

# A NOTE ON THE EIGENVALUE RIGIDITY OF HYPERBOLIC SURFACES IN THE RANDOM COVER MODEL

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ABSTRACT. Let  $X$  be a compact connected orientable hyperbolic surface and  $X_n$  be a degree  $n$  random cover. We show that, with high probability, the distribution of eigenvalues of the Laplacian on  $X_n$  converges to the spectral measure of the hyperbolic plane with polynomially decaying error. This is analogous to the eigenvalue rigidity property for graphs [HY24] and improves the logarithmic bound of [Mon22]. Our proof relies on the Selberg trace formula and a variant of the polynomial method.

## 1. INTRODUCTION

Let  $X$  be a compact orientable hyperbolic surface and  $\lambda_j(X)$  be its Laplacian eigenvalues. Following recent developments of random regular graphs (see also [Nau26]), it is reasonable to conjecture that the eigenvalue spacing of a random hyperbolic surface with large genus follows that of GOE matrices. Although this conjecture remains out of reach, there has been a recent breakthrough for random  $d$ -regular graphs by Huang, McKenzie, and Yau [HMY24], in which they prove that the eigenvalue distribution at the edge of the spectrum converges to the Tracy–Widom distribution. One important step in their proof is *optimal rigidity*, which states that with high probability, the  $j$ -th eigenvalue  $\lambda_j(G)$  of a random regular graph  $G$  of size  $N$  is close to the expected position  $\lambda_j$  prescribed by the spectrum of the regular tree (i.e., the position prescribed by the Kesten–McKay law). In particular,

$$|\lambda_j(G) - \lambda_j| \lesssim N^{-2/3+\epsilon} (\min\{j, N - j + 1\})^{-1/3}. \quad (1.1)$$

The bound (1.1) is expected to be optimal (up to  $N^\epsilon$ ) as the size of the right-hand side (the oscillation) is of the same size as the eigenvalue spacing.

The purpose of this note is to study the eigenvalue rigidity property for random hyperbolic surfaces. In this paper, we consider the random cover model. More precisely, we take degree  $n$  covers of the closed hyperbolic surface  $X$  uniformly at random. Using the formulas from [AM23] and [HMT25b], the same method applies to the Weil–Peterson model. This will be discussed in a future paper.

We are not able to prove optimal rigidity as in (1.1). But we are able to show a *polynomial* bound. This improves earlier work of Monk [Mon22], who proved a logarithmic bound under

the (weaker) assumption of Benjamini–Schramm convergence. The following is the main result of our paper.

**Theorem 1.** *Let  $X$  be a compact connected orientable hyperbolic surface of genus  $g \geq 2$ . For any  $\epsilon > 0$ , there exist  $\alpha = \alpha(g, \epsilon) > 0$  and  $C = C(X, \epsilon) > 0$  such that the following is true. Let  $X_n$  be a degree  $n$  cover of  $X$  taken uniformly at random and let  $\lambda_j(X_n)$  be the  $j$ -th Laplacian eigenvalue on  $X_n$ . Then with probability  $1 - n^{-1/10}$ , for any  $\Lambda \in [1/4, \infty)$  and any  $\lambda_j(X_n) \in [1/4, \Lambda]$ , we have*

$$|\lambda_j(X_n) - \lambda_j| \leq C\Lambda^{1/2+\epsilon}n^{-\alpha}, \quad n \in \mathbb{N}, \quad (1.2)$$

where  $\lambda_j \geq 1/4$  is defined by

$$\int_0^{\sqrt{\lambda_j - 1/4}} r \tanh(\pi r) dr = \frac{j}{n(2g - 2)}. \quad (1.3)$$

Moreover, we have the following Weyl law for  $N_{X_n}(\Lambda) := \#\{j : \lambda_j(X_n) \leq \Lambda\}$ :

$$N_{X_n}(\Lambda) = (2g - 2)n \int_0^{\sqrt{\Lambda - 1/4}} r \tanh(\pi r) dr + O_{X, \epsilon}(n^{1-\alpha}\Lambda^{1/2+\epsilon}), \quad \Lambda \in [1/4, \infty). \quad (1.4)$$

Since the  $\lambda_j$ 's are evenly distributed in  $[1/4, \infty)$  according to the spectral measure of the hyperbolic plane  $\mathbb{H}$ , (1.2) shows that the eigenvalues  $\lambda_j(X_n)$  are also evenly distributed up to an  $O(n^{-\alpha}\Lambda^{1/2+\epsilon})$ -oscillation. In particular, the multiplicity of eigenvalues is bounded by  $Cn^{1-\alpha}\Lambda^{1/2+\epsilon}$  for a typical random cover, see [GLMST21; Mon22; LM24; HWZ25] for related results.

We note that Hide, Macera, and Thomas [HMT25a, Theorem 1.1] show that there exists  $b, c > 0$  depending only on the genus of  $X$  such that a uniformly random degree  $n$  cover  $X_n$  of  $X$  has

$$\lambda_1^{\text{new}}(X_n) \geq \frac{1}{4} - cn^{-b}, \quad (1.5)$$

with probability tending to 1 as  $n \rightarrow \infty$ . Here,  $\lambda_1^{\text{new}}(X_n)$  denotes the smallest eigenvalue of  $X_n$  that is not an eigenvalue of  $X$ , accounting for multiplicities. Moreover, from (1.4), it is easy to see that there are at most  $Cn^{1-\alpha}$  eigenvalues of  $\Delta_{X_n}$  below  $1/4$ . Therefore, the new eigenvalues of  $X_n$  below  $1/4$  also satisfy (1.2) with  $\alpha$  replaced by  $\min(2\alpha/3, b)$  and probability tending to 1 as  $n \rightarrow \infty$ .

Theorem 1 can be thought as an analogue of the spectral graph theory result of Huang and Yau [HY24], which shows the rigidity estimate as in (1.1) with a polynomial bound. We also note that, compared to [HY24], we do not obtain estimates on the eigenfunctions.

1.1. **Previous work.** We briefly review prior results on the low energy spectral theory (i.e., the restriction to bounded eigenvalues) of random hyperbolic surfaces. Most of such work focused on the *spectral gap*. The spectral gap appears when studying the error term for counting closed geodesics of bounded length, the error term for the hyperbolic lattice counting problem, and the error term for the rate of mixing of the geodesic flow. The spectral gap on hyperbolic surfaces can be thought of as an analog of the Alon–Boppana bound on graphs. For additional context on spectral gaps, see the survey [MN26].

In [Hub74], Huber proved that for any sequence of compact hyperbolic surfaces  $X_n$  with genera  $g(X_n)$  tending to infinity,  $\limsup_{i \rightarrow \infty} \lambda_1(X_n) \leq \frac{1}{4}$ . More recently, in [MNP22], Magee, Naud, and Puder studied covers of compact hyperbolic surfaces. They showed that for all  $\varepsilon > 0$ , with high probability, a degree  $n$  covering surface  $X_n$  of  $X$  has no new eigenvalues below  $\frac{3}{16} - \varepsilon$ . In [MPvH25] Magee, Puder, and van Handel improved  $\frac{3}{16}$  to  $\frac{1}{4}$ . Hide, Macera, and Thomas [HMT25a] used [MPvH25] to obtain a spectral gap result with polynomial error. Specifically, they showed  $\lambda_1(X_n) \geq \frac{1}{4} - O(n^{-b})$  for uniformly random degree  $n$  covers  $X_n$  of a closed hyperbolic surface  $X$ . The work of [MNP22] was generalized to surfaces with variable negative curvature in [HMN25] by Hide, Moy, and Naud. In the even more general case of covers of closed Riemannian surfaces with Anosov geodesic flow, Moy [Moy26] proved the existence of a spectral gap for Pollicott–Ruelle resonances. In constant curvature, a spectral gap for the Laplacian implies a spectral gap for the Pollicott–Ruelle spectrum. However, this correspondence does not hold in variable curvature.

