ON 0-TH ORDER PSEUDO-DIFFERENTIAL OPERATORS ON THE CIRCLE

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Abstract. In this paper we consider 0-th order pseudodifferential operators on the circle. We show that inside any interval disjoint from the critical values of the principal symbol, the spectrum is absolutely continuous with possibly finitely many embedded eigenvalues. We also give an example of embedded eigenvalues.

1. Introduction

The study of 0-th order pseudodifferential operators has recently attracted new attention because of its connections to fluid mechanics (see the work of Colin de Verdière-Saint-Raymond [CdVS20], [CdV18] and Dyatlov-Zworski [DyZw19]). In this note we address the special case of the circle. We start with a general result valid for any compact manifold.

Theorem 1. *Let M be a compact smooth manifold. Suppose $H \in \Psi^0_{1,0}(M)$ is self-adjoint with the principal symbol $a = \sigma(H) \in S^0_{1,0}(T^*M)$. Then*

$$\mathrm{Spec}_{\mathrm{ess}}(H) = \{\lambda \in \mathbb{C} | \textit{there exists } (x_j, \xi_j) \in T^*M, |\xi_j| \to \infty, \textit{ such that} \lim_{j\to\infty} a(x_j, \xi_j) = \lambda\}.$$

This is a slight generalization of [CdV18, Theorem 2.1] where the operators are assumed to have classical symbol (in particular, a is homogeneous of degree 0).

We then specialize to the case of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and assume the principal symbol is homogeneous of degree 0. With the notation reviewed in Section 2, we have

Theorem 2. *Suppose $H \in \Psi^0_{1,0}(\mathbb{T}) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is self-adjoint and the principal symbol $a = \sigma(H)$ is homogeneous of degree 0. Let $\mathcal{N} = \{a(x, \pm 1) : \partial_x a(x, \pm 1) = 0\}$, $a_\pm = \min a(x, \pm 1)$, $a^\pm = \max a(x, \pm 1)$, then*

(1) $\mathrm{Spec}_{\mathrm{sc}}(H) \subseteq \mathcal{N}$,
(2) $\forall\, I \Subset \mathbb{R} \setminus \mathcal{N}$, $|\mathrm{Spec}_{\mathrm{pp}}(H) \cap I| < \infty$,
(3) $\mathrm{Spec}_{\mathrm{ac}}(H) \setminus \mathcal{N} = [a_-, a^-] \cup [a_+, a^+] \setminus \mathcal{N}$.

This means that inside an interval disjoint from the critical values, the spectrum is absolutely continuous with possibly finitely many embedded eigenvalues.

We conclude this introduction with an example of an operator with embedded eigenvalues.

Example 1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ where we can identify functions as periodic functions on $\mathbb{R}$ and do global Weyl quantization as in $\mathbb{R}$ by

$$\mathrm{Op}^w(a)u(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} a\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\xi}u(y)dyd\xi. \tag{1.1}$$

Let $a(x,\xi) = \sin(2\pi x)(1-\chi(\xi))$ and $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi((2k-1)\pi) = \chi((2k+1)\pi) = 1$ for some integer k, then the operator $H = a^w(x,D) \in \Psi^0_{1,0}(\mathbb{T}) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ has continuous spectrum near 0 by Theorem 2. We claim that H has an embedded eigenvualue at 0 with the corresponding eigenfunction $u_k = e^{2k\pi ix}$. To prove this, we first compute the standard quantization of a symbol b for tempered distributions

$$\begin{aligned} b(x,D)u_k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} b(x,\xi)e^{ix\xi}\mathcal{F}(u_k)(\xi)d\xi \\ &= \int_{\mathbb{R}} b(x,\xi)e^{ix\xi}\delta(\xi-2k\pi)d\xi \\ &= b(x,2k\pi)u_k(x). \end{aligned} \tag{1.2}$$

