NOTES ON RANDOM PERTURBATION OF NON-SELF-ADJOINT OPERATORS

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Contents

1. Introduction

This is the notes from Professor Maciej Zworski's Spring 2021 topics course at Berkeley. The primary reference is Sjöstrand's book [Sj19].

1.1. **Motivation from differential equations.** One central problem of PDEs is the stability of the equation under perturbation, in particular, the nonlinear perturbation.

Example 1. *Consider the equation*

$$\partial_t u = Au + F(u), \quad A \in M_{N\times N}(\mathbb{C}),\ F(u) = \mathcal{O}(|u|^\varepsilon).$$

Here F is considered as a small perturbation of the ODE. If $F = 0$, then as long as $\sigma(A)$ has negative real parts, the system is stable. However, let $A = J_N - 1/2$ where J_N is the Jordan block matrix. Let

$$F(u) = \begin{pmatrix} u_1^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then for initial value

$$u_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \varepsilon \end{pmatrix},$$

the system will blow up for $\varepsilon \sim \left(\frac{3}{4}\right)^N$. [Lack of proof or reference here.]

Example 2. *Here is a PDE version of our previous example. Consider the following PDE*

$$\partial_t u = \frac{1}{ih} Pu + au^2$$

where $P = \frac{h}{i}\partial_x + ig(x)$ and $g(x)$ is a real valued smooth function on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For the linear problem, we can simply solve it and get the eigenvalues of P.

$$z = kh + i\hat{g}, \quad k \in \mathbb{Z},\ \hat{g} = \frac{1}{2\pi}\int_{\mathbb{T}} g(x)dx.$$

Suppose $\hat{g} < 0$, then the linear problem is stable. However, for the nonlinear (quadratic) perturbation, the system will blow up. Let us write the equation as

$$(\partial_t + \partial_x)u = \frac{1}{h} g(x)u + bu^2$$

where $b = -\frac{ia}{h}$. *Let*

$$G(a,b) = \int_b^a g(\tau)d\tau,$$

the solution is

$$u(t,x) = \frac{e^{\frac{1}{h}G(x,x-t)}u_0(x-t)}{1 - be^{\frac{1}{h}G(x,x-t)}u_0(x-t)\int_0^t e^{\frac{1}{h}G(x-s,x)}ds}.$$

If $g(x_0) > 0$ *and* bu_0 *is a bump function at* x_0, *we find a blow up at time* $t \sim t^\delta$ *for initial data* u_0 *of size* $e^{-h^{-1+\delta}}$.

Example 3. *Here is another example by.*

$$\partial_t u = (-\partial_x^2 + \partial_x + \frac{1}{8})u + u^2.$$

[Lack of proof or reference here.]

The nonlinear instability is related to the linear stability as shown in the following theorem by Hager.

Theorem 1. *Let*

$$P_\delta = hD_x + ig(x) + \delta Q$$

on $\mathbb{T}^1$ *where* Q *is a random operator with*

$$Qu = \sum_{|k|,|l|\lesssim h^{-1}} \alpha_{k,l}(\omega)\langle u, e_k\rangle e_l(x), \quad e_l(x) = (2\pi)^{-1/2}e^{2\pi ilx}$$

and $\alpha_{k,l}$ *are i.i.d standard Gaussian random variables. Let* $\Gamma \Subset \Omega = \{z \in \mathbb{C} : \min \mathrm{Re} g < \mathrm{Re} z < \max \mathrm{Re} g\}$, *then for* $e^{-h^{-1+\varepsilon}} \leq \delta \leq h^4$, *we have*

$$\sharp(\sigma(P_\delta) \cap \Gamma) \sim \frac{1}{2\pi h}|p^{-1}(\Gamma)|$$

where $p(x,\xi) = \xi + ig(x)$.

2. Random perturbation of Jordan blocks

In this section we are going to prove the following theorem.

Theorem 2. *Let* $J_N : \mathbb{C}^N \to \mathbb{C}^N$ *be the Jordan block matrix, and* Q *is a matrix with entries i.i.d. standard Gaussian random variables. Let* $e^{-N^{1-\varepsilon}} \leq \delta \leq N^{-4}$, *then*

$$\frac{1}{N}\sum_{\lambda\in\sigma(J_N+\delta Q)} \delta_\lambda \to \frac{1}{2\pi}\delta_{S^1}.$$

2.1. **Review of Spectral theory.** In this section we breifly review the basics of spectral theory.

Definition 1. *Let A be a bounded operator on a Banach space, then the resolvant set is*

$$\rho(A) = \{z \in \mathbb{C} : A - z \text{ is inertible } \}$$

and the spectrum of A is

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Roughly speaking, spectral theory is the study of spectrum of linear operators. An important observation from linear algebra suggests some operators behave better from the spectral point of view.

Definition 2. *An bounded operator A on a Hilbert space $\mathcal{H}$ is called*

- *a self-adjoint operator if $A = A^*$,*
- *a unitary operator if $AA^* = A^*A = I$,*
- *a normal operator if $[A, A^*] = AA^* - A^*A = 0$.*

For self-adjoint operators, we have the following psectral theorem.

Theorem 3. *If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then there exists a projection-valued measure $dE(\lambda)$ such that*

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

This is also true for normal operators, since we can always write any normal operator $A = \mathrm{Re}A + i\mathrm{Im}A$, where

$$\mathrm{Re}A = \frac{1}{2}(A + A^*), \quad \mathrm{Im}A = \frac{1}{2i}(A - A^*).$$

Also, self-adjoint operators only have real spectrum. If A is self-adjoint, then by spectral theorem we obtain

$$\|(A - z)^{-1}\| = \frac{1}{d(z, \sigma(A))}.$$

But this is dramatically not true for non-self-adjoint operators, as the following example shows.

Example 4. *Let $J_N \in M_{N\times N}(\mathbb{C})$ be the Jordan block matrix, then*

$$\begin{aligned}(J_N - z)^{-1} &= -z^{-1}\left(1 - \frac{J_N}{z}\right)^{-1} \\ &= -z^{-1}\sum_{k=0}^{N-1} J_N^k z^{-k}\end{aligned}$$

and

$$\|(J_N - z)^{-1}\| \geq |z|^{-N}.$$

To study non-self-adjoint operators, there is a more 'stable' version of spectrum: pseudospectrum.

Definition 3. *Let A be a bounded operator on a Hilbert space, define the ε-pseudospectrum of A as*

$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : \|(A-z)^{-1}\| > \frac{1}{\varepsilon} \right\} \cup \sigma(A).$$

We have the following direct properties.

Proposition 4.
- $\sigma(A) + D(0,\varepsilon) \subset \sigma_\varepsilon(A)$
- *When A is a normal operator, $\sigma_\varepsilon(A) = \sigma(A) + D(0,\varepsilon)$.*

Also, we have the following equivalent definitions.

