FRAGILE TOPOLOGY ON SOLID GROUNDS AND THE THOULESS CONJECTURE

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Abstract. This paper provides a mathematical perspective on *fragile topology* in condensed matter physics. In dimension $d \leq 3$, vanishing Chern classes characterize the topological phases of periodic media that exhibit localized Wannier functions. However, for special symmetries I of the system, such as $C_{2z}T$ (space-time reversal), the existence of a localized set of Wannier functions that is also invariant under this symmetry I may not always exist. In systems with fragile topology, Wannier functions cannot respect both the symmetry I and the localization constraints. Nevertheless, this obstruction can be lifted by adding Chern-trivial line bundles, invariant under the I-symmetry. This allows for the construction of localized Wannier functions that respect symmetry. In the last section, we take on a broader perspective and obtain Wannier localization in dimensions $d = 2, 3$ without symmetry constraints for Chern nontrivial Bloch bundles. For $d = 2$, we obtain the Wannier decay with Thouless exponent $\mathcal{O}(|x|^{-2})$; for $d = 3$, we obtain the decay rate $\mathcal{O}(|x|^{-7/3})$.

1. Introduction

It is commonly accepted that topological features of the Bloch bundle in condensed matter physics such as the Hall conductance and the localization of Wannier functions are robust against topologically trivial perturbations of the system. Fragile topology challenges this narrative: Consider a periodic Hamiltonian that is invariant by the space-time reversal symmetry, which we refer to as the I-symmetry $Iu(x) = \overline{u(-x)}$ in this article. This symmetry naturally gives rise to a real sub-bundle $\mathcal{E}_0$ of the Bloch bundle $\mathcal{E}$, where the Bloch bundle is induced by spectral projections satisfying Assumption 1 in the presence of band gaps. If the real sub-bundle $\mathcal{E}_0 \subset \mathcal{E}$ is of rank two with Euler class $e_1(\mathcal{E}_0) \neq 0$, then there is no exponentially localized Wannier basis that is compatible with the I-symmetry (cf. Definition 1.3). Fragile topology, as introduced in [APY19, PWV18], claims that the topology of $\mathcal{E}_0$ is fragile in the sense that adding trivial bands (in the sense of taking the direct sum with trivial real line bundles) allows for the choice of an exponentially localized Wannier basis compatible with the I-symmetry. This topological feature has received attention because it prominently appears in twisted bilayer graphene [Po*19] and its flat bands in relation to strongly correlated electron phenomena [Pe*21, SEB20].

In this article, we prove a more general result that any real Bloch bundle $\mathcal{E}_0$ over $\mathbb{T}^2$ or $\mathbb{T}^3$ with rank $r \geq 3$ induced by the I-symmetry admits an exponentially localized Wannier basis compatible with the I-symmetry. We start by introducing some main concepts.

Assumption 1. *Let Γ be a lattice in $\mathbb{R}^d$ and $\mathcal{P} := (P(k))_{k\in\mathbb{R}^d}$ be a family of orthogonal projections with finite constant rank r acting on some Hilbert space $\mathcal{H}$ that depends real* *analytically on the parameter* $k \in \mathbb{R}^d$ *and satisfies for all* $k \in \mathbb{R}^d$ *and* $\gamma \in \Gamma^*$

$$P(k+\gamma) = \tau(\gamma)^{-1} P(k) \tau(\gamma), \quad \tau \in C^\omega(\mathbb{R}^d, U(\mathcal{H})) \text{ a unitary operator.} \tag{1.1}$$

In practice, one often has $\tau(\gamma)(x) := e^{-i\langle \gamma, x\rangle}$. For a family of projections as in Assumption 1, we define a vector bundle by introducing the equivalence relation

$$(k, \varphi) \sim (k', \varphi') \Leftrightarrow (k', \varphi') = (k+\gamma, \tau(\gamma)\varphi).$$

We then define the total space

$$E := \{(k, \varphi) \in \mathbb{R}^d \times \operatorname{ran}(P(k))\}/\sim$$

and the base space $B := \mathbb{R}^d/\Gamma^*$. Thus the projection map $\pi : E \to B$ induces a complex vector bundle $\mathcal{E}_{\mathbb{C}}$ (cf. Definition 2.1), which is called a *Bloch bundle*.

Definition 1.1. *For the orthogonal projections* $\mathcal{P} := (P(k))_{k \in \mathbb{R}^d}$ *satisfying Assumption 1, we say that* $\Phi : \mathbb{R}^d \to \mathcal{H}^r$ *is a* global Bloch frame*, if*

$$\Phi : \mathbb{R}^d \to \mathcal{H} \oplus \cdots \oplus \mathcal{H} = \mathcal{H}^r, \quad k \mapsto (\phi_1(k), \ldots, \phi_r(k)),$$

is τ-equivariant*:*

$$\phi_i(k+\gamma) = \tau(\gamma)\phi_i(k) \quad \text{for all } k \in \mathbb{R}^d,\ \gamma \in \Gamma^*,\ i \in \{1, \ldots, r\}, \tag{1.2}$$

and for a.e. $k \in \mathbb{R}^d$*, the set* $\{\phi_1(k), \ldots, \phi_r(k)\}$ *is an orthonormal basis spanning* $\operatorname{Ran} P(k)$*. Moreover, we say that a global Bloch frame is*

- continuous *(respectively* smooth, analytic*) if the maps* $\phi_i : \mathbb{R}^d \to \mathcal{H}$ *are continuous (respectively smooth, analytic) for all* $i \in \{1, \ldots, r\}$*;*
- H^s-regular *if the maps* $\phi_i : \mathbb{R}^d \to \mathcal{H}$ *lie in the corresponding local Sobolev space* $H^s_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{H})$ *for all* $i \in \{1, \ldots, r\}$.

Remark 1. *In terms of the language of vector bundles, each* $\phi_i(k)$ *is a normalized section of the underlying vector bundle, and a Bloch frame* Φ *is a family of orthonormal sections of the Bloch bundle* $\mathcal{E}_{\mathbb{C}}$ *that span the Bloch bundle over* $\mathbb{C}$.

To make the τ-equivariance (1.2) explicit, it is natural to introduce

$$\mathcal{H}_\tau = \{\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{H}) : \phi(k+\gamma, \bullet) = \tau(\gamma)\phi(k, \bullet) \text{ for all } \gamma \in \Gamma^*\}.$$

The τ-equivariance boundary condition is the natural boundary condition when using the Bloch transform. Alternatively, one may obtain periodic boundary conditions by considering $\tilde{P}(k) := \tau(k)P(k)\tau(k)^{-1}$. It is known that for $d \leq 3$, one can always find an H^s-regular global Bloch frame Φ for any $s < 1$ (see [Mo*18, Theorem 2.4]). From now on, we assume

$$\mathcal{H} = L^2(\mathbb{R}^d/\Gamma) \text{ and } \operatorname{ran} P(k) \subset C^\infty(\mathbb{R}^d/\Gamma) \text{ for each } k \in \mathbb{R}^d.$$

We now define Wannier functions using the so-called *Bloch transform*

$$(\mathcal{U}\psi)(k, x) = \sum_{\gamma \in \Gamma} e^{-i\langle x+\gamma, k\rangle} \psi(x+\gamma), \quad \psi \in L^2(\mathbb{R}^d), \tag{1.3}$$

where the Bloch transform gives the isometry

$$\mathcal{U} : L^2(\mathbb{R}^d) \longrightarrow \mathcal{H}_\tau = \{\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^d; L^2(\mathbb{R}^d/\Gamma)) : \phi(k+\gamma, x) = \tau(\gamma)\phi(k, x)\}.$$

Definition 1.2. *Let $\varphi(k,x) \in \mathcal{H}_\tau$ be a normalized section of the Bloch bundle satisfying Assumption 1, the* Wannier function $w(\varphi) \in L^2(\mathbb{R}^d)$ *is defined by*

$$w(\varphi)(x) := \frac{1}{|\mathbb{R}^d/\Gamma^*|}\int_{\mathbb{R}^d/\Gamma^*} e^{i\langle x,k\rangle}\varphi(k,x)\ dk = \mathcal{U}^{-1}(\varphi). \tag{1.4}$$

One can recover the Bloch function from the Wannier function by the Bloch transform

$$\varphi(k,x) = \sum_{\gamma\in\Gamma} e^{-i\langle x+\gamma,k\rangle} w(\varphi)(x+\gamma). \tag{1.5}$$

As $\varphi(k,x)$ is a normalized section, the shifted Wannier functions $\{w(\varphi)(\bullet-\gamma)\}_{\gamma\in\Gamma}$ form an orthonormal basis of the space

$$\Pi_\varphi L^2(\mathbb{R}^d) := \left\{\int_{\mathbb{R}^d/\Gamma^*} f(k)e^{i\langle k,x\rangle}\varphi(k,x)\,dk : f\in L^2(\mathbb{R}^d/\Gamma^*)\right\}.$$

Hence, we also refer it as *Wannier basis*. See [TaZw23] for a detailed discussion.

Note that in quantum mechanics, the Wannier function associated with a state φ is not uniquely defined, since $\varphi(x,k)$ and $e^{i\theta(k)}\varphi(x,k)$ for $\theta(k)\in\mathbb{R}$ are equivalent states, but give rise to different Wannier functions with possibly different decay properties. In fact, one has the following correspondence

$$\begin{aligned}\langle\bullet\rangle^m\psi \in L^2(\mathbb{R}^d),\ m\geq 0 &\Longleftrightarrow \mathcal{U}\psi\in\mathcal{H}_\tau\cap H^m_{\mathrm{loc}}(\mathbb{R}^d,L^2(\mathbb{R}^d/\Gamma))\\ e^{\varepsilon\langle\bullet\rangle}\psi\in L^2(\mathbb{R}^d),\ \text{for some } \varepsilon>0 &\Longleftrightarrow \mathcal{U}\psi\in\mathcal{H}_\tau\cap C^\omega(\mathbb{R}^d,L^2(\mathbb{R}^d/\Gamma)).\end{aligned} \tag{1.6}$$

Thus, the property stated below Remark 1 implies that, for any Bloch bundle with base dimension $d\leq 3$, there exist Wannier functions such that $\int_{\mathbb{R}^d}|w(\varphi)|^2\langle x\rangle^{2s}\,dx<\infty$ for any $s<1$. Moreover, the existence of exponentially decaying Wannier functions is equivalent to the triviality of the Bloch bundle [Mo*18].