The changing quantization formula [Zw12, Theorem 4.13] and (1.2) tells us that the Weyl quantization would be

$$a^w(x,D)u_k(x) = (e^{\frac{i}{2}\langle D_x,D_\xi\rangle}a)(x,D)u_k(x) = (e^{\frac{i}{2}\langle D_x,D_\xi\rangle}a)(x,2k\pi)u_k(x).$$

From [Zw12, Theorem 4.8] and the Poisson summation formula, we have

$$\begin{aligned} a^w(x,D)u_k(x) &= \frac{u_k(x)}{\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{-2iy\eta}a(x+y,2k\pi+\eta)dyd\eta \\ &= \frac{u_k(x)}{\pi}\int_{\mathbb{T}\times\mathbb{R}}\sum_{l\in\mathbb{Z}} e^{-2i(y+l)\eta}a(x+y,2k\pi+\eta)dyd\eta \\ &= u_k(x)\int_{\mathbb{T}\times\mathbb{R}}\sum_{l\in\mathbb{Z}}\delta(\eta+l\pi)e^{-2iy\eta}a(x+y,2k\pi+\eta)dyd\eta. \end{aligned}$$

A simplification gives

$$\begin{aligned} a^w(x,D)u_k(x) &= u_k(x)\sum_{l\in\mathbb{Z}}\int_{\mathbb{T}} e^{-2\pi ily}a(x+y,2k\pi+l\pi)dy \\ &= u_k(x)\sum_{l\in\mathbb{Z}} e^{2\pi ilx}\int_{\mathbb{T}} e^{-2\pi ilv}a(v,2k\pi+l\pi)dv. \end{aligned}$$

In the final sum the terms with $l \neq 1,-1$ vanish thanks to orthogonality between $e^{-2\pi ilv}$ and $\sin(2\pi v)$. We conclude that $a^w(x,D)u_k(x) = 0$ by $\chi((2k-1)\pi) = \chi((2k+1)\pi) = 1$ when l takes ± 1.

Example 2. The above example can be used to produce an operator on $\mathbb{T}^2$ which has an embedded eigenvalue. In [DyZw19], it was shown that the operator $H_0 = \langle D\rangle^{-1}D_{x_2} + 2\sin(2\pi x_1)$ on $\mathbb{T}^2$ has absolutely continuous spectrum near 0. We show that a $\Psi^{-\infty}$ perturbation of H_0 may have an embedded eigenvalue at 0. In fact, choose $\chi(\xi_1, \xi_2) \in C_0^\infty(\mathbb{R}^2)$ such that $\chi((2k-1)\pi, 0) = \chi((2k+1)\pi, 0) = 1$. Let $b(x, \xi) = 2\sin(2\pi x_1)\chi(\xi)$, then $u_{k,0} = e^{2k\pi i x_1}$ satisfies

$$(H_0 - b^w(x, D))u_{k,0} = (2\sin(2\pi x_1) - b^w(x, D))e^{2k\pi i x_1} = 0.$$

For numerical implementations of this construction, see Galkowski-Zworski [GaZw19] and Wang [Wa19]. Those papers also discuss embedded eigenvalues as limits of viscosity eigenvalues (that is, eigenvalues of $P + i\nu\Delta$, $\nu \to 0^+$).

Acknowledgements. This note is based on an undergraduate research project supervised by Maciej Zworski in Berkeley in the spring of 2019. I would like to thank him for introducing this topic and a lot of helpful discussions. I would also like to thank Xi'an Jiaotong University to provide me with the opportunity to study at Berkeley. Thanks also to the anonymous referee for many comments which improved the paper. The research was supported in part by the National Science Foundation grant DMS-1500852.

2. Preliminaries

We include some preliminaries for microlocal analysis in this section. For details, see [GrSj94, Chapter 3,4] or [Zw12, Chapter 4].

