SPECTRAL GAP FOR SURFACES OF INFINITE VOLUME WITH NEGATIVE CURVATURE

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Abstract. We prove that the imaginary parts of scattering resonances for negatively curved asymptotically hyperbolic surfaces are uniformly bounded away from zero and provide a resolvent bound in the resulting resonance-free strip. This provides an essential spectral gap without the pressure condition. This is done by adapting the methods of [NSZ11], [Vas13a] and [Vac22] and answers a question posed in [DyZa16].

1. Introduction

In a seminal paper Bourgain–Dyatlov [BoDy18] showed that a convex cocompact hyperbolic surface has an essential spectral gap between the unitarity axis and the set of scattering resonances. This means the Selberg zeta function has only finitely many zeros in the region $\operatorname{Re} s > 1/2 - \beta$ for some $\beta > 0$. This holds without any assumptions on the Hausdorff dimension of the trapped set, in particular without a "pressure condition" which in this case goes back to the works of Patterson and Sullivan [Pa76, Su79]. The purpose of this note is to generalize this result to negatively curved surfaces which are asymptotically hyperbolic in a sense described below. This is done by combining the quantum monodromy method of Nonnenmacher–Sjöstrand–Zworski [NSZ11] and Vasy's method for meromorphic continuation [Vas13a, Zw16, DyZw19] with the recent work of Vacossin [Vac22]. It answers a question posed by Dyatlov–Zahl [DyZa16] in the first paper on the fractal uncertainty principle.

Let X be an even asymptotically hyperbolic manifold. This means that X has a compactification $\overline{X}$, which is a manifold with smooth boundary ∂X, and the metric on X near the boundary takes the form

$$g = \frac{dx_1^2 + g_1(x_1^2)}{x_1^2}, \quad x_1|_{\partial X} = 0, \quad dx_1|_{\partial X} \neq 0 \tag{1.1}$$

where $g_1(x_1^2)$ is a smooth family of metrics on ∂X. See [DyZw19, §5.1] for a discussion of the invariance of this definition. Let Δ be the (negative) Laplacian on X. We prove

Theorem. *Suppose X has dimension 2 and (strictly) negative curvature. Then there exist $C_0, \beta > 0$ such that the resolvent*

$$R(\lambda) = (-\Delta - 1/4 - \lambda^2)^{-1} : L^2_{\mathrm{comp}}(X) \to L^2_{\mathrm{loc}}(X) \tag{1.2}$$

continues holomorphically from $\operatorname{Im}\lambda > 1$ to $\{|\lambda| > C_0, \operatorname{Im}\lambda > -\beta\}$. Moreover, for any $\chi \in C_c^\infty(X)$, we have the resolvent bound

$$\|\chi(-\Delta - 1/4 - \lambda^2)^{-1}\chi\|_{L^2 \to L^2} \leqslant C|\lambda|^{-1-C_1 \min(0, \operatorname{Im}\lambda)} \log|\lambda| \tag{1.3}$$

for $\operatorname{Im}\lambda > -\beta$ and $|\lambda| > C_0$.

The proof of the main Theorem follows from [Vac22] by reducing the problem to quantum monodromy maps using [NSZ11]. Although [Vac22] also uses [NSZ11], our approach is different by replacing the application of microlocal weight functions with propagation estimates. This approach simplifies some aspects of [NSZ11] and allows a seamless application of Vasy's method [Vas13a]. The geometric component comes from the now classical work of Eberlein [Eb72] which shows the trapped set is topologically one dimensional in our setting (see §2.3).

The spectral gap for open hyperbolic quantum systems has been studied since Ikawa [Ik88] in mathematics and Gaspard–Rice [GaRi89] in physics – if the topological pressure (an object from thermodynamical formalism) satisfies $P(1/2) < 0$, the statement of the theorem above holds with $\beta < -P(1/2)$. For an experimental manifestation of this gap, see [B*13]. A general spectral gap under the pressure condition was proved by Nonnenmacher–Zworski [NoZw09a, NoZw09b]. The first advances in the direction of improving the pressure gaps were made by Naud [Na05] (in the setting of constant curvature surfaces and complex dynamics) and Petkov–Stoyanov [PeSt10] (in the setting of obstacle scattering). These results were based on Dolgopyat's method [Do98]. However, it was conjectured by Zworski [Zw17, Conjecture 3] that the pressure condition is not necessary. Dyatlov–Zahl [DyZa16] made the first breakthrough showing a spectral gap without the pressure condition by introducing the *fractal uncertainty principle*. Bourgain–Dyatlov [BoDy18] proved the fractal uncertainty principle for any porous set in dimension one and showed that any (noncompact) convex cocompact hyperbolic surface has an essential spectral gap. For recent advances on the fractal uncertainty principle, see [BLT23, Co23]. The spectral gap was generalized to open quantum maps in dimension 2 by Vacossin [Vac22, Vac23] by combining the method of [BoDy18] and [DJN22]. That provided an essential spectral gap for classes of obstacles and semiclassical scattering problems.

Acknowledgement. We thank Maciej Zworski for many helpful discussions, and for his encouragement to write this note. We thank Stéphane Nonnenmacher for helpful discussions on Vacossin's paper [Vac22]. The author was partially supported by the NSF grant DMS-1952939 and by the Simons Targeted Grant Award No. 896630.

2. Microlocal preliminaries

2.1. **Review of concepts.** We use the terminology of [DyZw19, Appendix E]. For the reader's convenience, we review some concepts.

Let X be a smooth manifold without boundary (not necessarily compact). The polyhomogeneous symbols $S^m_h(T^*X)$ of order m on T^*X are defined in [DyZw19, Definition E.3] and their quantizations are semiclassical pseudodifferential operators $\Psi^m_h(X)$ on X defined in [DyZw19, Definition E.12]. For $a \in S^m_h(T^*X)$ there is a non-canonical construction of $\mathrm{Op}_h(a) \in \Psi^m_h(X)$, see [DyZw19, Proposition E.15]. Conversely, for $A \in \Psi^m_h(X)$, there is a canonical principal symbol $\sigma_h(A) \in S^m(T^*X)$ defined in [DyZw19, Proposition E.14].