Now we make a further assumption that the Bloch bundle has the following I-symmetry $Iu(x)=\overline{u(-x)}$ such that

$$IP(k)=P(k)I.$$

This condition holds for a natural class of Hamiltonians, e.g., Bloch transformed Schrödinger operators $H_k=(-i\nabla-k)^2+V$ with $V(-x)=\overline{V(x)}$ and the Bistrizter–Macdonald Hamiltonian of twisted bilayer graphene [BiMa11]. The I-symmetry acts as a real vector bundle homomorphism (i.e., each fiber is preserved). Since $I^2=1$, it has eigenvalue 1 and -1 on each fiber. The I-symmetry naturally induces a real subbundle of the Bloch bundle given by the eigenspace of 1, see (3.1). We want to study the existence of exponentially decaying Wannier functions that are also invariant under the I-symmetry. To this end, we define the compatiblity of Wannier functions with the I-symmetry:

Definition 1.3. *We say the Wannier basis $\{\varphi_{a,\gamma}(x) := \varphi_a(x-\gamma) : \gamma\in\Gamma, a\in\{1,\cdots,r\}\}$ is compatible with the I-symmetry with rescaled Wannier centers $\mathfrak{c}_1,\cdots,\mathfrak{c}_r\in\Gamma/(2\Gamma)$ if*

$$I\varphi_{a,\gamma}=\varphi_{a,\mathfrak{c}_a-\gamma},\ a\in\{1,\cdots,r\}. \tag{1.7}$$

For Bloch functions, by taking the Bloch transform (1.3), this translates into

$$I\varphi_a(k,x) = e^{i\langle \mathfrak{c}_a, k\rangle}\varphi_a(k,x),\ a \in \{1,\cdots r\}. \tag{1.8}$$

Hence, each $\varphi_a(k,\cdot)$ is a section of the real subbundle of the Bloch bundle

$$\mathcal{E}_{\mathfrak{c}_a} = \{(k,v) \in \mathcal{E}_{\mathbb{C}} : Iv = e^{i\langle \mathfrak{c}_a, k\rangle} v\}, \tag{1.9}$$

where $\mathcal{E}_{\mathfrak{c}}$ is a subbundle of the complex vector bundle $\mathcal{E}_{\mathbb{C}} \cong \mathcal{E}_{\mathfrak{c}} \otimes \mathbb{C}$. Given the real subbundle $\mathcal{E}_{\mathbb{R}} := \{(k,v) \in \mathcal{E}_{\mathbb{C}} : Iv = v\}$ of the Bloch bundle induced by the I-symmetry, we have

Theorem 1 (Fragile topology). *Let $\mathcal{E}_{\mathbb{R}}$ be a real subbundle of the Bloch bundle with base dimension ≤ 3 induced by the I-symmetry. Let r be the rank of the bundle.*

- *If $r \neq 2$, the bundle $\mathcal{E}_{\mathbb{R}}$ always admits an exponentially localized Wannier basis compatible with the I-symmetry.*
- *If $r = 2$ and the bundle $\mathcal{E}_{\mathbb{R}}$ is oriented, then it admits an exponentially localized Wannier basis compatible with the I-symmetry if and only if the Euler class vanishes.*

The proof of this theorem is presented in Subsection 3.6. Thus, an obstruction of constructing exponentially localized Wannier functions compatible with the I-symmetry may exist only for vector bundles of rank 2. As a direct consequence, one may take the real subbundle $\mathcal{E}_{\mathbb{R}}$ of the Bloch bundle of rank 2 and add to it another real line bundle L, then the new bundle $\mathcal{E}_{\mathbb{R}} \oplus L$ automatically admits an exponentially localized Wannier basis compatible with the I-symmetry. This is the phenomenon of fragile topology.

In the last section, we obtain results on Wannier decay without symmetry constraints. We recall that [Mo*18, Theorem 2.4] showed that one can always find a Bloch frame in H^s for all $s < 1$, independent of the Chern number of the Bloch bundle. Thus, under the Bloch transform, the corresponding Wannier functions $w(\varphi)$ satisfy the decay $\|\langle \bullet \rangle^s w(\varphi)\|_{L^2(\mathbb{R}^d)} < \infty$ for any $s < 1$ as in (1.6) and the discussion below.

Thouless [Th84] conjectured that for non-trivial Bloch bundles, Wannier functions should be able to attain some optimal decay rate $|x|^{-2}$ for $|x| \gg 1$ when $d = 2$. His argument rests on the Bloch function representation (1.5). Thus, if $w(\varphi) = \mathcal{O}(\langle \bullet \rangle^{-d-\varepsilon})$ for $\varepsilon > 0$, then the series converges uniformly

$$|\varphi(k,x)| \leq \sum_{\gamma\in\Gamma} |w(\varphi)(x+\gamma)| \lesssim \sum_{\gamma\in\Gamma} \langle x+\gamma\rangle^{-d-\varepsilon} < \infty.$$

This shows that $k \mapsto \varphi(k,x)$ is a continuous normalized global section of the Bloch bundle, and therefore the Bloch bundle is trivial. This leads Thouless to conjecture that the pointwise decay rate $\mathcal{O}(1/|x|^2)$ may be achieved by some Wannier functions for $d = 2$.

In [Li*24], the authors construct a distinguished section of the Bloch bundle that matches the decay $|x|^{-2}$. This is a refinement that implies the result of [Mo*18, Theorem 2.4]. This construction, for example, shows that there exists a Wannier function with decay rate $\|\langle \bullet \rangle (\log\langle \bullet \rangle)^{-s} w(\varphi)\|_{L^2(\mathbb{R}^2)} < \infty$ for any $s > 1/2$. Our first result is a refinement of [Li*24] showing that all Wannier functions may decay exponentially except one that exhibits the Thouless decay rate with a complete asymptotic expansion:

Theorem 2 (Wannier decay; $d = 2$). *Let $\mathcal{E}_{\mathbb{C}}$ be a Bloch bundle of rank r with Chern number $m \neq 0$ over $\mathbb{T}^2$, then there exists a Wannier basis $\{w(\varphi_{a,\gamma})(x) := w(\varphi_a)(x-\gamma) : \gamma \in \Gamma, a \in \{1, \cdots, r\}\}$ such that $w(\varphi_a)(x)$ decays exponentially for $a \in \{1, \cdots, r-1\}$ and*

$$w(\varphi_r)(x) \sim \frac{1}{|\mathbb{R}^2/\Gamma^*|} \sum_{\alpha \in \mathbb{N}^2} c_{m,\alpha}(\varphi_x)|x|^{-2-|\alpha|}\partial_k^\alpha \Phi(0,x)$$

with $c_{m,0}(\varphi_x) = 2\pi(-1)^m m e^{-im\varphi_x}$, $\sin\varphi_x = x_1/|x|$, $\cos\varphi_x = x_2/|x|$, and $\Phi(k,x)$ is a normalized smooth local section near $k = 0$.

Comparing the result of Thouless and [Mo*18] with the H^1-condition (that is $\|\langle\bullet\rangle w\|_{L^2(\mathbb{R}^2)} < \infty$) which implies the triviality of the bundle, we propose the following

Question 1. *For a non-trivial Bloch bundle over $\mathbb{T}^2$, does there exist a Wannier function $w(\varphi)$ that decays $o(1/|x|^2)$ as $x \to \infty$?*

The existing thresholds, that is, Thouless' continuity argument and the H^1-regularity that implies the triviality of the bundle, appear to not rule out, for instance, a pointwise decay $1/(|x|^2(\log|x|)^{1/2})$ as $x \to \infty$.

Interestingly, Thouless' simple continuity argument proves to be effective for $d = 2$, but fails to capture a sharp uniform decay in $d = 3$. Our construction can be carried over to Bloch bundles over $\mathbb{T}^3$ (see Section 5.2) to construct Bloch frames in H^s for all $s < 1$ with the corresponding Wannier function decaying like $\mathcal{O}((1+|x_1|+|x_2|)^{-2}\langle x_3\rangle^{-\infty})$ as $x \to \infty$. In fact, we obtain a uniform decay rate better than the decay rate for Wannier functions on $\mathbb{T}^2$.

Theorem 3 (Wannier decay; $d = 3$). *Let $\mathcal{E}_{\mathbb{C}}$ be a nontrivial complex Bloch bundle of rank r, then there exists a Wannier basis $\{w(\varphi_{a,\gamma})(x) := w(\varphi_a)(x-\gamma) : \gamma \in \Gamma, a \in \{1, \cdots, r\}\}$ such that $w(\varphi_a)(x)$ decays exponentially for $a \in \{1, \cdots, r-1\}$ and*

$$w(\varphi_r)(x) = \mathcal{O}(|x|^{-\frac{7}{3}}), \text{ as } |x| \to \infty.$$

It is reasonable to ask whether a better uniform decay in all directions is possible. Since the function $\langle x\rangle^{-\frac{5}{2}}$ satisfies $\|\langle\bullet\rangle w\|_{L^2(\mathbb{R}^3)} = \infty$ and $\|\langle\bullet\rangle^s w\|_{L^2(\mathbb{R}^3)} < \infty$ for $s < 1$, we propose

Question 2. *For a non-trivial Bloch bundle over $\mathbb{T}^3$, what is the optimal pointwise decay rate of the Wannier functions as $|x| \to \infty$? Can one construct a Wannier function $w(\varphi)$ such that $w(\varphi)(x) = \mathcal{O}(|x|^{-\frac{5}{2}})$ as $|x| \to \infty$?*

We can ask the following more general question.

Question 3. *Given a smooth vector bundle $\mathcal{E}_{\mathbb{C}}$ of rank r over a smooth manifold M. What is the largest $\alpha \in \mathbb{R}$ such that there exists a (measurable) normalized frame s_i, $i = 1, 2, \cdots, r$ with respect to a smooth metric, such that in any local trivialization over an open subset $U \subset M$ and for any smooth cutoff $\chi \in C_c^\infty(U)$, we have*

$$\mathcal{F}(\chi s_i)(\xi) = \mathcal{O}(|\xi|^{-\alpha})$$

as $|\xi| \to \infty$?

The only general thing we can say is that α can be arbitrarily large when $\mathcal{E}_{\mathbb{C}}$ is trivial, and $\alpha \le d := \dim M$ when $\mathcal{E}_{\mathbb{C}}$ is nontrivial. As we see above, there could be finer restrictions on α, and it is related to the sharp Fourier decay under certain topological constraints.

Related works. Different invariance conditions (1.8) for the Bloch and Wannier functions have been considered in [FMP16a, FMP16b]. The observed effects are fairly different, and fragile topology does not appear in these settings. To compare our framework with the setting studied by [FMP16a, FMP16b], we can compare the $C_{2z}T$-symmetry $I_{C_{2z}T}u(x) = \overline{u(-x)}$, with the time-reversal symmetry $I_T u(x) = \overline{u(x)}$ and notice that they act differently on Bloch functions. Indeed, while they are both bosonic in the terminology of the aforementioned works as they square to the identity, we have $[I_{C_{2z}T}, (-i\nabla - k)^2] = 0$ whereas $I_T(-i\nabla - k)^2 = (-i\nabla + k)^2 I_T$ for the time reversal symmetry. That is, time-reversal symmetry maps k to $-k$, while we consider symmetries that leave k unchanged; see Definition 1.3. See also [KLW16] for an account of the relation between the $\mathbb{Z}_2$ topological invariants (cf. [KM05, FK06, FKM07]) and the Stiefel–Whitney class in the setting of [FMP16b], and [Ah*19] for classification of topological phases in bands with the I-symmetry using Stiefel–Whitney class from a physics perspective.

The classical question on the existence of exponentially localized Wannier functions does not involve any symmetries and can be fully understood in terms of the Chern number [Pa07, Br07, Mo*18] according to Definition 1.3. Recently, results on Wannier basis localization have been extended to non-periodic systems [LSW22, MMP23, LuSt24, RoPa24].

Notations and conventions. Let $\mathcal{H}$ be a separable Hilbert space. Let $\Gamma = \sum_{i=1}^{d} \mathbb{Z}w_i \subset \mathbb{R}^d$, for linearly independent w_i, be a lattice, and $\Gamma^* := \{x \in \mathbb{R}^d : \langle x, w\rangle \in 2\pi\mathbb{Z} \text{ for all } w \in \Gamma\} = \sum_{i=1}^{d} \mathbb{Z}v_i$ be the dual lattice. We denote by $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$ and by $L(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$.

The lattices Γ and Γ^* naturally give rise to a torus as well as a dual torus, that is, $\mathbb{R}^d/\Gamma$ and $\mathbb{R}^d/\Gamma^*$. However, for simplicity, we will often just say that we have a vector bundle over a torus $\mathbb{T}^d$, which then in the physics setting corresponds to the dual torus $\mathbb{R}^d/\Gamma^*$.

Structure of the paper. In Section 2, we review the (properties of) characteristic classes for real and complex vector bundles that are relevant to fragile topology. In Section 3, we prove the topological background underlying the effect of fragile topology. In Section 4, we study the fragile topology in twisted bilayer graphene. In Section 5, we discuss the optimal decay of Wannier functions for a given non-trivial Chern class and prove Theorem 2 and 3.

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2. Characteristic classes and bundle classification

In this section, we discuss the classification of complex vector bundles $\mathcal{E}_{\mathbb{C}}$ and real vector bundles $\mathcal{E}_{\mathbb{R}}$ over a manifold of dimension $d \leq 3$. We also recall the basic properties of Chern classes, Euler classes, and Stiefel–Whitney classes.

2.1. **Bundles.** We briefly recall the definition of vector bundles, the notion of orientation, and the classification of line bundles.