Definition 1. *Let X be an open set in $\mathbb{R}^n$, $0 \le \rho, \delta \le 1$. The space of symbols $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ is defined to be the space of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for any compact $K \Subset X$ and $\alpha, \beta \in \mathbb{N}^N$, there is a constant $C_1 = C_{K,\alpha,\beta}(a)$ such that*

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \xi)| \le C_1(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}, (x, \xi) \in K \times \mathbb{R}^N.$$

We also define $\bar{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$ to be the space of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ such that for any $\alpha, \beta \in \mathbb{N}^N$, there is a constant $C_2 = C_{\alpha,\beta}(a)$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \xi)| \le C_2(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}, (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^N.$$

Definition 2. *Let $a \in S^m_{\rho,\delta}(T^*\mathbb{R}^n)$, $u \in \mathcal{E}'(\mathbb{R}^n)$, we define*

$$\mathrm{Op}_t(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(tx + (1-t)y, \xi) e^{i(x-y)\xi} u(y) dy d\xi.$$

Define the standard quantization to be $a(x, D) = \mathrm{Op}(a) = \mathrm{Op}_1(a)$, and the Weyl quantization to be $a^w(x, D) = \mathrm{Op}_{\frac{1}{2}}(a)$. Elements in the space $\Psi^m_{\rho,\delta}(\mathbb{R}^n) := \{\mathrm{Op}(a) | a \in S^m_{\rho,\delta}(T^*\mathbb{R}^n)\}$ are called pseudodifferential operators.

Remark 1. *We have a bijective map*

$$\begin{aligned} \sigma : \Psi^m_{1,0}(\mathbb{R}^n)/\Psi^{m-1}_{1,0}(\mathbb{R}^n) &\to S^m_{1,0}(T^*\mathbb{R}^n)/S^{m-1}_{1,0}(T^*\mathbb{R}^n) \\ A &\mapsto \sigma(A) := e^{-ix\xi} A(e^{ix\xi}) \end{aligned}$$

where $\sigma(A)$ is called the principal symbol of A.

The following proposition tells us that 0-th order pseudodifferential operators are bounded on the L^2 space.

Proposition 3. *Let $a \in \bar{S}^m_{1,0}(T^*\mathbb{R}^n)$, then $\operatorname{Op}(a) : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is bounded.*

We also have similar definitions for manifolds, see for example [Zw12, Section 14.2]. In particular, we recall

Proposition 4. *Let M be a compact manifold, $H \in \Psi^m_{1,0}(M)$ with $m < 0$. Then $H : L^2(M) \to L^2(M)$ is compact.*

3. General result on the essential spectrum

In this section we would like to study the essential spectrum of a self-adjoint pseudodifferential operator. The basic example of such operator is the Weyl quantization of a real function. Following Colin de Verdière's broad outline [CdV18, Theorem 2.1], we provide the following detailed proof.

Proof of Theorem 1. For convenience we set

$$\mathcal{A} = \{\lambda \in \mathbb{C} | \text{ there exists } (x_j, \xi_j) \in T^*M, |\xi_j| \to \infty, \text{ such that } \lim_{j\to\infty} a(x_j, \xi_j) = \lambda\}.$$

It suffices to prove $\operatorname{Spec}_{\text{ess}}(H) = \mathcal{A}$.

First we prove $\operatorname{Spec}_{\text{ess}}(H) \subseteq \mathcal{A}$. If $\lambda \notin \mathcal{A}$, then there exist $\varepsilon, N > 0$ such that $|a(x,\xi) - \lambda| > \varepsilon$ for $|\xi| > N$. Thus, there exists $b \in C^\infty(T^*M)$ such that $b(x,\xi) = \frac{1}{a(x,\xi)-\lambda}$ for $|\xi| > N$ and this gives

$$(H - \lambda) \operatorname{Op}(b) = I - \operatorname{Op}(c)$$

for some $c \in S^{-1}_{1,0}$. By Proposition 4, $\operatorname{Op}(c)$ is compact, so $H - \lambda$ has a right inverse up to a compact operator. Similarly, it has a left inverse up to a compact operator. $H - \lambda$ is therefore a Fredholm operator, and hence its spectrum near 0 is discrete. We see that λ is not in the essential spectrum, proving $\operatorname{Spec}_{\text{ess}}(H) \subseteq \mathcal{A}$.