Proposition 5. *The following are equivalent.*

- *(a) $z \in \sigma_\varepsilon(A)$;*
- *(b) There exists u with $\|u\| = 1$ and $\|(A-z)u\| < \varepsilon$;*
- *(c) There exists an operator B with $\|B\| < 1$ such that $z \in \sigma(A + \varepsilon B)$.*

Proof. Only $(b) \Rightarrow (c)$ is not trivial. But taking

$$Bv = -\frac{(A-z)u}{\varepsilon}(v,u)u$$

would work. □

We also have another property of pseudospectrum.

Proposition 6. *If U is a bounded component of $\sigma_\varepsilon(A)$, then $U \cap \sigma(A)$ is nonempty.*

Proof. We recall a function on $\Omega \subset \mathbb{C}$ is called subharmonic if it is upper semi-continuous and for any h harmonic in $K \Subset \Omega$, $u \leq h$ on ∂K implies $u \leq h$ in K.

By writing

$$\|(A-z)^{-1}\| = \sup_{\|u\|=\|v\|=1} \operatorname{Re}\langle (A-z)^{-1}u, v\rangle$$

as the supremum of a family of harmonic functions, we obtain $\|(A-z)^{-1}\|$ is a sub-harmonic function in z.

If $U \subset \rho(A)$, then $\|(A-z)^{-1}\|$ is subharmonic in U. Since $\|(A-z)^{-1}\| = \varepsilon^{-1}$ on ∂U, by subharmonicity we have

$$\|(A-z)^{-1}\| \leq \frac{1}{\varepsilon} \text{ on } U$$

This is a contradiction. □

2.1.1. *Properties of matrix exponentials.* There is a general theorem by Trefethen-Embree relating matrix exponentials and its spectrum.

Theorem 4.

$$\lim_{t\to+\infty} t^{-1}\log\|e^{tA}\| = \alpha(a) := \max \mathrm{Re}\sigma(A). \tag{2.1}$$

$$\lim_{t\to 0^+} t^{-1}\log\|e^{tA}\| = \omega(a) := \max \sigma(\mathrm{Re}A). \tag{2.2}$$

$$e^{t\alpha(A)} \leq \|e^{tA}\| \leq e^{t\omega(A)}, \quad t \geq 0. \tag{2.3}$$

Proof. To prove (2.1), we write the Jordan normal form $A = VJV^{-1}$, then

$$e^{tA} = Ve^{tJ}V^{-1}.$$

Therefore,

$$\|e^{tA}\| \leq \|V\|\|V^{-1}\|\|e^{tJ}\|$$

and

$$t^{-1}\log\|e^{tA}\| \leq t^{-1}\log(\|V\|\|V^{-1}\|) + t^{-1}\log(t^K e^{t\max\mathrm{Re}\sigma(A)}).$$

Let $t \to +\infty$, we obtain (2.1).

To prove (2.2), we write

$$\lim_{t\to 0^+} t^{-1}\log\|e^{tA}\| = \frac{d}{dt}\|e^{tA}\|_{t=0}.$$

Since

$$\begin{aligned}\|e^{tA}\| &= \|e^{tA}e^{tA^*}\|^{1/2}\\ &= \|1 + t(A+A^*) + \mathcal{O}(t^2)\|^{1/2}\\ &= 1 + t\max\sigma(\mathrm{Re}A) + \mathcal{O}(t^2).\end{aligned}$$

Taking the derivative gives (2.2).

To prove (2.3), suppose $Av = \mu v$ where $\mathrm{Re}\mu = \max\mathrm{Re}\sigma(A)$, then

$$e^{tA}v \leq e^{t\mu}v$$

and then

$$\|e^{tA}\| \geq e^{t\mu}.$$

On the other hand,

$$\begin{aligned}\|e^{tA}\| &\leq \|e^{tA/M}\|^M \\ &= \left(1 + \frac{t}{M}\omega(A) + \mathcal{O}\left(\left(\frac{t}{M}\right)^2\right)\right)^M.\end{aligned}$$

Let $M \to \infty$, we get

$$\|e^{tA}\| \leq e^{t\omega(A)}.$$

□

We conclude this part by a comment that Kreiss matrix theorem gives a surprising bound for the matrix exponentials.

Theorem 5. *Let $A \in M_{N\times N}(\mathbb{C})$ and*

$$K(A) = \sup_{\mathrm{Re} z>0} \mathrm{Re}\|(A-z)^{-1}\|,$$

then

$$K(A) \leq \sup_{t\geq 0} \|e^{tA}\| \leq eNK(A).$$

2.2. **Grushin problem.** An important method in spectral theory is the following Schur's complement formula.

Theorem 6. *Suppose*

$$\begin{pmatrix} P & R_- \\ R_+ & R_{+-} \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}^{-1} : X_1 \times X_- \to X_2 \times X_+ \tag{2.4}$$

are bounded operators on Banach spaces, then P is invertible if and only if E_{-+} is invertible. Moreover, we have

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = R_{+-} - R_+ P^{-1} R_-.$$

Proof. The proof is direct. If E_{-+} is invertible, then

$$PE + R_- E_- = I, \quad PE_+ + R_- E_{-+} = 0,$$

and then

$$PE - PE_+ E_{-+}^{-1} E_- = I.$$

Similarly, since

$$EP + E_+ R_+ = I, \quad E_- P + E_{-+} R_+ = 0,$$

we get

$$EP - E_+ E_{-+}^{-1} E_- P = I.$$

We conclude that P is invertible and $P^{-1} = E - E_+ E_{-+}^{-1} E_-$. The proof for the other case is similar. □

If $R_{+-} = 0$, we have the following observation.

Proposition 7. *If $R_{+-} = 0$ in(2.4), then R_+ and E_- are surjective, and R_- and E_+ are injective.*

Proof. This is because we have

$$R_+ E_+ = I, \quad E_- R_- = I.$$

□

We will call the $R_{+-} = 0$ case a Grushin problem, i.e.

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}^{-1} : X_1 \times X_- \to X_2 \times X_+ \tag{2.5}$$

Perturbation of Grushin problems are stable due to the Neumann series argument.

Proposition 8. *Suppose (2.5) is true, and suppose $A : X_1 \to X_2$ satisfies*

$$\|EA\|_{X_1 \to X_1}, \|AE\|_{X_2 \to X_2} < 1,$$

then the Grushin problem

$$\mathcal{P}_A = \begin{pmatrix} P + A & R_- \\ R_+ & 0 \end{pmatrix}$$

is still well-posed with inverse

$$\begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix}$$

where

$$F_{-+} = E_{-+} + \sum_{k=1}^{\infty} (-1)^k E_- A (EA)^{k-1} E_+.$$

Proof. Let

$$\mathcal{P} = \mathcal{E}^{-1} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}$$

then

$$\mathcal{P}_A = \mathcal{P} \left(1 + \mathcal{E} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right)$$

and

$$\begin{aligned}\mathcal{P}_A^{-1} &= \left(1+\mathcal{E}\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}\right)^{-1}\mathcal{P}^{-1} \\ &= \sum_{k=0}^{\infty}(-1)^k\left(\mathcal{E}\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}\right)^k\mathcal{E} \\ &= \mathcal{E}+\sum_{k=1}^{\infty}(-1)^k\begin{pmatrix} (EA)^k & 0 \\ E_-A(EA)^{k-1} & 0\end{pmatrix}\mathcal{E}.\end{aligned}$$

□

The Grushin problem is closely related to the Fredholm property. The proof is taken from [DyZw19, Appendix C].