Much recent progress has also been made in the regime of spectral gaps for covers of finite-area noncompact hyperbolic surfaces  $X$ . Hide and Magee studied this model in [HM23]. They showed for any  $\varepsilon > 0$ , with high probability, a uniformly random degree  $n$  cover of  $X$  has no new eigenvalues below  $\frac{1}{4} - \varepsilon$ . In [Hid23], Hide strengthened the work of [HM23] to replace  $\varepsilon$  with  $c \frac{(\log \log \log n)^2}{\log \log n}$ . In [Moy25], Moy studied spectral gaps in the case of covers of noncompact, geometrically finite surfaces with pinched sectional curvature. In [BMP25], Ballmann, Mondal, and Polymerakis studied random covers of a complete connected Riemannian manifold with Ricci curvature bounded from below, under certain conditions of the fundamental group.

Additionally, we discuss prior work on spectral gaps for Weil–Petersson random hyperbolic surfaces. In this regime, Wu and Xue [WX22] and Lipnowski and Wright [LW24] showed that with high probability,  $\lambda_1(X) \geq \frac{3}{16} - o(1)$ . The series of works [AM23], [AM24a], [AM24b], [AM25] by Anantharaman and Monk concluded that for all  $\varepsilon > 0$ ,  $\lambda_1(X) \geq \frac{1}{4} - \varepsilon$  for a Weil–Petersson random hyperbolic surface with high probability. Recently, Hide, Macera, and Thomas [HMT25b] used the polynomial method developed in [CGVTvH24;

[MPvH25] to give a polynomial error term in the result of Anantharaman and Monk. Specifically, they show that there exists a constant  $c > 0$  such that a genus  $g$  closed hyperbolic surface satisfies  $\lambda_1(X) \geq \frac{1}{4} - O(\frac{1}{g^c})$  with high probability. In [HWX26], He, Wu, and Xue show a uniform spectral gap for hyperbolic surfaces with genus  $g$  and  $n = O(g^\alpha)$  cusps for  $\alpha \in [0, \frac{1}{2})$ . For the Brooks–Markover model, Shen and Wu [SW25] proved the nearly optimal spectral gap  $1/4 - \epsilon$  using the polynomial method.

There has also been some other work outside of spectral gaps. In [Nau26], Naud studied random degree  $n$  covers of compact hyperbolic surfaces. He showed that the variance of their smooth spectral statistics obeys the universal law of either GOE or GUE random matrices in the large  $n$  limit and small window limit. This work was generalized to negatively curved surfaces by Moy [Moy24].

**1.2. Proof idea.** For simplicity, we assume  $\Lambda$  and  $\Lambda_0$  are bounded. In §2.2 and §3.1, we construct a test function  $h_\Lambda \circ f_{\Lambda_0}$  that approximates the indicator function of an interval of eigenvalues up to scale  $q^{-1}$ . The function  $h_\Lambda \circ f_{\Lambda_0}$  is sufficiently nice so that we can apply the Selberg trace formula and the polynomial method. For our purpose,  $q$  will be taken to be  $q = n^c$  for some  $0 < c < 1$ . Then with some work involving Chebyshev’s inequality in §3.1, we reduce the proof of Theorem 1 to following statement of Proposition 3.1:

$$\mathbb{E} \left( \frac{1}{n} \text{tr} \left( (h_\Lambda \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n}} - 1/4 \right) \right) - \frac{\text{Vol}(X_n)}{2\pi n} \int_0^\infty (h_\Lambda \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \lesssim \Lambda_0^2 \frac{q^{2\kappa}}{n}.$$

We prove Proposition 3.1 in §3.2. The starting point of the proof of Proposition 3.1 is the twisted Selberg trace formula for a random cover  $X_n \rightarrow X$ :

$$\begin{aligned} \sum_j \hat{\phi} \left( \sqrt{\lambda_j(X_n)} - \frac{1}{4} \right) &= \frac{n \text{Vol}(X)}{2\pi} \int_0^\infty \hat{\phi}(r) r \tanh(\pi r) dr \\ &+ \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^\infty \frac{\ell_\gamma(X)}{2 \sinh \left( \frac{k\ell_\gamma(X)}{2} \right)} \phi(k\ell_\gamma(X)) \text{tr}(\rho(\gamma^k)), \end{aligned} \tag{1.6}$$

where  $\rho$  is a permutation representation of the fundamental group of  $X$  that encodes the cover  $X_n \rightarrow X$ . For the complete statement of the formula, see Lemma 2.1. From the trace formula (1.6), we see that, in order to prove Proposition 3.1, it suffices to show

$$E_X(1/n) := \mathbb{E} \left( \frac{1}{n} \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^\infty \frac{\ell_\gamma(X)}{2 \sinh \left( \frac{k\ell_\gamma(X)}{2} \right)} (h_\Lambda \circ f_{\Lambda_0})^\vee(k\ell_\gamma(X)) \text{tr}(\rho(\gamma^k)) \right)^2 \lesssim \Lambda_0^2 \frac{q^{2\kappa}}{n}. \tag{1.7}$$

The main technical ideas in the proof of (1.7) come from the generalized polynomial method of [MPvH25]. For more on the polynomial method, see [vHan25]. Roughly speaking, we

treat the expectation  $E_X(1/n)$  in (1.7) as a polynomial of  $1/n$  and use Markov brothers' inequality (3.19) to prove a bound on the derivative of this polynomial. More specifically, our proof uses the following three steps.

(1) In §3.2.1, we use the spectral side of the Selberg trace formula (1.6) to show

$$E_X(1/n) \lesssim \Lambda_0^2. \quad (1.8)$$

(2) In §3.2.2, we estimate  $\mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1}))\text{tr}(\rho(\gamma_2^{k_2}))]$  to prove that for  $n \geq q^\kappa$  for some  $\kappa > 2$ ,  $n^2 E_X(1/n)$  is well-approximated by a polynomial  $p(1/n)$  with  $\deg p(x) \lesssim q$ .

(3) Finally, in §3.2.3, we apply the Markov brothers' inequality (3.19) to conclude

$$\sup_{[0, \frac{1}{2q^\kappa}]} |(x^2 p(x))'| \lesssim q^{2\kappa} \sup_{n > q^\kappa} \left| \frac{1}{n^2} p(1/n) \right| \lesssim \Lambda_0^2 q^{2\kappa}, \quad n > q^\kappa$$

and

$$E_X(1/n) \approx \frac{1}{n^2} p(1/n) \lesssim \frac{1}{n} \sup_{[0, \frac{1}{2q^\kappa}]} |(x^2 p(x))'| \lesssim \Lambda_0^2 \frac{q^{2\kappa}}{n}, \quad n > 2q^\kappa.$$

This implies (1.7) for  $n > 2q^\kappa$ . On the other hand, the case for  $n \leq 2q^\kappa$  follows from (1.8).