Then we turn to prove $\mathcal{A} \subseteq \operatorname{Spec}_{\text{ess}}(H)$. If $\lambda \in \mathcal{A}$, then there exists $(x_0, \xi_j), |\xi_j| \to \infty$ such that $a(x_0, \xi_j) \to \lambda$ by the compactness of M. Fix a coordinate chart U and a volume density on M, for $0 \le \chi \in C_0^\infty(U)$ with $\int_M \chi^2(x) = 1$, we have (see Remark 1)

$$(H - \lambda)(\chi e^{ix\cdot\xi_j}) = (a(x, \xi_j) - \lambda)\chi(x)e^{ix\cdot\xi_j} + o(1). \tag{3.1}$$

For $0 < \varepsilon < 1$, we choose χ with $\operatorname{diam}(\operatorname{supp}(\chi)) < \varepsilon/3\|\partial_x a\|_{L^\infty}$, which gives

$$|a(x, \xi) - a(x_0, \xi)| \le \|\partial_x a\|_{L^\infty}|x - x_0| < \frac{\varepsilon}{3}$$

on $\mathrm{supp}(\chi)$. For sufficiently large j such that $|a(x_0,\xi_j)-\lambda| < \frac{\varepsilon}{3}$, we have $|a(x,\xi_j)-\lambda| < \frac{2\varepsilon}{3}$ on $\mathrm{supp}(\chi)$ and

$$\|(a(x,\xi_j)-\lambda)\chi(x)e^{ix\cdot\xi_j}\|_{L^2} < \frac{2\varepsilon}{3}.$$

By (3.1), there exists N_0 such that for any $j > N_0$, we have $\|(H-\lambda)\chi e^{ix\cdot\xi_j}\|_{L^2} < \varepsilon$, where $\|\chi e^{ix\cdot\xi_j}\|_{L^2} = 1$. Hence $H-\lambda$ is not invertible and $\lambda \in \mathrm{Spec}(H)$.

If $\lambda \in \mathrm{Spec}(H)\backslash\mathrm{Spec}_{\mathrm{ess}}(H) = \mathrm{Spec}_{\mathrm{d}}(H)$, then there exists $\delta > 0$ such that $\mathrm{Spec}(H) \cap (\lambda-\delta,\lambda+\delta) = \{\lambda\}$. Make an orthogonal decomposition

$$V = L^2(M) = V_1 \oplus V_2 = \ker(H-\lambda) \oplus \ker(H-\lambda)^{\perp}.$$

Let $P : V \to V_1$ be the corresponding projection and $Q = I-P$. Now $H-\lambda$ is invertible on V_2 with $\|(H-\lambda)^{-1}Q\| \le C = \delta^{-1}$. Given $N \in \mathbb{N}$, select $0 \le \chi_1, ..., \chi_N \in C_0^\infty(U)$ such that $\mathrm{supp}(\chi_k) \cap \mathrm{supp}(\chi_l) = \emptyset(\ k \ne l)$ and

$$\mathrm{diam}(\mathrm{supp}(\chi_k)) < \frac{\varepsilon}{3\|\partial_x a\|_{L^\infty}},$$

$$\int_M \chi_k^2(x) = 1, k = 1, 2, \dots, N.$$

Let $u_k = \chi_k e^{ix\cdot\xi_j}$ for sufficiently large j such that $\|(H-\lambda)u_k\|_{L^2} < \varepsilon$. We have