Definition 2.1 (Vector bundle)**.** *Let E, X be topological spaces. $\pi : E \to X$ is called a (complex or real)* vector bundle *of rank r if for any $x \in X$, $\pi^{-1}(x)$ is a (complex or real) vector space of dimension r, and there exists a covering $\{U_i\}$ of X such that there is a homeomorphism, called the trivialization, which is linear on each fiber $\pi^{-1}(x)$, such that the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\cong"] \arrow[d, "\pi"] & U_i \times \mathbb{C}^r \text{ or } U_i \times \mathbb{R}^r \arrow[dl, "pr_1"] \\ U_i & \end{tikzcd}$$

Here E is also called the total space *and X is called the* base*. A vector bundle of rank* 1 *is called a line bundle.*

Remark 2. *In the above definition, if E, X are C^s manifolds, π is C^s and the trivialization is a C^s map, then the vector bundle is called a C^s vector bundle, where $s \in \mathbb{N}$ or $s = \infty$ (smooth) or $s = \omega$ (real analytic). We note that the classifications of C^s vector bundles are equivalent for $s \in \mathbb{N}$ or $s = \infty$ or $s = \omega$, see* [Sh64, Theorem 5]*.*

Replacement of the vector space with more general objects leads to the definition of fiber bundles.

Definition 2.2 (Fiber bundle)**.** *A fiber bundle is a quadruple (E, B, π, F), where E is the total space, B is the base space, $\pi : E \to B$ is a continuous projection map, and F is the typical fiber, a topological space, satisfying the following conditions:*

(1) For every $b \in B$, the preimage $\pi^{-1}(\{b\})$ is homeomorphic to F,
(2) There exists an open cover $\{U_i\}_{i \in I}$ of B and homeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ such that $\pi(e) = \mathrm{pr}_1(\phi_i(e))$, where $\mathrm{pr}_1 : U_i \times F \to U_i$ is the projection onto the first component.

From the definition of a fiber bundle, we can now define the principal G-bundles:

Definition 2.3 (G-bundle)**.** *A principal G-bundle, where G is a topological group, is a fiber bundle such that G acts continuously from the right on E. In addition, the action of the group must leave the fibers $\pi^{-1}(x)$ invariant and act freely and transitively on them.*

The extension of this concept to the smooth or real analytic category is straightforward by requiring that G has a smooth/real analytic structure and the action is smooth/real analytic.

The reason for introducing the principal G-bundles is that they allow the classification of complex or real vector bundles with symmetries. Complex or real vector bundles naturally correspond to principal G-bundles, where G is the general linear group (real or complex). One can extend the correspondence to include other groups G, by adding more structure to the vector bundle, such as, for instance, orientation:

Definition 2.4 (Orientation)**.** *An* orientation *of a real vector space is an equivalence class of ordered bases, where two ordered bases are equivalent if the invertible matrix taking the first basis to the second has a positive determinant. A vector bundle is called* orientable *if there exists a continuous choice of orientation on each fiber.*

We now briefly recall the classification of real or complex line bundles, which shall be used in later sections.

(1) For a structure group G, G-principal bundles are classified by the classifying space BG, which is a principal G-bundle whose total space EG is contractible:
$$\{\text{principal } G\text{-bundles over } M\}/\text{isomorphism} \cong [M, BG]$$
where $[M, BG]$ is the set of homotopy classes of continuous maps from M to BG.
(2) In particular, complex vector bundles of rank r are classified by $BU(r)$. Real vector bundles of rank r are classified by $BO(r)$ and oriented real vector bundles of rank r are classified by $BSO(r)$.
(3) We have $BU(1) = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, $BO(1) = \mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$, where $K(A, n)$ is the Eilenberg–Maclane space which satisfies
$$\pi_n(K(A, n)) = A, \quad \pi_m(K(A, n)) = 1, \quad m \neq n.$$
It has the universal property
$$[M, K(A, n)] \cong H^n(M; A).$$
(4) The element of $[M, BU(1)] \cong H^2(M; \mathbb{Z})$ is the first Chern class c_1. The element of $[M, BO(1)] \cong H^1(M; \mathbb{Z}/2)$ is the first Stiefel–Whitney class w_1.

2.2. **Classification of complex vector bundles.** For a complex vector bundle $\mathcal{E}_{\mathbb{C}}$ over a manifold M, the Chern class $c_k(\mathcal{E}_{\mathbb{C}})$ is an element of $H^{2k}(M; \mathbb{Z})$. If the vector bundle has a smooth connection with curvature Ω which is a $\mathfrak{gl}(r, \mathbb{C})$-valued 2-form, the Chern classes can be computed by
$$\det\left(I + \frac{\sqrt{-1}}{2\pi} t[\Omega]\right) = \sum_{j \geq 0} c_j(\mathcal{E}_{\mathbb{C}}) t^j.$$

Over any manifold M, complex line bundles are classified up to isomorphisms by the first Chern class $c_1 \in H^2(M; \mathbb{Z})$ as discussed at the end of Subsection 2.1 (see also [Ch79, p. 34] for a different proof). For higher-rank complex vector bundles, we have the following classification:

Proposition 2.5. *Complex vector bundles $\mathcal{E}_{\mathbb{C}}$ over a manifold M of dimension ≤ 3 are classified by the first Chern class $c_1 \in H^2(M;\mathbb{Z})$. Furthermore, $\mathcal{E}_{\mathbb{C}}$ admits a decomposition into line bundles*

$$\mathcal{E}_{\mathbb{C}} = \left(\bigoplus_{i=1}^{rk(\mathcal{E}_{\mathbb{C}})-1} L_i \right) \oplus L, \tag{2.1}$$

where all line bundles L_i are trivial apart from possibly L for which $c_1(\mathcal{E}_{\mathbb{C}}) = c_1(L)$.

Proof. Let $\mathcal{E}_{\mathbb{C}}$ be a complex vector bundle of rank $r \geq 2$ over M. By the Thom transversality theorem, there exists a non-vanishing section s of $\mathcal{E}_{\mathbb{C}}$. Since if s intersects the zero section 0_M transversally at $(x,0) \in \mathcal{E}_{\mathbb{C}}$, we have

$$ds(T_xM) + T_{(x,0)}0_M = T_{(x,0)}\mathcal{E}_{\mathbb{C}}.$$

Thus, the dimension of the intersection of the tangent space

$$\dim_{\mathbb{R}}(ds(T_xM) \cap T_{(x,0)}0_M) = \dim_{\mathbb{R}}(ds(T_xM)) + \dim_{\mathbb{R}}(T_{(x,0)}0_M) - \dim_{\mathbb{R}}(T_{(x,0)}\mathcal{E}_{\mathbb{C}}) < 0,$$

ensures that the intersection is empty. Hence, s induces a trivial line bundle L_1, and $\mathcal{E}_{\mathbb{C}}$ splits as a Whitney sum:

$$\mathcal{E}_{\mathbb{C}} \cong L_1 \oplus \mathcal{E}'_{\mathbb{C}},$$

where $\mathcal{E}'_{\mathbb{C}}$ is a complex vector bundle of rank $r-1$. By induction, $\mathcal{E}_{\mathbb{C}}$ is isomorphic to the Whitney sum of $r-1$ trivial line bundles with a complex line bundle L. The Chern number of L satisfies $c_1(L) = c_1(\mathcal{E}_{\mathbb{C}})$, which completes the proof. $\square$

Remark 3. *When $\mathcal{E}_{\mathbb{C}}$ is real analytic, the decomposition in the previous Proposition can be chosen so that L_i and L are real analytic line bundles. This is because (2.1) holds in the topological category, and Remark 2 implies a real analytic isomorphism*

$$\mathcal{E}_{\mathbb{C}} \cong \left(\bigoplus_{i=1}^{rk(\mathcal{E}_{\mathbb{C}})-1} L_i \right) \oplus L.$$

2.3. **Stiefel–Whitney class.** For a real vector bundle $\mathcal{E}_{\mathbb{R}}$ of finite rank on a base space M, the i-th Stiefel–Whitney class $w_i(\mathcal{E}_{\mathbb{R}})$ is an element of $H^i(M;\mathbb{Z}/2)$. An axiomatic definition of the Stiefel–Whitney class can, for instance, be found in [Hu94, Theorem 5.4]. Some basic properties of the Stiefel–Whitney classes that will be relevant to us are:

(1) $w_i(\mathcal{E}_{\mathbb{R}}) = 0$ if $i > \operatorname{rank}(\mathcal{E}_{\mathbb{R}})$ or $i > \dim M$.
(2) $\mathcal{E}_{\mathbb{R}}$ is orientable if and only if $w_1(\mathcal{E}_{\mathbb{R}}) = 0$ [Hu94, Theorem 2.1].
(3) Let

$$w_t(\mathcal{E}_{\mathbb{R}}) = 1 + w_1(\mathcal{E}_{\mathbb{R}})t + \cdots + w_r(\mathcal{E}_{\mathbb{R}})t^r, \quad w_i(\mathcal{E}) \in H^i(M;\mathbb{Z}/2).$$

Then the Stiefel–Whitney class of the Witney sum is given by

$$w_t(\mathcal{E}_{\mathbb{R}} \oplus \mathcal{F}_{\mathbb{R}}) = w_t(\mathcal{E}_{\mathbb{R}}) \smile w_t(\mathcal{F}_{\mathbb{R}}),$$

where the cup product induces a bilinear operation on cohomology via the binomial formula:

$$H^i(M;\mathbb{Z}/2)\times H^{k-i}(M;\mathbb{Z}/2)\to H^k(M;\mathbb{Z}/2).$$

Consequently, we have $w_k(\mathcal{E}_\mathbb{R}\oplus\mathcal{F}_\mathbb{R})=\sum_{i=0}^k w_i(\mathcal{E}_\mathbb{R})\smile w_{k-i}(\mathcal{F}_\mathbb{R})$ and in particular

$$w_1(\mathcal{E}_\mathbb{R}\oplus\mathcal{F}_\mathbb{R})=w_1(\mathcal{E}_\mathbb{R})+w_1(\mathcal{F}_\mathbb{R}), \tag{2.2}$$

$$w_2(\mathcal{E}_\mathbb{R}\oplus\mathcal{F}_\mathbb{R})=w_2(\mathcal{E}_\mathbb{R})+w_1(\mathcal{E}_\mathbb{R})\smile w_1(\mathcal{F}_\mathbb{R})+w_2(\mathcal{F}_\mathbb{R}). \tag{2.3}$$

(4) When tensoring two real line bundles L_1, L_2, the first Stiefel–Whitney class is given by $w_1(L_1\otimes L_2)=w_1(L_1)+w_1(L_2)$. In general, we can assume that both bundles are direct sums of line bundles and state a general formula. This formula still holds in general by the "splitting principle" (see [BoTu13, Section 21]).

For example, if $\mathcal{E}_\mathbb{R}$ is a real vector bundle of rank r and L is a real line bundle, then

$$w_1(\mathcal{E}_\mathbb{R}\otimes L)=w_1(\mathcal{E}_\mathbb{R})+rw_1(L). \tag{2.4}$$

In particular, the tensor bundle $L\otimes L$ is always a trivial line bundle.

2.4. **The Euler class.** Let $\mathcal{E}_\mathbb{R}$ be a real oriented vector bundle of rank $2k$ over a smooth manifold M, the Euler class $e(\mathcal{E}_\mathbb{R})$ is an element of $H^{2k}(M;\mathbb{Z})$.

If $\mathcal{E}_\mathbb{R}$ has an orthogonal connection (with respect to a bundle metric) with curvature Ω (an $\mathfrak{so}(2k)$-valued 2 form), the Euler class is given by

$$e(\mathcal{E}_\mathbb{R})=\frac{1}{(2\pi)^k}[\mathrm{Pf}(\Omega)]\in H^{2k}_{\mathrm{dR}}(M;\mathbb{R}),$$

where $\mathrm{Pf}(\Omega)$ is the Pfaffian of Ω.