We will consider h-tempered distributions on X, that is $u_h \in \mathcal{D}'(X)$ such that for any $\chi \in C_c^\infty(X)$, there exists N with $\|\chi u\|_{H_h^{-N}} \leqslant Ch^{-N}$. For an h-tempered distribution, the semiclassical wavefront set $\mathrm{WF}_h(u) \subset \overline{T}^*X$ is defined as the complement of the union of open sets $U \times V$ in $\overline{T}^*X$ such that

$$\mathcal{F}_h(\chi u)(\xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty}), \quad \chi \in C_c^\infty(U), \xi \in V \cap \mathbb{R}^d_\xi.$$

The semiclassical wavefront set of an operator $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ with h-tempered Schwartz kernel $\mathcal{K}_A$ is defined as (see [DyZw19, Definition E.36])

$$\mathrm{WF}'_h(A) := \{(x, \xi; y, \eta) \in \overline{T}^*(X \times Y) : (x, \xi; y, -\eta) \in \mathrm{WF}_h(\mathcal{K}_A)\}.$$

There is also a notation of wavefront set of a pesudodifferential operator $A \in \Psi^m_h(X)$, defined in [DyZw19, Definition E.27], which has the property that if $A = \mathrm{Op}_h(a)$ then $\mathrm{WF}_h(A)$ is the complement of the union of open sets $W \Subset \overline{T}^*X$ such that

$$a(x, \xi) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty}), \quad (x, \xi) \in W \cap T^*X.$$

An operator $C_c^\infty(X) \to \mathcal{D}'(X)$ is called compactly supported if its Schwartz kernel is compactly supported. We define the compactly microlocalized operators $\Psi_h^{\mathrm{comp}}(X)$ as compactly supported operators $A \in \Psi^m_h(X)$ such that $\mathrm{WF}_h(A)$ is a compact subset of T^*X, see [DyZw19, Definition E.28]. For two h-tempered distributions $u, v \in \mathcal{D}'(X)$, we say $u = v$ microlocally in some open subset $U \subset \overline{T}^*X$ if $\mathrm{WF}_h(u - v) \cap U = \varnothing$.

If X is compact, we define the semiclassical Sobolev space $H^s_h(X)$ by the norm

$$\|u\|^2_{H^s_h} = \sum_j \|\langle hD \rangle^s \varphi_j^*(\chi_j u)\|^2_{L^2},$$

where χ_j is a partition of unity and $\varphi_j : \mathbb{R}^d \supset U \to V \subset X$ are coordinate charts such that $\operatorname{supp}\chi_j \subset V$. When X is not compact, then we can similarly define the local semiclassical Sobolev space $H^s_{h,\mathrm{loc}}(X)$ and the compactly supported semiclassical Sobolev spaces $H^s_{h,\mathrm{comp}}(X)$ as in [DyZw19, Definition E.20].

If $\overline{X}$ is a compact manifold with boundary ∂X and interior X, we embed it into a compact manifold X_{ext} without boundary, and define $\bar{H}^s_h(X)$ by restrictions of functions in $H^s_h(X_{\mathrm{ext}})$.

Finally we recall the microlocal estimates in [DyZw19, Thereom E.33, E.47]. For $A \in \Psi^m_h(X)$, the elliptic set $\mathrm{ell}_h(A)$ is defined as $\{(x,\xi) \in \overline{T}^*X : \langle\xi\rangle^{-m}\sigma_h(A)(x,\xi) \neq 0\}$.

Proposition 2.1. *Let $P \in \Psi^m_h(X)$ be properly supported such that $\operatorname{Im}\langle\xi\rangle^{-m}\sigma_h(P) \leqslant 0$.*

(1) Suppose $A, B_1 \in \Psi^0_h(X)$ are compactly supported and $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(P) \cap \mathrm{ell}_h(B_1)$. Then there exists $\chi \in C^\infty_c(X)$ such that for any N,

$$\|Au\|_{H^s_h} \leqslant C\|BPu\|_{H^{s-m}_h} + \mathcal{O}(h^\infty)\|\chi u\|_{H^{-N}_h}. \tag{2.1}$$

(2) Suppose $A, B, B_1 \in \Psi^0_h(X)$ are compactly supported, and for any $(x,\xi) \in \mathrm{WF}_h(A)$ there exists $T \geqslant 0$ such that for $p = \operatorname{Re}\sigma_h(P)$,

$$\exp(-T\langle\xi\rangle^{1-m}H_p)(x,\xi) \in \mathrm{ell}_h(B), \quad \exp(-t\langle\xi\rangle^{1-m}H_p)(x,\xi) \in \mathrm{ell}_h(B_1) \text{ for all } t \in [0,T]. \tag{2.2}$$

Then there exists $\chi \in C^\infty_c(X)$ such that for any N,

$$\|Au\|_{H^s_h} \leqslant C\|Bu\|_{H^s_h} + Ch^{-1}\|B_1Pu\|_{H^{s-m+1}_h} + \mathcal{O}(h^\infty)\|\chi u\|_{H^{-N}_h}.$$

We will also use the sharp Gårding inequality [DyZw19, Proposition E.23].

Proposition 2.2. *If $A \in \Psi^{2m+1}_h(X)$ is compactly supported and $\operatorname{Re}\sigma_h(A) \geqslant 0$. Then*

$$\operatorname{Re}\langle Au, u\rangle_{L^2} \geqslant -Ch\|u\|^2_{H^m_h}. \tag{2.3}$$

2.2. **Vasy's method revisited.** Vasy [Vas13a] provided a very general method for showing meromorphic continuation of the resolvent for asymptotically hyperbolic systems. One can also look at [Zw16] for an elementary introduction to Vasy's method and Dyatlov–Zworski [DyZw19, Chapter 5] for a more detailed presentation. We start by recalling [DyZw19, Theorem 5.30, Theorem 5.33].