**Notation.** We say  $A \lesssim B$  or  $A = O(B)$  if there exists a constant  $C > 0$  such that  $A \leq CB$ . To emphasize the dependence of the constant on a parameter  $\alpha$ , we write  $\lesssim_\alpha$  or  $O_\alpha(\cdot)$ . In this paper, the constant  $C$  usually depends on the hyperbolic surface  $X$ , but does not depend on the degree of the cover,  $n \in \mathbb{N}$ . The value of  $C$  may vary from line to line. We use  $\text{tr}A$  for the trace of a matrix or a linear map  $A$ . We do not use the normalized trace in this paper.

For  $f \in \mathcal{S}(\mathbb{R})$ , we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

and its inverse Fourier transform by

$$\check{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\xi) e^{ix\xi} d\xi.$$

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## 2. PRELIMINARIES

We recall the necessary preliminaries in this section.

**2.1. Surface group.** Let  $X = \Gamma_g \backslash \mathbb{H}$  be a closed orientable surface of genus  $g$ . Then  $\Gamma_g$  is a surface group with  $2g$  generators:

$$\Gamma_g = \langle a_1, a_2, \dots, a_{2g-1}, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] = 1 \rangle.$$

We now define the degree  $n$  covers of  $X$ . Set

$$\mathbb{X}_{g,n} := \text{Hom}(\Gamma_g, S_n).$$

This set is finite and endowed with the uniform probability measure. Let  $\varphi_n : \Gamma_g \rightarrow S_n$  be a permutation representation, where  $S_n$  is the symmetric group on  $[n] := \{1, \dots, n\}$ . The action of  $\Gamma_g$  on  $\mathbb{H} \times [n]$  is given by

$$\gamma \cdot (z, i) = (\gamma z, \varphi_n(\gamma)(i)).$$

Then

$$X_n := (\Gamma_g)_{\varphi_n} \backslash (\mathbb{H} \times [n])$$

is a *degree  $n$  covering surface*.

Now define  $V_n := \ell^2([n])$ . Denote by  $\text{std}_n$  the standard representation of the symmetric group  $S_n$  of permutation matrices. Note that  $\text{std}_n$  acts on  $V_n$ . We compose  $\varphi_n$  and  $\text{std}_n$  to obtain a representation of  $\Gamma_g$  on  $V_n$ :

$$\rho_{\varphi_n} := \text{std}_n \circ \varphi_n : \Gamma_g \rightarrow \text{End}(V_n). \quad (2.1)$$

Oftentimes, we will drop the  $\varphi_n$  subscript and use  $\rho = \rho_{\varphi_n}$ .

**2.2. Selberg Trace Formula.** Every closed oriented geodesic  $\gamma$  in  $X$  determines a non-trivial conjugacy class  $[\tilde{\gamma}] \subset \Gamma$ .

**Notation 1.** By abuse of notation, we use  $\gamma$  to denote both the geodesic and the element of the conjugacy class  $[\tilde{\gamma}]$  with the shortest representing word in the generators.

We have the following twisted Selberg trace formula. For completeness, we include a proof here.

**Lemma 2.1.** *Suppose  $\varphi_n : \Gamma \rightarrow S_n$  is a permutation representation with  $\rho = \rho_{\varphi_n}$ . Let  $X$  be a compact hyperbolic surface and  $X_\rho = (\Gamma_g)_\rho \backslash (\mathbb{H} \times [n])$  a degree  $n$  covering surface. Let  $\{\lambda_j^\rho\}$  be the eigenvalues of  $X_\rho$ . Then for any even function  $\phi \in C_c^\infty(\mathbb{R})$ ,*

$$\begin{aligned} \sum_j \hat{\phi} \left( \sqrt{\lambda_j^\rho - \frac{1}{4}} \right) &= \frac{n \operatorname{Vol}(X)}{2\pi} \int_0^\infty \hat{\phi}(r) r \tanh(\pi r) dr \\ &+ \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^\infty \frac{\ell_\gamma(X)}{2 \sinh\left(\frac{k\ell_\gamma(X)}{2}\right)} \phi(k\ell_\gamma(X)) \operatorname{tr}(\rho(\gamma^k)), \end{aligned} \quad (2.2)$$

where  $\mathcal{P}(X)$  denotes the set of primitive oriented closed geodesics on  $X$ ,  $\hat{\phi}$  denotes the Fourier transform of  $\phi$ , and  $\ell_\gamma(X)$  denotes the length of the closed geodesic  $\gamma$  on  $X$ .

*Proof.* We recall the standard Selberg trace formula on  $X_\rho$  (see, for example, [Bor16, Theorem 3.4]).

$$\begin{aligned} \sum_j \hat{\phi} \left( \sqrt{\lambda_j^\rho - \frac{1}{4}} \right) &= \frac{\operatorname{Vol}(X_\rho)}{2\pi} \int_0^\infty \hat{\phi}(r) r \tanh(\pi r) dr \\ &+ \sum_{\delta \in \mathcal{P}(X_\rho)} \sum_{m=1}^\infty \frac{\ell_\delta(X_\rho)}{2 \sinh\left(\frac{m\ell_\delta(X_\rho)}{2}\right)} \phi(m\ell_\delta(X_\rho)). \end{aligned}$$

Let  $\pi : X_\rho \rightarrow X$  be the natural covering map. It suffices to show that for all  $\gamma \in \mathcal{P}(X)$  and  $k \in \mathbb{N}$ ,

$$\sum_{\substack{\delta^m \in \pi^{-1}(\gamma^k), \\ \delta \in \mathcal{P}(X_\rho)}} \ell_\delta(X_\rho) = \ell_\gamma(X) \operatorname{tr}(\rho(\gamma^k)).$$

If  $\gamma \in \mathcal{P}(X)$ ,  $\delta \in \mathcal{P}(X_\rho)$  and  $\pi(\delta^m) = \gamma^k$ , then  $m$  divides  $k$ . Suppose  $\delta$  starts at a point  $y_0$  with  $\pi(y_0) = x_0$ . Then  $\delta$  goes through  $k/m$  copies of  $\mathbb{H}$  since  $X_\rho$  is the quotient of  $n$  copies of  $\mathbb{H}$  by the action of  $\Gamma$  encoded by  $\rho$ . Therefore, the number of such  $\delta$ 's corresponds to the number of period  $k/m$  orbits in the action  $\varphi_n(\gamma) \in S_n$  acting on  $n$  points. We have

$$\#\{\text{length } k/m \text{ orbits in the action } \varphi_n(\gamma) \in S_n\} = \#\{\delta \in \mathcal{P}(X_\rho) \mid \pi(\delta^m) = \gamma^k\}.$$

Therefore,

$$\begin{aligned}
\mathrm{tr}(\rho(\gamma^k)) &= \#\mathrm{Fix}(\varphi_n(\gamma^k)) = \sum_{m|k} \sum_{\substack{k/m \text{ orbits in the} \\ \text{action } \varphi_n(\gamma) \in S_n}} \frac{k}{m} \\
&= \sum_{m|k} \sum_{\substack{\delta \in \mathcal{P}(X_\rho), \\ \pi(\delta^m) = \gamma^k}} \frac{\ell_\delta(X_\rho)}{\ell_\gamma(X)} \\
&= \sum_{\delta^m \in \pi^{-1}(\gamma^k)} \frac{\ell_\delta(X_\rho)}{\ell_\gamma(X)},
\end{aligned}$$

which completes the proof.  $\square$

The work on spectral gaps in [MNP22] looked at a similar Selberg trace formula, subtracting the eigenvalues from the base surface  $X$ . As we are studying the eigenvalue counting function instead of the first new eigenvalue on  $X_\rho$ , we don't need to subtract the eigenvalues from the base surface.