Example 1. *In particular, for $M=\mathbb{R}^2/\Gamma^*$, $k=1$, $d=2$ and a connection $\Omega=\begin{pmatrix}0 & \Omega_{12}\\ -\Omega_{12} & 0\end{pmatrix}$, the Euler number is*

$$\chi(\mathcal{E}_\mathbb{R}):=\int_{\mathbb{R}^2/\Gamma^*}e(\mathcal{E}_\mathbb{R})=\frac{1}{2\pi}\int_{\mathbb{R}^2/\Gamma^*}\Omega_{12}.$$

There is a natural homomorphism $\sigma: H^{2k}(M;\mathbb{Z})\to H^{2k}(M;\mathbb{Z}/2)$ that maps the Euler class to the top Stiefel–Whitney class [MiSt74, Prop. 9.5]

$$w_{2k}(\mathcal{E}_\mathbb{R})=\sigma(e(\mathcal{E}_\mathbb{R})). \tag{2.5}$$

The Euler class is also related to the Chern class. In fact, by [MiSt74, Lemma 14.1] for a complex vector bundle $\mathcal{E}_\mathbb{C}$ with rank k, its underlying real vector bundle $\mathcal{E}_\mathbb{R}$ of rank $2k$ has a canonical orientation. The top Chern class $c_k(\mathcal{E}_\mathbb{C})\in H^{2k}(M;\mathbb{Z})$ then coincides with the Euler class $e(\mathcal{E}_\mathbb{R})$ (cf. [MiSt74, Section 14.2]).

2.5. **Classification of real vector bundles.** Over any manifold M, real line bundles are classified by the first Stiefel–Whitney class $w_1 \in H^1(M;\mathbb{Z}/2)$ as discussed at the end of Subsection 2.1.

Suppose M has dimension ≤ 3, we can classify real vector bundles over M.

Proposition 2.6. *Real vector bundles of rank r over a manifold M with dimension ≤ 3 are classified by the Stiefel–Whitney classes $w_1 \in H^1(M;\mathbb{Z}/2)$ and $w_2 \in H^2(M;\mathbb{Z}/2)$ when $r \neq 2$. When $r = 2$, oriented real vector bundles are classified by the Euler class $e_2 \in H^2(M;\mathbb{Z})$.*

Proof. Then rank 2 oriented real bundles are classified by the Euler class

$$e_2 \in [M, BSO(2)] = [M, K(\mathbb{Z}, 2)] = H^2(M;\mathbb{Z}).$$

For rank r real vector bundles with $r \geq 3$, we may assume $r = 3$ because for $r \geq 4$ we can always split off a trivial subbundle as in the proof of Proposition 2.5. Moreover, we may assume it is orientable by tensoring with a line bundle, cf. (2.4). Now oriented rank 3 line bundles are classified by $[M, BSO(3)]$. Since $\pi_1 BSO(3) = \pi_0 SO(3) = 1$, $\pi_2 BSO(3) = \pi_1 SO(3) = \mathbb{Z}/2$, $\pi_3 BSO(3) = \pi_2 SO(3) = 1$, they are classified by the second Stiefel–Whitney class w_2:

$$[M, BSO(3)] = [M, K(\mathbb{Z}/2, 2)] = H^2(M;\mathbb{Z}/2) \ni w_2.$$

Since the Stiefel–Whitney class does not depend on the orientation, this finishes the classification of rank 3 vector bundles. ∎

3. Fragile topology

In this section, we consider a complex Bloch bundle $\mathcal{E}_{\mathbb{C}}$ of rank $r \geq 2$ over $\mathbb{T}^2$ or $\mathbb{T}^3$ equipped with an I-symmetry, i.e. an antilinear involution on $\mathcal{E}_{\mathbb{C}}$. Thus we get a real vector subbundle $\mathcal{E}_{\mathbb{R}}$ with (real) rank $r \geq 2$, i.e.

$$\mathcal{E}_{\mathbb{R}} := \{(x, v) \in \mathcal{E}_{\mathbb{C}} : Iv = v\}, \quad \mathcal{E}_{\mathbb{C}} = \mathcal{E}_{\mathbb{R}} \otimes \mathbb{C}. \tag{3.1}$$

We show that for $r = 2$ there is an exponentially localized Wannier basis compatible with the I-symmetry if and only if the Euler class $e(\mathcal{E}) = 0$, whereas for $r \geq 3$ we show that there is always an exponentially localized Wannier basis compatible with the I-symmetry.

3.1. **Chern numbers and localization dichotomy.** Let $\Omega(k) = \sum_{i,j} \Omega_{ij}(k)\, dk_i \wedge dk_j$ be the curvature form of the Berry connection:

$$\Omega_{ij}(k) := \operatorname{tr}_{\mathcal{H}}(P(k)[\partial_i P(k), \partial_j P(k)]).$$

Here, $P(k)$ denotes the family of orthogonal projections parametrized by $k \in \mathbb{R}^d$ that satisfy the equivariance conditions (1.1) of the reciprocal lattice Γ^*. Then, the first Chern class is defined as

$$c_1(\mathcal{E}_{\mathbb{C}}) = \tfrac{i}{2\pi}[\Omega] \in H^2_{\mathrm{dR}}(\mathbb{R}^d/\Gamma^*;\mathbb{R}),$$

where $[\Omega]$ represents the de Rham cohomology class of the curvature form Ω.

To compute the first Chern class explicitly, we use the de Rham isomorphism. For the d-dimensional torus $\mathbb{R}^d/\Gamma^*$, the second homology group satisfies

$$H_2(\mathbb{R}^d/\Gamma^*) \cong \mathbb{Z}^{\binom{d}{2}}.$$

This group is generated by the independent 2-cycles $(B_{ij})_{1\le i<j\le d}$, where $B_{ij} := (\mathbb{R}v_i + \mathbb{R}v_j)/(\mathbb{Z}v_i + \mathbb{Z}v_j)$ and $(v_i)_{i=1}^d$ are basis vectors of Γ^*.

Definition 3.1. *A family of projections $\mathcal{P}$ that satisfy Assumption 1 is called* Chern trivial *if for $d \in \{2,3\}$ the Chern numbers*

$$C_1(\mathcal{E}_{\mathbb{C}})_{ij} := \frac{i}{2\pi}\int_{B_{ij}} \operatorname{tr}_{\mathcal{H}}(P(k)[\partial_i P(k), \partial_j P(k)])\, dk_i \wedge dk_j$$

vanish for all $1 \le i < j \le d$.

By the de Rham isomorphism, Chern triviality implies the vanishing of the first Chern class $c_1(\mathcal{E}_{\mathbb{C}}) = 0$. The first Chern class has direct implications on the structure of Wannier functions. This is the *localization dichotomy* as stated, for instance, in [Mo*18]:

- **Case 1:** If $\mathcal{P}$ is Chern-trivial, then there exists a Bloch frame Φ that is real analytic in the parameter k, such that all Wannier functions $w(\varphi_i := \Phi e_i)$ decay exponentially.
- **Case 2:** If $\mathcal{P}$ is not Chern-trivial, then there does not exist Bloch frame that is H^1_{loc} in k. In addition, for any Bloch frame Φ, there exist Wannier functions $w(\varphi_i := \Phi e_i)$ for some i such that $xw(\varphi_i) \notin L^2$.

It is instructive to compare this dichotomy with Theorem 1. If $\mathcal{E}_{\mathbb{C}}$ is not Chern-trivial and $\mathcal{E}'_{\mathbb{C}}$ is Chern-trivial, then the direct sum $\mathcal{E}_{\mathbb{C}} \oplus \mathcal{E}'_{\mathbb{C}}$ remains not Chern-trivial. Hence, the obstruction to the existence of exponentially localized Wannier basis imposed by the Chern number cannot be removed by adding trivial bundles. This highlights the distinction between *classical topology* in condensed matter physics, where obstructions cannot be lifted by modifications by topologically trivial elements, and *fragile topology*, which depends on the rank and can be lifted by changing it.

3.2. **Wannier centers.** We continue with a more detailed discussion on Wannier centers. We recall the definition of

$$\mathcal{E}_{\mathfrak{c}_a} = \{(k,v) \in \mathcal{E}_{\mathbb{C}} : Iv = e^{i\langle \mathfrak{c}_a, k\rangle} v\},$$

from (1.9). We claim

$$\mathcal{E}_{\mathfrak{c}} \cong \mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathfrak{c}} \tag{3.2}$$

where $\mathcal{L}_{\mathfrak{c}}$ is the real line bundle, corresponding to the Stiefel–Whitney class $\mathfrak{c} \in \Gamma/(2\Gamma) \cong H^1(\mathbb{R}^n/\Gamma^*; \mathbb{Z}/2)$. In other words, $\mathcal{L}_{\mathfrak{c}}$ is a real line bundle contained in the trivial complex line bundle $\mathbb{T}^d \times \mathbb{C}$ satisfying

$$\mathcal{L}_{\mathfrak{c}} := \{(k,z) \in \mathbb{T}^d \times \mathbb{C} : \overline{z} = e^{i\langle \mathfrak{c}, k\rangle} z\}. \tag{3.3}$$

The isomorphism (3.2) follows from identifying $\mathcal{E}_{\mathbb{C}}$ with $\mathcal{E}_{\mathbb{R}} \otimes \mathbb{C}$, where I acts trivially on $\mathcal{E}_{\mathbb{R}}$ and acts as a conjugation on $\mathbb{C}$. In particular, $\mathcal{E}_{\mathfrak{c}} \cong \mathcal{E}_{\mathfrak{c}'}$ if $\mathfrak{c} - \mathfrak{c}' \in 2\Gamma$ as $\mathcal{L}_{\mathfrak{c}}$'s are classified by the first Stiefel–Whitney class $w_1 \in H^1(\mathbb{R}^n/\Gamma^*; \mathbb{Z}/2) \cong \Gamma/(2\Gamma)$.

We now give a more physics relevant definition of Wannier center.

Definition 3.2. *Let* $\langle \bullet \rangle^{1/2} w(\varphi) \in L^2(\mathbb{R}^d)$. *The* Wannier center $\mathfrak{c}(w(\varphi))$ *is defined as the expectation value of the position operator*

$$\mathfrak{c}(w(\varphi)) := \int_{\mathbb{R}^d} |w(\varphi)|^2 x \, dx \in \mathbb{R}^d.$$

Since Wannier centers are the expectation values of the position of charge carriers, they are directly related to the polarization where q is the electric charge of the particles [MaVa97, Ma*12]

$$P = q\mathfrak{c}(w(\varphi)).$$

We now clarify the relation between the two definitions of (rescaled) *Wannier centers* as position expectation values in Definitions 3.2 and phase factors (of the I-symmetry) in Definition 1.3. In particular, the Wannier centers give rise to the first Stiefel–Whitney class of the associated real line bundle (1.9).

Proposition 3.3. *Let* $u \in \mathcal{H}_\tau \cap H^{\frac{1}{2}}_{loc}(\mathbb{R}^d; L^2(\mathbb{R}^d/\Gamma))$ *be a normalized section. Suppose* $I u_k = e^{i\langle \mathfrak{c}, k\rangle} u_k$ *(as in Definition 1.3) is anti-unitary. Then for* $\mathfrak{c}(w(u))$ *as in Definition 3.2,*

$$\mathfrak{c}(w(u)) = \frac{\mathfrak{c}}{2}.$$

Proof. For simplicity, we shall give our proof in the dimension $d = 2$. We start by recalling that after conjugating by the Bloch–Floquet transform $\mathcal{U} x \mathcal{U}^{-1} = i\nabla_k$ which shows that

$$\begin{aligned}\mathfrak{c}(w(u)) = \langle w(u), x w(u)\rangle_{L^2(\mathbb{R}^d)} &= \frac{\langle \mathcal{U} w(u), (\mathcal{U} x \mathcal{U}^{-1})\mathcal{U} w(u)\rangle_{L^2(\mathbb{R}^d/\Gamma \times \mathbb{R}^d/\Gamma^*)}}{|v_1 \wedge v_2|} \\ &= -\frac{i\langle u, \nabla_k u\rangle_{L^2(\mathbb{R}^d/\Gamma\times\mathbb{R}^d/\Gamma^*)}}{|v_1 \wedge v_2|},\end{aligned}$$

where $\Gamma^* = \mathbb{Z}v_1 + \mathbb{Z}v_2$. With the Berry connection $A(k) = -i\langle u_k, \nabla_k u_k\rangle_{L^2(\mathbb{R}^d/\Gamma)}$ and unit vector $e_v = \frac{v}{\|v\|}$

$$\begin{aligned}\langle \mathfrak{c}(w(u)), e_{v_1}\rangle &= \frac{|v_1 \wedge v_2|}{|v_1 \wedge v_2|}\int_0^1\int_0^1 \langle A(t_1 v_1 + t_2 v_2), e_{v_1}\rangle \, dt_1 \, dt_2 \\ &= \int_0^1\int_0^1 \langle A(t_1 v_1 + t_2 v_2), e_{v_1}\rangle \, dt_1 \, dt_2 \\ &= \frac{1}{\|v_1\|}\int_0^1\int_0^1 \langle A(t_1 v_1 + t_2 v_2), v_1\rangle \, dt_1 \, dt_2 = \frac{1}{\|v_1\|}\int_0^1 \gamma_{v_1}(t_2)\, dt_2,\end{aligned}$$

where γ_{v_1} is the Berry phase in v_1 direction

$$\gamma_{v_1}(t_2) = \int_0^1 \langle A(t_1 v_1 + t_2 v_2), v_1\rangle dt_1.$$

The same computation applies to e_{v_2} with 1 replaced by 2 everywhere.