Definition 9. *A bounded linear operator $P : X_1 \to X_2$ between two Banach spaces is called a Fredholm operator if the kernel and cokernel of P are both finite dimensional. The index of a Fredholm operator is defined as*

$$\mathrm{ind}P = \dim\,\ker P - \dim\,\mathrm{coker}P.$$

Theorem 7. *(i) Suppose $P : X_1 \to X_2$ is a Fredholm operator. Then there exists finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$ and $R_+ : X_1 \to X_+$ such that the Grushin problem (2.5) is well-posed. In particular, the image of P is closed.*

(ii) Suppose the Grushin problem (2.5) is well-posed, then P is a Fredholm operator if and only if E_{-+} is a Fredholm operator, and

$$\mathrm{ind}P = \mathrm{ind}E_{-+}.$$

Proof. (i) Let $n_+ = \dim\,\ker P$ and $n_- = \dim\,\mathrm{coker}P$. Let $X_\pm = \mathbb{C}^\pm$. Suppose $\ker P$ is spanned by $x_1, \cdots, x_{n_+}$, by Hahn-Banach theorem there exists $x_j^* : X_1 \to \mathbb{R}$ such that $x_j^*(x_i) = \delta_{ij}$. We then define

$$R_+ : X_1 \to \mathbb{C}^{n_+}, \quad x \mapsto (x_1^*(x), \cdots, x_{n_+}^*(x)).$$

On the other hand, choose a representative $y_1, \cdots, y_{n_-}$ of $\mathrm{coker}P$ and define

$$R_- : \mathbb{C}^{n_-} \to X_2, \quad (a_1, \cdots, a_{n_-}) \mapsto \sum_{j=1}^{n_-} a_j y_j.$$

We claim the operator

$$\begin{pmatrix} P & R_- \\ R_+ & 0\end{pmatrix}$$

is bijective. First, if

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = 0,$$

then since the range of P and R_- does are disjoint, we have $Pu = R_-u_- = 0$, so $u_- = 0$ and $u \in \ker P$. By $R_+u = 0$ we conclude $u = 0$. So it is injective. On the other hand, $(R, R_-) : X_1 \times X_- \to X_2$ is surjective by definition. Since modifying $u \in \ker P$ does not affect value of Pu, we conclude the whole matix is also surjective.

Finally, PX_1 can be viewed as the image of the closed subspace $(X_1, 0)$ under the invertible map $(P, R_+) \pmod{\ker P}$. So the image of P is closed.

(ii) Take $u_- = 0$, we observe that

$$Pu = v \iff u = Ev + E_+v_+,\ 0 = E_-v + E_{-+}v_+. \tag{2.6}$$

So $E_- : PX_1 \to E_{-+}X_+$ and induces

$$E_-^\sharp : X_2/PX_1 \to X_-/E_{-+}X_+.$$

By Proposition 7, E_- is surjective, so $E_-^\sharp$ is surjective. On the other hand, $E_-v \in E_{-+}X_+$ will give us $v \in PX_1$ by (2.6), so $E_-^\sharp$ is also injective. We conclude

$$\dim\ \mathrm{coker} P = \dim\ \mathrm{coker} E_{-+}.$$

Now we look at

$$E_+ : \ker E_{-+} \to \ker P.$$

It is injective by 7. Moreover, if $u \in \ker P$, then by (2.6) we get $v_+ \in \ker E_{-+}$ such that $E_+v_+ = u$, so E_+ is also surjective. We conclude

$$\dim\ \ker P = \dim\ \ker E_{-+}.$$

This finishes the proof of (ii). □

Corollary 10.
- *The family of Fredholm operators is open and the index map is locally constant, i.e.*

$$\mathrm{ind} : \pi_0(\mathrm{Fred}(\mathcal{H}_1, \mathcal{H}_2)) \to \mathbb{Z}.$$

- *If K is a compact operator, then* $\mathrm{ind}\,(I + K) = 0$.
- *Fredholm operator has closed image.*

2.2.1. *Fredholm theory.* Here we provide several examples of Fredholm operators.

Example 5. *Suppose $P : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear map between two finite dimensional spaces, then P is Fredholm with*

$$\mathrm{ind}\, P = \dim \mathcal{H}_1 - \dim \mathcal{H}_2.$$

Example 6. *Let K be a compact operator on $\mathcal{H}$, then $I + K$ is a Fredholm operator.*

- *The unit ball in* $\ker(I+K)$ *is compact, so* $\ker(I+K)$ *is finite dimensional.*
- *For* $w \in \ker(I+K)^{\perp}$, $\|(I+K)w\| \geq C\|w\|$. *Suppose the opposite, then there exists* $\|w_n\| = 1$ *with* $\|(I+K)w_n\| \leq 1/n$. *Suppose* $Kw_n \to v$, *this gives* $w_n \to v$ *and* $v \in \ker(I+K)$, *a contradiction.*
- $\operatorname{im}(I+K)$ *is closed.*
- $\operatorname{coker}(I+K)$ *is finite dimensional since*
$$\dim \operatorname{coker}(I+K) = \dim \operatorname{im}(I+K)^{\perp} = \dim \ker(I+K^*) < \infty.$$
- *We hace proved* $I+K$ *is compact. There are many examples f compact examples in PDEs, e.g. finite rank operators,* $H^1(M) \to L^2(M)$ *compact embedding.*

Proposition 11. *Suppose* K *is a compact operator on a infinite dimensional Hilbert space* $\mathcal{H}$, *then* $\sigma(K) = \{\lambda_j\} \cup \{0\}$ *where* $\lambda_j \to 0$.

Proof. If K is invertible, then $\mathcal{H}$ is finite dimensional. So $0 \in \sigma(K)$. It suffices to prove the spectrum is isolated outside $\{0\}$.

Suppose $\lambda_0 \in \sigma(K) \setminus \{0\}$, then
$$K - \lambda_0 = -\lambda_0(I - \frac{1}{\lambda_0}K)$$
and
$$(K-\lambda)^{-1} = E(\lambda) - E_+(\lambda)E_{-+}(\lambda)^{-1}E_-(\lambda)$$
if $\det E_{-+}(\lambda)$ is not zero. But this is a meromorphic function, so there is only two possibilities: either vanish in an isolated set of points, or vanish identically. If it vanishes identically, then $\sigma(K)$ is the whole $\mathbb{C}$, contradictory to that K is bounded. So the only possibility is it is isolated. □

Example 7. $P - z : D_x + q(x) - z : H^1(S^1) \to L^2(S^1)$ *is a Fredholm operator since*
$$P - z = (I + (q - z + i)(D_x - i)^{-1})(D_x - i)$$
is a composition of Fredholm operators.