For our application of the Selberg trace formula (2.2), we want to use a  $\hat{\phi}$  that approximates an interval of eigenvalues and is suited to the polynomial method. We now construct such a  $\hat{\phi}$ .

We begin with the function  $f : \mathbb{R} \cup i\mathbb{R} \rightarrow \mathbb{R}$  from [HMT25b, §2]. Specifically,  $f$  has the following properties:

- (1)  $f$  is smooth.
- (2)  $f$  is non-negative on  $\mathbb{R} \cup i\mathbb{R}$ .
- (3)  $\check{f}$  is smooth, even, non-negative, and supported in  $[-1, 1]$ . Therefore,  $f$  is strictly increasing on  $t \in [0, \infty) \mapsto f(ti)$ ,  $f(0) > 0$ , and  $0 \leq f([0, \infty)) \leq f(0)$ .
- (4) From the smoothness of  $\check{f}$ , we know

$$|f(x)| \leq C_N(1 + |x|)^{-N}, \quad x \in \mathbb{R} \tag{2.3}$$

for any  $N \in \mathbb{N}$ .

- (5) Since

$$f''(0) = \int_{\mathbb{R}} -\xi^2 \check{f}(\xi) d\xi < 0,$$

there exists  $c_0 \in (0, 1/2)$  such that

$$\begin{aligned}
-Cx \leq f'(x) \leq -c_0x, \quad 0 < f(4c_0) \leq f(x) \leq f(0) - c_0x^2, \quad x \in [0, 4c_0], \\
f(x) \leq f(4c_0), \quad x > 4c_0.
\end{aligned} \tag{2.4}$$

We will use a rescaled version of  $f$ :

$$f_{\Lambda_0}(x) := f(c_0\Lambda_0^{-1/2}x), \quad \Lambda_0 \in [1/4, \infty).$$

Note that  $f_{\Lambda_0}(\sqrt{\Delta_{X_n} - \frac{1}{4}})$  is a bounded operator. The spectrum of  $f_{\Lambda_0}(\sqrt{\Delta_{X_n} - \frac{1}{4}})$  is contained in  $[0, f_{\Lambda_0}(\frac{i}{2})]$ , where  $[f_{\Lambda_0}(0), f_{\Lambda_0}(\frac{i}{2})]$  corresponds to the eigenvalues of  $\Delta_{X_n}$  below  $\frac{1}{4}$  and  $f_{\Lambda_0}([0, \infty))$  corresponds to the eigenvalues of  $\Delta_{X_n}$  above  $\frac{1}{4}$ .

When we apply (2.2), we will set  $\hat{\phi}(x) = h \circ f_{\Lambda_0}(x)$ , where  $h(x)$  is a polynomial of degree  $q \leq n$  such that  $h(x) = x\tilde{h}(x)$ . We use the notation

$$\|\tilde{h}\| := \sup_{x \in [0, f(i/2)]} |\tilde{h}(x)|. \quad (2.5)$$

For later use, we prove the following lemma.

**Lemma 2.2.** *We have*

$$\frac{1}{n} \left| \text{tr} h \circ f_{\Lambda_0}(\sqrt{\Delta_{X_n} - 1/4}) \right| \lesssim \frac{1}{n} \left| \text{tr} f_{\Lambda_0}(\sqrt{\Delta_{X_n} - 1/4}) \right| \|\tilde{h}\| \lesssim \Lambda_0 \|\tilde{h}\|. \quad (2.6)$$

*Proof.* Since  $h(x) = x\tilde{h}(x)$ , we have  $h \circ f_{\Lambda_0} = f_{\Lambda_0} \cdot (\tilde{h} \circ f_{\Lambda_0})$ . Since the norm of  $\tilde{h} \circ f_{\Lambda_0}(\sqrt{\Delta_{X_n} - 1/4})$  is bounded by  $\|\tilde{h}\|$ , it suffices to show

$$\frac{1}{n} \left| \text{tr} f_{\Lambda_0}(\sqrt{\Delta_{X_n} - 1/4}) \right| \lesssim \Lambda_0.$$

Due to the twisted Selberg trace formula (2.2), it suffices to show that

$$\int_0^\infty f_{\Lambda_0}(r) r \tanh(\pi r) dr \lesssim \Lambda_0 \quad (2.7)$$

and

$$\frac{1}{n} \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^\infty \frac{\ell_\gamma(X)}{2 \sinh\left(\frac{k\ell_\gamma(X)}{2}\right)} \check{f}_{\Lambda_0}(k\ell_\gamma(X)) \text{tr}(\rho(\gamma^k)) \lesssim \Lambda_0^{1/2}. \quad (2.8)$$

We note that (2.7) follows from the rapid decay of  $f$  in (2.3) and (2.8) follows from

$$\text{tr}(\rho(\gamma^k)) \leq n, \quad |\check{f}_{\Lambda_0}| \lesssim \Lambda_0^{1/2}, \quad \text{supp } \check{f}_{\Lambda_0} \subset [-c_0\Lambda_0^{-1/2}, c_0\Lambda_0^{-1/2}]. \quad (2.9)$$

□

### 3. PROOF OF THEOREM 1

In this section, we prove Theorem 1 following the outline in § 1.2.

**3.1. The proof of Theorem 1.** We prove Theorem 1 assuming the following proposition. Recall the definition of  $f_{\Lambda_0}$  from §2.2 and that  $h$  is a polynomial of degree  $q \leq n$  such that  $h(x) = x\tilde{h}(x)$ . Using the notation (2.5), we denote

$$\|\tilde{h}\|_{C^\kappa} = \sum_{0 \leq j \leq \kappa} \|\partial_x^j \tilde{h}\|.$$

**Proposition 3.1.** *There exists  $\kappa = \kappa(g) > 2$  and  $C = C(X) > 0$  such that, for any  $\Lambda_0 \in [C, \infty)$ , we have*

$$\mathbb{E} \left( \frac{1}{n} \operatorname{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \leq C \frac{\Lambda_0^2 q^{2\kappa}}{n} \|\tilde{h}\|^2. \quad (3.1)$$

**Remark 3.2.** *The estimate (3.1) can be formulated differently, see (3.3) and (3.5) below.*

We prove Proposition 3.1 in §3.2.

*Proof of Theorem 1.* We first rephrase (3.1) in a slightly different form. Let  $h(x) = x\tilde{h}(x)$  be a polynomial of degree  $q$ . By (2.9) we have

$$\operatorname{supp}(h \circ f_{\Lambda_0})^\vee \subset [-c_0 q \Lambda_0^{-1/2}, c_0 q \Lambda_0^{-1/2}]. \quad (3.2)$$

For  $C \leq \Lambda_0 \leq \left( \frac{c_0 q}{\min \ell_\gamma(X)} \right)^2$  and  $K \geq 0$ , by Proposition 3.1 we have

$$\begin{aligned} \mathbb{E} \left( \frac{1}{n} \operatorname{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\ \leq C_K \frac{q^{2(\kappa+2+K)}}{n \Lambda_0^K} \|\tilde{h}\|^2. \end{aligned} \quad (3.3)$$

On the other hand, using the Selberg trace formula (2.2) and (3.2), for  $\Lambda_0 > \left( \frac{c_0 q}{\min \ell_\gamma(X)} \right)^2$  we have

$$\frac{1}{n} \operatorname{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) = \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr. \quad (3.4)$$

Therefore, (3.3) holds for all  $\Lambda_0 \in [1/4, \infty)$ .