More can be said about the Berry phase when symmetries are enforced. For $Iu_k = e^{i\phi(k)}u_k$ and $\phi(k) := \langle \mathfrak{c}_a, k\rangle$, a simple computation shows that

$$\begin{aligned}\langle u_k, i\nabla_k u_k\rangle &= -i\langle u_k, \nabla_k u_k\rangle = \left\langle \frac{1}{i}\nabla_k(Iu_k), Iu_k\right\rangle \\ &= \left\langle \frac{1}{i}\nabla_k(e^{i\phi(k)}u_k), e^{i\phi(k)}u_k\right\rangle = \nabla\phi(k) - i\langle \nabla_k u_k, u_k\rangle.\end{aligned}$$

Using that $0 = \nabla_k\|u_k\|^2 = \langle \nabla_k u_k, u_k\rangle + \langle u_k, \nabla_k u_k\rangle$, we find

$$2\gamma_{v_1} = \int_0^1 \langle \nabla\phi(t_1v_1 + t_2v_2), v_1\rangle \, dt_1 = \langle \mathfrak{c}_a, v_1\rangle \tag{3.4}$$

such that

$$\langle \mathfrak{c}(w(u)), e_{v_j}\rangle = \frac{\langle \mathfrak{c}_a, v_j\rangle}{2\|v_j\|} = \frac{\langle \mathfrak{c}_a, e_{v_j}\rangle}{2}, \; j = 1, 2 \Longrightarrow \mathfrak{c}(w(u)) = \frac{\mathfrak{c}_a}{2}. \qquad \square$$

Remark 4. *It follows the proof that the Wannier center $\mathfrak{c}_a$ in (1.9) is completely determined by the Berry phase by (3.4).*

3.3. **Wannier basis and line bundles.** We first prove an equivalent statement of the existence of an exponentially localized Wannier basis compatible with the I-symmetry.

Proposition 3.4. *The real subbundle $\mathcal{E}_{\mathbb{R}}$ in (3.1) over $\mathbb{T}^d$ can be split into a direct sum of analytic line bundles if and only if there is an exponentially localized Wannier basis $\{\varphi^a_\gamma(x) : \gamma \in \Gamma, a \in \{1, \cdots, r\}\}$ (with some Wannier centers) compatible with the I-symmetry.*

Proof. If $\mathcal{E}_{\mathbb{R}} = L_1 \oplus \cdots \oplus L_r$ is a direct sum of analytic line bundles, we may take the line bundles to be orthogonal. Let $\mathfrak{c}_a \in \Gamma$ be a representative of the Stiefel–Whintney class $w_1(L_a) \in H^1(\mathbb{R}^d/\Gamma; \mathbb{Z}/2) = \Gamma/2\Gamma$. As $L_a \otimes \mathcal{L}_{\mathfrak{c}_a}$ is a trivial line bundle in $\mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathfrak{c}_a}$ and by orthogonality of $\{L_a\}_{1\le a\le r}$, there exist analytic orthonormal sections s_a of the tensor bundle $\mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathfrak{c}_a} = \mathcal{E}_{\mathfrak{c}_a}$, for $a = 1, 2, \cdots, r$. Taking the Bloch transform of s_a, we can construct exponentially localized Wannier functions satisfying

$$I\varphi^a_\gamma = \varphi^a_{\mathfrak{c}_a - \gamma}, \quad a \in \{1, \cdots, r\}.$$

On the other hand, if there exists an exponentially localized Wannier basis with Wannier centers at $\{\mathfrak{c}_a\}_{1\le a\le r}$, then the inverse Bloch transform would give analytic orthonormal sections s_a of bundles $\mathcal{E}_{\mathfrak{c}_a} = \mathcal{E}_{\mathbb{R}} \otimes \mathcal{L}_{\mathfrak{c}_a}$, $a = 1, 2, \cdots, r$. Now we claim $\mathcal{E}_{\mathbb{R}} \cong \mathcal{L}_{\mathfrak{c}_1} \oplus \cdots \oplus \mathcal{L}_{\mathfrak{c}_r}$.

We can split off a trivial bundle $L_1 \cong \mathbb{R}s_1$:

$$\mathcal{E}_{\mathfrak{c}_1} = L_1 \oplus \mathcal{E}'_{\mathfrak{c}_1}.$$

Tensoring $\mathcal{E}_{\mathfrak{c}_1}$ with $\mathcal{L}_{\mathfrak{c}_1}$ yields that

$$\mathcal{E}_{\mathbb{R}} = (L_1 \otimes \mathcal{L}_{\mathfrak{c}_1}) \oplus (\mathcal{E}'_{\mathfrak{c}_1} \otimes \mathcal{L}_{\mathfrak{c}_1}) \cong \mathcal{L}_{\mathfrak{c}_1} \oplus (\mathcal{E}'_{\mathfrak{c}_1} \otimes \mathcal{L}_{\mathfrak{c}_1}).$$

Repeating this construction gives the desired direct sum decomposition. $\square$

3.4. **Band topology for rank two vector bundles.** In this section, we show that an oriented rank two real Bloch bundle $\mathcal{E}_{\mathbb{R}}$ over the torus in dimension $d \leq 3$ exhibits an exponentially localized Wannier basis compatible with the I-symmetry if and only if the Euler class $e(\mathcal{E}_{\mathbb{R}}) = 0$. In fact, we have the following more general result.

Proposition 3.5. *A rank two oriented real vector bundle $\mathcal{E}_{\mathbb{R}}$ over $\mathbb{T}^d$ can be split to the direct sum of line bundles if and only if $e(\mathcal{E}_{\mathbb{R}}) = 0$. In particular, $\mathcal{E}_{\mathbb{R}}$ is a trivial bundle when it splits.*

Proof. Suppose $\mathcal{E}_{\mathbb{R}} = L_1 \oplus L_2$, where L_1 and L_2 are orthogonal line bundles to each other. Then there exists a section s_1 of L_1 that vanishes on a subtorus γ_1 representing $w_1(L_1)$. Similarly, there exists another section s_2 of L_2 that vanishes on a subtorus γ_2 representing $w_1(L_2)$. Since $L_1 \oplus L_2$ is orientable, $w_1(L_1) = w_1(L_2)$. Therefore, we can choose γ_1 and γ_2 to be parallel subtori so that $s_1 + s_2$ is nonvanishing everywhere. Thus, $\mathcal{E}_{\mathbb{R}}$ contains a non-vanishing section and is trivial. $\square$

Remark 5. *This proposition is not true over $\mathbb{RP}^2$. Let L be the tautological line bundle over $\mathbb{RP}^2$, then $L \oplus L$ is not trivial.*

3.5. **Fragile topology for higher rank vector bundles.** In this section, we consider the so-called fragile topology for three or more Bloch bands over $\mathbb{T}^2$ or $\mathbb{T}^3$. In view of Proposition 3.4, it suffices to show the corresponding real Bloch bundle splits into line bundles.

Proposition 3.6. *Let M be $\mathbb{T}^2$ or $\mathbb{T}^3$ and $\mathcal{E}_{\mathbb{R}}$ be a real vector bundle of rank ≥ 3 over M. Then $\mathcal{E}_{\mathbb{R}}$ can be written as a direct sum of real line bundles.*

Proof. It suffices to consider the case $r = 3$, as any real vector bundle with rank ≥ 4 over M can split off a trivial line bundle by applying the Thom transversality theorem as in Proposition 2.5. Without loss of generality, we may also assume $\mathcal{E}_{\mathbb{R}}$ is orientable. Otherwise, we may tensor it with a line bundle to make it orientable (see (2.4)), then we can tensor it with the same line bundle after splitting to obtain the decomposition of the original bundle.

First, we discuss bundles over $\mathbb{T}^2$. The orientablity of $\mathcal{E}_{\mathbb{R}}$ yields $w_1 = 0 \in H^1(\mathbb{T}^2; \mathbb{Z}/2) = \mathbb{Z}/2e_1 + \mathbb{Z}/2e_2$. We assume $w_2 = ae_1 \smile e_2 \in H^2(\mathbb{T}^2; \mathbb{Z}/2) = \mathbb{Z}/2e_1 \smile e_2$. We can choose three line bundles with $w_1 = ae_1, ae_2, ae_1 + ae_2$ respectively. The bundle we get from direct sum has the total Stiefel–Whitney class

$$(1 + ae_1)(1 + ae_2)(1 + ae_1 + ae_2) = 1 + ae_1 \smile e_2,$$

which agrees with the total Stiefel–Whitney class of $\mathcal{E}_{\mathbb{R}}$.

Now we discuss bundles over $\mathbb{T}^3$. Again, the orientablity of $\mathcal{E}_{\mathbb{R}}$ yields $w_1 = 0 \in H^1(\mathbb{T}^3; \mathbb{Z}/2) = \mathbb{Z}/2e_1 + \mathbb{Z}/2e_2 + \mathbb{Z}/2e_3$. We assume

$$w_2 = ae_1 \smile e_2 + be_2 \smile e_3 + ce_3 \smile e_1 \in H^2(\mathbb{T}^3; \mathbb{Z}/2) = \mathbb{Z}/2e_1 \smile e_2 + \mathbb{Z}/2e_2 \smile e_3 + \mathbb{Z}/2e_3 \smile e_1.$$

There are four cases we need to consider

- When $a = b = c = 0$, this is a trivial bundle.

- When one of a, b, c is nonzero, say $a = 1, b = c = 0$, we choose three line bundles with $w_1 = e_1, e_2, e_1 + e_2$ respectively, and use
$$(1 + e_1)(1 + e_2)(1 + e_1 + e_2) = 1 + e_1 \smile e_2.$$
- When two of a, b, c are nonzero, say $a = b = 1, c = 0$, we choose three line bundles with $w_1 = e_1 + e_3, e_2, e_1 + e_2 + e_3$ respectively, and use
$$(1 + e_1 + e_3)(1 + e_2)(1 + e_1 + e_2 + e_3) = 1 + e_1 \smile e_2 + e_2 \smile e_3.$$
- When all of a, b, c are nonzero, i.e. $a = b = c = 1$, we choose three line bundles with $w_1 = e_1 + e_3, e_2 + e_3, e_1 + e_2$ respectively, and use
$$(1 + e_1 + e_3)(1 + e_2 + e_3)(1 + e_1 + e_2) = 1 + e_1 \smile e_2 + e_2 \smile e_3 + e_3 \smile e_1.$$

In each case, $\mathcal{E}_{\mathbb{R}}$ can be decomposed into the direct sum of three line bundles given by total Stiefel–Whitney class in the LHS of the equalities. □

One may furthermore take the three line bundles to be orthogonal. For real Bloch bundles, this gives the Wannier centers and Wannier functions compatible with the I-symmetry.

