1/|z|)} \leqslant e^{(C(M+1)-\Re s)\ell^{na}(\gamma)/2}\frac{|sz^{M+1}|}{\log(1/|z|)},\end{aligned}$$

where to obtain the second to last inequality, we use the inequality

$$|1-e^z| \lesssim |z|, \quad \text{for } |\Re z| \leqslant 1,$$

because Eq. (4.14), and the assumption $\Re s < 10\log(1/|z|)$ and $\ell^{na}(\gamma) \leqslant \epsilon_0 \log(1/|z|)$ give

$$|\Re s(e^{C\ell^{na}(\gamma)}|z|)^{M+1}|/\log(1/|z|) \leqslant 10e^{C\ell^{na}(\gamma)}|z| \leqslant 1.$$

Meanwhile, for $\log(1/|z|) \geqslant C(M+1)$ and $\ell^{na}(\gamma) > \epsilon_0 \log(1/|z|)$, by Lemma 3.5 and Lemma 4.8 we have

$$|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)})| \lesssim e^{-c\Re s\ell^{na}(\gamma)}, \quad |\log(1-e^{-s\ell_M(\gamma,z)})| \lesssim e^{-c\Re s\ell^{na}(\gamma)}, \tag{4.16}$$

where we use $|\log(1-e^{-z})| \lesssim e^{-\Re z}$ for $\Re z > 1$. Thus for $\log(1/z) \geqslant C(M+1)$, and for $(M+1)s_0 \leqslant \Re s < 10\log(1/|z|)$ and $|\Im s| \leqslant C_0|z|^{-M}$, we have

$$\begin{aligned}\sum_{[\gamma]\in\mathcal{P}} |\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1-e^{-s\ell_M(\gamma,z)})| &\lesssim \frac{|sz^{M+1}|}{\log(1/|z|)}\left(\sum_{\ell^{na}(\gamma)\leqslant\epsilon_0\log(1/|z|)} e^{(C(M+1)-\Re s)\ell^{na}(\gamma)/2}\right) \\ &+ \sum_{\ell^{na}(\gamma)>\epsilon_0\log(1/|z|)} e^{-c\Re s\ell^{na}(\gamma)} \lesssim \frac{|sz^{M+1}|}{\log(1/|z|)} + |z| \lesssim_{C_0} |z|,\end{aligned} \tag{4.17}$$

where we used the counting result Eq. (3.7) to obtain the last inequality.

Now, we treat the part of $k \geqslant 1$ terms in the Selberg zeta function. We only provide the details for the $SL_2(\mathbb{C})$-case here, and $SL_2(\mathbb{R})$-case is similar. For $\Re s \gg 1$, Lemma 3.5 and Eq. (4.13) allow us to use Eq. (4.15) with $z_2 = 0$ to obtain

$$\sum_{k\geqslant 1}(k+1)|\log(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)})| \lesssim \sum_{k\geqslant 1}(k+1)e^{-(\Re s/\log(1/|z|)+k)\ell(\gamma_z)} \leqslant 2e^{-(\Re s/\log(1/|z|)+1)\ell(\gamma_z)}/(1-e^{-\ell(\gamma_z)})^2. \tag{4.18}$$

Let $\log(1/|z|) \gg C$ and $\Re s \gg C$. We have the following two cases.

- For $\ell^{na}(\gamma) \leqslant \epsilon_0 \log(1/|z|)$ with $\epsilon_0 > 0$ sufficiently small, by Eq. (4.11), we have

$$\ell(\gamma_z) \geqslant \log(1/|z|)\left(\ell^{na}(\gamma) - \frac{|a_0(\gamma)|}{\log(1/|z|)} + O\left(\frac{e^{C\ell^{na}(\gamma)}|z|}{\log(1/|z|)}\right)\right) \geqslant \left(1 - \frac{2C}{\log(1/|z|)}\right)\log(1/|z|)\ell^{na}(\gamma).$$

  This gives [13]

$$(\Re s/\log(1/|z|)+1)\ell(\gamma_z) \geqslant (\Re s/\log(1/|z|)+1)\left(1-\frac{2C}{\log(1/|z|)}\right)\log(1/|z|)\ell^{na}(\gamma) \geqslant \log(1/|z|)\ell^{na}(\gamma).$$

- For $\ell^{na}(\gamma) > \epsilon_0 \log(1/|z|)$, we can continue to apply Lemma 3.5 to Eq. (4.18) to obtain an upper bound in terms of $\ell^{na}(\gamma)$.

[13] For $\Re s, \log(1/|z|) \geqslant 4C$, we have $(\Re s/\log(1/|z|)+1)(1-2C/\log(1/|z|)) \geqslant 1 + 2C/\log(1/|z|) - 8C^2/\log(1/|z|)^2 \geqslant 1$.

In conclusion, for $\log(1/|z|) \gg C$ and $\Re s \gg C$, we obtain

$$\sum_{[\gamma]\in\mathcal{P}} \sum_{k\geqslant 1} (k+1)|\log(1-e^{-(s/\log(1/|z|)+k)\ell(\gamma_z)})| \lesssim \sum_{\ell^{na}(\gamma)\leqslant \epsilon_0 \log(1/|z|)} e^{-\ell^{na}(\gamma)\log(1/|z|)} + \sum_{\ell^{na}(\gamma)>\epsilon_0\log(1/|z|)} e^{-c(\Re s+\log(1/|z|))\ell^{na}(\gamma)} \lesssim |z|, \tag{4.19}$$

where we used again the counting result Eq. (3.7) to obtain the last inequality.

For $\log(1/|z|) \geqslant C(M+1)$ and $(M+1)s_0 < \Re s < 10\log(1/|z|)$, we obtain Eq. (4.10) by combining Eq. (4.17) and Eq. (4.19). For $\Re s \geqslant 10\log(1/|z|)$, it remains to bound the right-hand side of the following inequality by $|z|$:

$$\left|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)}) - \log(1-e^{-s\ell_M(\gamma,z)})\right| \leqslant \left|\log(1-e^{-s\ell(\gamma_z)/\log(1/|z|)})| + |\log(1-e^{-s\ell_M(\gamma,z)})\right|.$$

This can be achieved by an argument similar to the proof of Eq. (4.19). □

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. Fix any $C > 1$, $M \in \mathbb{Z}_{\geqslant 0}$, and $\epsilon > 0$. Recall the constants $s_0 > 0$ and $C_\Gamma > 0$ in Proposition 4.9.

Set $R = (M+1)s_0 + 1$, and let $C_\epsilon > 1$ be a constant satisfying $C_\epsilon > \max\{10C, R\}$ and

$$\log(R+10C)/\log(R+C_\epsilon) < \epsilon.$$

It suffices to prove the proposition for $\log(1/|z|) > C_\Gamma(M+1)$. This is because the zeta functions depend continuously on the parameters z and s, thus uniformly bounded for $s \in [-C,C] + i[-C|z|^{-M}, C|z|^{-M}]$ and $\log(1/|z|) \leqslant C_\Gamma(M+1)$ depending on C, M, ϵ.

For any $K_T := [-C,C] + i[T, T+1]$ with $T \in [-C|z|^{-M}, C|z|^{-M}]$, we choose three concentric discs $D_1 \subset D_2 \subset D_3$ with radii $r_1 < r_2 < r_3$ such that $D_1 \subset \{s : \Re s > (M+1)s_0, |\Im s| \leqslant (C+2)|z|^{-M}\}$, $D_2 \supset K_T$ and $D_3 \subset \{s : \Re s > -C_\epsilon, |\Im s| \leqslant (C+R+C_\epsilon+1)|z|^{-M}\}$ (see Figure 4.1). In particular, we can take

$$D_1 = D(a, R-(M+1)s_0), \quad D_2 = D(a, R+10C), \quad D_3 = D(a, R+C_\epsilon)$$

where $a = R + iT$.

By Corollary 5.8, the function

$$F_z(s) = Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)$$

is an entire function. By Proposition 4.9, $F_z(s)$ satisfies the bound $|F_z(s)| \lesssim_C |z|$ for $s \in D_1$ and $s \in D_3 \cap \{s : \Re s > (M+1)s_0\}$. By Proposition 3.12 and Proposition 3.15, $Z(\Gamma_z, s/\log(1/|z|))$ is uniformly bounded on $D_3 \cap \{s : \Re s \leqslant (M+1)s_0\}$. Meanwhile, Corollary 5.8 shows that $Z_M(\Gamma, z, s)$ is uniformly bounded on $D_3 \cap \{s : \Re s \leqslant (M+1)s_0\}$. Therefore the function $F_z(s)$ is uniformly bounded on D_3, and from Proposition 4.7 we conclude

$$|F_z(s)| \lesssim_{C,M,\epsilon} |z|^{\log(r_3/r_2)/\log(r_3/r_1)}, \quad s \in \partial D_2.$$

The same bound also holds for $s \in D_2$ by the maximal modulus principle since $F_z(s)$ is an entire function. The statement follows from

$$\log(r_3/r_2)/\log(r_3/r_1) = 1 - \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} = 1 - \log(R+10C)/\log(R+C_\epsilon) > 1-\epsilon. \qquad \square$$

Proof of Theorem 4.1. Take $M = 0$ in Theorem 4.4, we have for $0 < |z| < 1/e$,

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_0(\Gamma, z, s)| \lesssim_C |z|^{1/2}, \quad \text{for } |\Re s| \leqslant C,\ |\Im s| \leqslant C.$$

It suffices to prove that

$$|Z_0(\Gamma, z, s) - Z_I(\Gamma, s)| \lesssim_K \frac{1}{\log(1/|z|)}, \quad s \in K.$$

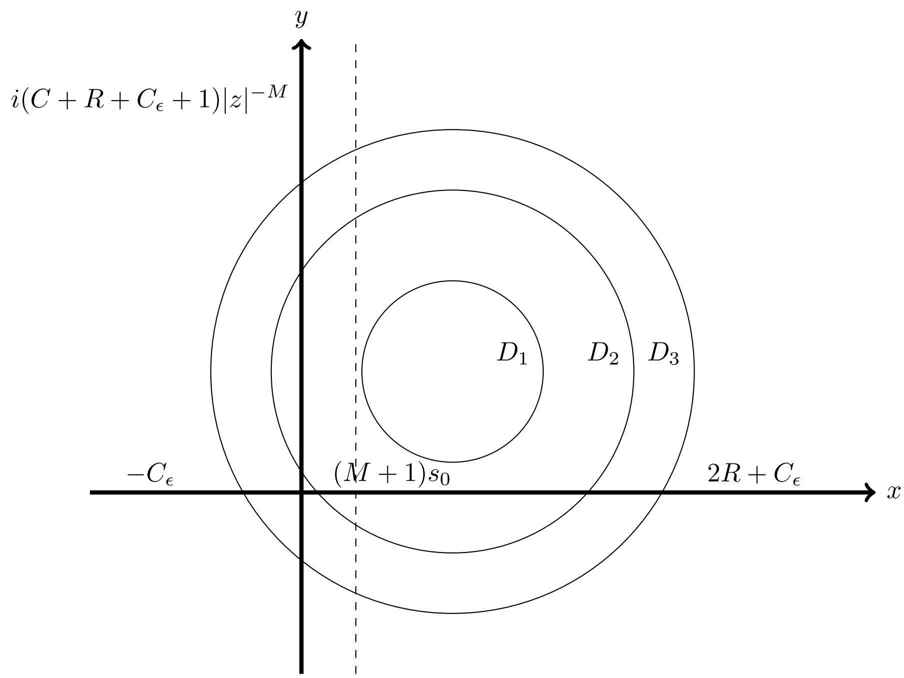

Figure 4.1: Choice of discs

By Corollary 5.8,

$$Z_0(\Gamma, z, s) = P_0(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)})$$

where each $\mu_j(z) = a_j + a_{j,0}/\log(1/|z|)$. By Corollary 5.9,

$$Z_I(\Gamma, s) = P_0(e^{sa_1}, \cdots, e^{sa_J}).$$

Since P_0 is smooth, we conclude that

$$|P_0(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)}) - P_0(e^{sa_1}, \cdots, e^{sa_J})| \lesssim_K \sup_j |\mu_j(z) - a_j| \lesssim_K \frac{1}{\log(1/|z|)}, \quad s \in K.$$

The proof is complete. □

Proof of Corollary 4.5. Given a disc D_R that satisfies the assumption, let ρ^{na} be a zero of $Z_I(\Gamma, s)$ in D_R of multiplicity m. Then there exists a small disc D_0 centered at ρ^{na} such that ρ^{na} is the only zero of $Z_I(\Gamma, s)$ in $\bar{D}_0$. We can write

$$Z_I(\Gamma, s) = (s - \rho^{na})^m f(s)$$

where $f(s)$ is an entire function non-vanishing on D_0.