$$\|Qu_k\| \le C\|(H-\lambda)Qu_k\| = C\|(H-\lambda)u_k\| < C\varepsilon,$$

and then

$$\|u_k\|^2 = \|Pu_k\|^2 + \|Qu_k\|^2 < \|Pu_k\|^2 + C^2\varepsilon^2,$$

which gives $\|Pu_k\|^2 > 1 - C^2\varepsilon^2$. On the other hand, we have

$$|\langle Pu_k, Pu_l\rangle| = |\langle Qu_k, Qu_l\rangle| \le C^2\varepsilon^2$$

for $l \ne k$. Therefore, for any $l \le (1-C^2\varepsilon^2)/C^2\varepsilon^2$, the matrix $(\langle Pu_i, Pu_j\rangle)_{1\le i,j\le l}$ is strictly diagonally dominant and thus nonsingular, i.e. every l elements of $\{Pu_k\}$ form a linearly independent subset of V_1. It follows that

$$\mathrm{dim}V_1 \ge \min\left\{N, \left[\frac{(1-C^2\varepsilon^2)}{C^2\varepsilon^2}\right]\right\},$$

i.e. $\dim\ker H$ can be arbitrarily large. Thus $\dim\ker H = \infty$, contradictory to that $\lambda \in \mathrm{Spec}_{\mathrm{d}}(H)$. □

The essential spectrum has regular structures by the connectedness of the sphere.

Corollary 5. *Let M be a connected compact manifold. Suppose $H \in \Psi^0_{1,0}(M)$ is self-adjoint with the principal symbol $a = \sigma(H) \in S^0_{1,0}(T^*M)$. If the dimension $n \ge 2$, then $\mathcal{A} = \mathrm{Spec}_{\mathrm{ess}}(H)$ is a closed interval. If $n = 1$, then $\mathcal{A} = \mathrm{Spec}_{\mathrm{ess}}(H)$ is the union of two closed intervals.*

Proof. First we consider the case $n \geq 2$. Let $\alpha, \beta \in \mathcal{A}$ and $\alpha < \lambda < \beta$. We will show $\lambda \in \mathcal{A}$.

For any $k \in \mathbb{N}$, there exist $|\xi_{j_k}| > k$, $|\eta_{j_k}| > k$ and $x_0, x_1 \in M$ such that $|a(x_0, \xi_{j_k}) - \alpha| < \frac{1}{k}$ and $|a(x_1, \eta_{j_k}) - \beta| < \frac{1}{k}$ by Theorem 1. Now we can find a continuous path $(\kappa_k(t), \gamma_k(t))$ on T^*M connecting (x_0, ξ_{j_k}) and (x_1, η_{j_k}) with $|\gamma_k(t)| > k$. Let $\alpha + \frac{1}{k} < \lambda < \beta - \frac{1}{k}$, then by connectedness we have $a(\kappa_k(t_k), \gamma_k(t_k)) = \lambda$ for some t_k. Hence we get $|\gamma_k(t_k)| \to \infty$ and $\lim\limits_{k\to\infty} a(\kappa_k(t_k), \gamma_k(t_k)) = \lambda$, which gives $\lambda \in \mathcal{A}$ as desired. Thus $\mathcal{A}$ is a closed interval.

For $n = 1$, we have $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$ with

$$\mathcal{A}_\pm = \{\lambda \in \mathbb{C} | \text{there exists } (x_j, \xi_j) \in T^*M, \xi_j \to \pm\infty, \text{ such that } \lim_{j\to\infty} a(x_j, \xi_j) = \lambda\}.$$

Following the same arguments, each subset described in this union is a closed interval. □

The result is a little different in dimension one since jumping between large positive and negative values for ξ_j cannot cover all intermediate values as in the higher dimensional case since S^0 is disconnected.

Corollary 6. *Under the same assumptions with Corollary 5, and for $n \geq 2$, if there is a $\lambda \in \mathcal{A}$ which is not a limit point of the spectrum, then $\mathcal{A} = \{\lambda\}$.*

In this case we have $\lim\limits_{|\xi|\to\infty} \sup\limits_{x\in M} |a(x,\xi) - \lambda| = 0$. By the spectral theory of self-adjoint operators, $H - \lambda$ has finite rank with $\dim\ker(H - \lambda) = \infty$.