Let  $\tilde{h}(x)$  be a polynomial of degree  $q - 1$ . We write  $\tilde{h}(x) = \sum_{j=0}^{q-1} a_j T_j(2f(i/2)^{-1}x - 1)$ , where  $T_j$  is the  $j$ -th Chebyshev polynomial as in [CGVTvH24, (4.1)]. Then we have

$$\begin{aligned} & \mathbb{E} \left( \frac{1}{n} \operatorname{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\ & \leq \frac{C_K}{n\Lambda_0^K} \left( \sum_{j=0}^{q-1} (j+1)^{\kappa+2+K} |a_j| \right)^2 \\ & \leq \frac{C_K}{n\Lambda_0^K} \|\tilde{h}\|_{C^{\kappa+3+K}}^2, \end{aligned} \quad (3.5)$$

where the first step is due to the Minkowski inequality and (3.3), and the second step is due to [CGVTvH24, Corollary 4.5]. By approximating smooth functions by polynomials, we note that (3.5) works for general smooth functions  $h(x) = x\tilde{h}(x)$ ,  $\tilde{h}(x) \in C^\infty([0, f(i/2)])$ .

Let  $\alpha_0 = \frac{1}{3(\kappa+3+K)}$  where  $K = \lfloor \frac{\kappa+5}{2\epsilon} \rfloor + 1$ . Let  $\Lambda \geq 1/4$  and  $\epsilon > 0$ . We take  $\Lambda_0 = \Lambda$  for  $\Lambda \geq C$  and  $\Lambda_0 = C$  for  $\Lambda \in [1/4, C]$ . By [DZ16, Lemma 3.3], we can take a smooth cutoff  $h_{\Lambda, \epsilon}(x)$  such that

$$h_{\Lambda, \epsilon}(x) = \begin{cases} 1, & x \in [f_{\Lambda_0}(\sqrt{\Lambda - 1/4}), f(i/2)], \\ 0, & x \in [0, f_{\Lambda_0}(\sqrt{\Lambda + \Lambda^{1/2+\epsilon}n^{-\alpha_0} - 1/4})] \end{cases} \quad (3.6)$$

and moreover,

$$0 \leq h_{\Lambda, \epsilon}(x) \leq 1, \quad h_{\Lambda, \epsilon}(x) = x\tilde{h}_{\Lambda, \epsilon}(x), \quad |\tilde{h}_{\Lambda, \epsilon}^{(j)}(x)| \leq C_j \Lambda^{j(1/2-\epsilon)} n^{j\alpha_0}, \quad x \in [0, f(i/2)], \quad (3.7)$$

where the last estimate is due to (2.4) and the mean value theorem:

$$\begin{aligned} & f_{\Lambda_0}(\sqrt{\Lambda - 1/4}) - f_{\Lambda_0}(\sqrt{\Lambda + \Lambda^{1/2+\epsilon}n^{-\alpha_0} - 1/4}) \\ & \geq C^{-1} \Lambda_0^{-1/2} \sqrt{\Lambda - 1/4} \left( \Lambda_0^{-1/2} \sqrt{\Lambda + \Lambda^{1/2+\epsilon}n^{-\alpha_0} - 1/4} - \Lambda_0^{-1/2} \sqrt{\Lambda - 1/4} \right) \\ & \geq C^{-1} \Lambda^{-1/2+\epsilon} n^{-\alpha_0}. \end{aligned}$$

Applying (3.5) with  $K = \lfloor \frac{\kappa+5}{2\epsilon} \rfloor + 1$ , we have

$$\begin{aligned} & \mathbb{E} \left( \frac{1}{n} \operatorname{tr} \left( (h_{\Lambda, \epsilon} \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h_{\Lambda, \epsilon} \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\ & \leq \frac{C}{n\Lambda_0^K} \|\tilde{h}_{\Lambda, \epsilon}\|_{C^{\kappa+3+K}}^2 \leq \frac{C}{n\Lambda_0^K} \Lambda^{2(\kappa+3+K)(1/2-\epsilon)} n^{2(\kappa+3+K)\alpha_0} \leq C \Lambda^{-2} n^{-1/3}, \end{aligned}$$

where the second to last step follows from (3.7).

By Chebyshev's inequality, there is a set of degree  $n$  covers  $\Omega_n(X, \Lambda, \epsilon) \subset \mathbb{X}_{g,n}$  with probability (where  $C_1 > 0$  will be determined later in (3.12))

$$\mathbb{P}(\Omega_n(X, \Lambda, \epsilon)) \geq 1 - (C_1 \Lambda)^{-2} n^{-1/9}, \quad (3.8)$$

such that for  $X_n \in \Omega_n(X, \Lambda, \epsilon)$ ,

$$\left| \frac{1}{n} \operatorname{tr} \left( (h_{\Lambda, \epsilon} \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h_{\Lambda, \epsilon} \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right| \lesssim n^{-1/9}. \quad (3.9)$$

By (3.9), we have

$$\begin{aligned} \frac{1}{n} N_{X_n}(\Lambda) &\leq \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h_{\Lambda, \epsilon} \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr + C n^{-1/9} \\ &\leq (2g-2) \int_0^{\sqrt{\Lambda + \Lambda^{1/2} + \epsilon n^{-\alpha_0} - 1/4}} r \tanh(\pi r) dr + C n^{-1/9} \\ &\leq (2g-2) \int_0^{\sqrt{\Lambda - 1/4}} r \tanh(\pi r) dr + C \Lambda^{1/2 + \epsilon} n^{-\alpha_0} + C n^{-1/9}. \end{aligned} \quad (3.10)$$

In particular, we know (1.4) for  $\Lambda \leq 1/4 + n^{-\alpha_0}$ . On the other hand, for  $\Lambda \geq 1/4 + n^{-\alpha_0}$ , by (3.9) we have

$$\begin{aligned} \frac{1}{n} N_{X_n}(\Lambda) &\geq \frac{\operatorname{Vol}(X_n)}{2\pi n} \int_0^\infty (h_{\Lambda - \Lambda^{1/2} + \epsilon n^{-\alpha_0}, \epsilon} \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr - C n^{-1/9} \\ &\geq (2g-2) \int_0^{\sqrt{\Lambda - \Lambda^{1/2} + \epsilon n^{-\alpha_0} - 1/4}} r \tanh(\pi r) dr - C n^{-1/9} \\ &\geq (2g-2) \int_0^{\sqrt{\Lambda - 1/4}} r \tanh(\pi r) dr - C \Lambda^{1/2 + \epsilon} n^{-\alpha_0} - C n^{-1/9}. \end{aligned} \quad (3.11)$$

Combining (3.10) and (3.11), we conclude (1.4) for  $X_n \in \Omega_n(X, \Lambda, \epsilon)$ . Now we take  $\Lambda(j) := C2^j$  for  $j \geq 1$ , and  $\Lambda(0) = C$ . Let

$$\Lambda(j, \ell) := \Lambda(j) + \ell n^{-0.01} \Lambda(j)^{1/2} \in [\Lambda(j), \Lambda(j+1)], \quad 0 \leq \ell \leq L_j := \left\lceil n^{0.01} \frac{\Lambda(j+1) - \Lambda(j)}{\Lambda(j)^{1/2}} \right\rceil,$$

and

$$\Omega_n(X, \epsilon) := \bigcap_{j=0}^{\infty} \bigcap_{\ell=0}^{L_j} \Omega_n(X, \Lambda(j, \ell), \epsilon).$$

Taking  $C_1$  in (3.8) sufficiently large, we have

$$\begin{aligned}
\mathbb{P}(\Omega_n(X, \epsilon)) &\geq 1 - \sum_{j=0}^{\infty} \sum_{\ell=0}^{L_j} (C_1 \Lambda(j, \ell))^{-2} n^{-1/9} \\
&\geq 1 - \sum_{j=0}^{\infty} (C_1 \Lambda(j))^{-2} C n^{0.01} 2^{j/2} n^{-1/9} \\
&\geq 1 - n^{-1/10}.
\end{aligned} \tag{3.12}$$

Now (1.4) holds for all  $\Lambda(j, \ell)$  and  $X_n \in \Omega_n(X, \epsilon)$ . It follows that (1.4) holds for all  $\Lambda$ , and the estimate (1.2) is a direct corollary of (1.4).  $\square$

The remainder of this paper is dedicated to proving Proposition 3.1.