Proposition 2.3. *Let X be an even asymptotically hyperbolic manifold of dimension $d+1$ with negative curvature, then there exist a compact manifold X_1 with boundary ∂X_1, containing X as an open subset, and a second-order semiclassical differential operator $P(z)$ on X_1 with the following properties.*

- *For* $z \in [-h,h] + i[-C_0h, Ch]$ *and* $s > C_0 + 1/2$,

$$P(z) : D_h^s = \{u \in \bar{H}_h^s(X_1) : P(0)u \in \bar{H}_h^{s-1}\} \to \bar{H}_h^{s-1}(X_1) \tag{2.4}$$

is a holomorphic Fredholm family of index 0.
- *The set of poles of* $P(z)^{-1}$ *contains the set of poles of* $(h^2(-\Delta - d^2/4) - (1+z)^2)^{-1}$ *(with multiplicity).*

We sketch the proof of Proposition 2.3 and refer the readers to [DyZw19, Chapter 5] for more details and to [Zw12] for preliminaries on semiclassical analysis. Recall the metric on X is of the form (1.1) with some boundary defining function x_1. First one needs to change the smooth structure on $\overline{X}$ so that $\mu = x_1^2$ becomes a boundary defining function. Then we conjugate the operator $h^2(-\Delta - d^2/4) - (1+z)^2$ to another operator $P(z)$, which has the form

$$P(z) = \mu^{-1-d/4+i(z+1)/2h}\left(h^2\left(-\Delta - \frac{d^2}{4}\right) - (1+z)^2\right)\mu^{d/4-i(z+1)/2h} \tag{2.5}$$

near the boundary ∂X. Then $P(z)$ is well-defined with smooth coefficients up to the boundary, and we can extend it over the boundary to some slightly larger manifold X_1. The Fredholm property of $P(z)$ follows from the propagation estimates and radial estimates, see [DyZw19, §5.5].

From the construction of $P(z)$ and the propagation estimates, we have the following properties. We recall the trapped set is $K_0 := \Gamma_+ \cap \Gamma_-$ where the outgoing/incoming sets are defined as

$$\Gamma_\pm := \{(x,\xi) \in T^*X\backslash 0 : \exp(tH_p)(x,\xi) \text{ remains bounded as } t \to \mp\infty\} \cap p^{-1}(0).$$

- ([DyZw19, Theorem 5.34] and Proposition 2.5) There exists $Q \in \Psi_h^{\mathrm{comp}}(X)$ such that $P(z) - ihQ : D_h^s \to \bar{H}_h^{s-1}(X_1)$ is invertible for $0 < h < h_0$, with the bound

$$\|(P(z) - ihQ)^{-1}\|_{\bar{H}_h^{s-1} \to \bar{H}_h^s} \leqslant Ch^{-1}. \tag{2.6}$$

- ([DyZw19, §5.3]) $P(z)$ has real principal symbol. For any pre-fixed neighbourhood V_0 of the trapped set K_0, we can require $\mathrm{WF}_h(Q) \subset V_0$.
- ([DyZw19, Theorem 5.35] and Proposition 2.5) Let $p = \sigma_h(P(0))$ be the principal symbol and $\varphi^t = \exp(t\langle\xi\rangle^{-1}H_p)$, then

$$\mathrm{WF}_h'((P(z) - ihQ)^{-1}) \cap \overline{T}^*(X \times X) \subset \Delta_{\overline{T}^*X} \cup \Omega_+ \cup \Omega_\Gamma \tag{2.7}$$

where $\Delta_{\overline{T}^*X} := \{(x,\xi,x,\xi) : (x,\xi) \in \overline{T}^*X\}$,

$$\Omega_+ := \{(\varphi^t(y,\eta), y, \eta) : (y,\eta) \in T^*X,\ p(y,\eta) = 0,\ t \geqslant 0\}$$

is the positive flowout and $\Omega_\Gamma = \Gamma_+ \times \Gamma_-$.

Note we use $(P(z) - ihQ)^{-1}$ instead of $(P(z) - iQ)^{-1}$ to get a slightly better estimate in (3.7). This works because of the following Lemma.

Lemma 2.4. *Let $P \in \Psi_h^m(X)$ and $Q \in \Psi_h^{\mathrm{comp}}(X)$. Suppose $\operatorname{Im}\sigma_h(P) \leqslant 0$ and $\operatorname{Re}\sigma_h(Q) > 0$ near a compact set $K \subset T^*X$. Then there exists $Y_1, Y_2, Z \in \Psi_h^{\mathrm{comp}}(X)$ and $M > 0$ such that*

$$Y_1 Y_2 = I + \mathcal{O}(h^\infty) \text{ near } K, \quad K \subset \mathrm{ell}_h(Z)$$

and we have the following estimate for any $N > 0$ with some $\chi \in C_c^\infty(X)$ and sufficiently small $h > 0$:

$$\operatorname{Im}\langle Y_1(P - iMhQ)Y_2 u, u\rangle_{L^2} \leqslant -h\|Zu\|_{L^2}^2 + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

We remark that similar modifications are also used in [JiTa23, Proposition 2.7].

Proof. Let $Z \in \Psi_h^{\mathrm{comp}}$ be a microlocal cutoff to a neighbourhood of K. Since $\operatorname{Im}\sigma_h(P) \leqslant 0$ near $\mathrm{WF}_h(Z)$, by (2.3) we have for some constant $C > 0$,

$$\operatorname{Im}\langle PZu, Zu\rangle_{L^2} \leqslant Ch\|Zu\|_{L^2}^2 + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

By assumption $\sigma_h(Q) > c > 0$ near $\mathrm{WF}_h(Z)$, by (2.3) we have

$$\operatorname{Re}\langle QZu, Zu\rangle_{L^2} \geqslant c\|Zu\|_{L^2}^2 - Ch\|Zu\|_{L^2}^2 - \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

Consequently,

$$\operatorname{Im}\langle (P - iMhQ)Zu, Zu\rangle_{L^2} \leqslant (C - cM)h\|Zu\|_{L^2}^2 + \mathcal{O}(h^2)\|Zu\|_{L^2}^2 + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

Taking $Y_1 = Z^*$, $Y_2 = Z$ and $Mc > C + 10$ finishes the proof. □

Using Lemma 2.4, we have the following estimate similar to [DyZw19, Theorem 5.34] but with Q replaced by MhQ.