Since $Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s)$ uniformly for $s \in \partial D_0$, there exists $t_1 > 0$ such that for $0 < |z| < t_1$, we have

$$|Z_I(\Gamma, s)| > |Z_I(\Gamma, s) - Z(\Gamma_z, s/\log(1/|z|))| \quad \text{for} \quad s \in \partial D_0.$$

By Rouché's theorem, $Z_I(\Gamma, s)$ and $Z(\Gamma_z, s/\log(1/|z|))$ have the same number of zeros in D_0. We can write

$$Z(\Gamma_z, s/\log(1/|z|)) = (s - \rho_1) \cdots (s - \rho_m) f_z(s)$$

where $f_z(s)$ is an entire function non-vanishing on D_0. We claim that $f_z(s) \to f(s)$ uniformly on $\bar{D}_0$. Write $D_0 = D(\rho^{na}, r)$. By the uniform convergence $Z(\Gamma_z, s/\log(1/|z|)) \to Z_I(\Gamma, s)$ on $\bar{D}_0$, for any $\epsilon > 0$, there exists $t(\epsilon) > 0$ such that for $0 < |z| < t(\epsilon)$, we have

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_I(\Gamma, s)| < \min\left\{\epsilon, \inf_{\epsilon \leqslant |s-\rho^{na}| \leqslant r} |Z_I(\Gamma, s)|\right\} \quad \text{for} \quad s \in \bar{D}_0.$$

Therefore $Z(\Gamma_z, s/\log(1/|z|))$ has no zero in $\{s : \epsilon \leqslant |s-\rho^{na}| \leqslant r\}$, which implies $|\rho_j - \rho^{na}| < \epsilon$ for each ρ_j. For $s \in \bar{D}_0$,

$$|f_z(s) - f(s)| \leqslant \sup_{s\in\partial D_0} |Z(\Gamma_z, s/\log(1/|z|))/((s-\rho_1)\cdots(s-\rho_m)) - Z_I(\Gamma, s)/(s-\rho^{na})^m| \lesssim \epsilon.$$

As a consequence, there exist $C > 1$ and $t_2 > 0$ such that for $0 < |z| < t_2$, we have $C^{-1} < |f_z(s)| < C$ for $s \in D_0$.

Similarly, since $Z_M(\Gamma, z, s) \to Z_I(\Gamma, s)$, there exists $t_3 > 0$ such that for $0 < |z| < t_3$, $Z_M(\Gamma, z, s)$ has m zeros in D_0, and we can write

$$Z_M(\Gamma, z, s) = (s-\rho_1^M)\cdots(s-\rho_m^M) f_{M,z}(s)$$

where $f_{M,z}(s)$ is an entire function non-vanishing on D_0.

For $0 < |z| < \min\{t_1, t_2, t_3\}$, consider the following functions on D_0:

$$g_z(s) = (s-\rho_1)\cdots(s-\rho_m) \quad \text{and} \quad h_z(s) = g_z(s) - Z_M(\Gamma, z, s)/f_z(s)$$

Thus, we can complete the proof by applying the following Lemma 4.10 to $g_z(s)$ and $h_z(s)$. ☐

Lemma 4.10. *Let $g(s) = s^d + a_1 s^{d-1} + \cdots + a_d = (s-\rho_1)\cdots(s-\rho_d)$ be a polynomial such that all of its zeros $\rho_1, \ldots, \rho_d$ are in a disc $D(\rho, \epsilon_0)$. Let $h(s)$ be a holomorphic function on $D(\rho, 3\epsilon_0)$ such that $|h(s)| \leqslant \epsilon \leqslant 4^{-d^3-10d}\epsilon_0^d$ for $z \in \partial D(\rho, 2\epsilon_0)$. Then $g(s) + h(s)$ also has d zeros $\rho'_1, \cdots, \rho'_d$ in $D(\rho, 2\epsilon_0)$, and we can order the zeros such that for each $j \in [1, d] \cap \mathbb{N}$,*

$$|\rho_j - \rho'_j| < 4^{d^2+3}\epsilon^{1/d}.$$

Proof. Since $|g(s)| \geqslant \epsilon_0^d$ on $\partial D(\rho, 2\epsilon_0)$, and $|h(s)| \leqslant \epsilon < \epsilon_0^d$, by Rouché's theorem $g(s)$ and $g(s) + h(s)$ have the same number of zeros in $D(\rho, 2\epsilon_0)$.

We claim there exist finitely many discs D_ℓ of radius $C\epsilon^{1/d}$ for some $C \in [2, 4^{d^2+2}] \cap \mathbb{N}$ such that they cover all the ρ_j's, and $2D_\ell \cap 2D_{\ell'} = \varnothing$ [14] for $\ell \neq \ell'$. Let $S = \{|\rho_i - \rho_j| : i, j = 1, 2, \cdots, d\} \subset [0, 2\epsilon_0]$. Then $\#S \leqslant d^2$. By the pigeonhole principle, there exists $C \in [2, 4^{d^2+2}] \cap \mathbb{N}$ such that $[C\epsilon^{1/d}, 4C\epsilon^{1/d}] \cap S = \varnothing$ (note the assumption $\epsilon \leqslant 4^{-d^3-10d}\epsilon_0^d$ is used here to ensure $4^{d^2+3}\epsilon^{1/d} < \epsilon_0$). Then for any pair ρ_i, ρ_j, we have either $|\rho_i - \rho_j| < C\epsilon^{1/d}$ or $|\rho_i - \rho_j| > 4C\epsilon^{1/d}$. Let $D_1 = D(\rho_1, C\epsilon^{1/d})$. Choose a zero $\rho_{j_2} \notin D_1$, and let $D_2 = D(\rho_{j_2}, C\epsilon^{1/d})$. Using the inequality $|\rho_1 - \rho_{j_2}| > 4C\epsilon^{1/d}$, we obtain $2D_1 \cap 2D_2 = \varnothing$. We repeat this process finitely many times to obtain the desired family of discs.

The fact $\epsilon \leqslant 4^{-d^3-10d}\epsilon_0^d$ ensures $2D_\ell \subset D(\rho_*, 2\epsilon_0)$. Note

$$|h(s)| \leqslant \epsilon < (C\epsilon^{1/d})^d \leqslant |g(s)| \quad \text{for} \quad s \in \partial(2D_\ell).$$

By Rouché's theorem, $g(s)+h(s)$ and $g(s)$ have the same number of zeros in $2D_\ell$, and the proof is complete. ☐

Proof of Corollary 4.6. By Rouché's theorem, it suffices to show

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| < |Z_M(\Gamma, z, s)|, \quad z \in \partial D.$$

Since we assume $|Z_M(\Gamma, z, s)| \geqslant |z|^{1-\epsilon}$, it follows from Theorem 4.4 that

$$|Z(\Gamma_z, s/\log(1/|z|)) - Z_M(\Gamma, z, s)| \lesssim_{C,M,\epsilon} |z|^{1-\epsilon/2} < |z|^{1-\epsilon}$$

for $|z|$ sufficiently small. ☐

Proof of Theorem 1.3 and Theorem 1.1. Theorem 1.3 is a direct consequence of Theorem 4.1 and Theorem 4.4. The estimate on the Hausdorff dimension of the limit set in Theorem 1.1 is a consequence of Corollary 4.5 and the computation of the intermediate zeta function $Z_0(\Gamma, z, s)$ for symmetric three-funnel hyperbolic surfaces in Example 6.7. ☐

[14] Here, if $D_\ell = D(a, r)$, then $2D_\ell = D(a, 2r)$.

5 Analytic extension of intermediate zeta functions

Recall the intermediate zeta functions introduced in Eq. (4.8):

$$Z_M(\Gamma, z, s) = \prod_{[\gamma]\in\mathcal{P}} (1 - e^{-s\ell_M(\gamma,z)}) \quad \text{for} \quad \Re s \gg 1. \tag{5.1}$$

The goal of this section is to show that $Z_M(\Gamma, z, s)$ admits an analytic extension to $\mathbb{C}$.

This can be seen as a generalization of the analytic continuation of the Ihara zeta function. We need to find good combinatorial relations between $\ell_M(\gamma, t)$'s and then obtain a determinant formula similar to the Ihara zeta function.

We will first review the proof of the determinant formula for a weighted version of the Ihara zeta function. Then, we use derivative cocycles from the Schottky group to get a good combinatorial relation and apply the determinant formula.

5.1 Weighted Ihara zeta function

We briefly review the proof of analytic continuation of the weighted Ihara zeta function. For more details, see, for example, [HST06, Section 3].

Let $G = (V, E)$ be a finite graph, where $E = \{e_1, \cdots, e_{2J}\}$ is the set of oriented edges of G with e_j and e_{j+J} in opposite direction. We write $e_{j+J} = e_j^{-1}$ to refer to their relation. For each edge e_j, we associate it a complex number h_j. Define $W(s)$ to be a $(2J) \times (2J)$-matrix by

$$W(e_j, e_k)(s) = e^{-s(h_j+h_k)/2}$$

if the endpoint of e_j is the beginning point of e_k and $e_j \neq e_k^{-1}$, and $W(e_j, e_k)(s) = 0$ otherwise.

For a non-backtracking loop $P = (e_{i_1}, \cdots, e_{i_n})$, define $\ell_h(P) = \sum_{j=1}^{n} h_{i_j}$. The weighted Ihara zeta function is defined as a product over primitive loops P:

$$Z_I(G, h, s) = \prod_{[P]\in\mathcal{P}} (1 - e^{-s\ell_h(P)}).$$

Recall that we use P to denote a non-backtracking loop with a beginning point at a vertex, and $[P]$ to denote a loop forgetting the starting point. For s with large real part, the product is absolute convergent. The following Proposition 5.1 gives the analytic extension of $Z_I(G, h, s)$ to the entire complex plane.

Proposition 5.1. *For $\Re s \gg 1$, we have $Z_I(G, h, s) = \det(I - W(s))$.*

Proof. We expand $\log Z_I(G, h, s)$:

$$\begin{aligned} \log Z_I(G, h, s) &= -\sum_{j=1}^{\infty} \sum_{[P]\in\mathcal{P}} \frac{1}{j} e^{-js\ell_h(P)} \\ &= -\sum_{j=1}^{\infty} \sum_{P\in\mathcal{P}} \frac{1}{j\#(P)} e^{-js\ell_h(P)}, \\ &= -\sum_{n=1}^{\infty} \sum_{\substack{P\in\mathcal{P} \\ j:j\#(P)=n}} \frac{1}{n} e^{-js\ell_h(P)} \end{aligned}$$

where the sum $\sum_{P\in\mathcal{P}}$ is over primitive non-backtracking loops P and $\#(P)$ is the number of edges in the primitive loop P.

We can also expand $\log\det(I - W(s))$:

$$\log\det(I - W(s)) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} W(s)^n.$$

We finish the proof by using the equality $\mathrm{tr} W(s)^n = \sum_{\substack{P\in\mathcal{P} \\ j:j\#(P)=n}} e^{-js\ell_h(P)}$. ☐

5.2 Mth-expansion of lengths and derivative cocycles

In this section we prove the intermediate zeta function $Z_M(\Gamma, z, s)$ has an analytic extension and is computable for concrete examples. We will use the geometry of Schottky basis. Throughout this section, we fix generators $\mathfrak{a}_1, \cdots, \mathfrak{a}_g$ and a Schottky basis $D_{\mathfrak{a}_j}$ satisfying Proposition 3.20.

For computational convenience, we will moreover assume $D_{\mathfrak{a}_j}$ are contained in $D(0, C)$, i.e. uniformly bounded away from ∞. This is always possible by choosing an appropriate coordinate.

The key observation is that the leading terms of the derivative is locally constant, from which we can compute the intermediate zeta function.