**3.2. Proof of Proposition 3.1.** By (2.2), we need to study

$$\begin{aligned}
&\mathbb{E} \left( \frac{1}{n} \text{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\text{Vol}(X_n)}{2\pi n} \int_0^{\infty} (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\
&= \mathbb{E} \left( \frac{1}{n} \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{\ell_{\gamma}(X)}{2 \sinh \left( \frac{k \ell_{\gamma}(X)}{2} \right)} (h \circ f_{\Lambda_0})^{\vee}(k \ell_{\gamma}(X)) \text{tr}(\rho(\gamma^k)) \right)^2 \\
&= \frac{1}{n^2} \sum_{\gamma_1, \gamma_2 \in \mathcal{P}(X)} \sum_{k_1, k_2=1}^{\infty} \left( \frac{\ell_{\gamma_1}(X) \ell_{\gamma_2}(X)}{4 \sinh \left( \frac{k_1 \ell_{\gamma_1}(X)}{2} \right) \sinh \left( \frac{k_2 \ell_{\gamma_2}(X)}{2} \right)} (h \circ f_{\Lambda_0})^{\vee}(k_1 \ell_{\gamma_1}(X)) \right. \\
&\quad \left. \cdot (h \circ f_{\Lambda_0})^{\vee}(k_2 \ell_{\gamma_2}(X)) \mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1})) \text{tr}(\rho(\gamma_2^{k_2}))] \right).
\end{aligned} \tag{3.13}$$

**3.2.1. Uniform bound.** We first claim

$$(3.13) \lesssim \Lambda_0^2 \|\tilde{h}\|^2. \tag{3.14}$$

Using the spectral side of (3.13), we have

$$\begin{aligned}
(3.13) &= \mathbb{E} \left( \frac{1}{n} \text{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) - \frac{\text{Vol}(X)}{2\pi} \int_0^{\infty} (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\
&\leq 2\mathbb{E} \left( \frac{1}{n} \text{tr} \left( (h \circ f_{\Lambda_0}) \left( \sqrt{\Delta_{X_n} - 1/4} \right) \right) \right)^2 + 2\mathbb{E} \left( \frac{\text{Vol}(X)}{2\pi} \int_0^{\infty} (h \circ f_{\Lambda_0})(r) r \tanh(\pi r) dr \right)^2 \\
&\lesssim \Lambda_0^2 \|\tilde{h}\|^2.
\end{aligned}$$

The final inequality follows from the fact that the first term is bounded by Lemma 2.2 and the second term is uniformly bounded by

$$\left( \frac{\text{Vol}(X)}{2\pi} \int_0^\infty f_{\Lambda_0}(r) r \tanh(\pi r) dr \right)^2 \|\tilde{h}\|^2 \lesssim \Lambda_0^2 \|\tilde{h}\|^2.$$

3.2.2. *Polynomial expansion.* In [MPvH25, §4], the authors examined  $\mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1}))]$ . We adapt the arguments from [MPvH25, §4] to study

$$\mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1})) \text{tr}(\rho(\gamma_2^{k_2}))].$$

**Lemma 3.3.** *There exists  $C = C(g) > 2$  such that for  $n^{-1} \in [0, (Cq)^{-C}]$  and  $\gamma_1, \gamma_2 \in \Gamma_g \setminus \{\text{id}\}$  with word length  $\leq q$ ,*

$$\mathbb{E}[\text{tr}(\rho(\gamma_1)) \text{tr}(\rho(\gamma_2))] = \frac{Q_{\gamma_1, \gamma_2}(1/n)}{Q_{\text{id}}(1/n)} + O((Cq)^{Cq} n^{-q}),$$

where  $Q_{\gamma_1, \gamma_2}, Q_{\text{id}}$  are polynomials with  $\deg(Q_{\gamma_1, \gamma_2}) \leq 9q(4g+1)$ ,  $\deg(Q_{\text{id}}) \leq 9q(4g+1) + 1$  and  $Q_{\text{id}}(1/n) \in [C^{-1}, C]$  for  $n \geq q^C$ .

*Proof.* 1. Recall that  $\mathbb{X}_{g,n} := \text{Hom}(\Gamma_g, S_n)$  and  $\Gamma_g$  is a surface group with  $2g$  generators:  $\Gamma_g = \langle a_1, a_2, \dots, a_{2g-1}, a_{2g} \mid [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] = 1 \rangle$ .

Let  $\gamma_1 = a_{i_1}^{\alpha_1} a_{i_2}^{\alpha_2} \cdots a_{i_m}^{\alpha_m}$ ,  $\gamma_2 = a_{j_1}^{\beta_1} a_{j_2}^{\beta_2} \cdots a_{j_l}^{\beta_l} \in \Gamma_g$ , where  $i_k, j_k \in [2g]$  and  $\alpha_k, \beta_k \in \{-1, 1\}$ . We assume that the above expressions of  $\gamma_1, \gamma_2$  in terms of the generators are cyclically reduced. We no longer assume that the corresponding closed geodesics are primitive; they may be powers of primitive geodesics. Let the word length of these expressions be  $|\gamma_1|, |\gamma_2| \leq q$ .

We define the graph  $C_{\gamma_1, \gamma_2}$  to be the graph of two disjoint cycles of length  $m$  and  $l$ , where the cycle of length  $m$  has directed edges labeled by  $a_{i_1}^{\alpha_1}, a_{i_2}^{\alpha_2}, \dots, a_{i_m}^{\alpha_m}$  and the cycle of length  $l$  has edges labeled by  $a_{j_1}^{\beta_1}, a_{j_2}^{\beta_2}, \dots, a_{j_l}^{\beta_l}$ . Specifically, the cycle of length  $l$  has an edge labeled by  $\alpha_k$  from the  $k$ -th to the  $(k+1)$ -th vertex when  $\alpha_k = 1$  and from the  $(k+1)$ -th to the  $k$ -th vertex when  $\alpha_k = -1$ . The cycle of length  $m$  is labeled using the same method.

The paper [MPvH25] uses  $C_\gamma$ , a graph consisting of a single loop, in lieu of  $C_{\gamma_1, \gamma_2}$ . Our use of  $C_{\gamma_1, \gamma_2}$  is what permits the study of  $\mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1})) \text{tr}(\rho(\gamma_2^{k_2}))]$  instead of  $\mathbb{E}[\text{tr}(\rho(\gamma_1^{k_1}))]$ .