4. Absence of singular continuous spectrum for the circle

By decoding the positivity of commutators, the method of Mourre estimates is very effective in studying the absence of singular spectrum of operators. In this section we would like to study the spectrum of self-adjoint pseudodifferential operators by the following form of Mourre estimates (see for example Cycon-Froese-Kirsch-Simon [CFKS08, Section 4.3] or Perry-Sigal-Simon [PSS81, Theorem 1.1]).

Lemma 7. *Let $H : V \to V$ be a bounded self-adjoint operator on the Hilbert space V. If there exists a self-adjoint operator A such that $[A, H]$ and $[A, [A, H]]$ are bounded, and*

$$\chi(H)[iA, H]\chi(H) \geq \chi_1(H) + K$$

for some compact operator K and nonnegative functions $\chi, \chi_1 \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(\chi_1) \subseteq \operatorname{supp}(\chi)$, then for any interval $[\alpha, \beta] \subseteq \{x : \chi_1(x) > 0\}$, H has at most finitely many eigenvalues in $[\alpha, \beta]$, each with finite multiplicity, and H has no singular continuous spectrum in $[\alpha, \beta]$.

We will also need the following lemmas on almost analytic continuation to help us deal with functional calculus (see Dimassi-Sjöstrand [DiSj99, (8.1), (8.2), Theorem 8.1] respectively).

Lemma 8. *Let $\chi \in C_0^\infty(\mathbb{R})$, then there exists an almost analytic function $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ such that $\tilde{\chi}|_{\mathbb{R}} = \chi$ and $|\bar{\partial}\tilde{\chi}| \leq C_N |\operatorname{Im} z|^N$ for any $N > 0$.*

Proposition 9. *Let H be a self-adjoint operator on a Hilbert space, $\chi \in C_0^2(\mathbb{R})$, $\tilde{\chi} \in C_0^1(\mathbb{C})$ be an extension of χ such that $\bar{\partial}\tilde{\chi} = O(|\operatorname{Im} z|)$, then*

$$\chi(H) = -\frac{1}{\pi}\int \bar{\partial}\tilde{\chi}(z)(z-H)^{-1}dz.$$

The method of Mourre estimates apply perfectly to the dimension one case if we add a homogeneity assumption to the principal symbol as in the statement of Theorem 2.

Proof of Theorem 2. Let $a \in S^0_{1,0}(\mathbb{T}\times\mathbb{R})$ such that $H = \mathrm{Op}(a)$ is self-adjoint. By our assumption a has the form $a(x,\xi) = a_0(x,\xi) + a_1(x,\xi)$, where $a_0(x,\xi) = a_0(x, \frac{\xi}{|\xi|})$ for $|\xi| \geq 1$, and $a_1 \in S^{-1}_{1,0}(\mathbb{T}\times\mathbb{R})$. Let $[\alpha,\beta] \subseteq \mathbb{R}\setminus\mathcal{N}$. Since the essential spectrum is treated in the previous section, if suffices to prove that the spectrum of H inside $[\alpha,\beta]$ is absolutely continuous with possibly finitely many embedded eigenvalues.

The point here is that a_0 has only two parts (corresponding to two directions on $\mathbb{R}$) $a_0(x,1)$ and $a_0(x,-1)$, so that it is easy to construct the Mourre estimate. Since H is self-adjoint we can assume a_0 is a real function. Let $(\alpha,\beta) \Subset (\alpha',\beta') \Subset \mathbb{R}\setminus\mathcal{N}$ be intervals that exclude critical values of the principal symbol.