Consider pairs of the form (U, f) where U is an open neighborhood of 0, and f is meromoprhic on U and holomorphic on $U\backslash\{0\}$. Two such pairs (U, f) and (V, g) are equivalent if there exists an open neighborhood W of 0, contained in $U \cap V$, such that $f|_W = g|_W$. A germ of meromorphic functions at 0 is an equivalence class of such a pair, and it has a representative that is a convergent complex Laurent series about 0, $t^n \sum_{j=0}^{\infty} a_j t^j$ with $n \in \mathbb{Z}, a_0 \neq 0$. Let $\mathcal{M}_0$ be the ring of germs of meromorphic functions at 0, which is isomorphic to the ring of all convergent complex Laurent series about 0. The ring $\mathcal{O}_0$ of germs of holomorphic functions at 0 is defined analogously, which is isomorphic to the ring of all convergent complex power series about 0.

Fix any $M \in \mathbb{Z}_{\geqslant 0}$. We introduce several combinatorial operations.

- For $a = t^n \sum_{j=0}^{\infty} a_j t^j \in \mathcal{M}_0$ with $n \in \mathbb{Z}, a_0 \neq 0$, we let $\mathrm{lt}_M(a)$ be the first M leading terms of a:

$$\mathrm{lt}_M(a) := t^n \sum_{j=0}^{M} a_j t^j.$$

- We define the formal expansion of logarithm $\mathrm{plog} : \mathcal{M}_0 \to \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z})$ by mapping $a = a_0 t^n (1 + \sum_{j=1}^{\infty} a_j t^j) \in \mathcal{M}_0$ with $n \in \mathbb{Z}$ and $a_0 \neq 0$ to

$$\mathrm{plog}(a) := -n\log(1/t) + \log a_0 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\sum_{j=1}^{\infty} a_j t^j \right)^m. \tag{5.2}$$

  Here $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (\sum_{j=1}^{\infty} a_j t^j)^m$ is the power series expansion of the holomorphic function $\log(1 + \sum_{j=1}^{\infty} a_j t^j)$ about 0, which is uniquely determined.

- We define $\mathrm{lt}'_M : \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z}) \to \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z})$ by mapping $a = n\mathrm{log}(1/t) + \sum_{j=0}^{\infty} a_j t^j$ to its first M leading terms

$$\mathrm{lt}'_M(a) := n\log(1/t) + \sum_{j=0}^{M} a_j t^j.$$

  And we define $\mathrm{lt}'_{-1} : \mathbb{Z}\log(1/t) \oplus (\mathcal{O}_0/2\pi i\mathbb{Z}) \to \mathbb{Z}\log(1/t)$ by mapping $a = n\mathrm{log}(1/t) + \sum_{j=0}^{\infty} a_j t^j$ to its (-1)-leading term

$$\mathrm{lt}'_{-1}(a) := n\log(1/t).$$

Lemma 5.2. *For any $M \in \mathbb{Z}_{\geqslant 0}$, and for $x, y \in \mathcal{M}_0$, we have*

$$\begin{aligned} &\mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(x) = \mathrm{lt}'_M \circ \mathrm{plog}(x), \\ &\mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(xy) = \mathrm{lt}'_M \circ \mathrm{plog}(x) + \mathrm{lt}'_M \circ \mathrm{plog}(y). \end{aligned} \tag{5.3}$$

Proof. We recall the definition of plog Eq. (5.2). The element $x \in \mathcal{M}_0$ can be written uniquely as $x = x_0(1 + x_1)$ with $x_1 = \sum_{j \geqslant 1} b_j t^j$, $x_0 = b_0 t^n$ for some $b_0 \neq 0$, $n \in \mathbb{Z}$. Then $\mathrm{plog} x = -n\log(1/t) + \log b_0 + (x_1 - x_1^2/2 + x_1^3/3 - \cdots)$. From this expansion, we see that $\mathrm{lt}'_M \circ \mathrm{plog}(x)$ is determined by x_0 and the first $M - 1$ term of x_1, and hence determined by $\mathrm{lt}_M(x)$. This completes the proof of the first equality.

The second equality uses the observation that for two convergent power series $\sum_{j \geqslant 1} a_j t^j$, $\sum_{l \geqslant 1} b_l t^l$, we have $\log((1 + \sum_{j \geqslant 1} a_j t^j)(1 + \sum_{l \geqslant 1} b_l t^l)) = \log(1 + \sum_{j \geqslant 1} a_j t^j) + \log(1 + \sum_{l \geqslant l} b_l t^l)$ in an open neighborhood of 0. □

In this subsection, it suffices to consider the action of $\mathrm{SL}_2(\mathbb{C}((t)))$ on $\mathbb{P}^1_{\mathbb{C}((t))}$, the set of classical points. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}((t)))$ and $x \neq \gamma^{-1}\infty \in \mathbb{C}((t))$, the derivative of γ at x is given by

$$\gamma' x = 1/(cx+d)^2.$$

Lemma 5.3. *Fix $M \in \mathbb{Z}_{\geqslant 0}$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}((t)))$ with $\gamma\infty \neq \infty$ and $x, y \in \mathbb{C}((t)) \backslash \{\gamma^{-1}\infty\}$ with $|x-y|_{na} < e^{-M}|x-\gamma^{-1}\infty|_{na}$, we have*

$$\mathrm{lt}_M(\gamma' x) = \mathrm{lt}_M(\gamma' y).$$

Proof. It is sufficient to prove that $|\gamma'(x) - \gamma'(y)|_{na} < e^{-M}|\gamma'(x)|_{na}$, which is equivalent to $\mathrm{lt}_M(\gamma' x) = \mathrm{lt}_M(\gamma' y)$.

We have

$$\gamma' x - \gamma' y = \frac{1}{(cx+d)^2} - \frac{1}{(cy+d)^2} = \frac{c(y-x)(c(x+y)+2d)}{(cx+d)^2(cy+d)^2}.$$

Due to $\gamma\infty \neq \infty$, we know $c \neq 0$. Therefore

$$\frac{|\gamma' x - \gamma' y|_{na}}{|\gamma' x|_{na}} = \left| \frac{c(x-y)(c(x+y)+2d)}{(cy+d)^2} \right|_{na} = \left| \frac{(x-y)(c(x+y)+2d)}{(y+d/c)(cy+d)} \right|_{na}.$$

Due to $|x-y|_{na} < e^{-M}|x-\gamma^{-1}\infty|_{na}$, we have that

$$|y+d/c|_{na} = |y-\gamma^{-1}\infty|_{na} = |x-\gamma^{-1}\infty|_{na} > e^M|x-y|_{na}$$

and $|cx+d+cy+d|_{na} \leqslant \max\{|cx+d|_{na}, |cy+d|_{na}\} = |cy+d|_{na}$. The proof is complete. □

Let Γ be a Schottky group in $\mathrm{SL}_2(\mathbb{C}((t)))$. For $\gamma \in \Gamma \backslash \{id\}$, let $\lambda_1(\gamma),\ \lambda_2(\gamma) \in \mathbb{C}((t))$ be its eigenvalues with $|\lambda_1(\gamma)|_{na} > |\lambda_2(\gamma)|_{na}$. Then there exists $\eta > 0$ depending on the group Γ, such that for any $0 < |z| < \eta$, the evaluation of $\lambda_1(\gamma)$ at z satisfies $\lambda_1(\gamma)(z) = \lambda_1(\gamma_z)$. Hence, $\lambda_1(\gamma)$ is a meromorphic function at 0 (see the proof of Proposition 4.2). Using Eq. (4.7) and the definition of plog, we obtain the following lemma.

Lemma 5.4. *There exists a constant $C > 0$ that depends only on Γ such that for any $\gamma \in \Gamma \backslash \{id\}$, we have for any $0 < |z| < e^{-C\ell^{na}(\gamma)}$,*

$$2\, \Re\left(\mathrm{plog}(\lambda_1(\gamma))(z)\right) = \ell^{na}(\gamma) \log(1/|z|) + \Re\left(\sum_{j \geqslant 0} a_j(\gamma) z^j\right),$$

where $\mathrm{plog}(\lambda_1(\gamma))(z)$ is to evaluate the series $\mathrm{plog}(\lambda_1(\gamma))$ at z, and $a_j(\gamma)$'s are the complex coefficients given in Eq. (4.1).

For each $\gamma \in \Gamma \backslash \{id\}$, we define

$$L_M(\gamma) := \mathrm{lt}'_M(2\ \mathrm{plog}(\lambda_1(\gamma))).$$

Then we have that for any $z \in \mathbb{D}^*$, the evaluation of $L_M(\gamma)$ at z satisfies

$$\Re\left(L_M(\gamma)(z)\right) = \ell_M(\gamma, z) \log(1/|z|) \tag{5.4}$$

where $\ell_M(\gamma, z)$ is defined in Eq. (4.8), and hence we obtain $\mathrm{lt}'_{-1}(L_M(\gamma)) = \ell^{na}(\gamma)\log(1/t)$.

Proposition 5.5. *Fix any $M \in \mathbb{Z}_{\geqslant 0}$. There exists $C > 0$ depending only on Γ such that for $N > C(M+1)$, we have for any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$,*

$$\mathrm{lt}_M(\mathfrak{a}'_{i_1}(x)) = \mathrm{lt}_M(\mathfrak{a}'_{i_1}(y)) \quad \textit{for any} \quad x, y \in \mathbb{C}((t)) \cap D_{\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}}.$$

Denote the common value by $\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}))$*. Moreover, for any cyclically reduced word* $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ *with* $n > N$*, we have*

$$-L_M(\gamma) = l_M(\mathfrak{a}_{i_1},(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N})) + l_M(\mathfrak{a}_{i_2},(\mathfrak{a}_{i_3}\cdots\mathfrak{a}_{i_{N+1}})) + \cdots + l_M(\mathfrak{a}_{i_n},(\mathfrak{a}_{i_{n+1}}\cdots\mathfrak{a}_{i_{n+N-1}})), \tag{5.5}$$

where $i_{k+j} := i_{k+j-n}$ *when* $k+j > n$*, and*

$$l_M(\mathfrak{a}_{i_k},(\mathfrak{a}_{i_{k+1}}\cdots\mathfrak{a}_{i_{k+N-1}})) := \mathrm{lt}'_M\circ\mathrm{plog}\circ\mathrm{lt}_M(\mathfrak{a}'_{i_k}(\mathfrak{a}_{i_{k+1}}\cdots\mathfrak{a}_{i_{k+N-1}})).$$

Remark 5.6. An easy consequence of Eq. (5.5) is that

$$\mathrm{lt}'_{-1}(-L_M(\gamma)) = \mathrm{lt}'_{-1}\left(\sum_{k=1}^{n} l_M(\mathfrak{a}_{i_k},(\mathfrak{a}_{i_{k+1}}\cdots\mathfrak{a}_{i_{k+N-1}}))\right) = -\ell^{na}(\gamma)\log(1/t),$$

which is useful for the holomorphic extension of the Ihara zeta function.

Remark 5.7. Proposition 5.5 gives another proof of the lower of the leading coefficients. In general, it is hard to get a lower bound estimate of the leading coefficients. Similar questions in Archimedean case are important and hard. In the case of Schottky groups, the geometry and the hyperbolicity enable us to obtain such lower bound.

Proof of Proposition 5.5. By assumption, ∞ is not contained in any Schottky discs. Due to the properties of the Schottky group, there exists c_0 such that we have a lower bound

$$d_{na}(\mathfrak{a}_i^{-1}\infty,(D_{\bar{\mathfrak{a}}_i})^c) \geqslant c_0$$

for all generators $\mathfrak{a}_i$. Moreover, the size of Schottky discs $D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}}$ tends uniformly to zero as the word length N goes to infinity (Lemma 3.3 for non-Archimedean norm). Therefore, we can find N such that the radius of any disc $D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}}$ is less than $e^{-M}c_0$. By Lemma 3.3, we can take $N > C(M+1)$ for a uniform constant C depending only on Γ.