Let  $\mathcal{R}$  denote the collection of surjective labeled-graph morphisms

$$r : C_{\gamma_1, \gamma_2} \twoheadrightarrow W_r$$

such that

- (1)  $W_r$  is *folded*, i.e., every vertex has at most one incoming  $a$ -labeled edge and at most one outgoing  $a$ -labeled edge for each  $a \in \{a_1, \dots, a_{2g}\}$ .

- (2) Every path in  $W_r$  that spells out an element of the free group  $F_{2g}$  in the kernel of  $F_{2g} \rightarrow \Gamma_g$  is closed.

Note that  $\#\mathcal{R} \leq (|\gamma_1| + |\gamma_2|)!$ .

2. Let  $\phi \in \mathbb{X}_{g,n}$ . We denote the Schreier graph by

$$X_\phi := \text{Schreier}(\{\phi(a_1), \dots, \phi(a_{2g})\}, [n]).$$

This is the graph on  $n$  vertices labeled  $1, \dots, n$ , where there is a directed edge from  $i$  to  $j$  exactly when there exists  $f \in \{a_1, \dots, a_{2g}\}$  such that  $\phi(f)[i] = j$ . Such an edge is labeled by  $f$ .

By construction, it is clear that any morphism of folded labeled graphs  $C_{\gamma_1, \gamma_2} \rightarrow X_\phi$  factors uniquely as a surjective morphism followed by an injective one, namely  $C_{\gamma_1, \gamma_2} \rightarrow W_r \hookrightarrow X_\phi$  for unique  $r \in \mathcal{R}$ .

For  $\phi \in \mathbb{X}_{g,n}$ , let  $\text{Fix}(\phi(\gamma))$  denote the number of fixed elements of  $[n]$  under the action of  $\phi(\gamma)$ . Therefore,

$$\begin{aligned} \mathbb{E}[\text{tr}(\rho(\gamma_1))\text{tr}(\rho(\gamma_2))] &= \frac{1}{\#\mathbb{X}_{g,n}} \sum_{\phi \in \mathbb{X}_{g,n}} [\text{Fix}(\phi(\gamma_1))\text{Fix}(\phi(\gamma_2))] \\ &= \frac{1}{\#\mathbb{X}_{g,n}} \sum_{\phi \in \mathbb{X}_{g,n}} [\# \text{ of morphisms } C_{\gamma_1, \gamma_2} \rightarrow X_\phi] \\ &= \frac{1}{\#\mathbb{X}_{g,n}} \sum_{r \in \mathcal{R}} \sum_{\phi \in \mathbb{X}_{g,n}} [\# \text{ of injective morphisms } W_r \hookrightarrow X_\phi]. \end{aligned}$$

Otherwise put, for

$$\mathbb{E}_n^{\text{emb}}(W_r) := \mathbb{E}_{\phi \in \mathbb{X}_{g,n}} [\# \text{ of injective morphisms } W_r \hookrightarrow X_\phi],$$

$$\mathbb{E}[\text{tr}(\rho(\gamma_1))\text{tr}(\rho(\gamma_2))] = \sum_{r \in \mathcal{R}} \mathbb{E}_n^{\text{emb}}(W_r).$$

3. We now estimate  $\mathbb{E}_n^{\text{emb}}(W_r)$ , closely following the arguments of [MPvH25, §4.4].

Let  $\zeta(s; S_n)$  be the Witten zeta function of  $S_n$ . Its definition is given on [MPvH25, pg. 18]. It is known that  $\zeta(s; S_n) \rightarrow 2$  as  $n \rightarrow \infty$ . From [MP23, (1.3)], we have that  $\#\mathbb{X}_{g,n} = (\#S_n)^{2g-1} \zeta(2g-2, S_n)$ .

It is shown on [MPvH25, pg. 26] that for  $|\gamma_1|, |\gamma_2| \leq q$  and  $n \geq 28q^2$ ,

$$\mathbb{E}_n^{\text{emb}}(W_r) = \frac{2}{\zeta(2g-2, S_n)} \left( \frac{p_r(n)}{(n)_{9q}^{1+4g}} + O((Cq)^{Cq} n^{-q}) \right).$$

Here  $(n)_{9q} = n(n-1) \cdots (n-9q+1)$  is the falling Pochhammer symbol and  $p_r$  is a polynomial in  $n$  with  $\deg(p_r) \leq 9q(4g+1) + 2$ . As noted above, [MPvH25] uses  $C_\gamma$  instead of  $C_{\gamma_1, \gamma_2}$ . However, their proofs hold as written when  $C_\gamma$  is replaced by  $C_{\gamma_1, \gamma_2}$ .

Since  $\#\mathcal{R} \leq (|\gamma_1| + |\gamma_2|)!$ , we conclude

$$\mathbb{E}[\mathrm{tr}(\rho(\gamma_1))\mathrm{tr}(\rho(\gamma_2))] = \frac{2}{\zeta(2g-2, S_n)} \left( \frac{p_{\gamma_1, \gamma_2}(n)}{(n)_{9q}^{1+4g}} + O((Cq)^{Cq} n^{-q}) \right), \quad (3.15)$$

where  $p_{\gamma_1, \gamma_2}$  is a polynomial in  $n$  with  $\deg(p_{\gamma_1, \gamma_2}) \leq 9q(4g+1) + 2$ .

4. It remains to transform (3.15) into a rational function in  $1/n$  with the same error term.

Let

$$g_q(t) := \prod_{k=0}^{9q-1} (1 - kt)^{1+4g}.$$

For  $t := n^{-1}$  and a polynomial  $P_{\gamma_1, \gamma_2}(t)$  of degree at most  $\deg(p_{\gamma_1, \gamma_2})$ ,

$$\frac{p_{\gamma_1, \gamma_2}(n)}{(n)_{9q}^{1+4g}} = t^{9q(1+4g)+2-\deg(p_{\gamma_1, \gamma_2})} \frac{P_{\gamma_1, \gamma_2}(t)}{t^2 g_q(t)} =: \frac{Q_{\gamma_1, \gamma_2}(t) + a_{-1}(\gamma_1, \gamma_2)t^{-1} + a_{-2}(\gamma_1, \gamma_2)t^{-2}}{g_q(t)}.$$

Note that for  $t \in [0, (Cq)^{-C}]$ ,  $C > 2$ , we have  $C^{-1} \leq g_q(t) \leq C$ . By [Nau26, Proposition 3.1], we have

$$\mathbb{E}[\mathrm{tr}(\rho(\gamma_1))\mathrm{tr}(\rho(\gamma_2))] \lesssim_q 1, \quad n \rightarrow \infty.$$

Therefore,  $a_{-1}(\gamma_1, \gamma_2) = a_{-2}(\gamma_1, \gamma_2) = 0$  and

$$Q_{\gamma_1, \gamma_2}(t) = O(1), \quad \text{for } t = n^{-1} \in [0, (Cq)^{-C}]. \quad (3.16)$$

Also note that  $\deg(Q_{\gamma_1, \gamma_2}) \leq 9q(4g+1)$ .