We now take $b = \phi(x,\xi)\xi \in S^1_{\mathrm{cl}}(\mathbb{T}\times\mathbb{R})$ with the real-valued function $\phi = \partial_x a_0(x,\xi)$. On $\mathbb{T}$ we quantize symbols by (1.1) as in Example 1. Let $A = b^w(x,D)$ be the corresponding self-adjoint operator, then

$$\sigma([iA,H]) = \phi(x,\xi)\partial_x a_0(x,\xi) = |\partial_x a_0(x,\xi)|^2 \geq C > 0$$

on $\{x \in \mathbb{T} : a_0(x,\pm 1) \in (\alpha',\beta')\}$. Then we select $\chi \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\chi) \subseteq (\alpha',\beta')$ and $\chi = 1$ on (α,β). We would like to prove there exists a compact operator K such that $\chi(H)[iA,H]\chi(H) \geq C\chi^2(H) + K$.

Let $\tilde{\chi}$ be the almost analytic approximation of χ as in Lemma 8. Pseudo-differential calculus tells us $((z-a_0)^{-1})(x,D) - (z - a(x,D))^{-1}$ is of order -1 and thus compact by Proposition 4. Then Proposition 9 gives

$$\begin{aligned}\chi(H)u &= -\frac{1}{\pi}\int \bar{\partial}\tilde{\chi}(z)(z-a(x,D))^{-1}u dz \\ &= -\frac{1}{\pi}\int \bar{\partial}\tilde{\chi}(z)((z-a_0)^{-1})(x,D)u dz \\ &+ \frac{1}{\pi}\int \bar{\partial}\tilde{\chi}(z)(((z-a_0)^{-1})(x,D)-(z-a(x,D))^{-1})u dz \\ &= (\chi\circ a_0)(x,D)u + Ku,\end{aligned}$$

for some compact operator K. Therefore

$$\sigma(\chi(H)[iA,H]\chi(H)) = (\chi\circ a_0)^2|\partial_x a_0|^2$$

and $\sigma(\chi^2(H)) = (\chi\circ a_0)^2$. Since $\operatorname{supp}(\chi)$ avoids the critical values of $a_0(x,\pm 1)$, we have $|\partial_x a_0(x,\pm 1)|^2 \geq C > 0$ on $\operatorname{supp}(\chi\circ a_0)$ and then get

$$\chi(H)[iA,H]\chi(H) \geq C\chi^2(H) + K$$

for some compact operator K.

Finally, since $[A,H]$ and $[A,[A,H]]$ are 0-th order pseudodifferential operators, we can apply Lemma 7 to finish the proof. □

5. Absence of singular continuous spectrum for unitary operators

There is a close relationship between self-adjoint operators and unitary operators by Cayley transform. Fernández, Richard and de Aldecoa [FRA13, Theorem 2.7] utilized the method of Mourre estimates to characterize the spectrum of unitary operators.

Lemma 10. *Let U be a unitary operator on a Hilbert space V. Suppose there exist a self-adjoint operator A, a compact operator K, an open interval $\Theta \subseteq S^1$ and $C > 0$ such that $[A,U]$ and $[A,[A,U]]$ are bounded and*

$$E^U(\Theta)U^*[A,U]E^U(\Theta) \geq CE^U(\Theta) + K,$$

where E^U is the spectral projection. Then U has at most finitely many eigenvalues in Θ, each with finite multiplicity, and U has no singular continuous spectrum in Θ.

We can apply this lemma to get an analogy of Theorem 2 for unitary pseudodifferential operators.

Theorem 3. *Assume there exists $a \in S^0_{1,0}(\mathbb{T}\times\mathbb{R})$ such that $U = \operatorname{Op}(a)$ is unitary. Suppose the principal symbol $a_0(x,\xi) = \sigma(U)$ is homogeneous with respect to ξ. Let $\Theta \Subset S^1 \setminus (\{a_0(x,1) : \partial_x a_0(x,1) = 0\} \cup \{a_0(x,-1) : \partial_x a_0(x,-1) = 0\})$ be an open interval, then the spectrum of H inside Θ is absolutely continuous with possibly finitely many eigenvalues.*