For any cyclically reduced word $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_n}$ with $n > N$, recall $\lambda_1(\gamma),\lambda_2(\gamma)\in\mathbb{C}(\!(t)\!)$ are its eigenvalues with $|\lambda_1(\gamma)|_{na} > |\lambda_2(\gamma)|_{na}$, and let γ_+ be its attracting fixed point, which belongs to $\mathbb{C}(\!(t)\!)\subset\mathbb{P}^1_{\mathbb{C}(\!(t)\!)}$. Then we have that $\gamma'(\gamma_+) = \lambda_2(\gamma)^2$. It follows from Lemma 5.4 that

$$\begin{aligned}-L_M(\gamma) = \mathrm{lt}'_M\circ\mathrm{plog}(\gamma'(\gamma_+)) &= \mathrm{lt}'_M\circ\mathrm{plog}\circ\mathrm{lt}_M(\gamma'(\gamma_+))\\ &= \mathrm{lt}'_M\circ\mathrm{plog}\circ\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\gamma_+)\cdots\mathfrak{a}'_{i_n}(\gamma_+))\\ &= \mathrm{lt}'_M\circ\mathrm{plog}\circ\mathrm{lt}_M\left(\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n}\gamma_+))\cdots\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\gamma_+))\right),\end{aligned}$$

where for the last equality we use the fact that for any $x,y\in\mathbb{C}(\!(t)\!)$,

$$\mathrm{lt}_M(xy) = \mathrm{lt}_M(\mathrm{lt}_M(x)\mathrm{lt}_M(y)).$$

Since γ is cyclically reduced, the attracting fixed point γ_+ is in the disc $D_\gamma\subset D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}}$, and $\gamma^{-1}\infty\in D_{\mathfrak{a}_{i_n}^{-1}}$. Due to the choice of N and Lemma 5.3, we obtain for any $y\in D_{\mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_{N-1}}}\cap\mathbb{C}(\!(t)\!)$,

$$\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\gamma_+)) = \mathrm{lt}_M(\mathfrak{a}'_{i_n}(y))$$

For any $\mathfrak{a}_{i_j}$, we have $\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma_+\in\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}D_\gamma\subset D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma}\subset D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_{j+N-1}}}$, and $a_{i_j}^{-1}\infty\in D_{a_{i_j}}$. Therefore, for any $y\in D_{\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_{j+N-1}}}\cap\mathbb{C}(\!(t)\!)$,

$$\mathrm{lt}_M(\mathfrak{a}'_{i_j}(\mathfrak{a}_{i_{j+1}}\cdots\mathfrak{a}_{i_n}\gamma_+)) = \mathrm{lt}_M(\mathfrak{a}'_{i_j}(y)).$$

In conclusion, we obtain that

$$-L_M(\gamma) = \mathrm{lt}'_M\circ\mathrm{plog}\circ\mathrm{lt}_M\left(\mathrm{lt}_M(\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N})\cdots\mathrm{lt}_M(\mathfrak{a}'_{i_n}(\mathfrak{a}_{i_n}\cdots\mathfrak{a}_{n+N-1}))\right)$$

The proof is complete by applying Eq. (5.3) to the right hand side of the above equation. ☐

As a corollary, we obtain the following.

Corollary 5.8. *For any $M \in \mathbb{Z}_{\geqslant 0}$, the intermediate zeta function $Z_M(\Gamma, z, s)$ has a holomorphic extension to the entire complex plane $s \in \mathbb{C}$. More precisely, there exists a polynomial $P_M(x_1, \cdots, x_J)$ such that for any $s \in \mathbb{C}$ and $0 < |z| < 1/e$,*

$$Z_M(\Gamma, z, s) = P_M(e^{s\mu_1(z)}, e^{s\mu_2(z)}, \cdots, e^{s\mu_J(z)}) \tag{5.6}$$

where for $j = 1, 2, \cdots, J$, $\mu_j(z)$ is of the form

$$\mu_j(z) = a_j + \frac{1}{\log(1/|z|)} \Re \left(\sum_{k=0}^{M} a_{j,k} z^k \right) \quad \text{with } a_j \in \mathbb{R},\ a_{j,k} \in \mathbb{C} \text{ for } k = 0, \ldots, M. \tag{5.7}$$

Proof. For $\gamma \in \Gamma \backslash \{id\}$, recall from Eq. (5.4) that for $z \in \mathbb{D}^*$, $\Re(L_M(\gamma)(z)) = \ell_M(\gamma, z) \log(1/|z|)$.

We introduce an edge matrix $W(s)$ with coordinates given by reduced words of length N, that is, words of form $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, where $N > 0$ is the constant given in Proposition 5.5. The entries are given by

$$W(\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}, \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_{N+1}})(s) = \exp\left(\frac{s}{2\log(1/|z|)} \Re\left(l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))(z) + l_M(\mathfrak{a}_{i_2}, (\mathfrak{a}_{i_3} \cdots \mathfrak{a}_{i_{N+1}}))(z) \right) \right),$$

and the other entries are all equal to zero. Using the definition of $Z_M(\Gamma, z, s)$, Proposition 5.1, and Proposition 5.5, we can show that $\det(I - W(s))$ is the holomorphic extension of $Z_M(\Gamma, z, s)$ to entire complex plane.

It follows from Proposition 5.5 that for any reduced word $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$,

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(\mathfrak{a}'_{i_1}(x)) \quad \text{for any } x \in D_{\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}} \cap \mathbb{C}(\!(t)\!).$$

Note $\mathfrak{a}'_{i_1}(x)$ is of the form $t^m \sum_{j \geqslant 0} b_j t^j$ with $m \in \mathbb{Z}$, $b_0 \neq 0$. Using the definitions of lt'_M, plog, lt_M, we can show that $\det(I - W(s))$ is a polynomial described in the statement of the corollary. ☐

Remark. We define a graph $G_{N,g}$ with vertices given by

$$\{\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}} : \mathfrak{a}_{i_j} \mathfrak{a}_{i_{j+1}} \neq 1\}$$

and there is a (directed) edge between two vertices of the form $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}$ and $\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}$. We consider the weighted Ihara zeta function of this graph equipped the edge between $\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}$ and $\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}$ with weight given by $-\frac{1}{\log(1/|z|)} \Re l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$. By a similar proof as Proposition 5.1, we have $Z_M(\Gamma, z, s) = \det(I - V(s))$ where

$$V(\mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_{N-1}}, \mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})(s) = \exp\left(\frac{s}{\log(1/|z|)} \Re l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) \right),$$

which is smaller and has the advantage of being easier to compute.

Continuing with Corollary 5.8, we prove that any two intermediate zeta functions are related in the following way.

Corollary 5.9. *Fix any $M \in \mathbb{Z}_{\geqslant 0}$. Let $\mu_j(z)$ be given as in Eq. (5.7) for $j = 1, \ldots, J$. Then we have for any $M' \in \mathbb{Z}_{\geqslant 0}$ with $M' \leqslant M$, the holomorphic extension of $Z_{M'}(\Gamma, z, s)$ to $\mathbb{C}$ is given by*

$$Z_{M'}(\Gamma, z, s) = P_M(e^{s\mu_1^{M'}(z)}, \cdots, e^{s\mu_J^{M'}(z)}). \tag{5.8}$$

where each $\mu_j^{M'}(z)$ is the first M' leading terms of $\mu_j(z)$:

$$\mu_j^{M'}(z) = a_j + \frac{1}{\log(1/|z|)} \Re \left(\sum_{k=0}^{M'} a_{j,k} z^k \right)$$

with a_j, $a_{j,k} \in \mathbb{C}$ defined in Eq. (5.7). Similarly, the holomorphic extension of the Ihara zeta function to $\mathbb{C}$ is given by

$$Z_I(\Gamma, s) = P_M(e^{sa_1}, \cdots, e^{sa_J}). \tag{5.9}$$

Proof. Recall that in the proof of Proposition 5.5, we showed that for any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, $l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$ is of the following form:

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M} b_j t^j.$$

with $m \in \mathbb{Z}$ and $b_0 \neq 0$. We have the following general formula for combinatorial operations: for $x \in \mathbb{C}(\!(t)\!)$,

$$\mathrm{lt}'_{M'} \circ \mathrm{plog} \circ \mathrm{lt}_{M'}(x) = \mathrm{lt}'_{M'} \circ \mathrm{plog} \circ \mathrm{lt}_M(x) = \mathrm{lt}'_{M'} \circ \mathrm{lt}'_M \circ \mathrm{plog} \circ \mathrm{lt}_M(x),$$

where the first equality is due to Eq. (5.3). By definition of l_M, this implies $l_{M'}(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = \mathrm{lt}'_{M'} l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$, and hence

$$l_{M'}(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M'} b_j t^j.$$

To obtain Eq. (5.8), when applying the proof of Corollary 5.8 to $Z_{M'}(\Gamma, z, s)$, we consider an edge matrix $W(s)$ similar to the one for $Z_M(\Gamma, z, s)$ with l_M replaced by $l_{M'}$.

For the holomorphic extension of the Ihara zeta function, recall that for any $\gamma \in \Gamma \backslash \{id\}$, $\ell^{na}(\gamma) \log(1/t) = \mathrm{lt}'_{-1}(L_M(\gamma))$. Remark 5.6 allows us to apply the proof of Corollary 5.8 to $Z_I(\Gamma, s)$ with $l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))$ replaced by $\mathrm{lt}'_{-1}\left(l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}))\right)$, which yields Eq. (5.9). ☐

Proposition 5.10. *There exists $C > 0$ depending on Γ such that for any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_N}$, we have*

$$\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N}(\infty)) = t^m \sum_{j=0}^{\infty} a_j t^j, \tag{5.10}$$

where a_j's are complex numbers satisfying

$$|a_0| \geqslant e^{-C} \quad \text{and} \quad |a_j| \leqslant e^{C(N+j)} \quad \text{for} \quad j \in \mathbb{Z}_{\geqslant 0}.$$

Moreover, we have for any $M \in \mathbb{Z}_{\geqslant 0}$ and $N \geqslant C_1(M+1)$ (from Proposition 5.5),

$$l_M(\mathfrak{a}_{i_1}, (\mathfrak{a}_{i_2} \cdots \mathfrak{a}_{i_N})) = m \log(1/t) + \sum_{j=0}^{M} b_j t^j, \tag{5.11}$$

where b_j's are complex numbers satisfying

$$|b_j| \leqslant e^{C(N+1)j} \quad \text{for} \quad j = 0, \ldots, M.$$

For the proof, please see Section 7.2.

Proof of Lemma 4.8. By Eq. (5.4) and Proposition 5.5, there exists $C_1 > 0$ depending only on Γ such that for $N = \lceil C_1(M+2) \rceil$ and for any cyclically reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n}$ with $n > N$, we have

$$\log(1/|z|)\ell_M(\gamma, z) = -\Re\left(\sum_{k=1}^{n} l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}}))(z)\right).$$

By Proposition 5.10, there exists $C_2 > 0$ depending only on Γ, such that for $k = 1, \ldots, n$

$$l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}})) - \mathrm{lt}'_{-1}\left(l_M(\mathfrak{a}_{i_k}, (\mathfrak{a}_{i_{k+1}} \cdots \mathfrak{a}_{i_{k+N-1}}))\right) = \sum_{j=0}^{M} b_j t^j$$

where each b_j satisfies $|b_j| \leqslant e^{C_2(N+1)j}$. This implies there exists $C_3 > 0$ depending only on Γ, such that for any $|z| < e^{-2C_2(N+1)} \leqslant e^{-2C_2(C_1(M+2)+1)}$, we have

$$\left| \sum_{j=0}^{M} b_j z^j \right| \leqslant C_3.$$

Recall Remark 5.6. Combining these together, we obtain $C > 0$ depending only on Γ such that for any $0 < |z| < e^{-C(M+1)}$,

$$\log(1/|z|)\ell_M(\gamma, z) \geqslant \log(1/|z|)\ell^{na}(\gamma) - nC_3 \geqslant \frac{1}{2}\log(1/|z|)\ell^{na}(\gamma),$$

where the second inequality is due to Lemma 3.7.

For any reduced word $\gamma = \mathfrak{a}_{i_1} \cdots \mathfrak{a}_{i_n}$ with $n \leqslant N = \lceil C_1(M+1) \rceil$, we can prove the statement for γ by using Lemma 3.7 and Proposition 4.2. □

6 Examples

Recall that $\Gamma < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying (★).