3.6. **Proof of Theorem 1.** Now, we are ready to give the proof of Theorem 1:

Proof of Theorem 1. By Proposition 3.4, the wannierizability with respect to the I-symmetry is equivalent to the decomposability of the real Bloch bundle $\mathcal{E}_{\mathbb{R}}$ into a direct sum of line bundles. Proposition 3.6 shows that this is always the case for real bundles with rank ≥ 3. The case of bundles of rank two is discussed in Proposition 3.5. □

4. Applications: Fragile topology in the chiral TBG at magic angles

In this section, using the fragile topology framework developed in previous sections, we determine the Wannier centers of the exponentially localized Wannier functions arising from the topologically non-trivial flat bands of twisted bilayer graphene (TBG). In particular, we consider the so-called chiral limit of the TBG. See [TKV19, Be*22, Zw23] for more detailed discussions on the chiral TBG and [BiMa11, BeZw23, Be*24] for the general Bistritzer–Macdonald Hamiltonian.

The Bloch transformed Hamiltonian is given by
$$H_k(\alpha) = \begin{pmatrix} 0 & D(\alpha)^* + \bar{k} \\ D(\alpha) + k & 0 \end{pmatrix} \text{ with } D(\alpha) = \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}$$
with a potential $U \in C^\infty(\mathbb{C})$ satisfying for $\gamma \in \Lambda := \mathbb{Z} + \omega\mathbb{Z}$ with $\omega = e^{2\pi i/3}$, $K = \frac{4\pi}{3}$, and $\langle a, b \rangle = \mathrm{Re}(\bar{a}b)$
$$U(z + \gamma) = e^{i\langle \gamma, K \rangle} U(z), \ U(\omega z) = \omega U(z) \text{ and } \overline{U(\bar{z})} = -U(-z),$$
such that H_k commutes with the symmetry $I = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}$ with $Qu(z) = \overline{u(-z)}$. The operator acts on the Hilbert space
$$L^2_0 := \{v \in L^2_{\mathrm{loc}}(\mathbb{C}; \mathbb{C}^4); \mathscr{L}_\gamma u = u \text{ for all } \gamma \in \Lambda\},$$

where $L_\gamma u(z) = \operatorname{diag}(\omega^{\gamma_1+\gamma_2}, 1, \omega^{\gamma_1+\gamma_2}, 1)u(z+\gamma)$, with $\gamma = \gamma_1 + \omega\gamma_2$ and $(\gamma_1, \gamma_2) \in \mathbb{Z}^2$.

A *magic angle* in this model is defined as a parameter $\alpha \in \mathbb{C}$ such that

$$0 \in \bigcap_{k\in\mathbb{C}} \operatorname{Spec}_{L^2_0}(H_k(\alpha)).$$

By Fredholm theory, this implies that $\dim\ker_{L^2_0}(D(\alpha)^* + \bar{k}) = \dim\ker_{L^2_0}(D(\alpha) + k) \neq 0$ for all $k \in \mathbb{C}$. If $\dim\ker_{L^2_0}(D(\alpha) + k) = 1$ for all $k \in \mathbb{C}$, we call the magic angle *simple* and if $\dim\ker_{L^2_0}(D(\alpha)+k) = 2$ for all $k \in \mathbb{C}$, we call it *two-fold degenerate*. The existence of simple and two-fold degenerated magic angles are proved in [BHZ23a, BHZ23b, WaLu21].

Recall that flat bands and the corresponding eigenspaces are given by a family of orthogonal projections that satisfy Assumption 1 due to the existence of band gaps between flat bands and other bands [BHZ23a, BHZ23b]. We consider $V(k) := \ker_{L^2_0}(D(\alpha) + k) \subset L^2_0$. This allows us to define a trivial bundle $\pi : \tilde{E} \to \mathbb{C}$, where

$$\tilde{E} := \{(k, v) : v \in V(k)\} \subset \mathbb{C} \times L^2_0(\mathbb{C}/\Lambda; \mathbb{C}^2).$$

To define a vector bundle over the torus $\mathbb{C}/\Lambda^*$, we introduce

$$(k, u) \sim (k + p, \tau(p)u), \;\; \tau(p)u(z) = e^{i\langle z,p\rangle}u(z),$$

for $p \in \Lambda^*$. This way $\mathcal{E}_{\mathbb{C}} := \tilde{E}/{\sim} \to \mathbb{C}/\Lambda^*$ is a holomorphic vector bundle. At simple magic angles $\operatorname{rank}_{\mathbb{C}}(\mathcal{E}_{\mathbb{C}}) = 1$, while at two-fold degenerate magic angles $\operatorname{rank}_{\mathbb{C}}(\mathcal{E}_{\mathbb{C}}) = 2$.

We now define the complex Bloch bundle over $\mathbb{C}/\Lambda^*$ (corresponding to the flat band):

$$\mathcal{F} := \{(k, \phi) : (\mathbb{C} \times L^2_0(\mathbb{C}/\Lambda; \mathbb{C}^4)/{\sim} : \phi \in 1\!\!1_0(H_k(\alpha))\}, \quad (k, \phi) \sim (k + p, \tau(p)\phi) \text{ for } p \in \Lambda^*.$$

and the corresponding real Bloch bundle $\mathcal{F}_0 := \{\varphi \in \mathcal{F} : I\varphi = \varphi\}$. Note that such Bloch bundles corresponding to flat bands can be defined using spectral projections due to the existence of band gaps at magic angles (see [BHZ24, BHZ23b]).

To identify the complex bundle $\mathcal{E}_{\mathbb{C}}$ with an oriented real bundle of twice the rank, we take some basis of every fiber u_1 (for simple magic angles) or u_1, u_2 (for degenerate magic angles). Focusing now exclusively on two-fold degenerate magic angles, to streamline the presentation, u_1, iu_1, u_2, iu_2 then defines an oriented basis. This one is always consistently oriented by general concepts [MiSt74, Lemma 14.1]. To obtain an oriented basis of the bundle $\mathcal{F}_0$ associated with the I-symmetry that commutes with the Hamiltonian, we just use the symmetry $Qu(z) = \overline{u(-z)}$ and define

$$(u_1, Qu_1), (iu_1, -iQ(u_1)), (u_2, Q(u_2)), (iu_2, -iQ(u_2)).$$

Since $\mathcal{F}_0 \cong \mathcal{E}_{\mathbb{C}}$ and the Euler class equals the top Chern class, we conclude that $w_1(\mathcal{F}_0) = 0$ and $e(\mathcal{F}_0) = c_1(\mathcal{E}_{\mathbb{C}})$. Thus the Euler number of $\mathcal{F}_0$ can be computed by

$$\chi(\mathcal{F}_0) = \int_{\mathbb{R}^2/\Gamma^*} e(\mathcal{F}_0) = \int_{\mathbb{R}^2/\Gamma^*} c_1(\mathcal{E}_{\mathbb{C}}).$$

4.1. **TBG with two flat bands.** At a simple magic angle, we take the real bundle $\mathcal{F}_0$. We may add a line bundle and define: $\mathcal{F}_0' := \mathcal{F}_0 \oplus L$.

By Proposition 3.6, we can write $\mathcal{F}_0' = \bigoplus_{i=0}^2 L_i$. As the real Bloch bundle $\mathcal{F}_0$ is orientable, i.e. $w_1(\mathcal{F}_0) = 0$, we have

$$w_1(L) = w_1(\mathcal{F}_0') = \sum_{i=0}^{2} w_1(L_i), \tag{4.1}$$

as well as

$$w_2(\mathcal{F}_0) = w_2(\mathcal{F}_0') = \sum_{i<j} w_1(L_i) \smile w_1(L_j). \tag{4.2}$$

Recall that for a line bundle L_i, the rescaled Wannier center $\mathfrak{c}_i \in \Gamma$ is the same as the first Stiefel–Whitney class $w_1(L_i) \in H^1(\mathbb{R}^2/\Gamma^*; \mathbb{Z}/2) = \Gamma/2\Gamma$. Assume $w_1(L) = \mathfrak{c} \in \Gamma/(2\Gamma)$, then the first condition yields that

$$\sum_{i=0}^{2} \mathfrak{c}_i = \mathfrak{c}.$$

Let ω be the generator of $H^2(\mathbb{R}^2/\Gamma^*; \mathbb{Z}) \cong \mathbb{Z}$, then the Euler class of $\mathcal{F}_0$ is given by $e(\mathcal{F}_0) = -\omega$, as the Chern number of the associated complex line bundle $\mathcal{E}_{\mathbb{C}}$ is -1. Thus, by equation (2.5), the second Stiefel–Whitney class is given by $w_2(\mathcal{F}_0) = e_1 \smile e_2$ with $e_1 \smile e_2$ being the generator of $H^2(\mathbb{R}^2/\Gamma^*, \mathbb{Z}/2) \cong \mathbb{Z}/2$ as in the proof of Proposition 3.6. By the proof of Proposition 3.6, we obtain

Corollary 4.1. *If L is a real line bundle from an isolated band, then we may decompose the real Bloch bundle $\mathcal{F}_0' = \mathcal{F}_0 \oplus L$ into a direct sum of three real line bundles $\mathcal{F}_0' = \bigoplus_{i=0}^2 L_i$ with Wannier centers*

$$\mathfrak{c}_0 = e_1 + \mathfrak{c}, \ \ \mathfrak{c}_1 = e_2 + \mathfrak{c}, \ \ \textit{and } \mathfrak{c}_2 = e_1 + e_2 + \mathfrak{c}.$$

.

4.2. **TBG with four flat bands.** In case of TBG at a two-fold degenerate magic angle, the real vector bundle $\mathcal{F}_0$ can be written as a direct sum of real line bundles $\mathcal{F}_0 = \bigoplus_{i=0}^3 L_i$. Using the orientability of $\mathcal{F}_0$, we find that the first Stiefel–Whitney class satisfies

$$\sum_{i=0}^{3} w_1(L_i) = w_1(\mathcal{F}_0) = 0.$$

As the Chern number of the two-fold degenerate flat band is -1 (cf. [BHZ23b, Theorem 5]), we also obtain

$$\sum_{i<j} w_1(L_i) \smile w_1(L_j) = w_2(\mathcal{F}_0) = e_1 \smile e_2.$$

Splitting a trivial bundle off the rank four real Bloch bundle $\mathcal{F}_0$ and using the proof of Proposition 3.6, we obtain the following

Corollary 4.2. *We can decompose $\mathcal{F}_0$ into a direct sum of four real line bundles $\mathcal{F}_0 = \bigoplus_{i=0}^3 L_i$ with Wannier centers at*

$$\mathfrak{c}_0 = e_1, \ \ \mathfrak{c}_1 = e_2, \ \ \mathfrak{c}_2 = e_1 + e_2, \ \ \textit{and } \mathfrak{c}_3 = 0.$$

5. Asymptotics of Wannier functions

In this section, we compute the asymptotics of Wannier functions when the Chern number is nonzero for a complex line bundle $\mathcal{E}_{\mathbb{C}}$ over $\mathbb{T}^2$ and $\mathbb{T}^3$. We verified in the (proof of) Proposition 2.5 that a general complex vector bundle can be decomposed into the Whitney sum of $r-1$ trivial line bundles L_i and a complex line bundle L such that $c_1(L) = c_1(\mathcal{E}_{\mathbb{C}})$. Trivial bundles L_i always give rise to exponentially localized Wannier functions. Thus, the optimal Wannier decay of a general complex vector bundle over $\mathbb{T}^2$ or $\mathbb{T}^3$ is reduced to the optimal Wannier decay rate of a single complex line bundle, which we will investigate next.

5.1. **Wannier function asymptotics on $\mathbb{T}^2$.**

The asymptotics of Wannier functions is related to possible singularities of the normalized section. We simplify the discussion in [Li*24] about the movability of zeros and first prove the following proposition.