Corollary 12. *An operator* $P : \mathcal{H}_1 \to \mathcal{H}_2$ *is Fredholm if and only if it is invertible modulo compact operators.*

Proof. If P is Fredholm, then by the Grushin problem we get
$$PE = I - R_-E_-, \quad EP = I - E_+R_+,$$
i.e. P is invertible modulo finite rank operators. On the other hand, if
$$PE = I + K_1, \quad EP = I + K_2,$$
then
$$\operatorname{im}(I+K_1) \subset \operatorname{im}P, \quad \ker P \subset \ker(I+K_2),$$

which tells us P is Fredholm. $\square$

Remark 1. *If $P - z : \mathcal{H}_1 \to \mathcal{H}_2$ is Fredholm for all $z \in \mathbb{C}$, then $\sigma(P)$ is either empty or the whole $\mathbb{C}$.*

We can use Grushin problem to simplify the question. Let us give an example.

Example 8. *Let $P = J_N$ be the N-dimensional Jordan block matrix. Then the Grushin problem*

$$\begin{pmatrix} J_N & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is well-posed for

$$R_- = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad R_+ = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For $J_N - z$ we define

$$O(z) = \begin{pmatrix} -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ 1 & 0 & 0 & \cdots & 0 & -z \end{pmatrix} = \tilde{J}_{N+1} - z, \quad \tilde{J}_{N+1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Since $\tilde{J}_{N+1}$ is unitary, $\tilde{J}_{N+1} - z$ is invertible for $|z| < 1$, and

$$E_{-+}(z) = \frac{z^N}{1 - z^{N+1}}.$$

There is a lemma we will use.

Lemma 13. *Suppose we have a family of operators*

$$\mathcal{P}(z) = \begin{pmatrix} P(z) & R_-(z) \\ R_+(z) & R_{+-}(z) \end{pmatrix} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}^{-1} : X_1 \times X_- \to X_2 \times X_+ \tag{2.7}$$

well-posed on $\overline{\Omega} \subset \mathbb{C}$, $\mathcal{P}(z)^{-1} = \mathcal{E}(z)$. Suppose moreover $P(z)$ is invertible on $\partial\Omega$, then

$$\operatorname{tr} \int_{\partial\Omega} P(z)^{-1} dP(z) = \operatorname{tr} \int_{\partial\Omega} E_{-+}(z)^{-1} dE_{-+}(z).$$

Proof. When $\mathcal{P}(z)$ is holomorphic, there is an easy proof. Since $\partial_z\mathcal{P}(z) = -\mathcal{P}(z)\partial_z\mathcal{E}(z)\mathcal{P}(z)$, we have

$$\begin{aligned}
\mathrm{tr}\int_{\partial\Omega} P(z)^{-1}dP(z) &= \mathrm{tr}\int_{\partial\Omega}(E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z))dP(z) \\
&= \mathrm{tr}\int_{\partial\Omega} -E_+(z)E_{-+}(z)^{-1}E_-(z)(P(z)dE(z)P(z) + R_-(z)dE_-(z)P(z) \\
&+ P(z)dE_+(z)R_+(z) + R_-(z)dE_{-+}(z)R_+(z)).
\end{aligned}$$

The first three terms vanish because e.g.

$$E_-(z)P(z) + E_{-+}(z)R_-(z) = 0$$

gives

$$\mathrm{tr}E_{-+}(z)^{-1}E_-(z)P(z) = -\mathrm{tr}R_-(z)$$

so that $E_{-+}(z)^{-1}$ is eliminated. By similar methods and

$$E(z)P(z) + E_+(z)R_+(z) = I, \quad P(z)E(z) + R_-(z)E_-(z) = I,$$

the last term becomes

$$\mathrm{tr}\int_{\partial\Omega} E_{-+}(z)^{-1}dE_{-+}(z).$$

Now we prove for the general case. We define a contour deformation by

$$\begin{aligned}
\tilde{\mathcal{P}}(z,s) &= \begin{pmatrix} P(z) & \cos sR_-(z) \\ \cos sR_+(z) & \sin^2 sE_{-+}(z)^{-1} + \cos^2 sR_{+-}(z)\end{pmatrix} \\
&= \begin{pmatrix} P(z) & 0 \\ \cos sR_+(z) & I\end{pmatrix}\begin{pmatrix} I & \cos sP(z)^{-1}R_-(z) \\ 0 & E_{-+}(z)^{-1}\end{pmatrix}, \quad 0 \le s \le \frac{\pi}{2}.
\end{aligned}$$

Then

$$\tilde{\mathcal{P}}\left(z, \frac{\pi}{2}\right) = \begin{pmatrix} P(z) & 0 \\ 0 & E_{-+}(z)^{-1}\end{pmatrix}, \quad \tilde{\mathcal{P}}(z, 0) = \mathcal{P}(z).$$

By the following crucial algebraic property

$$\mathrm{tr}d(P(z)^{-1}dP(z)) = 0$$

for any family of operators, we obtain

$$\begin{aligned}
0 &= \mathrm{tr}\int_{\partial\Omega\times[0,\pi/2]} d(\tilde{\mathcal{P}}^{-1}d\tilde{\mathcal{P}}) \\
&= \mathrm{tr}\int_{\partial\Omega}\tilde{\mathcal{P}}\left(z,\frac{\pi}{2}\right)^{-1}d\tilde{\mathcal{P}}\left(z,\frac{\pi}{2}\right) - \mathrm{tr}\int_{\partial\Omega}\tilde{\mathcal{P}}(z,0)^{-1}d\tilde{\mathcal{P}}(z,0),
\end{aligned}$$

i.e.

$$\mathrm{tr}\int_{\partial\Omega}\mathcal{P}^{-1}d\mathcal{P} = \mathrm{tr}\int_{\partial\Omega}\begin{pmatrix} P(z) & 0 \\ 0 & E_{-+}(z)^{-1}\end{pmatrix}^{-1} d\begin{pmatrix} P(z) & 0 \\ 0 & E_{-+}(z)^{-1}\end{pmatrix}.$$

The left hand side vanishes since $\mathcal{P}$ is invertible, and a simple computation gives

$$\operatorname{tr}\int_{\partial\Omega} P(z)^{-1}dP(z) = \operatorname{tr}\int_{\partial\Omega} E_{-+}(z)^{-1}dE_{-+}(z).$$

□

2.3. **Review of Probability theory.** In the section we review basics of probability theory.

Definition 14. *A probability space is a triple $(\Omega, \mathcal{M}, \mu)$ where Ω is a set, $\mathcal{M}$ is a σ-algebra of Ω, and μ is a measure on $\mathcal{M}$ such that $\mu(\Omega) = 1$ (we will call it a probability measure).*

Like Tao pointed out in [Ta12, Section 1.1], probability theory are considering concepts which are preserved under extension. Here an extension of the probability space $(\Omega, \mathcal{M}, \mu)$ is another probabillity space $(\Omega', \mathcal{M}', \mu')$ along with a measurable map $\pi : \Omega' \to \Omega$, such that $\pi_*\mu' = \mu$. For example, we define

Definition 15. *A random variable X on the probability space $(\Omega, \mathcal{M}, \mu)$ is a measurable map from Ω to another measure space $(R, \mathcal{R})$. When $(R, \mathcal{R}) = (\mathbb{R}_+, \mathcal{B})$ ($\mathcal{B}$ is the Borel algebra), we define the expectation to be*

$$\mathbb{E}X = \int_\Omega X(\omega)d\mu(\omega).$$

The famous Borel-Cantelli lemma is an important tool to prove, e.g., convergence.