6.1 Two-generator case

In this section, we discuss the case when Γ is generated by two generators to illustrate the 0-th intermediate zeta function. In this case, we can obtain explicit expressions for the leading coefficient $A_0(\gamma)$ using the Fricke relation of $\mathrm{SL}_2(\mathbb{C}(\!(t)\!))$:

$$\mathrm{tr}(gh) + \mathrm{tr}(gh^{-1}) = \mathrm{tr}(g)\mathrm{tr}(h), \quad g, h \in \mathrm{SL}_2(\mathbb{C}(\!(t)\!)). \tag{6.1}$$

For convenience, we denote the leading term of the trace of $\gamma \in \Gamma \backslash \{id\}$ as

$$\mathrm{lt}(\gamma) := \mathrm{lt}_0(\mathrm{tr}(\gamma)) = A_0(\gamma)t^{-\ell^{na}(\gamma)/2}$$

(see Eq. (4.3) for the Laurent expansion of $\mathrm{tr}(\gamma)$). Recall the 0-th expansion of length, $\ell_0(\gamma, t)$ defined in Eq. (4.9).

Lemma 6.1. *For any $\gamma \in \Gamma \backslash \{id\}$, we have*

$$\ell_0(\gamma, z) = 2\log|\mathrm{lt}(\gamma)| / \log(1/|z|),$$

and hence

$$Z_0(\Gamma, z, s) = \prod_{[\gamma] \in \mathcal{P}} (1 - e^{-2s\log|\mathrm{lt}(\gamma)|/\log(1/|z|)}).$$

Proof. Given any $\gamma \in \Gamma \backslash \{id\}$, recall Proposition 4.3 gives

$$\mathrm{tr}(\gamma) = t^{-\ell^{na}(\gamma)/2}(A_0(\gamma) + A_1(\gamma)t + \cdots),$$

and Proposition 4.2 and Eq. (4.7) give

$$\ell(\gamma_z)/\log(1/|z|) = \ell^{na}(\gamma) + 2\log|A_0(\gamma)|/\log(1/|z|) + \Re\left(\sum_{j \geqslant 1} a_j(\gamma)z^j / \log(1/|z|)\right).$$

The proof is complete. □

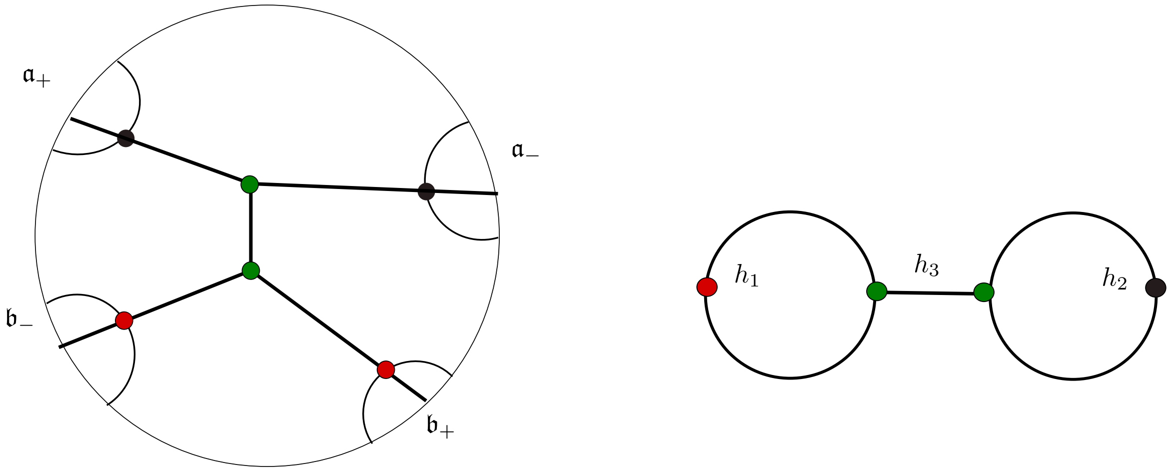

Figure 6.1: Dumbbell graph

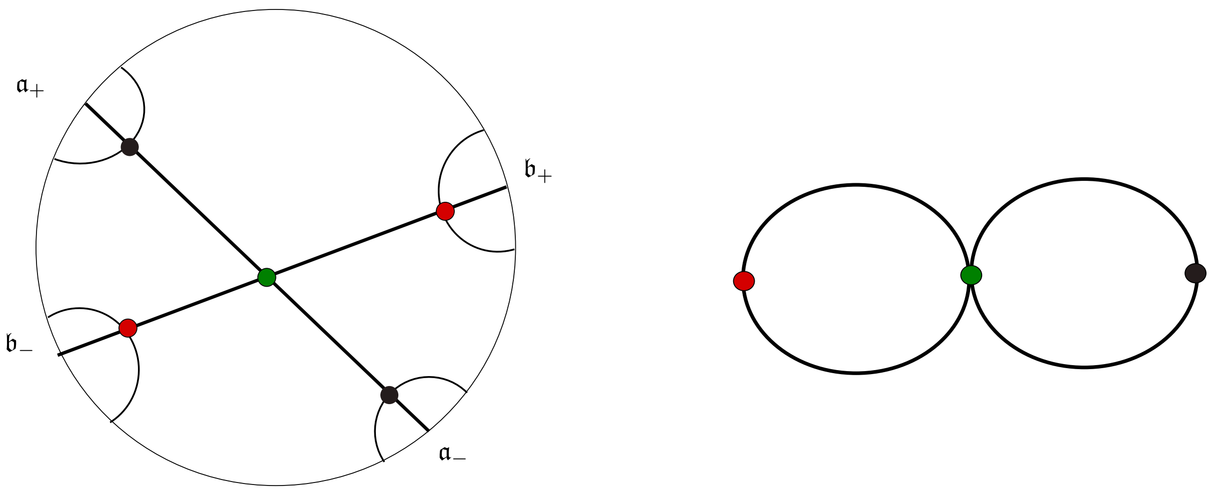

Figure 6.2: Figure eight graph

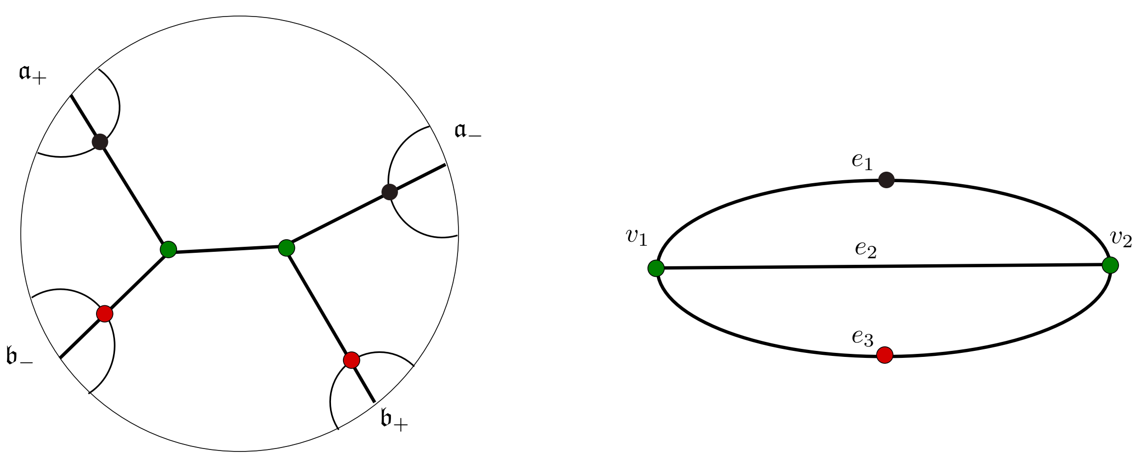

Figure 6.3: Figure Θ graph

When the Schottky group Γ is generated by two generators, i.e. $\Gamma = \langle \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \rangle$ where $\mathfrak{a}_1\mathfrak{a}_3 = \mathfrak{a}_2\mathfrak{a}_4 = id$, we have three different possibilities (Figs. 6.1 to 6.3) for the corresponding finite graph Σ_X that we will discuss below. (In the Figs. 6.1 to 6.3, we use $\mathfrak{a}$ and $\mathfrak{b}$ to denote two generators and $\mathfrak{a}_\pm$ for its attracting and repelling fixed points in $\mathbb{P}^1_{\mathbb{C}((t))}$. The tree on the left is the subtree connecting these points in $\mathbb{P}^{1,an}_{\mathbb{C}((t))}$. The graph on the right is the corresponding Mumford curve of the Schottky group.)

We will need

Lemma 6.2. *Assume $\Gamma = \langle \mathfrak{a}_1, \ldots, \mathfrak{a}_4 \rangle < \mathrm{SL}_2(\mathbb{M}(\mathbb{D}))$ is a family of Schottky groups satisfying ($\bigstar$).*[15] *The following identity holds for any cyclically reduced word:*

$$\mathrm{lt}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_n})^2 = \prod_{j=1}^{n} \mathrm{lt}(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}) \tag{6.2}$$

with the convention that $i_{n+1} = i_1$.

Proof. Suppose we have a cyclically reduced word $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$. By Eq. (6.1), we have

$$\mathrm{tr}(\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2) = \mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\mathfrak{a}_1\gamma_2) - \mathrm{tr}(\gamma_2^{-1}\gamma_1).$$

Claim: We have

$$|\mathrm{tr}(\gamma_2^{-1}\gamma_1)|_{na} < |\mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\gamma_2^{-1}\mathfrak{a}_1^{-1})|_{na} = |\mathrm{tr}(\mathfrak{a}_1\gamma_1)\mathrm{tr}(\mathfrak{a}_1\gamma_2)|_{na}. \tag{6.3}$$

Proof of the claim: The claim only works for the two-generator case, and we give an ad-hoc proof.

For the figure eight graph, by using the graph and $2\log|\mathrm{tr}(\gamma)|_{na} = \ell^{na}(\gamma)$, the length of the non-backtracting loop of γ, we have

$$|\mathrm{tr}(\mathfrak{a}_1\gamma_1)|_{na} > |\mathrm{tr}(\gamma_1)|_{na}.$$

Therefore

$$\begin{aligned}|\mathrm{tr}(\gamma_2^{-1}\gamma_1)|_{na} &= \exp(\ell^{na}(\gamma_2^{-1}\gamma_1)/2) \leqslant \exp((\ell^{na}(\gamma_2^{-1}) + \ell^{na}(\gamma_1))/2)\\ &= |\mathrm{tr}(\gamma_1)|_{na}|\mathrm{tr}(\gamma_2^{-1})|_{na} < |\mathrm{tr}(\mathfrak{a}_1\gamma_1)|_{na}|\mathrm{tr}(\gamma_2^{-1}\mathfrak{a}_1^{-1})|_{na}.\end{aligned}$$

For the figure Θ graph (Fig. 6.3): let h_1, h_2, h_3 be the lengths of the three edges e_1, e_2, e_3 connecting two vertices v_1, v_2. For each generator, we use a loop with the starting point v_1 in the graph. For example the loops of $\mathfrak{a}_1, \mathfrak{a}_2$ are represented by $e_1e_5 = e_1e_2^{-1}$ and $e_2e_6 = e_2e_3^{-1}$, respectively. For a cyclically reduced word γ, we connect the loops corresponding to each letters through v_1 to get a loop of γ. The only possible backtracking part in the loop is the middle edge in the graph Θ, which corresponds to the pairs $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_4\mathfrak{a}_3$ in γ. We have that

$$\ell^{na}(\gamma) = n_1(\gamma)(h_1 + h_2) + n_2(\gamma)(h_2 + h_3) - n_3(\gamma)(2h_2), \tag{6.4}$$

with $n_1(\gamma)$ the number of $\mathfrak{a}_1, \mathfrak{a}_3$ in γ, $n_2(\gamma)$ the number of $\mathfrak{a}_2, \mathfrak{a}_4$, and $n_3(\gamma)$ the number of pairs $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_4\mathfrak{a}_3$ in γ[16].