Proposition 5.1. *Let $\mathcal{E}_{\mathbb{C}}$ be a smooth complex line bundle over $\mathbb{T}^2$. If the Chern number $c_1(\mathcal{E}_{\mathbb{C}}) = m \in \mathbb{Z}$, then there exists a smooth section $s : \mathbb{T}^2 \to \mathcal{E}_{\mathbb{C}}$ that vanishes at a single point and is of the form z^m in local coordinates.*

Proof. Pick a point $p \in \mathbb{T}^2$. Let $D \subset \mathbb{T}^2$ be a small disc around p where $\mathcal{E}_{\mathbb{C}}$ is trivialized, so we can identify the bundle locally with $D \times \mathbb{C}$. In this trivialization, define a section s_{local} by $s_{\text{local}}(z) = z^m$, where z is a complex coordinate on D with $z = 0$ corresponding to p. This section has a zero of order m at p.

Now extend the domain of consideration to a slightly larger disc $D' \supset D$ with boundary $\partial D'$. On the complement $\mathbb{T}^2 \setminus D'$, the line bundle $\mathcal{E}_{\mathbb{C}}$ is trivial because $\mathbb{T}^2 \setminus D'$ is homotopically equivalent to $\mathbb{S}^1 \vee \mathbb{S}^1$, and every complex line bundle over $\mathbb{S}^1 \vee \mathbb{S}^1$ is trivial as $H^2(\mathbb{S}^1 \vee \mathbb{S}^1, \mathbb{Z}) = 0$. Therefore, there exists a smooth, non-vanishing section s_{outer} defined on $\mathbb{T}^2 \setminus D'$ (see also [BoTu13, Proposition 11.14]). Note that in particular the winding number of the section s_{outer} on $\partial D'$ equals the Chern number of $\mathcal{E}_{\mathbb{C}}$ by [BoTu13, Theorem 11.16], as s_{outer} can be view as a section of a S^1-bundle since it is non-vanishing.

Now we have a section s_{local} on $\overline{D}$ and a nonvanishing section s_{outer} on $\mathbb{T}^2 \setminus D'$ such that their winding number on the boundary agrees. We want to glue them smoothly on $D' \setminus D$. Since they have the same winding number, there is a homotopy $H(t, \theta) : [0, 1] \times \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$ such that $H(0, \theta) = s|_{\partial D}$ and $H(1, \theta) = s'|_{\partial D'}$. Therefore, we can define a continuous gluing by

$$\widetilde{s}(r, \theta) = H\left(\frac{r - r_D}{r_{D'} - r_D}, \theta\right) \in \mathbb{C} \setminus \{0\},$$

where $r_D, r_{D'}$ are the radii of D and D', respectively. By convolving with a smooth approximation of identity near the gluing region, we can make it smooth and still nonvanishing.

By construction, the section s vanishes only at p, where it coincides with $s_{\text{local}}(z) = z^m$. Away from p, s is smooth and non-vanishing, completing the construction. $\square$

Now we compute the asymptotics of Wannier functions using the section constructed in Proposition 5.1.

Proof of Theorem 2. We first normalize the section. The normalized section is smooth everywhere, aside from the singularity at the origin given by $\frac{k^m}{|k|^m}\Phi(k,x)$ on a disc D, where $\Phi(k,x)$ is a local normalized smooth section. Here we think of k as a complex variable $k=k_1+ik_2$. Let $\chi\in C_c^\infty([0,\infty))$ be a smooth cutoff function such that $\chi\equiv 1$ near zero. We now use (1.4) to recover the Wannier function from the Bloch function and integrate over $\mathbb{R}^2$, due to the presence of the cutoff function. Modulo an $\mathcal{O}(|x|^{-\infty})$-error, which is given by taking the Bloch transform of the smooth part corresponding to the cutoff function $1-\chi$ away from the singularity, the Wannier function is given by

$$\begin{aligned}
w(\varphi)(x) &= \frac{1}{|\mathbb{R}^2/\Gamma^*|}\int_{\mathbb{R}^2} e^{i\langle k,x\rangle}\frac{k^m}{|k|^m}\chi(|k|)\Phi(k,x)\,dk+\mathcal{O}(|x|^{-\infty})\\
&= \frac{1}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho(x_1\cos\theta+x_2\sin\theta)}\chi(\rho)\Phi(k,x)\rho\,d\rho\,d\theta+\mathcal{O}(|x|^{-\infty})\\
&= \frac{\Phi(0,x)}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho(x_1\cos\theta+x_2\sin\theta)}\chi(\rho)\rho\,d\rho\,d\theta\\
&\quad+\frac{1}{|\mathbb{R}^2/\Gamma^*|}\int_{\mathbb{R}^2} e^{i\langle k,x\rangle}\frac{k^m}{|k|^m}\chi(|k|)(\Phi(k,x)-\Phi(0,x))\,dk+\mathcal{O}(|x|^{-\infty}).
\end{aligned}\tag{5.1}$$

The first term in the right-hand side of (5.1) is given by

$$\begin{aligned}
&\frac{\Phi(0,x)}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho(x_1\cos\theta+x_2\sin\theta)}\chi(\rho)\rho\,d\rho\,d\theta\\
=&\frac{\Phi(0,x)}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho|x|\sin(\theta+\varphi_x)}\chi(\rho)\rho\,d\rho\,d\theta\\
=&\frac{\Phi(0,x)e^{-im\varphi_x}}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho|x|\sin\theta}\chi(\rho)\rho\,d\rho\,d\theta
\end{aligned}$$

where $\sin\varphi_x = x_1/|x|$ and $\cos\varphi_x = x_2/|x|$.

We can compute the asymptotics as follows.

$$\begin{aligned}
&\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho|x|\sin\theta}\chi(\rho)\rho\,d\rho\,d\theta = |x|^{-2}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}\chi(\tfrac{\rho}{|x|})\rho\,d\rho\,d\theta\\
=&|x|^{-2}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}\rho\,d\rho\,d\theta-|x|^{-2}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}(1-\chi(\tfrac{\rho}{|x|}))\rho\,d\rho\,d\theta\\
=&|x|^{-2}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}\rho\,d\rho\,d\theta+\mathcal{O}(|x|^{-\infty}).
\end{aligned}\tag{5.2}$$

The second term is $\mathcal{O}(|x|^{-\infty})$ by integration by parts in ρ and θ. Here the integral does not converge in the usual sense of Riemann integral. They are defined in the distributional sense:

$$\int_0^\infty e^{i\rho\sin\theta}\rho d\rho := \lim_{\varepsilon\to 0+}\int_0^\infty e^{i\rho(\sin\theta+i\varepsilon)}\rho d\rho = -\frac{1}{(\sin\theta+i0)^2}.$$

By integration by parts,

$$-\int_0^{2\pi} e^{im\theta}(\sin\theta+i0)^{-2}d\theta = -im\int_0^{2\pi} e^{im\theta}\frac{\cos\theta}{\sin\theta+i0}d\theta.$$

Since $\frac{\cos\theta}{\sin\theta+i0} = p.v.\cot\theta - i\pi\delta_0(\theta) + i\pi\delta_\pi(\theta)$, we have

$$\int_0^{2\pi} e^{im\theta}\frac{\cos\theta}{\sin\theta+i0}d\theta = \int_0^{2\pi} e^{im\theta}p.v.\cot\theta d\theta + i\pi((-1)^m-1)$$
$$= i\int_0^{2\pi}\frac{\sin m\theta\cos\theta}{\sin\theta}d\theta + i\pi((-1)^m-1) = \pi i((-1)^m+1) + \pi i((-1)^m-1) = 2\pi i(-1)^m.$$

Therefore

$$\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}\rho\,d\rho\,d\theta = 2\pi m(-1)^m.$$

Another way to compute the integral is to use the Bessel function

$$J_m(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta+i\rho\sin\theta}\,d\theta$$

so that

$$\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho\sin\theta}\rho\,d\rho\,d\theta = 2\pi(-1)^m\int_0^\infty J_m(\rho)\rho d\rho.$$

The integral of Bessel function can be computed by differentiating the following Fourier integral for $\xi\in[0,1)$ (see [DLMF, (10.22.59)])

$$\int_0^\infty J_m(x)e^{ix\xi}\,dx = \frac{e^{im\arcsin(\xi)}}{(1-\xi^2)^{1/2}}.$$

In conclusion, the first term in the right-hand side of (5.1) is given by

$$\frac{\Phi(0,x)}{|\mathbb{R}^2/\Gamma^*|}\int_0^{2\pi}\int_0^\infty e^{im\theta}e^{i\rho(x_1\cos\theta+x_2\sin\theta)}\chi(\rho)\rho\,d\rho\,d\theta = \frac{2\pi(-1)^m me^{-im\varphi_x}}{|\mathbb{R}^2/\Gamma^*||x|^2}\Phi(0,x) + \mathcal{O}(|x|^{-\infty}).$$

For the second term in the right-hand side of (5.1), we recall

$$\Phi(k,x) = \Phi(0,x) + \sum_{0<|\alpha|\le N_0}\frac{k^\alpha}{\alpha!}\partial_k^\alpha\Phi(0,x) + \sum_{|\beta|=N_0+1}k^\beta R_\beta(k,x)$$

where $R_\beta(k,x)$ is smooth and $k^\alpha := k_1^{\alpha_1}k_2^{\alpha_2}$ for $k = k_1+ik_2$. By a similar estimate as above, we have

$$\int_{\mathbb{R}^2} e^{i\langle k,x\rangle}\frac{k^m}{|k|^m}k^\alpha\chi(|k|)dk = c_{m,\alpha}(\varphi_x)|x|^{-2-|\alpha|} + \mathcal{O}(|x|^{-\infty}).$$

Since $\frac{k^m}{|k|^m}k^\beta\chi(|k|)\in C^{N_0}$ for $|\beta| = N_0+1$, we have

$$\sum_{|\beta|=N_0+1}\int_{\mathbb{R}^2} e^{i\langle k,x\rangle}\frac{k^m}{|k|^m}k^\beta R_\beta(k,x)\chi(|k|)dk = \mathcal{O}(|x|^{-N_0}).$$

We conclude from (5.1) that

$$w(\varphi)(x) \sim \frac{1}{|\mathbb{R}^2/\Gamma^*|}\sum_{\alpha\in\mathbb{N}^2}c_{m,\alpha}(\varphi_x)|x|^{-2-|\alpha|}\partial_k^\alpha\Phi(0,x)$$

with $c_{m,0}(\varphi_x) = 2\pi(-1)^m me^{-im\varphi_x}$. □

5.2. **Decay of Wannier function on $\mathbb{T}^3$.** In this section, we consider decay of Wannier functions of non-trivial Bloch bundles over $\mathbb{T}^3$. For simplicity, we assume $\Gamma^* = \mathbb{Z}^3$ in this section. The first Chern class for complex vector bundles over $\mathbb{T}^3$ is given by

$$c_1 = m_1[dx_1 \wedge dx_2] + m_2[dx_2 \wedge dx_3] + m_3[dx_3 \wedge dx_1], \quad m_1, m_2, m_3 \in \mathbb{Z}.$$

To further simplify the presentation, we want to find a change of coordinates on $\mathbb{T}^3$ to simply the form of c_1, which corresponds to a transformation $y = Bx$ with a unimodular matrix $B \in \mathrm{SL}(3, \mathbb{Z})$. We have the following Lemma:

Lemma 5.2. *There exist $B \in \mathrm{SL}(3, \mathbb{Z})$ and new coordinates $y := Bx$ with $x \in \mathbb{R}^3/\mathbb{Z}^3$ such that the first Chern class is given by*

$$c_1 = m[dy_1 \wedge dy_2] \text{ where } m := \gcd(m_1, m_2, m_3).$$

Proof. We set

$$v_1 := \frac{1}{m}(m_1, m_2, m_3) \in \mathbb{Z}^3.$$

Since v_1 is a primitive vector, we can build a unimodular matrix $A := (v_1, v_2, v_3) \in \mathrm{SL}(3, \mathbb{Z})$, see [Sc60, Lemma 1].