Proposition 2.5. *Let $Q \in \Psi_h^{\mathrm{comp}}(X)$ such that $\sigma_h(Q) \geqslant 0$ everywhere and $\sigma_h(Q) > 0$ near the trapped set K_0. Then for sufficiently large $M > 0$, $0 < h < h_0$, $z \in [-h, h] + i[-C_0 h, Ch]$ and $s > C_0 + 1/2$, we have*

$$\|u\|_{\bar{H}_h^s(X_1)} \leqslant Ch^{-1}\|(P(z) - iMhQ)u\|_{\bar{H}_h^{s-1}(X_1)}. \tag{2.8}$$

Proof. First, by [DyZw19, Lemma 5.25], there exists $\chi_1 \in C_c^\infty(X_1)$ such that

$$\|u\|_{\bar{H}_h^s} \leqslant Ch^{-1}\|(P(z) - iMhQ)u\|_{\bar{H}_h^{s-1}} + C\|\chi_1 u\|_{H_h^s}.$$

The phase space dynamics on $\overline{T}^*X_1$ can be described as follows (see [DyZw19, §5.4]). There exist $\Sigma_\pm$ such that $\{\langle\xi\rangle^{-2}p = 0\} \subset \overline{T}^*X_1$ is the disjoint union of Σ_+ and Σ_-. Moreover, $\Sigma_+ \cap \overline{T}^*X = \varnothing$. For $(x, \xi) \in \Sigma_\pm$, we have two possibilities

- $\varphi^t(x,\xi) \to L_\pm$ as $t \to \pm\infty$, where $L_\pm = \{\mu = 0\} \cap \Sigma_\pm \cap \partial\overline{T}^* X_1$ are the radial sets.
- $\varphi^t(x,\xi) \to K_0$ as $t \to -\infty$.

By propagation estimates (Proposition 2.1), it suffices to estimate near $L_\pm$ and K_0. Near $L_\pm$ we use the radial estimate [DyZw19, Lemma 5.23]. Near K_0 we use our Lemma 2.4 which gives for some microlocal cutoff A to a neighbourhood of K_0 (see [DyGu16, Lemma 2.7])

$$\|Au\|_{H_h^s} \leqslant C\|A_1 u\|_{H_h^s} + Ch^{-1}\|(P(z) - ihQ)u\|_{\bar{H}_h^{s-1}} + Ch^{1/2}\|A_2 u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}},$$

where A_1 has the property that $\varphi^t(\mathrm{WF}_h(A_1)) \to L_-$ as $t \to -\infty$, $A_2 \in \Psi_h^{\mathrm{comp}}(X)$ and $\chi \in C_c^\infty(X)$. Then the $A_1 u$ term can be propagated to L_- and the $A_2 u$ term can be improved to $h^N\|\chi u\|_{H_h^{-N}}$ by iterating the estimate. □

The resolvent bound (2.6) follows from (2.8). The wavefront set estimate (2.7) follows from the proof of Proposition 2.5. In order to show (2.7), we need to show for any $(x_0,\xi_0;y_0,\eta_0) \in \overline{T}^*(X\times X)$ such that $(x_0,\xi_0) \notin (\Delta_{\overline{T}^*X} \cup \Omega_+ \cup \Omega_\Gamma)(y_0,\eta_0)$, there are open neighbourhoods U of (x_0,ξ_0) and V of (y_0,η_0) such that

$$\langle (P(z) - ihQ)^{-1}Bu, Av\rangle = \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}\|v\|_{H_h^{-N}} \tag{2.9}$$

for any compactly supported $A, B \in \Psi_h^0(X)$ with $\mathrm{WF}_h(A) \subset U$, $\mathrm{WF}_h(B) \subset V$ and any $u, v \in C_c^\infty(X)$. By elliptic estimate (2.1) we may assume $p(x_0,\xi_0) = p(y_0,\eta_0) = 0$. Note $T^*X \cap \{p = 0\} \subset \Sigma_-$, so $\varphi^t(x_0,\xi_0) \to L_-$ or K_0 as $t \to -\infty$. If $\varphi^t(x_0,\xi_0) \to L_-$ as $t \to -\infty$, we conclude (2.9) from the propagation estimate (2.2). If $\varphi^t(x_0,\xi_0) \to K_0$ as $t \to -\infty$, then $(x_0,\xi_0) \in \Gamma_+$. A dual estimate would then give (2.9) unless $(y_0,\eta_0) \in \Gamma_-$.

Now we state a forward solvability property (up to $\mathcal{O}(h^\infty)$ error) which will be crucial to our analysis.

Proposition 2.6. *Suppose $f \in H_{h,\mathrm{comp}}^{s-1}(X)$ and $\mathrm{WF}_h(f) \cap \Gamma_- = \varnothing$. Then there exists $u \in \bar{H}_h^s(X_1)$ such that $P(z)u = f + \mathcal{O}_{H_{h,\mathrm{comp}}^N}(h^\infty)$ for any N, and*

$$\|u\|_{\bar{H}_h^s} \leqslant Ch^{-1}\|f\|_{\bar{H}_h^{s-1}}, \quad \mathrm{WF}_h(u) \cap \overline{T}^*X \subset \bigcup_{t=0}^\infty \varphi^t(\mathrm{WF}_h(f)).$$

Proof. Since $\mathrm{WF}_h(f) \cap \Gamma_- = \varnothing$, we may choose a small microlocal cutoff $Q \in \Psi_h^{\mathrm{comp}}$ to a neighbourhood of K_0 so that the backward flow of $\mathrm{WF}_h(Q)$ does not intersect $\mathrm{WF}_h(f)$. Define $u = (P(z) - ihQ)^{-1}f$, then it suffices to show $Qu = \mathcal{O}_{H_{h,\mathrm{comp}}^N}(h^\infty)$. By (2.7), $\mathrm{WF}_h(u) \cap \overline{T}^*X \subset \bigcup_{t=0}^\infty \varphi^t(\mathrm{WF}_h(f)) \cup \Omega_\Gamma \circ \mathrm{WF}_h(f)$. Since $\mathrm{WF}_h(f) \cap \Gamma_- = \varnothing$, we have $\Omega_\Gamma \circ$

$\mathrm{WF}_h(f) = \varnothing$. Moreover, $\bigcup_{t=0}^{\infty} \varphi^t(\mathrm{WF}_h(f))$ does not intersect with $\mathrm{WF}_h(Q)$, thus $\mathrm{WF}_h(u) \cap \mathrm{WF}_h(Q) = \varnothing$ and $Qu = \mathcal{O}_{H^N_{h,\mathrm{comp}}}(h^\infty)$. □

3. Quantum monodromy without weights

In order to apply [Vac22] to obtain the spectral gap, we need to construct quantum monodromy maps as in [NSZ11] but in the asymptotically hyperbolic setting. In §3.1 we apply Vasy's method to construct such quantum monodromy maps without using weights for $d+1$-dimensional asymptotically hyperbolic manifolds whose trapped set has topological dimension 1. In §3.2 we use [Eb72] to verify this assumption for all asymptotically hyperbolic surfaces.