Lemma 6.3. *For cyclically reduced $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}$, we have* [17]

$$2\ell^{na}(\gamma) = \sum_{1\leqslant j\leqslant N} \ell^{na}(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}). \tag{6.5}$$

Proof. It suffices to check the equality for n_1, n_2, n_3 in Eq. (6.4). The corresponding equality for n_1, n_2 is trivial. For n_3, let n_4 be the number of pairs of $\mathfrak{a}_2\mathfrak{a}_1$ and $\mathfrak{a}_3\mathfrak{a}_4$ in γ. Let $I = \{\mathfrak{a}_1, \mathfrak{a}_4\}$ and $II = \{\mathfrak{a}_2, \mathfrak{a}_3\}$. A cyclically reduced word can be represented as loop on the graph with vertices $\{\mathfrak{a}_i, 1 \leqslant i \leqslant 4\}$ and all the edges are allowed except the edges $\mathfrak{a}_i\mathfrak{a}_{i+2}$. Then n_3 is the number of edges in the loop from vertices in I

[15] This lemma also works for a general Schottky group in $\mathrm{SL}_2(\mathbb{C}((t)))$.

[16] For example $\gamma = \mathfrak{a}_{i_1}\cdots\mathfrak{a}_{i_N}$, the pair $\mathfrak{a}_{i_N}\mathfrak{a}_{i_1}$ should also be considered.

[17] It is possible that the word is simply $\mathfrak{a}_{i_1}$, then we use that $2\ell^{na}(\mathfrak{a}_{i_1}) = \ell^{na}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_1})$.

to II. n_4 is the number of edges in the loop from vertices in II to I. Since γ represents a loop, we have $n_3(\gamma) = n_4(\gamma)$. Notice that $n_3(\mathfrak{a}_2\mathfrak{a}_1) = n_3(\mathfrak{a}_1\mathfrak{a}_2) = 1$. We have

$$2n_3(\gamma) = n_3(\gamma) + n_4(\gamma) = \sum_{1\leqslant j\leqslant N} n_3(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}),$$

where the second equality is because both sides represent the number of $\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{a}_4\mathfrak{a}_3$, $\mathfrak{a}_2\mathfrak{a}_1$, $\mathfrak{a}_3\mathfrak{a}_4$ in γ. The proof is complete. □

For a non-cyclically reduced word γ, we also have

$$2\ell^{na}(\gamma) \leqslant \sum_{1\leqslant j\leqslant N} \ell^{na}(\mathfrak{a}_{i_j}\mathfrak{a}_{i_{j+1}}), \tag{6.6}$$

since in order to obtain a cyclically reduced representation of γ, we are doing induction and we only need to treat the case such as $\cdots \mathfrak{b}_1\mathfrak{a}_1\mathfrak{a}_1^{-1}\mathfrak{b}_2\cdots$.

We go back to the claim Eq. (6.3). For any $\mathfrak{b}_1, \mathfrak{b}_2$ in the set of generators, we have

$$\ell^{na}(\mathfrak{b}_1\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_2) \geqslant \ell^{na}(\mathfrak{b}_1\mathfrak{b}_2), \tag{6.7}$$

which is true by considering all 16 possibilities of $\mathfrak{b}_1, \mathfrak{b}_2$, and the equality holds if and only if

$$\mathfrak{b}_1 = \mathfrak{a}_3 \text{ or } \mathfrak{b}_2 = \mathfrak{a}_1. \tag{6.8}$$

Therefore suppose $\mathfrak{a}_1\gamma_1 = \mathfrak{a}_1\mathfrak{b}_1\cdots\mathfrak{b}_2$ and $\gamma_2^{-1}\mathfrak{a}_1^{-1} = \mathfrak{b}_3\cdots\mathfrak{b}_4\mathfrak{a}_1^{-1}$ with $\mathfrak{b}_i$ from the set of generators, due to Eq. (6.5) and Eq. (6.6)

$$\begin{aligned}
&2\ell^{na}(\mathfrak{a}_1\gamma_1) + 2\ell^{na}(\gamma_2^{-1}\mathfrak{a}_1^{-1}) - 2\ell^{na}(\gamma_2^{-1}\gamma_1)\\
\geqslant\, &\ell^{na}(\mathfrak{a}_1\mathfrak{b}_1) + \ell^{na}(\mathfrak{b}_2\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_3) + \ell^{na}(\mathfrak{b}_4\mathfrak{a}_1^{-1}) - \ell^{na}(\mathfrak{b}_4\mathfrak{b}_1) - \ell^{na}(\mathfrak{b}_2\mathfrak{b}_3)\\
=\, &\ell^{na}(\mathfrak{b}_1\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_4) - \ell^{na}(\mathfrak{b}_1\mathfrak{b}_4) + \ell^{na}(\mathfrak{b}_2\mathfrak{a}_1) + \ell^{na}(\mathfrak{a}_1^{-1}\mathfrak{b}_3) - \ell^{na}(\mathfrak{b}_2\mathfrak{b}_3) \geqslant 0,
\end{aligned}$$

where the last inequality is due to Eq. (6.7). Since $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$ is cyclically reduced, we obtain $\mathfrak{b}_1, \mathfrak{b}_2 \neq \mathfrak{a}_3$ and $\mathfrak{b}_3, \mathfrak{b}_4 \neq \mathfrak{a}_1$. Therefore neither $(\mathfrak{b}_1, \mathfrak{b}_4)$ nor $(\mathfrak{b}_2, \mathfrak{b}_3)$ satisfies Eq. (6.8). Hence the equality of the last equation cannot hold and the proof is complete.

For the dumbbell graph (Fig. 6.1), we can prove by the same method as the figure Θ graph: first we show Eq. (6.5) holds, then Eq. (6.7) holds, finally we can conclude Eq. (6.3).

Back to the proof: Hence [18]

$$\mathrm{lt}(\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2) = \mathrm{lt}(\mathfrak{a}_1\gamma_1)\mathrm{lt}(\mathfrak{a}_1\gamma_2). \tag{6.9}$$

Moreover, if $\mathfrak{a}_1\gamma_1$ and $\mathfrak{a}_1\gamma_2$ satisfy Eq. (6.2), then by Eq. (6.9), the longer word $\mathfrak{a}_1\gamma_1\mathfrak{a}_1\gamma_2$ also satisfies Eq. (6.2).

For any cyclically reduced word of length greater than 5, we can find some generator that appears at least twice. Therefore we can use Eq. (6.9) to reduce the length of the word and do induction.

The only cyclically reduced words without any generator appearing twice are the commutator $\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4$ and $\mathfrak{a}_i\mathfrak{a}_j$, $\mathfrak{a}_i$. For the commutator, we have the well-known formula from the Fricke relation

$$\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}) = \mathrm{tr}(\mathfrak{a}_1)^2+\mathrm{tr}(\mathfrak{a}_2)^2+\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)^2-\mathrm{tr}(\mathfrak{a}_1)\mathrm{tr}(\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)-2 = \mathrm{tr}(\mathfrak{a}_1)^2+\mathrm{tr}(\mathfrak{a}_2)^2-\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2^{-1})-2$$

By checking the three graphs of two generator cases, we have $|\mathrm{tr}(\mathfrak{a}_1)^2|_{na} = |\mathrm{tr}(\mathfrak{a}_1^2)|_{na} < |\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{tr}(\mathfrak{a}_2^{-1}\mathfrak{a}_1)|_{na}$. Hence

$$\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}) = -\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2)\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2^{-1}).$$

The proof is complete. □

[18] This is not a general equation, for three generators with graph (||) consisting of two vertices and four edges between them, the sum of length of $\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{a}_2^{-1}\mathfrak{a}_3^{-1}$ equals the length of $\mathfrak{a}_1\mathfrak{a}_3^{-1}$.

6.1.1 Figure Θ graph

We first consider the case that the graph Σ_X is like Θ in Fig. 6.3. Without loss of generality, we suppose each edge of the graph Σ_X has length 2 (for example the symmetric three-funnel case). Suppose $\mathrm{lt}(\mathfrak{a}_1) = Az^{-2}$, $\mathrm{lt}(\mathfrak{a}_2) = Bz^{-2}$ and $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2) = Cz^{-2}$ with $A, B, C \in \mathbb{C}^*$. For the weighted graph, we can define three weights by

$$h_j = 2 + \frac{2\alpha_j}{\log(1/|z|)},$$

with α_j's determined by

$$\alpha_1 + \alpha_2 = \log|A|, \quad \alpha_2 + \alpha_3 = \log|B|, \quad \alpha_1 + \alpha_3 = \log|C|.$$

For a reduced word γ of word length two, from the choice of α_j we can verify that the weighted length ℓ_h satisfies

$$\ell_h(\gamma) = 2\log|\mathrm{lt}(\gamma)|/\log(1/|z|) = \ell_0(\gamma, z). \tag{6.10}$$

For example for the word $\mathfrak{a}_1\mathfrak{a}_2^{-1}$, from the Fricke relation, we have $\mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2^{-1}) = \mathrm{tr}(\mathfrak{a}_1)\mathrm{tr}(\mathfrak{a}_2) - \mathrm{tr}(\mathfrak{a}_1\mathfrak{a}_2)$, then $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2^{-1}) = \mathrm{lt}(\mathfrak{a}_1)\mathrm{lt}(\mathfrak{a}_2)$ which implies $\ell_0(\mathfrak{a}_1\mathfrak{a}_2^{-1}, z) = h_1 + 2h_2 + h_3 = \ell_h(\mathfrak{a}_1\mathfrak{a}_2^{-1})$. By Eq. (6.2), we know this relation also holds for any cyclically reduced γ. Therefore, the weighted Ihara zeta function of the weight h gives the intermediate zeta function Z_0.

For this example, by Proposition 5.1, the zeta function $Z_0(\Gamma, z, s)$ can be computed and is given by $(-2abc + a^2 + b^2 + c^2 - 1)(2abc + a^2 + b^2 + c^2 - 1)$ with $a = e^{-s(h_2+h_3)/2}, b = e^{-s(h_1+h_3)/2}, c = e^{-s(h_1+h_2)/2}$.

6.1.2 Dumbbell graph

The case that the graph Σ_X is like a dumbbell as in Fig. 6.1 is similar to Section 6.1.1. Suppose each edge of the graph has length two. For example, $\mathrm{lt}(\mathfrak{a}_1) = Az^{-1}$, $\mathrm{lt}(\mathfrak{a}_2) = Bz^{-1}$ and $\mathrm{lt}(\mathfrak{a}_1\mathfrak{a}_2) = Cz^{-4}$ with $A, B, C \in \mathbb{C}^*$. We define the three weights by

$$h_j = 2 + \frac{2\beta_j}{\log(1/|z|)},$$

with β_j's determined by

$$\beta_1 = \log|A|, \quad \beta_2 = \log|B|, \quad \beta_1 + 2\beta_3 + \beta_2 = \log|C|.$$

Similar to the previous case, the weighted Ihara zeta function equals the intermediate zeta function Z_0.

For this example, by Proposition 5.1 the zeta function $Z_0(\Gamma, z, s)$ can be computed and is given by $-(c-1)(c+1)(a-1)(a+1)(4a^2b^4c^2 - a^2c^2 + a^2 + c^2 - 1)$ with $a = e^{-sh_1/2}, b = e^{-sh_3/2}, c = e^{-sh_2/2}$. For $t = 0$, $s = \frac{1}{2}\log 2$ is the solution with the maximal real part.

Example 6.4. *If $A = B = C = 1$, then the intermediate zeta function coincides with the Ihara zeta function.*

6.1.3 Figure eight graph

The case when the graph Σ_X is like figure 8 (as shown in Fig. 6.2) is not directly applicable for weighted Ihara zeta function, since Eq. (6.10) does not hold. But using the identity Eq. (6.2), we can compute

$$\begin{aligned} -\log Z_0(\Gamma, z, s) &= \sum_{j=1}^{\infty} \sum_{[\gamma]\in\mathcal{P}} \frac{1}{j} e^{-2js\log|\mathrm{lt}(\gamma)|/\log(1/|z|)} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j\#(\gamma)=n} e^{-js\log|\mathrm{lt}(\gamma)^2|/\log(1/|z|)} \end{aligned}$$

Consider the matrix $W \in M_{12\times 12}$ given by $W(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2}, \mathfrak{a}_{i_2}\mathfrak{a}_{i_3}) = \exp\left(-s\frac{\log|\mathrm{lt}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2})| + \log|\mathrm{lt}(\mathfrak{a}_{i_2}\mathfrak{a}_{i_3})|}{2\log(1/|z|)}\right)$, then the above expression is equal to

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr}W^n = -\log\det(I - W).$$

Remark. This is indeed the path zeta function in Section 4 in [HST06]. Consider the matrix $V \in M_{4\times 4}$ given by $V(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}) = \exp\left(-s\frac{\log|\mathrm{lt}(\mathfrak{a}_{i_1}\mathfrak{a}_{i_2})|}{\log(1/|z|)}\right)$ for $\mathfrak{a}_{i_1}\mathfrak{a}_{i_2} \neq id$. Then the above expression is also equal to

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr} V^n = -\log\det(I - V).$$

Example 6.5. *Example of funneled torus in [Bor16, Section 16]. Here the matrices are given by*

$$S_1 = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}, \ S_2 = \begin{pmatrix} \cosh(\ell/2) - \cos\phi\ \sinh(\ell/2) & \sin^2\phi\ \sinh(\ell/2) \\ \sinh(\ell/2) & \cosh(\ell/2) + \cos\phi\ \sinh(\ell/2) \end{pmatrix}$$

with ϕ the angle between the two shortest geodesics. We do the change of variable with $z = e^{-\ell/2}$ and obtain a Schottky family. [19] *Moreover,*

$$\mathrm{lt}(S_1) = z^{-1}, \ \mathrm{lt}(S_2) = z^{-1}, \ \mathrm{lt}(S_1S_2) = (1 + \cos\phi)z^{-2}/2.$$

The graph is figure eight with edge length 2.