Lemma 16. *Suppose a sequence E_n satisfies*

$$\sum_{n=1}^{\infty} P(E_n) < \infty,$$

then $P(\limsup E_n) = 0$, i.e. any element appears in at most finitely many E_n, almost surely.

Now let us give several definitions of asymptotic validity of events.

Definition 17. *Suppose we have a sequence $E_n \in \mathcal{M}$.*

- *The events E_n holds almost surely (a.s.) if $P(E_n) = 1$*
- *E_n holds with overwhelming probability (w.o.p) if $P(E_n) > 1 - \mathcal{O}(n^{-\infty})$*
- *E_n holds with high probability (w.h.p) if there exists $\delta > 0$ such that $P(E_n) > 1 - \mathcal{O}(n^{-\delta})$*
- *E_n holds asymptotically if $P(E_n) \to 1$.*

Example 9. *(a) If $\mathbb{E}|X_n| \leq C$, then $|X_n| = \mathcal{O}(n^\varepsilon)$ with high probability.*

(b) If $\mathbb{E}|X_n|^k \leq C_K$ for each $k \in \mathbb{N}$, then $|X_n| = \mathcal{O}(n^\varepsilon)$ with overwhelming probability.

2.3.1. *Independence.*

Definition 18. *A family of random variables $\{X_\alpha\}$ is called jointly independent if the distribution of $\{X_\alpha\}$ is the product measure of individual X_α's.*

Example 10. *Let $M = M_{N\times N}(\mathbb{C})$ and $\mathcal{M}$ be the Borel algebra, then the following distribution*

$$d\mu_N = \prod_{i,j=1}^{N} \frac{1}{\pi} e^{-|a_{ij}|^2} dm(a_{ij})$$

gives a random matrix with independent elements.

2.3.2. *Convergence.*

Definition 19. *Suppose $X_n, X : M \to (R, d)$ are random variables with value in a σ-compact metric space. Define*

- *$X_n \to X$ almost surely (a.s.) if $P(\limsup d(X_n, x) \leq \varepsilon) = 0$ for any $\varepsilon > 0$;*
- *$X_n \to X$ in probability if $\liminf P(d(X_n, X) \leq \varepsilon) = 1$ for any $\varepsilon > 0$;*
- *$X_n \to X$ in distribution if $\mu_{X_n} \to \mu_X$ weakly.*

Proposition 20. *For the three kinds of convergence, we have* (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. If $X_n \to X$ a.s., then by Fatou's lemma

$$\liminf P(d(X_n, X) \leq \varepsilon) \geq \int \liminf \mathbb{1}_{d(X_n,X)\leq\varepsilon} = 1.$$

If $X_n \to X$ in probability, then for any $f \in C(R)$, $\varepsilon > 0$ suppose $d(x_n, x) < \delta$ gives $|f(x_n) - f(x)| < \varepsilon$.

$$\begin{aligned}\left|\int f d\mu_{X_n} - \int f d\mu\right| &= \left|\int (f(X_n) - f(X)) d\mu\right| \\ &\leq \varepsilon + \int_{d(X_n,X)>\delta} 2\|f\|_\infty d\mu.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \left|\int f d\mu_{X_n} - \int f d\mu\right| \leq \varepsilon$$

for any $\varepsilon > 0$, and this gives $\int f d\mu_{X_n} \to \int f d\mu$. □

Remark 2. *The space of probability measures can be given the Lévy-Prkhorov metric*

$$d(\mu, \nu) = \inf\{\alpha > 0 : \mu(A) \leq \nu(A + D(0, \alpha)) + \alpha,\ \ \nu(A) \leq \mu(A + D(0, \alpha)) + \alpha\}$$

which gives weak convergences.

We also have other metrics, e.g., Wasserstein distance or Kantorovich–Rubinstein metric.

Example 11. *For a sequence of events E_n,*

- $\mathbb{1}_{E_n} \to 0$ *a.s. if and only if* $P(E_n) \to 0$*;*
- $\mathbb{1}_{E_n} \to 0$ *in probabillity if and only if* $P(\cup_{n>k} E_k) \to 0$.

In probability theory, it is sometimes difficult to prove a.s. convergence directly. One useful tool is the Borel Cantelli lemma:

Proposition 21. *If for any $\varepsilon > 0$ we have*

$$\sum_{n\to\infty} P(d(X_n, X) \geq \varepsilon) < \infty,$$

then $X_n \to X$ a.s.

Proof. By Borel-Cantelli lemma,

$$P(\limsup\{d(X_n, X) \geq \varepsilon\}) = 0.$$

□

2.4. **Proof.** In this section we finish the proof of Theorem 2. For this purpose we need a tool called logrithmic potential.

Definition 22. *Let $\mathcal{P}(\mathbb{C})$ be the set of probability measures satisfying*

$$\int \log(1 + |z|^2) d\mu(z) < \infty,$$

we define the logrithmic potential of $\mu \in \mathcal{P}(\mathbb{C}$ to be

$$U_\mu(z) = \int \log|z - w| d\mu(w).$$

Proposition 23. *For $\nu \in \mathcal{P}(\mathbb{C})$, we have*

- $U_\nu \in L^1_{loc}(m)$*. In particular, $U_\nu(z) > -\infty$ a.e.*
- $\Delta U_\nu = 2\pi\nu$*.*

Proof. The first statament is simple. For any $R > 0$, we have

$$\int_{|z|<R} |U_\nu(z)| dm(z) \leq \int_{|w|<2R} \left(\int_{|z|<R} |\log|z - w|| \, dm(z) \right) d\nu(w) + C_R \int \log(1 + |w|^2) d\nu(w) < \infty.$$

For the second statement, we only need to prove for $\nu = \delta$, i.e., for any $\phi \in C_c^\infty(\mathbb{C})$ we have

$$\int \log|z - w| \Delta\phi(z) dm(z) = 2\pi\phi(w).$$

This is due to the fundamental solution of the 2-d Laplace equation, which can be checked as follows.

$$\begin{aligned}\int \log|z-w|\Delta\phi(z)dm(z) &= \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus D(0,\varepsilon)} (\log|z-w|\Delta\phi(z) - \Delta\log|z-w|\phi(z))dm(z)\\ &= \lim_{\varepsilon\to 0}\int_{D(0,\varepsilon)} (-\log|z-w|n\partial_x\phi(z) + \frac{(z-w)\cdot(z-w)}{|z-w|^3}\phi(z))dm(z)\\ &= 2\pi\phi(w).\end{aligned}$$

□

We can recover the probability measure from the logrithmic potential as follows.