3.1. **Construction of quantum monodromy maps.**

In this section, we reduce the problem to a quantum monodromy map following [NSZ11]. In particular we show the following.

Proposition 3.1. *Suppose we are in the setting of Proposition 2.3 and the trapped set K_0 has topological dimension 1. Then there exists a holomorphic family of matrices $M(z,h)$ acting on $\mathbb{C}^N$ with $N \sim h^{-d}$ for $z \in [-h,h] + i[-C_0h, Ch]$ so that the poles of $P(z)^{-1}$ are given by the zeros of* $\det(I - M(z,h))$.

The proof of this proposition follows from the construction of a Grushin problem. This construction proceeds in two steps. First one constructs a *microlocal* Grushin problem near the trapped set K_0, which is done in [NSZ11] and we can directly use it here. The second step is to construct a *global* Grushin problem. This is done in [NSZ11, §5] using weight functions. Here we apply a different construction by using propagation estimates alone. This allows a simple connection with Vasy's method and gives better bounds (see the remark after [Vac23, Corollary 1]).

3.1.1. *Dynamical preliminaries.* Suppose the trapped set K_0 is topologically one dimensional. Then by [BoWa72] (for the statement here we cite [NSZ11, Proposition 2.1]), there exist finitely many compact contractible smooth (up to boundary) hypersurfaces $\Sigma_j \subset p^{-1}(0)$ so that

- $\partial\Sigma_j \cap K_0 = \varnothing$, $\Sigma_j \cap \Sigma_{j'} = \varnothing$, $j \neq j'$;
- H_p is transversal to Σ_j up to the boundary;
- The Hamiltonian flow starting from K_0 touches $\cup_j \Sigma_j$ in both directions. Moreover, we may assume the successor of a point in $\Sigma_k \cap K_0$ is in Σ_j for some $j \neq k$.

We recall some notations from [NSZ11]. Let $\mathcal{T} = K_0 \cap \cup_j \Sigma_j$ be the reduced trapped set, and $\mathcal{T}_j = K_0 \cap \Sigma_j$. Let $f : \mathcal{T} \to \mathcal{T}$ be the Poincaré map restricted to $\mathcal{T}$ (see [NSZ11, §2.3.1]) and

$$\mathcal{D}_{jk} = \mathcal{T}_k \cap f^{-1}(\mathcal{T}_j), \quad \mathcal{A}_{jk} = \mathcal{T}_j \cap f(\mathcal{T}_k)$$

be the *departure* and *arrival* sets (note $f(\mathcal{D}_{jk}) = \mathcal{A}_{jk}$). We take (disjoint) neighbourhoods $D_{jk} \subset \Sigma_k$ ($A_{jk} \subset \Sigma_j$, respectively) of $\mathcal{D}_{jk}$ ($\mathcal{A}_{jk}$, respectively) and extend f to a local symplectomorphism

$$F_{jk} : D_{jk} \to A_{jk}.$$

We may assume D_{jk} and A_{jk} are mutually disjoint and denote

$$D_k = \bigcup_j D_{jk}, \quad A_j = \bigcup_k A_{jk}.$$

As in [NSZ11], we may choose $\widetilde{\Sigma}_j \subset T^*\mathbb{R}^d$ and smooth symplectomorphisms $\kappa_j : \widetilde{\Sigma}_j \to \Sigma_j$ up to the boundary (by taking Σ_j with small diameter). We then define $\widetilde{\mathcal{T}}_j$, $\widetilde{D}_{jk}$, $\widetilde{A}_{jk}$ and $\widetilde{F}_{jk}$ accordingly using κ_j's. Now the dynamics is encoded by the monodromy map $F = (F_{jk})$. One quantity that will be useful later is the minimal propagation time

$$t_0 := \min\{t > 0 : \text{ there exist } j \neq k \text{ and } (x,\xi) \in \Sigma_k \text{ such that } \varphi^t(x,\xi) \in \Sigma_j\} > 0. \tag{3.1}$$

3.1.2. *Microlocal Grushin problem.* We now recall the microlocal Grushin problem constructed in [NSZ11, §4]. Let $H(V)$ be the space of functions microlocalized in V. We will always assume $z \in [-h,h] + i[-C_0 h, Ch]$.

Lemma 3.2. *There exist neighbourhoods $V_0 \Subset V_1 \Subset V_2 \Subset X$ of K_0, and semiclassical Fourier integral operators $\widetilde{R}_-^j : L^2(\mathbb{R}^d) \to H(V_2)$, $\widetilde{R}_+^j : L^2(X) \to H(\widetilde{\Sigma}_j)$ such that for any v microlocalized in V_1, and any v_+^k microlocalized in $\widetilde{D}_k$, we can find u microlocalized in V_2, and u_-^k microlocalized in $\widetilde{D}_k \cup \widetilde{A}_k$, so that (u, u_-) solves*

$$\begin{pmatrix} \frac{i}{h}P(z) & \widetilde{R}_- \\ \widetilde{R}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$$

microlocally inside $V_1 \times (\times_k \widetilde{D}_k)$.

A more precise description of $\widetilde{R}_\pm$ is given as follows.