From the first paragraph of Section 6.1.3, with $\phi = \pi/2$ the middle zeta function can be computed and is given by

$$Z_0(\Gamma, z, s) = -(-a + 2b + 1)(a + 2b - 1)(a - 1)^2,$$

with $a = e^{-2s}$, $b = e^{-s(2-\log 2/\log(1/|z|))}$. Hence the Hausdorff dimension δ_z of the limit set is given by the first zero of

$$e^{-2s} + 2e^{-s(2-\log 2/\log(1/|z|)} - 1 = 0$$

dividing by $\log(1/|z|)$. *For the case $\ell = 10$ or $z = e^{-5}$, the first zero of $Z_0(\Gamma, e^{-5}, s)$ dividing by* $\log(1/|z|)$ *is close to* 0.115, *which is close to numerics in [Bor16, Fig 16.8]. The main term $\frac{\log 3}{10}$ of [DM24] gives* 0.1099.

Remark 6.6. In [Wei15, Theorem 1.1], the author considered the case of three funnels with width given by $n_1\ell, n_2\ell, n_3\ell$ with $n_1, n_2, n_3 \in \mathbb{N}$ and $\ell \to \infty$. Under an extra triangle condition $n_i + n_j > n_k$ with $\{i, j, k\} = \{1, 2, 3\}$, the convergence to the Ihara zeta function is proved in [Wei15]. Actually, the triangle condition says that the graph is figure Θ. If $n_1 + n_2 = n_3$, then the graph is figure eight. If $n_1 + n_2 < n_3$, then the graph is figure dumbbell. For all these three cases, by the change of variable with $z = e^{-\ell/2}$, our computation of Ihara zeta functions/intermediate zeta function Z_0 and the convergence all work.

We also remark that the type of graph (figure eight, dumbbell, and figure Θ) depends on the choice of the lengths of the generators, while the topological type of the surface depends on the configuration of the Schottky discs. In the case of two generators $\mathfrak{a}_1$ and $\mathfrak{a}_2$, if $D_{\mathfrak{a}_1}$ and $D_{\mathfrak{a}_1^{-1}}$ are adjacent (Figs. 6.1 and 6.3), then the hyperbolic surface is a three-funnel surface; if $D_{\mathfrak{a}_1}$ and $D_{\mathfrak{a}_1^{-1}}$ are separated by $D_{\mathfrak{a}_2}$ and $D_{\mathfrak{a}_2^{-1}}$ (Fig. 6.2), then the hyperbolic surface is a funneled torus.

6.2 Symmetric three-funnel surface

In this section, we discuss how our main theorem applies to the symmetric three-funnel surface. In particular, we recover the result of [PV19].

Example 6.7. *Recall $X(\ell)$ from introduction. In [Bor16, Section 16], an explicit form is given by*

$$S_1 = \begin{pmatrix} \cosh(\ell/2) & \sinh(\ell/2) \\ \sinh(\ell/2) & \cosh(\ell/2) \end{pmatrix}, \ S_2 = \begin{pmatrix} \cosh(\ell/2) & a^{-1}\sinh(\ell/2) \\ a\sinh(\ell/2) & \cosh(\ell/2) \end{pmatrix}$$

and $a \in \mathbb{R}$ such that $\mathrm{tr}(S_1S_2) = -2\cosh(\ell/2)$. We do the change of variable with $z = e^{-\ell/4}$, then the group becomes a Schottky family with

$$\mathrm{lt}(S_1) = \mathrm{lt}(S_2) = -\mathrm{lt}(S_1S_2) = z^{-2}.$$

This family satisfies condition (★) can be verified in the following way:

$$\mathrm{tr}(S_1S_2) = 2((z^{-2} + z^2)/2)^2 + (a + a^{-1})((z^{-2} - z^2)/2)^2 = -2(z^{-2} + z^2)/2.$$

[19] The verification of condition (★) can be done similarly as in Example 6.7.

So we obtain a solution

$$a^{-1} = -1 + 2z + O(|z^2|).$$

The fixed points of S_1 are at 1 (attracting) and -1 (repelling). The fixed points of S_2 are at a^{-1} and $-a^{-1}$. The cross-ratios of these four points are

$$\left(\frac{1 \pm a^{-1}}{1 - \pm a^{-1}}\right)^2,$$

which have order greater than -2. The contracting ratios of S_1 are S_2 are z^4. From these, we can verify that this family satisfies condition (★).

The graph is two vertices with three edge connecting them with each edge of length 2. From the discussion of the two generator case, the intermediate zeta function Z_0 with $M = 0$ is exactly the graph zeta function for symmetric three-funnel surface.

Since Theorem 4.4 also works for $\mathrm{SL}_2(\mathbb{R})$-case, combined with the computation of Z_0, we obtain the convergence of zeta functions in Theorem 1.1. The statement of Hausdorff dimension follows directly.

Example 6.8. *We want to compute the zeta function for other M and explain that we recover the result of [PV19].*

Continue from the previous example. The three-funnel surfaces can be described by four discs, each of radius z^2. Therefore the separation between $d_{na}(\mathfrak{a}_i\infty, (D_{\mathfrak{a}_i})^c) = e^{-2}$ for the generators.

After doing computations using the algorithm in Proposition 5.5 and Corollary 5.8, we obtain: Let $x_1 = e^{-2s}$, $x_2 = e^{sz^2/2\log(1/|z|)}$, $x_3 = e^{sz^4/2\log(1/|z|)}$. Then

$$Z_2(\Gamma, z, s) = ((x_2^4 + x_1^4(-1 + x_2^2)^2 + x_1^2(x_2^2 - 2x_2^4))^2(x_2^4 + x_1^4(-1 + x_2^2)^2 - 2x_1^2(x_2^2 + x_2^4)))/x_2^{12}$$

and

$$\begin{aligned} Z_4(\Gamma, z, s) = \frac{1}{x_3^{18}x_2^{12}}&(x_1^8 - 4x_3^2x_1^8 + x_3^8x_1^4(-1 + x_1^2)^2 + x_3^4(-2x_1^6 + 6x_1^8) + x_3^3x_1^4(-2 + x_1^2)x_2^2 - 2x_3^5x_1^4(-1 + x_1^2)x_2^2 \\ &+ x_3^7x_1^2(-1 + x_1^2)^2x_2^2 - x_3^6(-1 + x_1^2)(4x_1^6 + x_2^4 - x_1^2x_2^4))^2 \\ &(x_1^8 - 4x_3^2x_1^8 + x_3^8x_1^4(-1 + x_1^2)^2 + x_3^4(-2x_1^6 + 6x_1^8) + 4x_3^5x_1^4(-1 + x_1^2)x_2^2 \\ &- 2x_3^7x_1^2(-1 + x_1^2)^2x_2^2 - 2x_3^3x_1^4(1 + x_1^2)x_2^2 - x_3^6(-1 + x_1^2)(4x_1^6 + x_2^4 - x_1^2x_2^4)). \end{aligned}$$

In particular, factorizing Z_2, we can recover the result of [PV19] by Corollary 4.6. In Z_2, if we let $x_2 = 1$, then we obtain the Ihara zeta function of the symmetric three-funnel surface, which is predicted by Corollary 5.9.

Remark 6.9. From previous computations, we conjecture that the term x_1 in $Z_{2M}(\Gamma, z, s)$ has degree $6 \cdot 2^M$ with coefficient $(1 - x_{M+1}^2)^{3\cdot 2^M}$, where $x_{M+1} = e^{sz^{2M}/2\log(1/|z|)}$. This conjecture gives some support to the fractal Weyl law, that is the number of resonances in the region $\{s \in \mathbb{C} : \Re s > -C, |\Im s| \leqslant T\}$ for any fixed $C > 0$ grows asymptotic to $T^{1+\delta_\Gamma}$ with δ_Γ the critical exponent of the Schottky group Γ.

Below is a heuristic argument: In the region $\Re s \in [-C, \delta], \Im s \in [2T\pi, 2(T+1)\pi]$ with $T \approx |z|^{-2M}$, the leading coefficient $(1 - x_{M+1}^2)^{3\cdot 2^M}$ is of constant size. Since z is small, the terms $x_{j+1} = e^{sz^{2j}/2\log(1/|z|)}$ with $j \geqslant 1$ are almost constant. The only variable in Z_{2M} is x_1. Hence as a polynomial on x_1, Z_{2M} is of degree $6 \cdot 2^M$ and the ratio between the coefficients and the leading coefficient is bounded. Therefore this polynomial has $6 \cdot 2^M$ zeros of bounded size. Then $s = -\frac{1}{2}\log x_1$ is in the box $\{\Re s \in [-C, \delta],\ \Im s \in [2T\pi, 2(T+1)\pi]\}$. The predicted numbers of zeros by the fractal Weyl law in this region is

$$(|z|^{-2M})^{\delta} \approx (|z|^{-2M})^{\log 2/2\log(1/|z|)} = 2^M.$$

For the region $\Re s \in [-C, \delta], \Im s \in [2T\pi, 2(T+1)\pi]$ with $T \ll |z|^{-2M}$, the leading coefficient $(1-x_{M+1}^2)^{3\cdot 2^M}$ is so small, hence Z_{2M} may have large zeros of x_1 and the corresponding s is outside the region. Since $Z_{2M'}(s)$ for some $M' < M$ with $M' \approx \log T/2\log(1/|z|)$ is a good approximation to $Z_{2M}(s)$ in the region, the number of zeros of $Z_{2M}(s)$ can be computed from $Z_{2M'}(s)$, which is approximately $2^{M'} \approx T^{\delta}$.

In conclusion, with the intermediate zeta function Z_{2M}, for $T \leqslant |z|^{-2M}$, we observe the number of zeros in the region $\{\Re s \in [-C, \delta], \Im s \in [2T\pi, 2(T+1)\pi]\}$ is approximately T^{δ}, which fits the prediction of the fractal Weyl law.

7 Appendix

7.1 Critical exponent of the non-Archimedean Poincaré series

In this Appendix, we explain that the critical exponent of the Poincaré series is equal to the growth rate of the number of periodic geodesics for non-Archimedean k.

Let Γ be a Schottky group in $\mathrm{PGL}_2(k)$ of g generators. Recall from [PT21, Theorem II.3.18] the non-wandering domain O of Γ on $\mathbb{P}_k^{1,an}$, the Mumford curve $X = \Gamma\backslash O$ and their skeleta Σ_O and Σ_X. Then Σ_X is a graph of rank g. Let p be the quotient map from O to X, then $p^{-1}\Sigma_X = \Sigma_O$ and $\Sigma_X = p(\Sigma_O) = \Gamma\backslash\Sigma_O$. Recall that we have a distance d_a on Σ_O, the Γ action on Σ_O preserves the distance and we have a quotient distance on Σ_X.

Recall the definition of length periodic primitive geodesic in Lemma 2.17. Let $\mathcal{G}_\Gamma(L)$ be the number of periodic primitive geodesics of length less than L in Σ_X, which is also the least translation length of the corresponding conjugacy class on Σ_O. Let $\mathcal{N}_\Gamma(L, o)$ be the number of $\gamma \in \Gamma$ such that $d_a(o, \gamma o) \leqslant L$ for $o \in \Sigma_O$.