Lemma 24. *Let $\nu_n, \nu \in \mathcal{P}(\mathbb{C})$ be random measures, and* $\operatorname{supp}\nu_n \subset \Omega \Subset \Omega' \Subset \mathbb{C}$. *If for a.e. $z \in \Omega'$, $U_{\nu_n}(z) \to U_\nu(z)$ almost surely, then*

$$\nu_n \to \nu \quad a.s.$$

Proof. The crucial point is to notice U_{ν_n} is uniformly bounded in L^2,since

$$\int_{\Omega'} |U_\nu(z)|dm(z) \le \int\int_{\Omega'} (\log|z-w|)^2\, dm(z)d\nu(w) \le C.$$

The result then follows from the fact a.e. convergence + L^2 boundedness implies L^1 convergence. Then for any $\phi \in C_c^\infty(\Omega')$, we have

$$\int U_{\nu_n}\Delta\phi \to \int U_\nu\Delta\phi,$$

that is

$$\int \phi d\nu_n \to \int \phi d\nu.$$

□

Lemma 25. *Let Q be i.i.d standard Guassian $N\times N$ matrix, then*

$$P(\|Q\|_{HS} \ge CN) \le \exp((\log 2 - \frac{1}{2}C^2)N^2).$$

Proof.

$$\begin{aligned}P(\sum|Q_{ij}|^2 \ge (CN)^2) &\le \mathbb{E}(\exp(\frac{1}{2}(\sum|Q_{ij}|^2 - (CN)^2)))\\ &= e^{-\frac{1}{2}(CN)^2}\prod_{i,j}\mathbb{E}e^{\frac{1}{2}|Q_{ij}|^2}\\ &= e^{-\frac{1}{2}(CN)^2}2^{N^2}\\ &= \exp((\log 2 - \frac{1}{2}C^2)N^2).\end{aligned}$$

□

Corollary 26. • *For* $C \gg 1$, $\|Q\|_{HS} \le CN$ *w.o.p.*
- *For* $\delta \le N^{-2}$, $\nu_N = N^{-1} \sum_{\lambda \in \sigma(J_N+\delta Q)} \delta_\lambda$, *we have* $\operatorname{supp} \nu \subset D(0,2)$ *w.o.p.*

Define $E_N = \{z \in \mathbb{C} : d(z, S^1) > \frac{1}{N}\}$, then for $z \notin E_N$, we have

$$\begin{aligned} \|\mathcal{P}^\delta(z)\| &= \left\| \left(\mathcal{P}(z) + \begin{pmatrix} \delta Q & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1} \right\| \\ &= \left\| \left(I + \mathcal{P}(z)^{-1} \begin{pmatrix} \delta Q & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1} \mathcal{P}(z)^{-1} \right\| \\ &\le (1 - \delta \|\mathcal{P}(z)^{-1}\| \|Q\|)^{-1} \|\mathcal{P}(z)^{-1}\| \\ &\le (d(z, S^1))^{-1} (1 - \delta d(z, S^1) CN)^{-1} \\ &\le N(1 - C\delta N^2)^{-1}, \quad w.o.p. \end{aligned}$$

So $\|\mathcal{P}^\delta(z)^{-1}\| \le CN$ for $\delta \ll N^{-2}$, $z \notin E_N$, w.o.p. It follows that $|E_{-+}^\delta(z)| \le CN$ and

$$\frac{1}{N} \log |E_{-+}^\delta(z)| \lesssim \frac{\log N}{N}, w.o.p.$$

On the other hand, for $z \in D(0,2) \setminus E_N$ we have

$$E_{-+}^\delta(z) = E_{-+}(z) - \delta E_- Q E_+ - \delta \sum_{j=1}^{\infty} E_- Q (-\delta E Q)^j E_+,$$

which implies

$$\|E_{-+}^\delta(z)\| \ge \|E_{-+}(z) - \delta E_- Q E_+\| - \delta^2 C N^5 (1 - \delta C N^2)^{-1}.$$

Moreover,

$$\begin{aligned} E_- Q E_+ &= \left(\frac{1}{1 - z^{N+1}} \right)^{-2} \sum_{j,k=0}^{N-1} z^{j+k} \alpha_{jk} \\ &\sim \mathcal{N}_{\mathbb{C}} \left(0, \frac{(1 - |z|^{2N})^2}{|1 - z^{N+1}|^4 (1 - |z|^2)^2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(|E_{-+}(z) - \delta E_-(z) Q E_+(z)| \le t) &\le \mathbb{P}(|\mathrm{Re}(E_{-+}(z) - \delta E_-(z) Q E_+(z))| \le t) \\ &\le \mathbb{P}(|\mathrm{Re}(\delta E_-(z) Q E_+(z))| \le t) \\ &\le \frac{1}{\sqrt{\pi}} \int_{|x| \delta (1 - |z|^2)^{-1} \lesssim t} e^{-x^2} dx \\ &= \mathcal{O}\left(\frac{t}{\delta} \right). \end{aligned}$$

Let $\frac{t}{\delta} = N^{-2+\varepsilon}$, then

$$|E^{\delta}_{-+}(z)| \geq \delta N^{-2+\varepsilon} - \mathcal{O}(\delta^2 N^{-5}) = \delta(N^{-2+\varepsilon} - \mathcal{O}(\delta N^5)).$$

If we take $\delta \leq N^{-7}$, then $|E^{\delta}_{-+}(z)| \geq \delta N^{-2+\varepsilon}$ with probability $1 - \mathcal{O}(N^{-2+\varepsilon})$. Under these assumptions we get

$$\frac{1}{N} \log |E^{\delta}_{-+}(z)| \geq -N^{-\varepsilon}.$$

In conclusion, for $z \notin S^1$, we have

$$-N^{-\varepsilon} \leq \frac{1}{N} \log |E^{\delta}_{-+}(z)| \leq \frac{\log N}{N}$$

with probability $1 - \mathcal{O}(N^{-2+\varepsilon})$. By Borel-Cantelli lemma we get convergence a.s.

Finally, we observe that

$$\sigma(\tilde{J}_{N+1} + \delta Q) = \{\omega^k + \mathcal{O}(\delta \|Q\|)\},$$

and

$$\frac{1}{N} \sum_{k=0}^{N-1} \log |\omega^k + \mathcal{O}(\delta C N) - z| \to \frac{1}{2\pi} \int \log |e^{i\theta} - z| d\theta.$$

So

$$\frac{1}{N} \log |\det(J_N + \delta Q - z)| \to \frac{1}{2\pi} U_{\delta_{S^1}}(z), \quad a.s.$$

which means

$$\frac{1}{N} \sum_{\lambda \in \sigma(J_N + \delta Q)} \delta_\lambda \to \frac{1}{2\pi} \delta_{S^1}, \quad a.s.$$

3. Random perturbation of differential operators

3.1. **Unbounded operators.** There is a need to study unbounded operators (of course) in infinite dimensional spaces as the following example shows.