- $\mathrm{WF}_h'(\widetilde{R}_+^j) \subset \widetilde{\Sigma}_j \times V_2$ and $\mathrm{WF}_h'(\widetilde{R}_-^j) \subset V_2 \times \widetilde{\Sigma}_j$.
- For any prefixed $\epsilon > 0$, we can require

$$\begin{aligned} \mathrm{WF}_h'(\widetilde{R}_+^j) &\subset \{(x,\xi;\varphi^t(\kappa(x,\xi))) : (x,\xi) \in \widetilde{\Sigma}_j, |t| < \epsilon\}, \\ \mathrm{WF}_h'(\widetilde{R}_-^j) &\subset \{(\varphi^t(\kappa(x,\xi)); x,\xi) : (x,\xi) \in \widetilde{\Sigma}_j, t > -\epsilon\}. \end{aligned} \tag{3.2}$$

The microlocal Grushin problem is solved microlocally by

$$u = \widetilde{E}v + \widetilde{E}_+ v_+, \quad u_- = \widetilde{E}_- v + \widetilde{E}_{-+} v_+$$

where $\widetilde{E}, \widetilde{E}_\pm$ and $\widetilde{E}_{-+}$ are compactly supported operators with compact wavefront sets in $T^*(X \times X)$ and the following almost forward propagation properties:

$$\begin{aligned} \mathrm{WF}'_h(\widetilde{E}) &\subset \{(\varphi^t(x,\xi); x, \xi) : t > -\epsilon\}, \\ \mathrm{WF}'_h(\widetilde{E}^j_-) &\subset \{(x,\xi; \varphi^t(\kappa_j(x,\xi))) : (x,\xi) \in \widetilde{\Sigma}_j, t < \epsilon\}, \\ \mathrm{WF}'_h(\widetilde{E}^j_+) &\subset \{(\varphi^t(\kappa_j(x,\xi)); x, \xi) : (x,\xi) \in \widetilde{\Sigma}_j, t > -\epsilon\}, \\ \widetilde{E}^{jk}_{-+} &= \mathcal{M}_{jk} - \delta_{jk}, \end{aligned} \tag{3.3}$$

where $\mathcal{M}_{jk}$ is a Fourier integral operator quantizing $\widetilde{F}_{jk}$.

3.1.3. *Global Grushin problem.* The next step is to construct a global Grushin problem. Let $V^j \Subset \kappa_j^{-1}(V_1 \cap \Sigma_j)$ be small neighbourhoods of $\widetilde{\mathcal{T}}_j$ in $\widetilde{\Sigma}_j$, Q_0^j be the (self-adjoint) quantization of a cutoff which is positive in V^j and negative outside $\overline{V}^j$, and Π_j be the orthogonal projection defined by $\mathbf{1}_{>0}(Q_0^j)$. Denote $V = \cup_j V^j$. Consider the following Grushin problem.

$$\mathcal{P}(z) \begin{pmatrix} u \\ u_- \end{pmatrix} := \begin{pmatrix} \frac{i}{h}P(z) & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} : D_h^s(X_1) \times \mathcal{H}_h \to \bar{H}_h^{s-1}(X_1) \times \mathcal{H}_h. \tag{3.4}$$

where

$$\mathcal{H}_h = \bigtimes_j \mathcal{H}_h^j, \quad \mathcal{H}_h^j = \Pi_j H(\widetilde{D}_j), \quad R_-^j = \widetilde{R}_-^j \Pi_j, \quad R_+^j = \Pi_j \widetilde{R}_+^j. \tag{3.5}$$

Note $\mathcal{H}_h$ is a finite dimensional space with $\dim \mathcal{H}_h \sim h^{-d}$. Recall $\mathrm{WF}_h(Q) \subset V_0$ and $\mathrm{WF}_h(\Pi_j) \subset \overline{V}$. We will choose V_0 and V to be sufficiently small neighbourhoods in the following to conclude the well-posedness of the Grushin problem.

Lemma 3.3. *There exists*

$$\mathcal{E}(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} = \mathcal{O}(1) : \bar{H}_h^{s-1}(X_1) \times \mathcal{H}_h \to D_h^s(X_1) \times \mathcal{H}_h \tag{3.6}$$

solving the Grushin problem (3.4), *i.e.* $\mathcal{P}(z)\mathcal{E}(z) = I$.

Proof of Lemma 3.3. Since $\mathcal{P}(z)$ is a Fredholm operator of index 0, it suffices to construct a right inverse of $\mathcal{P}(z)$. In other words, given $(v, v_+) \in \bar{H}_h^{s-1}(X_1) \times \mathcal{H}_h$ with $\|v\|^2_{\bar{H}_h^{s-1}} + \|v_+\|^2_{L^2} \leqslant 1$, we want to find $(u, u_-) \in D_h^s(X_1) \times \mathcal{H}_h$ such that

$$\mathcal{P}(z) \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}, \quad \|u\|^2_{\bar{H}_h^s} + \|u_-\|^2_{L^2} = \mathcal{O}(1).$$

Step 1: Microlocalize to the trapped set.

In order to apply the microlocal Grushin problem in Lemma 3.2, we need to first microlocalize to a neighbourhood of the trapped set. Thus we take

$$u_0 = \frac{h}{i}(P(z) - ihQ)^{-1}v$$

so that

$$\frac{i}{h}P(z)u_0 = P(z)(P(z) - ihQ)^{-1}v = v + ihQ(P(z) - ihQ)^{-1}v. \tag{3.7}$$

Let

$$v_0 = -ihQ(P(z) - ihQ)^{-1}v, \quad v_{0+} = R_+u_0,$$

we aime to solve for $\mathcal{P}(z)(u, u_-)^T = (v_0, v_+ - v_{0+})^T$. At this point we can use the microlocal Grushin problem to find $(\widetilde{u}, \widetilde{u}_-) \in H(V_2) \times H(\widetilde{D} \cup \widetilde{A})$ such that

$$\frac{i}{h}P(z)\widetilde{u} + \widetilde{R}_-\widetilde{u}_- = v_0 - f, \quad R_+\widetilde{u} = v_+ - v_{0+} + \mathcal{O}_{L^2}(h^\infty)$$

where $\mathrm{WF}_h(f) \subset V_2 \backslash V_1$.

Step 2: Correct the error f using forward propagation.