Since Σ_O is a locally finite tree with a metric, it is CAT(-1). We can apply [Rob02] to obtain the critical exponent of the Poincaré series

$$\delta(\Gamma) = \lim_{L\to\infty} \frac{1}{L} \log \mathcal{N}_\Gamma(L, o).$$

Moreover, by [Rob02, Corollaire 2]

Proposition 7.1. *We have*

$$\lim_{L\to\infty} \frac{1}{L} \log \mathcal{G}_\Gamma(L) = \delta(\Gamma).$$

By [HST06, Theorem 2.10], we have

Corollary 7.2. *The first zero of Ihara zeta function of the graph is equal to the critical exponent.*

7.2 Proof of Proposition 5.10

Before the proof, we state three elementary lemmas about estimates of the coefficients of the Laurent series after multiplication, division, and logarithm.

Lemma 7.3. *Suppose $f(t), g(t) \in \mathbb{C}(\!(t)\!)$ are two formal Laurent series:*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j \quad \textit{with} \quad |f_j| \leqslant e^{C(j+1)} \textit{ for } j \in \mathbb{Z}_{\geqslant 0};$$

$$g(t) = t^{m'} \sum_{j=0}^{\infty} g_j t^j \quad \textit{with} \quad |g_j| \leqslant e^{C(j+1)} \textit{ for } j \in \mathbb{Z}_{\geqslant 0}.$$

Then $f(t)g(t) \in \mathbb{C}(\!(t)\!)$ has the Laurent series expansion

$$f(t)g(t) = t^{m+m'} \sum_{j=0}^{\infty} h_j t^j \quad \textit{with} \quad |h_j| \leqslant (j+1)e^{C(j+2)} \textit{ for } j \in \mathbb{Z}_{\geqslant 0}.$$

Proof. We have

$$|h_n| = \left| \sum_{j+k=n} f_j g_k \right| \leqslant (n+1)e^{C(n+2)}.$$

□

Lemma 7.4. *Suppose $f(t), g(t) \in \mathbb{C}(\!(t)\!)$ are two formal Laurent series with*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j, \quad f_0 \neq 0, \, |f_j| \leqslant e^{C(j+1)};$$

$$g(t) = t^{m'} \sum_{j=0}^{\infty} g_j t^j, \quad |g_0| \geqslant e^{-A},\, |g_j| \leqslant e^{C(j+1)}.$$

Then $f(t)/g(t) \in \mathbb{C}((t))$ has the Laurent series expansion

$$f(t)/g(t) = t^{m-m'} \sum_{j=0}^{\infty} h_j t^j, \quad |h_j| \leqslant e^{(3C+C_1+A)(j+1)}$$

for some universal constant C_1.

Proof. We may first take $\tilde{g}_0 = 1$ and $|\tilde{g}_j| = |g_j/g_0| \leqslant e^{C(j+1)+A}$. Then

$$\frac{g_0 t^{m'}}{g(t)} = \sum_{n=0}^{\infty} \left(-\sum_{j=1}^{\infty} \tilde{g}_j t^j \right)^n = \sum_{n=0}^{\infty} g'_n t^n$$

where

$$|g'_n| \leqslant \sum_{j_1+\cdots+j_k=n} |\tilde{g}_{j_1} \cdots \tilde{g}_{j_k}| \leqslant e^{C_0 n} e^{2Cn+An} \leqslant e^{(2C+C_0+A)n}$$

for some universal constant C_0. Finally we multiply by $f(t)$:

$$\frac{f(t)}{g(t)} = \frac{f(t)}{g_0 t^{m'}} \frac{g_0 t^{m'}}{g(t)} = g_0^{-1} t^{m-m'} \left(\sum_{j=0}^{\infty} f_j t^j \right) \left(\sum_{n=0} g'_n t^n \right) = t^{m-m'} \sum_{j=0}^{\infty} h_j t^j$$

where

$$|h_n| \leqslant |g_0|^{-1} \sum_{j+k=n} |f_j g'_k| \leqslant (n+1) e^A e^{C(n+1)} e^{(2C+C_0+A)n} \leqslant e^{(3C+C_1+A)(n+1)}$$

for some universal constant C_1. □

Lemma 7.5. *Suppose $f(t) \in \mathbb{C}((t))$ has the expansion*

$$f(t) = t^m \sum_{j=0}^{\infty} f_j t^j, \quad f_0 \neq 0, \quad |f_j| \leqslant e^{C(j+1)}.$$

Then as a formal series, $\log(f(t))$ *has the expansion*

$$\log(f(t)) = m \log t + \log(f_0) + \sum_{j=0}^{\infty} h_j f_0^{-j} t^j, \quad |h_j| \leqslant e^{(3C+C_0)j}$$

where C_0 is a universal constant.

Proof. We use the formula for log:

$$\begin{aligned} \log(f(t)) &= m \log t + \log(f_0) + \log \left(1 + \sum_{j=1}^{\infty} \frac{f_j}{f_0} t^j \right) \\ &= m \log t + \log(f_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\sum_{j=1}^{\infty} \frac{f_j}{f_0} t^j \right)^n. \end{aligned}$$

Therefore,

$$|h_n| \leqslant \sum_{j_1+\cdots+j_k=n} \frac{1}{k} |f_{j_1} \cdots f_{j_k}| \cdot |f_0|^{n-k} \leqslant e^{C_0 n + 3Cn}.$$

□

Proof of Proposition 5.10. First we check the estimate for $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))$. Each entry of $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}$ is a sum of at most 2^{N-1} terms, with each term a product of $N-1$ entries of generators. By subadditivity and submultiplicativity of the hybrid norm, for each entry $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(i,l)$ we have

$$\sup_{1\leqslant i,l\leqslant 2}\|\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(i,l)\|_{\text{hyb}}\leqslant 2^{N-1}\sup_{\substack{\mathfrak{a}\in\mathcal{A}\\ 1\leqslant i,l\leqslant 2}}\|\mathfrak{a}(i,l)\|_{\text{hyb}}^{N-1}\lesssim e^{CN}.$$

Suppose $\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. For some $0<r<1/e$ independent of N and $|t|<r$, the Schottky discs $D_{\mathfrak{a}}$ are disjoint for $\mathfrak{a}\in\mathcal{A}$. If $a=\sum_{n\geqslant m}a_nt^n\in\mathbb{C}(\!(t)\!)$ with $a_m\neq 0$, then $\|a\|_{\text{hyb}}\leqslant e^{CN}$ implies that $e^{-m}\leqslant\|a\|_{\text{hyb}}\leqslant e^{CN}$ and

$$|a_z|\leqslant\sum_{n\leqslant 0}|a_n|(r/2)^n+\sum_{n>0}|a_n|e^{-n}\leqslant\max\{(re/2)^m,1\}\|a\|_{\text{hyb}}\leqslant e^{C'N},\quad \frac{r}{2}\leqslant|z|\leqslant 1/e.$$

In the last step, we use $(re/2)^m=e^{m\log(r/2)}$ and $\log(r/2)<0$.

Let $f(z)=\sum_{j=0}^{\infty}c_jz^j$ where c_j is the coefficient in the Laurent series $c=t^m\sum_{j=0}^{\infty}c_jt^j\in\mathbb{C}(\!(t)\!)$ with $c_0\neq 0$. Since ∞ does not lie in $D_{\bar{\mathfrak{a}}_{i_N},z}$ and $a/c=\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(\infty)$, a_z/c_z is in the disc $D_{\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N},z}\subset D_{\mathfrak{a}_{i_2},z}$, which we assume is bounded uniformly from ∞. We conclude that $|a_z/c_z|\leqslant C$ and $c_z\neq 0$ for $|z|<r$. Thus $\log|f(z)|$ is harmonic in the disc $D(0,r)$. Since $ad-bc=1$, we have

$$1\leqslant|a_zd_z|+|b_zc_z|\leqslant C(|a_z|+|b_z|)|c_z|\leqslant e^{CN}|c_z|,\quad \frac{r}{2}\leqslant|z|<r.$$

From the Laurent series expansion $c=t^m\sum_{j=0}^{\infty}c_jt^j\in\mathbb{C}(\!(t)\!)$, we have

$$e^{-m}\leqslant\|c\|_{\text{hyb}}\leqslant e^{CN},$$

which implies $m\geqslant -CN$.

By mean value theorem on $D(0,r/2)$, we have

$$\log|c_0|=\log|f(0)|\geqslant\min_{|z|=r/2}\log|f(z)|=\min_{|z|=r/2}\log|c_z/z^m|\geqslant -CN-m\log(r/2)\geqslant -C'N.$$

In the last step we use $\log(r/2)<0$ and $m\geqslant -CN$. Therefore, $|c_0|\geqslant e^{-CN}$. Recall $\|a\|_{\text{hyb}}\leqslant e^{CN}$ and $\|c\|_{\text{hyb}}\leqslant e^{CN}$, we have a Laurent series expansion

$$\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))=\frac{a}{c}=t^{m'}\sum_{j=0}^{\infty}a'_jt^j\text{ with }|a'_j|\leqslant e^{C(N+1)j},\tag{7.1}$$

where in the last step we use Lemma 7.4.

Suppose $\mathfrak{a}_{i_1}=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$, then $\mathfrak{a}'_{i_1}(w)=(c'w+d')^{-2}$. We would like to show

$$\mathfrak{a}'_{i_1}(\mathfrak{a}_{i_2}(\cdots\mathfrak{a}_{i_{N-1}}(\mathfrak{a}_{i_N}(\infty))))=t^m\sum_{j=0}^{\infty}a_jt^j,\quad |a_j|\leqslant e^{C(N+1)j}.$$

By Lemma 7.3, it suffices to show a similar expansion for $(\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(\infty)+d'/c')^{-1}$. By Lemma 7.4 and Eq. (7.1), it suffices to show the leading coefficient π_0 in the expansion

$$\mathfrak{a}_{i_2}\cdots\mathfrak{a}_{i_N}(\infty)+d'/c'=t^m\sum_{j=0}^{\infty}\pi_jt^j,\quad \pi_0\neq 0$$

satisfies $|\pi_0|\geqslant e^{-CN}$. By Proposition 3.20, $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))\in\mathfrak{a}_{i_2}D_{\mathfrak{a}_{i_3},z}\subset|z|^{v'}D_{\mathfrak{a}_{i_2},z}$ for some $v'>0$ and $0<|z|<r$. Hence the point $\mathfrak{a}_{i_2}(\cdots(\mathfrak{a}_{i_N}(\infty)))$ is $|z|^v$ separated from $D_{\bar{\mathfrak{a}}_{i_1},z}$ (which contains $-d'/c'$) for

some $v > 0$ and $0 < |z| < r$. By considering the separation with non-Archimedean norm, we know $|m| \leqslant C$ is uniformly bounded. Consider the holomorphic function $f(z) = \sum_{j=0}^{\infty} \pi_j z^j$, then $\log|f(z)|$ is a harmonic function in a small disc $|z| < r$. By mean value theorem,

$$\log|\pi_0| \geqslant \min_{|z|=r/2} \log|f(z)| \geqslant \min_{|z|=r/2} \log|z^m f(z)| - m\log(r/2) \geqslant \nu \log(r/2) - m\log(r/2).$$

Since m and ν are uniform, we actually prove $|\pi_0| \geqslant e^{-C}$ holds for some uniform constant $C > 0$. Therefore Lemma 7.4 gives Eq. (5.10). Similarly, since the discs are bounded uniformly away from ∞, we have

$$\log|\pi_0| \leqslant \max_{|z|=r/2} \log|f(z)| \leqslant \max_{|z|=r/2} \log|z^m f(z)| - m\log(r/2) \leqslant \log C - m\log(r/2).$$

Thus $|\pi_0| \leqslant e^C$ for some uniform constant $C > 0$. Since we know the leading coefficient a_0 in Eq. (5.10) is given by π_0^{-2} times the leading coefficient of c'^{-2}. We conclude

$$|a_0| \geqslant e^{-2C'} |\pi_0|^{-2} \geqslant e^{-2C-2C'}.$$

Eq. (5.11) then follows from Eq. (5.10) and the lower bound $|a_0| \geqslant e^{-C}$. □

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