Example 12. *In quantum mechanics, we have the Heisenberg uncertainty principle:*

$$[A, B] = I.$$

This is impossible for bounded operators by

$$[A^n, B] = nA^{n-1}$$

and

$$n\|A^{n-1}\| \leq 2\|A\|\|B\|\|A^{n-1}\|.$$

Example 13. *Let us look at the operator*

$$P = p(x)\partial_x + q(x)$$

acting on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, *where* $p(x), q(x) \in C^\infty(S^1; \mathbb{C})$ *and* $p(x) \neq 0$. *Then the equation* $(P - z)u = f$ *has solution*

$$u = e^{z\alpha(x)-\beta(x)} \int_0^x e^{-z\alpha(y)-\beta(y)} f(y)dy + ce^{z\alpha(x)-\beta(x)}$$

if $z\alpha(2\pi) - \beta(2\pi) \notin 2\pi i\mathbb{Z}$, *where*

$$\alpha(x) = \int_0^x \frac{1}{p(y)} dy, \quad \beta(x) = \int_0^x \frac{q(y)}{p(y)} dy.$$

When $z\alpha(2\pi) - \beta(2\pi) \in 2\pi i\mathbb{Z}$, $e^{z\alpha(x)-\beta(x)}$ *is an eigenfunction of* P *with eigenvalue* z, *so*

$$\sigma(P) = \{z \in \mathbb{C} : z\alpha(2\pi) - \beta(2\pi) \in 2\pi i\mathbb{Z}\}.$$

When $\alpha(2\pi) \neq 0$, *the spectrum is given by* $\alpha(2\pi)^{-1}(\beta(2\pi) + 2\pi i\mathbb{Z})$. *When* $\alpha(2\pi) = 0$, *the spectrum is empty when* $\beta(2\pi) \notin 2\pi i\mathbb{Z}$ *and is* $\mathbb{C}$ *when* $\beta(2\pi) \in 2\pi i\mathbb{Z}$.

We now give the definition of an unbounded operator.

Definition 27. $P : H_1 \to H_2$ *is called an unbounded operator if there exists a linear subspace* $D(P) \subset H_1$ *and a linear map* $P : D(P) \to H_2$. P *is called densely defined if* $D(P)$ *is dense in* H_1.

We will be particlarly interested in closed operators defined as follows.

Definition 28. *The graph of an unbounded operator* $P : H_1 \to H_2$ *is*

$$G(P) = \{(x, Px) : x \in D(P)\} \subset H_1 \times H_2.$$

P *is closed if the graph is closed.* P *is closurable if* $\overline{G(P)}$ *is the graph of an operator* $\bar{P}$.

The closed graph theorem says a closed operator P with $D(P) = H_1$ is bounded. Now we can also define the adjoint of an operator.

Theorem 8. *Suppose* $P : H_1 \to H_2$ *is a densely defined operator. Then tehre exists* $P^* : H_2 \to H_1$ *with*

$$D(P^*) = \{v \in H_2 : \forall u \in D(P), u \mapsto \langle Pu, v\rangle \text{ is bounded }\},$$

and

$$\langle Pu, v\rangle = \langle u, P^*v\rangle, \quad u \in D(P), v \in D(P^*).$$

Example 14. *If $P = D_x + q$ on S^1 with $D(P) = H^1(S^1)$ has adjoint*

$$P^* = D_x + \bar{q}, \quad D(P^*) = H^1(S^1).$$

Definition 29. *Let A, B be two unbounded operators, say $A \subset B$ if $G(A) \subset G(B)$.*

Proposition 30. *Let A be densely defined, then $A \subset B \Rightarrow B^* \subset A^*$.*

Definition 31. *An unbounded operator A is symmetric if $A \subset A^*$. A is called self-adjoint if $A = A^*$.*

It is important to notice an unbounded operator may have different self-adjoint extensions.

Example 15. *Let $P = D_x$ with $D(P) = C_0^\infty((0,1))$, then*

$$D(P^*) = H^1((0,1)), \quad D(P^{**}) = \bar{P} = H_0^1((0,1))$$

are the maximal and minimal closed extensions. Then

$$D(P_\theta) = \overline{\{u \in C^\infty([0,1]) : u(1) = u(0)e^{2\pi i\theta}\}}$$

gives an infinite family of self-adjoint extensions. Those self-adjoint extensions are not unitarily equivalent since

$$\sigma(P_\theta) = 2\pi(\theta + \mathbb{Z}).$$

We have the following theorem by von Neumann.

Theorem 9. *Let T be closed, densely defined operator on a Hilbert space $\mathcal{H}$. Then the operator*

$$T^*T : D(T^*T) \to \mathcal{H}$$

given by

$$D(T^*T) = \{u : u \in D(T), Tu \in D(T^*)\}$$

is self-adjoint.

Definition 32. *Let T be closed, densly defined, T^* densly defined, we say T is normal if $TT^* = T^*T$.*

Proof.

Lemma 33. *Let*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}.$$

Let P be a densely defined operator on $\mathcal{H}$, then

$$J(G(P))^\perp = G(P^*).$$

Proof of the Lemma.

$$J(G(P))^{\perp} = \{(u_1, u_2) : \forall u \in D(P), \langle (u_1, u_2), (-Pu, u)\rangle = 0\}$$

gives $\langle u_2, u\rangle = \langle u_1, Pu\rangle$, which means $u_1 \in D(P^*)$ and $u_2 = P^* u_1$. So $(u_1, u_2) \in G(P^*)$. The other direction is obvious. □

Corollary 34. *If P is densely defined and closed, then*

$$\mathcal{H} \times \mathcal{H} = J(G(P)) \oplus G(P^*).$$

Now we decompose

$$(0, u) = (v - Tv', T^* v + v'), \quad v \in D(T^*), v' \in D(T).$$

Then $v = Tv'$, which means $v' \in D(T^*T)$. Let $S = I + T^*T$, then $u = Sv'$. So S has an inverse. Since S^{-1} is a bounded symmetric operator, it is self-adjoint.

Now we claim $D(T^*T) = \operatorname{Im} S$ is dense:

$$\begin{aligned}(\operatorname{Im} S)^{\perp} &= \{u : \langle S^{-1}v, u\rangle = 0\}\\ &= \{u : \langle v, S^{-1}u\rangle = 0\}\\ &= \{u : S^{-1}u = 0\}\\ &= 0.\end{aligned}$$

Finally we need to prove $D(S^*) = D(S)$:

$$D(S^*) = \{v \in \mathcal{H} : \forall u \in D(S) : |\langle Su, v\rangle| \lesssim \|u\|\}$$

we can find v_0 such that $Sv_0 = S^* v$. Moreover,

$$\langle Su, v_0\rangle = \langle Su, v\rangle$$

gives $v_0 = v$, so $D(S^*) = D(S)$. □

Theorem 10. *Suppose $P : \mathcal{H} \to \mathcal{H}$ is a densely defined self-adjoint operator, then*

$$\emptyset \neq \sigma(P) \subset \mathbb{R}, \quad \|(P - z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}.$$

Proof.

$$|\langle (P - z)u, u\rangle| \geq |\operatorname{Im} z| \|u\|^2, \quad |\langle (P - z)^* u, u\rangle| \geq |\operatorname{Im} z| \|u\|^2$$

implies

$$\|(P - z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}.$$

If $\sigma(P) = \emptyset$, then

$$(P^{-1} - z)^{-1} = z(z^{-1} - P)^{-1} P^{-1}$$

can only be singular at $z = 0$, so P^{-1} is a bounded operator with $\sigma(P^{-1}) = \{0\}$, a contradiction. □

3.2. **Hager's theorem.** Hager proves this theorem in her Ph.D. thesis.