Proof. We have $|a_0(x, \frac{\xi}{|\xi|})|^2 - 1 \in S^{-1}_{1,0}(\mathbb{T} \times \mathbb{R})$ since U is unitary. Let $\xi \to \infty$ we get $|a_0(x, \pm 1)|^2 = 1$, so we may assume $a_0(x, \xi)$ takes values on the unit circle $S^1 \subseteq \mathbb{C}$. By taking derivatives, we get $\bar{a}_0 \partial_x a_0 + a_0 \partial_x \bar{a}_0 = 0$. The function $\phi(x, \xi) = a_0(x, \xi)\partial_x \bar{a}_0(x, \xi) \in S^0_{1,0}(\mathbb{T} \times \mathbb{R})$ is then purely imaginary. It follows that $b(x, \xi) = i\phi(x, \xi)\xi \in S^1_{1,0}(\mathbb{T} \times \mathbb{R})$ is a real function, and $A = b^w(x, D)$ is a self-adjoint operator. Now

$$\sigma(U^*[A, U]) = |a_0|^2 |\partial_x a_0|^2 = |\partial_x a_0|^2.$$

We can then apply the same method as before to get the Mourre estimate

$$\chi(H)U^*[A, U]\chi(H) \geq C\chi^2(H) + K$$

for any nonnegative function $\chi \in C_0^\infty(\Theta)$ and some compact operator K. Since $[A, U]$ and $[A, [A, U]]$ are 0-th order pseudodifferential operators, this completes the proof of Theorem 3 by applying Lemma 10. $\square$

Nakamura [Na18, Section 4] considered the unitary scattering matrix $S(\lambda) \in \Psi^0_{1,0}(\mathbb{T} \times \mathbb{R})$ with the principal symbol

$$a(x, \xi) = \exp\left(\frac{ia\pi}{\sqrt{2\lambda}} \sin x \frac{\xi}{\langle \xi \rangle}\right).$$

It provides a good example of unitary operators with absolutely continuous spectrum. The method of Mourre estimates used here is the same as that in [Na18].

References

[CdV18] Y. Colin de Verdière, *Spectral theory of pseudo-differential operators of degree 0 and application to forced linear waves,* preprint, arXiv:1804.03367 [math-ph].

[CdVS20] Y. Colin de Verdière and L. Saint-Raymond, *Attractors for two dimensional waves with homogeneous Hamiltonians of degree 0,* Comm. Pure Appl. Math. **73** (2020): 421-462.

[CFKS08] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry,* Text and Monographs in Physics, Springer-Verlag, 2008.

[DiSj99] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit,* London Mathematical Society Lecture Notes Series **268**, Cambridge University Press, 1999.

[DyZw19] S. Dyatlov and M. Zworski, *Microlocal Analysis of Forced Waves,* Pure and Applied Mathematics **1** (2019), 359384.

[FRA13] C. Fernández, S. Richard, and R. T. de Aldecoa, *Commutator methods for unitary operators,* Journal of Spectral Theory. **3** (2013), 271–292.

[GaZw19] J. Galkowski and M. Zworski, *Viscosity limits for 0th order pseudodifferential operators,* preprint, arXiv:1912.09840 [math.AP].

[GrSj94] A. Grigis and J. Sjöstrand, *Microlocal Analysis for Differential Operators, An Introduction,* London Mathematical Society Lecture Note Series **196**, Cambridge University Press, 1994.

[Na18] S. Nakamura, *Remarks on scattering matrices for Schrödinger operators with critically long-range perturbations,* preprint, arXiv:1804.05489 [math-ph].

[PSS81] By P. Perry, I. M. Sigal and B. Simon, *Spectral analysis of N-body Schrödinger operators,* Annals of Mathematics. **114** (1981), 519–567.

[Wa19] J. Wang, *The scattering matrix for 0th order pseudodifferential operators,* preprint, arXiv:1909.06484 [math.AP].

[Zw12] M. Zworski, *Semiclassical Analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.

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