From [MPvH25, (4.19)], for a polynomial  $Q_{\mathrm{id}}(t)$  with  $\deg(Q_{\mathrm{id}}) \leq 9q(4g+1) + 1$ ,

$$\frac{2}{\zeta(2g-2, S_n)} = \frac{g_q(1/n)}{Q_{\mathrm{id}}(1/n)} (1 + O((Cq)^{Cq} n^{-q-1})).$$

Therefore,

$$\begin{aligned} \frac{1}{n} \mathbb{E}[\mathrm{tr}(\rho(\gamma_1))\mathrm{tr}(\rho(\gamma_2))] &= \frac{2}{\zeta(2g-2, S_n)} \left( \frac{Q_{\gamma_1, \gamma_2}(1/n)}{g_q(1/n)} + O((Cq)^{Cq} n^{-q}) \right) \\ &= \frac{Q_{\gamma_1, \gamma_2}(1/n)}{Q_{\mathrm{id}}(1/n)} (1 + O((Cq)^{Cq} n^{-q-1})) + O((Cq)^{Cq} n^{-q}) \\ &= \frac{Q_{\gamma_1, \gamma_2}(1/n)}{Q_{\mathrm{id}}(1/n)} + O((Cq)^{Cq} n^{-q}), \end{aligned}$$

which completes the proof.  $\square$

We conclude this subsection with the following corollary of Lemma 3.3.

**Corollary 3.4.** *There exist  $\kappa = \kappa(g) > 2$  and  $C = C(X) > 0$  such that the following holds. For  $n \geq q^\kappa$  and  $\Lambda_0 \geq C$ ,*

$$\begin{aligned} & \mathbb{E} \left( \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{\ell_{\gamma}(X)}{2 \sinh \left( \frac{k \ell_{\gamma}(X)}{2} \right)} (h \circ f_{\Lambda_0})^\vee(k \ell_{\gamma}(X)) \operatorname{tr} \rho(\gamma^k) \right)^2 \\ &= \frac{p(1/n)}{Q_{\text{id}}(1/n)} + O \left( \Lambda_0 (Cq)^{\kappa q} n^{-q} \|\tilde{h}\|^2 \right), \end{aligned}$$

where  $p$  is a polynomial of degree at most  $C \Lambda_0^{-1/2} q$  and  $Q_{\text{id}}(1/n) \in [C^{-1}, C]$  for  $n \geq q^\kappa$ .

*Proof.* Let

$$\begin{aligned} p(x) := & \sum_{\gamma_1, \gamma_2 \in \mathcal{P}(X)} \sum_{k_1, k_2=1}^{\infty} \frac{\ell_{\gamma_1}(X) \ell_{\gamma_2}(X)}{4 \sinh \left( \frac{k_1 \ell_{\gamma_1}(X)}{2} \right) \sinh \left( \frac{k_2 \ell_{\gamma_2}(X)}{2} \right)} \\ & \cdot (h \circ f_{\Lambda_0})^\vee(k_1 \ell_{\gamma_1}(X)) (h \circ f_{\Lambda_0})^\vee(k_2 \ell_{\gamma_2}(X)) Q_{\gamma_1^{k_1}, \gamma_2^{k_2}}(x). \end{aligned} \quad (3.17)$$

Recall Notation 1. From [MNP22, Lemma 2.3], we have the following relationship between length of geodesics and word length:

$$|\gamma| \leq K_1 \ell_{\gamma}(X) + K_2, \quad (3.18)$$

where  $K_1, K_2 > 0$  depend only on  $X$  and the choice of generators. Therefore, since  $\operatorname{supp}(h \circ f_{\Lambda_0})^\vee \subset [-c_0 \Lambda_0^{-1/2} q, c_0 \Lambda_0^{-1/2} q]$ , we know  $|\gamma^k| \leq C c_0 \Lambda_0^{-1/2} q$  for nonzero terms in the sum (3.17). Therefore,  $p(x)$  is a polynomial of degree at most  $C \Lambda_0^{-1/2} q$ .

By taking  $\Lambda_0 > C^2$  we may assume  $|\gamma^k| \leq C \Lambda_0^{-1/2} q \leq q$ . Then,

$$\begin{aligned} & \sum_{\gamma_1, \gamma_2 \in \mathcal{P}(X)} \sum_{k_1, k_2=1}^{\infty} \frac{\ell_{\gamma_1}(X) \ell_{\gamma_2}(X)}{4 \sinh \left( \frac{k_1 \ell_{\gamma_1}(X)}{2} \right) \sinh \left( \frac{k_2 \ell_{\gamma_2}(X)}{2} \right)} |(h \circ f_{\Lambda_0})^\vee(k_1 \ell_{\gamma_1}(X)) (h \circ f_{\Lambda_0})^\vee(k_2 \ell_{\gamma_2}(X))| \\ & \lesssim \#\{(\gamma, k) \in \mathcal{P}(X) \times \mathbb{N} : k \ell_{\gamma}(X) \leq q\}^2 \cdot \sup_{[0, q]} |(h \circ f_{\Lambda_0})^\vee(t)|^2 \\ & \lesssim e^{2q} \Lambda_0 \|\tilde{h}\|^2. \end{aligned}$$

Therefore, the error term in Lemma 3.3 remains of the same form up to a change of the constant  $C$ .  $\square$

3.2.3. *Markov brothers' inequality.* Finally, we finish the proof of Proposition 3.1 via the following elaboration of the Markov brothers' inequality [MPvH25, Lemma 2.1].

**Lemma 3.5.** *For every real polynomial  $P$  of degree at most  $q$  and every  $k \in \mathbb{N}$ ,*

$$\sup_{t \in [0, \frac{1}{2q^2}]} |P^{(k)}(t)| \leq \frac{2^{2k+1} q^{4k}}{(2k-1)!!} \sup_{n \geq q^2} \left| P\left(\frac{1}{n}\right) \right|, \quad (3.19)$$

where  $(2k-1)!! := (2k-1)(2k-3)\cdots 3 \cdot 1$ .

Now recall that in §3.2.1, we showed

$$\frac{1}{n^2} \mathbb{E} \left( \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{\ell_{\gamma(X)}}{2 \sinh\left(\frac{k\ell_{\gamma(X)}}{2}\right)} (h \circ f_{\Lambda_0})^\vee(k\ell_{\gamma(X)}) \text{tr} \rho(\gamma^k) \right)^2 \lesssim \Lambda_0^2 \|\tilde{h}\|^2. \quad (3.20)$$

From Corollary 3.4 and (3.20), we know

$$\frac{1}{n^2} |p(1/n)| \lesssim \Lambda_0^2 \|\tilde{h}\|^2, \quad n \geq Cq^\kappa. \quad (3.21)$$

We use the notation

$$\|f\|_{[a,b]} := \sup_{x \in [a,b]} |f(x)|.$$

From Corollary 3.4, Lemma 3.5, and (3.21), we know that

$$\|(x^2 p(x))'\|_{[0, \frac{1}{2Cq^\kappa}]} \lesssim q^{2\kappa} \sup_{n \geq Cq^\kappa} \frac{1}{n^2} |p(1/n)| \lesssim q^{2\kappa} \Lambda_0^2 \|\tilde{h}\|^2.$$

Taking the Taylor expansion, we have

$$\left| \frac{1}{n^2} \frac{p(1/n)}{Q_{\text{id}}(1/n)} \right| \lesssim \frac{1}{n} \|(x^2 p(x))'\|_{[0, \frac{1}{2Cq^\kappa}]} \lesssim \frac{q^{2\kappa} \Lambda_0^2}{n} \|\tilde{h}\|^2, \quad n \geq 2Cq^\kappa.$$

In other words, (3.13)  $\lesssim \frac{q^{2\kappa} \Lambda_0^2}{n} \|\tilde{h}\|^2$  for  $n \geq 2Cq^\kappa$ . This shows (3.1) for  $n \geq 2Cq^\kappa$ . On the other hand, for  $n \leq 2Cq^\kappa$ , (3.1) follows from (3.14). This finishes the proof of Proposition 3.1.

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