Thus, we have $Ae_1 = v_1$ and $(m_1, m_2, m_3)A^{-T} = (m, 0, 0)$. We want to find coordinates $y := Bx$ with $B \in \mathrm{SL}(3, \mathbb{Z})$ such that

$$\left(dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ dx_3 \wedge dx_1\right)^T = A^{-T} \left(dy_1 \wedge dy_2,\ dy_2 \wedge dy_3,\ dy_3 \wedge dy_1\right)^T.$$

Since

$$dy_i \wedge dy_j = \left(\sum_k B_{ik} dx_k\right) \wedge \left(\sum_l B_{jl} dx_l\right) = \sum_{k<l} \det \begin{pmatrix} B_{ik} & B_{il} \\ B_{jk} & B_{jl} \end{pmatrix} dx_k \wedge dx_l,$$

and the adjugate matrix $\mathrm{adj}(B) = B^{-1}$, we have

$$\left(dy_1 \wedge dy_2,\ dy_2 \wedge dy_3,\ dy_3 \wedge dy_1\right)^T = B^{-1} \left(dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ dx_3 \wedge dx_1\right)^T.$$

We take $B = A^{-T}$. The Chern class in the y coordinate is $c_1 = m[dy_1 \wedge dy_2]$. ∎

In the following we shall assume that the Chern class is of the form described in Lemma 5.2. Recall that by Proposition 2.5, complex vector bundles over $\mathbb{T}^3$ are also classified by the first Chern class. We now extend the construction of Wannier functions in Section 5.1 to non-trivial Bloch bundles over $\mathbb{T}^3$. As in Section 5.1, we will make the construction on a line bundle $\mathcal{E}_{\mathbb{C}}$ with the first Chern class $(m, 0, 0)$. We first construct a section of the complex line bundle $\mathcal{E}_{\mathbb{C}}$ over $\mathbb{T}^3$, using Proposition 5.1:

Proposition 5.3. *Let $\mathcal{E}_{\mathbb{C}}$ be a smooth complex line bundle over $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. If the Chern number is given by $c_1(\mathcal{E}_{\mathbb{C}}) = (m, 0, 0) \in \mathbb{Z}^3$, then there exists a smooth section $s : \mathbb{T}^3 \to \mathcal{E}_{\mathbb{C}}$ such that the section only vanishes on a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), t) = (a\cos(2\pi t), a\sin(2\pi t), t)$ for some sufficiently small $a > 0$. Moreover,*

$$s(k_1, k_2, t) = (k_1 + ik_2 - \gamma_1 - i\gamma_2(t))^m$$

near the curve $\gamma(t)$.

Proof. Since the line bundle $\mathcal{E}_{\mathbb{C}}$ has Chern number $(m, 0, 0)$, it is isomorphic to the pullback bundle $\pi^* L_m$ where $\pi : \mathbb{R}^3/\mathbb{Z}^3 \to \mathbb{R}^2/\mathbb{Z}^2$ is the projection onto the first two coordinates and L_m is the line bundle with Chern number m on $\mathbb{R}^2/\mathbb{Z}^2$. By Proposition 5.1, we can find a section of L_m that only vanishes at $(0, 0)$ and has the form k^m near $(0, 0)$. By pulling back this section to $\mathbb{R}^3/\mathbb{Z}^3$, we get a section s_{outer} of $\mathcal{E}_{\mathbb{C}}$ that only vanishes on the curve $(0, 0, t)$, $t \in \mathbb{R}/\mathbb{Z}$ and has the form k^m near $(0, 0, t)$. Note $\mathcal{E}_{\mathbb{C}}$ restricted to a tubular neighbourhood $D := \{(k_1, k_2, t) : |(k_1, k_2)| \leq 2c_0, t \in \mathbb{R}/\mathbb{Z}\}$ of the helix is trivial. Thus we construct a local section of the form

$$s_{\text{local}}(k_1, k_2, t) := (k_1 + ik_2 - \gamma_1(t) - i\gamma_2(t))^m, \quad (k_1, k_2, t) \in D.$$

We now take

$$s(k_1, k_2, t) = \begin{cases} s_{\text{local}}(k_1, k_2, t), & |k| \leq c_0 \\ \frac{2c_0 - |k|}{c_0} s_{\text{local}}(k_1, k_2, t) + \frac{|k| - c_0}{c_0} s_{\text{outer}}(k_1, k_2, t), & c_0 \leq |k| \leq 2c_0 \\ s_{\text{outer}}(k_1, k_2, t), & (k_1, k_2, t) \notin D. \end{cases}$$

Now we show that the section $s(k_1, k_2, t)$ vanishes only at the helix γ. Since

$$\begin{aligned} |s_{\text{local}}(k_1, k_2, t) - k^m| &= |k|^m \left| \left(1 - \frac{\gamma_1(t) + i\gamma_2(t)}{k_1 + ik_2}\right)^m - 1 \right| \\ &\leq |k|^m \max\left((1 - (1 - a/c_0)^m, (1 + a/c_0)^m - 1\right), \end{aligned}$$

by taking a sufficiently small, we have

$$|s(k_1, k_2, t)| \geq |k|^m - |k|^m \max\left((1 - (1 - a/c_0)^m, (1 + a/c_0)^m - 1)\right) > 0, \quad c_0 \leq |k| \leq 2c_0.$$

Convolving with a smooth approximation of identity near the gluing region gives a smooth section that only vanishes at the helix and has the form

$$s_{\text{local}}(k_1, k_2, t) := (k_1 + ik_2 - \gamma_1(t) - i\gamma_2(t))^m$$

near the helix. □

Now using the section constructed in Proposition 5.3, we construct Wannier functions and compute the decay of the Wannier functions for non-trivial complex line bundles over $\mathbb{T}^3$.

Proof of Theorem 3. Similar to the proof of Theorem 2, we first normalize the section constructed in Proposition 5.3. The normalized section is smooth everywhere away from the singularity at the helix curve γ given by

$$\frac{(k_1 + ik_2 - \gamma_1(t) - i\gamma_2(t))^m}{|k_1 + ik_2 - \gamma_1(t) - i\gamma_2(t)|^m} \Phi(k_1, k_2, t, x)$$

in a tubular neighborhood D of γ, where $\Phi(k_1, k_2, t, x)$ is a local normalized smooth section. Let $\chi \in C_c^\infty([0, \infty))$ be a smooth cutoff function such that $\chi = 1$ near 0. We again use (1.4) to recover the Wannier function from the Bloch function and reduce the integration over

$\mathbb{T}^2_{k_1,k_2}$ to the integration over $\mathbb{R}^2$, due to the presence of the cutoff function. The Wannier function is given by

$$\begin{aligned}
w(\varphi)(x) = &\int_0^1\int_{\mathbb{R}^2} e^{i(k_1x_1+k_2x_2+tx_3)}\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}\\
&\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)\Phi(k_1,k_2,t,x)\,dk_1\,dk_2\,dt+\mathcal{O}(|x|^{-\infty})\\
=&\int_0^1\int_{\mathbb{R}^2} e^{i(k_1x_1+k_2x_2+tx_3)}\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}\\
&\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)\Phi(\gamma(t),x)\,dk_1\,dk_2\,dt\\
+&\int_0^1\int_{\mathbb{R}^2} e^{i(k_1x_1+k_2x_2+tx_3)}\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}\\
&\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)(\Phi(k_1,k_2,t,x)-\Phi(\gamma(t),x))\,dk_1\,dk_2\,dt+\mathcal{O}(|x|^{-\infty})
\end{aligned}\tag{5.3}$$

We consider the Fourier transform

$$\begin{aligned}
\mathcal{I}_{\gamma,\chi,\Phi}(x) = &\int_0^1\int_{\mathbb{R}^2}\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}e^{i(k_1x_1+k_2x_2+tx_3)}\\
&\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)\Phi(\gamma(t),x)dk_1dk_2dt\\
=&\int_0^\infty\int_0^{2\pi}\int_0^1 e^{im\theta}e^{i(x_1(\gamma_1(t)+\cos\theta)+x_2(\gamma_2(t)+\sin\theta)+tx_3)}\chi(r)\Phi(\gamma(t),x)rdrd\theta dt\\
=&\int_0^\infty\int_0^{2\pi} e^{im\theta}e^{i(x_1\cos\theta+x_2\sin\theta))}\chi(r)rdrd\theta\cdot\int_0^1 e^{i(\gamma_1(t)x_1+\gamma_2(t)x_2+tx_3)}\Phi(\gamma(t),x)dt.
\end{aligned}$$

By (5.2), the first term is estimated by

$$\int_0^\infty\int_0^{2\pi} e^{im\theta}e^{i(x_1\cos\theta+x_2\sin\theta))}\chi(r)rdrd\theta=\mathcal{O}((1+|x_1|+|x_2|)^{-2}).\tag{5.4}$$

Consider

$$I_k(x)=\{t\in\mathbb{R}/\mathbb{Z}:\frac{d^k}{dt^k}(\gamma_1(t)x_1+\gamma_2(t)x_2+tx_3)\neq 0\},\quad k=1,2,3,\quad x\neq 0.$$

Since $\gamma'(t),\gamma''(t),\gamma'''(t)$ are linearly independent, we have $I_1(x)\cup I_2(x)\cup I_3(x)=\mathbb{R}/\mathbb{Z}$. Take a smooth partition of unity $1=\chi_1(t)+\chi_2(t)+\chi_3(t)$ on $\mathbb{R}/\mathbb{Z}$ with $\operatorname{supp}\chi_k(t)\subset I_k(x)$. Note this partition of unity may depend on x, but we can choose it locally uniformly in terms of $\frac{x}{|x|}$. By van der Corput lemma,

$$\int_0^1 e^{i(\gamma_1(t)x_1+\gamma_2(t)x_2+tx_3)}\chi_k(t)\Phi(\gamma(t),x)dt=\mathcal{O}(|x|^{-1/k}),\ k=1,2,3.$$

Moreover, when $|x_1|+|x_2|\ll|x_3|$, we have $\mathbb{R}/\mathbb{Z}=I_1(x)$ and nonstationary phase gives

$$\int_0^1 e^{i(\gamma_1(t)x_1+\gamma_2(t)x_2+tx_3)}\chi_1(t)\Phi(\gamma(t),x)dt=\mathcal{O}(|x|^{-\infty}).$$

Therefore we conclude

$$\mathcal{I}_{\gamma,\chi,\Phi}(x)=\mathcal{O}(|x|^{-7/3}).$$

For the second term in (5.3), we have

$$\Phi(k_1,k_2,t,x)-\Phi(\gamma(t),x)=\sum_{0<|\alpha|\le N_0}\frac{(k_1-\gamma_1(t))^{\alpha_1}(k_2-\gamma_2(t))^{\alpha_2}}{\alpha!}\partial_k^\alpha\Phi(\gamma(t),x)+R_{N_0}(k_1,k_2,t,x)$$

where $\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}R_{N_0}(k_1,k_2,t,x)\in C^{N_0}$. By the similar estimates as before, we have

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^2}e^{i(k_1x_1+k_2x_2+tx_3)}\frac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}\\&\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)(k_1-\gamma_1(t))^{\alpha_1}(k_2-\gamma_2(t))^{\alpha_2}\partial_k^\alpha\Phi(\gamma(t),x)\,dk_1\,dk_2\,dt\\=&\mathcal{O}(|x|^{-7/3-|\alpha|}).\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^2}e^{i(k_1x_1+k_2x_2+tx_3)}\tfrac{(k_1+ik_2-\gamma_1(t)-i\gamma_2(t))^m}{|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|^m}R_{N_0}(k_1,k_2,t,x)\chi(|k_1+ik_2-\gamma_1(t)-i\gamma_2(t)|)\,dk_1\,dk_2\,dt\\&=\mathcal{O}(|x|^{-N_0}).\end{aligned}$$

Taking $N_0=3$ finishes the proof. $\square$

If we allow anisotropic decay, then we may also construct a section of the complex line bundle $\mathcal{E}_{\mathbb{C}}$ vanishing along $\gamma(t)=(0,0,t),t\in[0,1]$ such that

$$s(k_1,k_2,t)=(k_1+ik_2)^m$$

near the curve $\gamma(t)$. By separation of variables, proof of Theorem 2, and non-stationary phase, we can conclude the following

Corollary 5.4. *There exists a Wannier function $w(\varphi)$ for a complex Bloch line bundle which exhibits $\mathcal{O}((1+|x_1|+|x_2|)^{-2}\langle x_3\rangle^{-\infty})$ decay.*

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