We want to correct the error f without affecting the projection R_+. For this we use the forward propagation property in Proposition 2.6 and the following important observation (see [DyGu16, Lemma 1.4]):

$$\begin{gathered}\text{Suppose } U \text{ is a neighbourhood of the trapped set } K_0.\\ \text{Then there exists a smaller neighbourhood } U_0 \subset U \text{ such that}\\ \text{if } (x,\xi) \in U_0 \text{ and } \varphi^t(x,\xi) \notin U \text{ for some } t > 0, \text{ then } (x,\xi) \notin \Gamma_-.\end{gathered} \tag{3.8}$$

Note (3.2) and (3.3) imply that any $(x,\xi) \in \mathrm{WF}_h(f)$ is of the form $\varphi^t(y,\eta)$ for some $(y,\eta) \in V_0 \cup \kappa(V)$ and $t > -2\epsilon$. By choosing V_0 and V sufficiently small, (3.8) implies $\mathrm{WF}_h(f) \cap \Gamma_- = \varnothing$. Consequently, by Proposition 2.6, there exists $u_1 \in D_h^s(X_1)$ such that

$$\frac{i}{h}P(z)u_1 = f + \mathcal{O}_{\bar{H}_h^{s-1}}(h^\infty), \quad \mathrm{WF}_h(u_1) \cap \overline{T}^*X \subset \bigcup_{t=0}^{\infty} \varphi^t(\mathrm{WF}_h(f)).$$

We then define $\mathcal{E}^{(1)}$ by

$$\begin{pmatrix} u^{(1)} \\ u_-^{(1)} \end{pmatrix} := \begin{pmatrix} \widetilde{u} + u_1 + u_0 \\ \Pi\widetilde{u}_- \end{pmatrix} = \begin{pmatrix} E^{(1)} & E_+^{(1)} \\ E_-^{(1)} & E_{-+}^{(1)} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix}.$$

Since u_1 is the forward solution and V_0, V are taken sufficiently small, we may also assume $R_+u_1 = \mathcal{O}_{L^2}(h^\infty)$. Thus

$$\frac{i}{h}P(z)u^{(1)} + R_-u_-^{(1)} = v + \widetilde{R}_-(\Pi - I)\widetilde{u}_- + \mathcal{O}_{\bar{H}_h^{s-1}}(h^\infty), \quad R_+u^{(1)} = v_+ + \mathcal{O}_{L^2}(h^\infty).$$

In other words

$$\mathcal{P}\mathcal{E}^{(1)} = I - \mathcal{R}, \quad \mathcal{R}\begin{pmatrix} v \\ v_+ \end{pmatrix} = \begin{pmatrix} \widetilde{R}_-(I-\Pi)\widetilde{E}_- & \widetilde{R}_-(I-\Pi)\mathcal{M} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_0 \\ v_+ - v_{0+} \end{pmatrix} + \mathcal{O}_{\bar{H}_h^{s-1}\times L^2}(h^\infty).$$

Step 3: Correct $\mathcal{R}$ by iteration.

Finally we need to remove the error $\mathcal{R}$. This is done by showing $\mathcal{R}$ is nilpotent modulo $\mathcal{O}(h^\infty)$. First we note $\mathcal{M}$ has a minimal propagation time $t_0 > 0$ defined in (3.1) which gives

$$\mathrm{WF}'_h(\mathcal{M}) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant t_0\}.$$

Moreover, since $\mathrm{WF}'_h(\widetilde{E}_- Q) \subset V_2 \times V_0$, for V_0 sufficiently small we have

$$\mathrm{WF}'_h((I-\Pi)\widetilde{E}_- Q) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant t_0\}.$$

So we conclude

$$\mathrm{WF}'_h(\mathcal{R}) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant t_0 - 2\epsilon\}.$$

Due to the projection $I - \Pi$, the wavefront set of $\mathcal{R}^N$ does not intersect Γ_- for $N \geqslant N_0$:

$$\mathrm{WF}'_h(\mathcal{R}^N) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant N(t_0 - 2\epsilon), (x,\xi) \notin \Gamma_-\}, \quad N \geqslant N_0.$$

Eventually $\mathrm{WF}_h(\mathcal{R}^N(v, v_+)^T) \cap \mathrm{WF}_h(\widetilde{R}_-) = \varnothing$ and thus there exists $N_1 \in \mathbb{N}$ with $\mathcal{R}^{N_1} = \mathcal{O}(h^\infty)$. Let

$$\mathcal{E}^{(2)} := \mathcal{E}^{(1)}(I + \mathcal{R} + \cdots + \mathcal{R}^{N_1 - 1}),$$

then $\mathcal{P}\mathcal{E}^{(2)} = I + \mathcal{O}(h^\infty)$. So we finally conclude the inverse

$$\mathcal{E}(z) = \mathcal{E}^{(2)}(I + \mathcal{O}(h^\infty))^{-1} = \mathcal{E}^{(1)}(I + \mathcal{R} + \cdots + \mathcal{R}^{N_1-1})(I + \mathcal{O}(h^\infty)). \tag{3.9}$$

One checks that each step is uniformly bounded in h. This finishes the proof of Lemma 3.3. □

Proof of Proposition 3.1. Let E_{-+} be the matrix component defined in (3.6). We define the matrices $M(z,h) := I + E_{-+}(z)$ and the statement follows from the Grushin problem (3.4). □

We remark that $M(z,h)$ has the form $M(z,h) = \Pi\mathcal{M}(z,h)\Pi + \mathcal{R}_1$ where $\mathcal{R}_1$ again satisfies

$$\mathrm{WF}'_h(\mathcal{R}_1^N) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant N(t_0 - 2\epsilon), (x,\xi) \notin \Gamma_-\}, \quad N \geqslant N_0$$

which implies $\mathcal{R}_1^{N_1} = \mathcal{O}(h^\infty)$. Moreover, a direct computation shows that $\mathcal{R}_1$ has the form $\mathcal{R}_1 = A(I-\Pi)\mathcal{M}(z,h)\Pi + \mathcal{O}(h^\infty)$ where A satisfies the forward propagation property

$$\mathrm{WF}'_h(A) \subset \{(\varphi^t(x,\xi), x, \xi) : t \geqslant 0\}.$$

3.2. **Structure of the trapped set.** In this section, we verify the dynamical assumption in Proposition 3.1: K_0 is topologically one dimensional. From now on we assume X is 2-dimensional.

Proposition 3.4. *Suppose X is a negatively curved (even) asymptotically hyperbolic surface, then the trapped set K_0 has topological dimension 1.*

Proof. The trapped set K_0 we defined before is the same as the trapped set defined by the geodesic flow $H_{\widetilde{p}}$ on $T^*X\backslash 0$ with $\widetilde{p}(x,\xi) := |\xi|^2 - 1$. So we can use knowledge of negative curved geometry.

Let $\widetilde{X}$ be the universal cover of X, then there is a natural compactification $\overline{\widetilde{X}}$, such that the boundary at infinity $\partial_\infty \widetilde{X}$ (which is topologically a circle) can be thought as equivalence classes of geodesic rays. The original manifold X is then a quotient of the universal cover $\widetilde{X}$ by a discrete group of isometries Γ. The *limit set* $\Lambda_\Gamma \subset \partial_\infty \widetilde{X}$ is defined as the accumulation points of any orbit of Γ in $\widetilde{X}$. The lift $\widetilde{K}_0$ of the trapped set K_0 to $S^*\widetilde{X}$ is given by the convex hull of the limit set. Using the Hopf parametrisation, we know $\widetilde{K}_0$ is homeomorphic to $((\Lambda_\Gamma \times \Lambda_\Gamma)\backslash\Delta) \times \mathbb{R}$.

In order to show K_0 has topological dimension 1, it suffices to show the limit set Λ_Γ is totally disconnected. By [Eb72, Theorem 2.5], the limit set is either nowhere dense or the full $\partial_\infty \widetilde{X}$. But the hyperbolic ends of X correspond to intervals in $\partial_\infty \widetilde{X}$ that does not belong to the limit set Λ_Γ. So the second case does not happen, and Λ_Γ has to be nowhere dense and hence totally disconnected. □

4. Proof of the spectral gap and the resolvent estimate

We conclude the proof of the main Theorem in this section. Let X be an (even) asymptotically hyperbolic surface with (strictly) negative curvature. Then (see [MaMe87, Gu05, Vas13a, Vas13b, Zw16]) the resolvent

$$R(s) = (-\Delta - 1/4 - \lambda^2)^{-1} : L^2_{\mathrm{comp}}(X) \to L^2_{\mathrm{loc}}(X) \tag{4.1}$$

has a meromorphic continuation to $\lambda \in \mathbb{C}$. After rescaling $\lambda = h^{-1}(1+z)$ we can apply Proposition 2.3 and study the poles of $P(z)^{-1}$ for $z \in [-h,h] + i[-C_0 h, Ch]$.

We have reduced the problem to quantum monodromy maps in §3.1. In the case of surfaces, [Vac22, Proposition 4.1] shows

Proposition 4.1. *There exist $c_0, h_0 > 0$ and $\gamma > 0$ such that for $0 < h < h_0$, $c_0 \log(1/h) \leqslant N \leqslant C \log(1/h)$, we have*

$$\|\mathcal{M}(z,h)^N\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant C h^\gamma h^{C_1 \min(0, h^{-1}\operatorname{Im} z)}.$$

From this we conclude

Proposition 4.2. *There exist $c_1, h_0 > 0$ and $\gamma > 0$ such that for $0 < h < h_0$, $c_1 \log(1/h) \leqslant N \leqslant C \log(1/h)$, we have*

$$\|M(z,h)^N\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant Ch^\gamma h^{C_1 \min(0, h^{-1}\operatorname{Im} z)}.$$

Proof. Consider

$$M(z,h)^{N+2N_2} = M(z,h)^{N_2}(\Pi\mathcal{M}(z,h)\Pi + \mathcal{R}_1)^N M(z,h)^{N_2}.$$

We claim this is equal to $M(z,h)^{N_2}\mathcal{M}(z,h)^N M(z,h)^{N_2}$ modulo $\mathcal{O}(h^\infty)$, if we take N_2 sufficiently large but fixed. It suffices to show

$$M(z,h)^{N_2}(\Pi\mathcal{M}(z,h)\Pi)^j \mathcal{R}_1 M(z,h)^{N-j-1+N_2} = \mathcal{O}(h^\infty), \quad j \in \{0, 1, \cdots, N-1\}$$

and

$$M(z,h)^{N_2}\mathcal{M}(z,h)^k(I-\Pi)\mathcal{M}(z,h)M(z,h)^{N-k-1+N_2} = \mathcal{O}(h^\infty), \quad k \in \{1, \cdots, N-1\}.$$

This follows from the observation that if (x,ξ) lies outside V (due to the projection $I - \Pi$ and the definition of $\mathcal{R}_1$), then it has be to disjoint from either Γ_+ or Γ_-, which implies that either $F^{N_2}(x,\xi)$ or $F^{-N_2}(x,\xi)$ has to escape V for N_2 sufficiently large. □

Proposition 4.2 already implies the spectral gap through Proposition 3.1. But we can further estimate the resolvent as below. By the Grushin problem (3.4), we have

$$P(z)^{-1} = \frac{i}{h}(E - E_+E_{-+}^{-1}E_-) \tag{4.2}$$

where E, E_-, E_+, E_{-+} are all $\mathcal{O}(1)$. Moreover, since $E_{-+}(z) = M(z,h) - I$,

$$\|E_{-+}^{-1}(z)\| \leqslant 1 + \|M(z,h)\| + \cdots + \|M(z,h)^{N-1}\| + \|M(z,h)^N\|\|E_{-+}^{-1}(z)\|.$$

For $c_1 \log(1/h) \leqslant N \leqslant C \log(1/h)$ and $\operatorname{Im} z \geqslant -C_1^{-1}\gamma h/2$, we conclude

$$\|E_{-+}^{-1}(z)\| \leqslant 2(1 + \|M(z,h)\| + \cdots + \|M(z,h)^{N-1}\|).$$

Similar to the proof of Proposition 4.2, we have for $2N_2 \leqslant j \leqslant C\log(1/h)$,

$$\|M(z,h)^j\| \leqslant \|M(z,h)^{N_2}\mathcal{M}(z,h)^{j-2N_2}M(z,h)^{N_2}\| + \mathcal{O}(h^\infty) \leqslant Ce^{-C_1 j \min(0, h^{-1}\operatorname{Im} z)}.$$

Since we have $c_1\log(1/h)$ many terms, we conclude

$$\|E_{-+}^{-1}(z)\| \leqslant C\log(1/h)h^{C_1\min(0,h^{-1}\operatorname{Im} z)}.$$

By (4.2), we have

$$\|P(z)^{-1}\|_{\bar{H}_h^{s-1}\to\bar{H}_h^s} \leqslant Ch^{-1}\log(1/h)h^{C_1\min(0,h^{-1}\operatorname{Im} z)}.$$

The resolvent bound (1.3) follows from the definition (2.5) of $P(z)$ and rescaling.

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