Theorem 11. *Suppose $P_\delta = hD_x + ig(x) + \delta Q$, where $g(x) \in C^\infty(S^1, \mathbb{R})$ having exactly two critical points and*

$$Q = \sum_{j,k \lesssim h^{-1}} \alpha_{jk}(\omega) e_j(x) e_k(x),$$

where $e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx}$ and $\alpha_{j,k}$ are i.i.d. standard Gaussian distributions, and $e^{-\frac{C}{h}} \leq \delta \leq h^K$ for some large K, then for $p(x,\xi) = x + i\xi$, $\varepsilon = h \log\left(\frac{1}{\delta}\right)$ and $\Omega \Subset p(\mathbb{C})$, we have

$$\sharp \sigma(P_\delta) \cap \Omega = \frac{1}{2\pi h} \text{Area}\, p^{-1}(\Omega) + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h})$$

wtih probability $\geq 1 - \mathcal{O}(\frac{\delta^2}{\sqrt{\varepsilon} h^2})$.

3.3. **Semiclassical analysis.** To prove Hager's theorem, we need a little bit of semiclassical analysis, which, roughly speaking, studies the following quantum-classical correspondence

$$p(x, \xi; h) = \sum_{k \leq m} a_k(x; h) \xi^k \mapsto P = \sum_{k \leq m} a_k(x; h)(hD_x)^k.$$

We assume a_k has an expansion

$$a_k(x; h) \sim \sum a_k^j(x) h^j$$

and define the principal symbol to be

$$\sigma(P) = \sum_{k \leq m} a_k^0(x) \xi^k.$$

Let us look at an example.

Example 16. *Let*

$$P = \begin{pmatrix} J_N & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be the Jordan block matrix, then

$$Pe_1 = 0, \quad [P, P^*]e_1 = e_1$$

and

$$P^* e_N = 0, \quad [P, P^*]e_N = -e_N.$$

This example suggests we should look for the solution of $Pu = 0$ at the place where $p(x, \xi) = 0$ and $\{\mathrm{Re}p, \mathrm{Im}p\} < 0$. This can also be seen from

$$\|Pu\|^2 = \|\mathrm{Re}Pu\|^2 + \|\mathrm{Im}Pu\|^2 + i\langle[\mathrm{Re}P, \mathrm{Im}P]u, u\rangle.$$

(Oppositely, this will give unique continuation results.)

Let us apply this intuition to our case.

Lemma 35. *For Hager's operator P and for any $z \in \Omega$ as in Hager's theorem, there exists $u \in C^\infty(S^1)$ supported near $x_+(z)$ with $\|u\|_{L^2} = 1$ such that*

$$(P - z)u = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Here $x_+(z)$ satisfies $g(x) = \mathrm{Im}z$ and $g'(x) < 0$.

Proof. We can write

$$(P - z)u = (hD_x + ig(x) - z)u = e^{i\phi/h}(hD_x)(e^{-i\phi/h}u)$$

where

$$\phi_+(x) = \int_{x_+(z)}^{x} (z - ig(y))dy.$$

Now since $\phi_+(x_+(z)) = \mathrm{Im}\phi_+'(x_+(z)) = 0$, we have

$$\mathrm{Im}\phi_+(x) \sim -g'(x_+(z))(x - x_+(z))^2$$

near $x_+(z)$. Now we put

$$\tilde{u}(x) = \chi(x - x_+(z))e^{i\phi_+(x)/h},$$

then

$$(P - z)\tilde{u} = e^{-\phi_+(x)/h}\frac{h}{i}\chi'(x - x_+(z)) = \mathcal{O}(e^{-\frac{1}{Ch}})$$

since $\mathrm{Im}\phi_+(x) > c > 0$ on $\mathrm{supp}\,\chi'(x)$.

Finally we need to estimate the L^2-norm of $\tilde{u}$, which follows form the following stationary phase lemma

Lemma 36. *Suppose $a \in C_0^\infty(\mathbb{R}), \phi \in C^\infty(\mathbb{R})$ such that*

- *$\phi(x) > 0$ for $x \neq 0$*
- *$\phi(0) = \phi'(0) = 0$*
- *$\psi''(0) > 0$*

then

$$\int a(x)e^{-\psi(x)/h}dx \sim \sqrt{\frac{2\pi h}{\psi''(0)}}(a_0 + b_1h + b_2h^2 + \cdots).$$

Proof. With out loss of generality we can assume $\operatorname{supp} a \subset (-\delta, \delta)$. Also write $\phi(x) = \frac{1}{2}f(x)^2$ and $f'(0) = \psi''(0)^{\frac{1}{2}}$. Then

$$\begin{aligned} I &= \int a(x(y))x'(y)e^{-\frac{y^2}{2h}}dy \\ &= \frac{1}{2\pi}\int \hat{b}(\xi)h^{\frac{1}{2}}\sqrt{2\pi}e^{h\xi^2/2}d\xi \\ &= \sqrt{\frac{h}{2\pi}}\sum \frac{h^k}{k!}\int \hat{b}(\xi)\left(-\frac{\xi^2}{2}\right)^k d\xi \\ &= \sqrt{\frac{h}{2\pi}}(2\pi b(0) + b_1 h + \cdots) \\ &= \sqrt{\frac{2\pi h}{\psi''(0)}}(a_0 + b_1 h + \cdots). \end{aligned}$$

□

□

Definition 37. *The approximate solution constructed in 35 is called WKB approximate solution.*

Now let $Q = (P-z)^*(P-z)$ and $\tilde{Q} = (P-z)(P-z)^*$ be self-adjoint operators on $L^2(S^1)$, where $D(Q) = D(\tilde{Q}) = H^2(S^1)$ and

$$(Q-i)^{-1}, (\tilde{Q}-i)^{-1} : L^2(S^1) \to H^2(S^1)$$

are isomorphisms. We conclude that $\sigma(Q), \sigma(\tilde{Q})$ are discrete, and tends to ∞. Moreover, Q and $\tilde{Q}$ are Fredholm oprators of index 0, so $1 \geq \dim \ker Q = \dim \ker(P-z) = \dim \ker(P-z)^* = \dim \ker \tilde{Q}$. Therefore, Q ahd $\tilde{Q}$ have the same spectrum at 0. They of course have same eigenvalues outside 0, so

$$\sigma(Q) = \sigma(\tilde{Q}) = \{t_0^2, t_1^2, \cdots\},$$

where $0 \leq t_0 < t_1 < \cdots$.

Proposition 38.

$$t_0 = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Proof. We know $Qe_{WKB} = \mathcal{O}(e^{-\frac{1}{Ch}})$, so

$$(0, \sigma(Q)) = \|Q^{-1}\|^{-1} = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

□

Proposition 39.

$$t_1^2 - t_0^2 \geq \frac{h}{C}.$$

Proof. Step 1 There exists some eigenfunction e_0 of the eigenvalue t_0^2 such that

$$\|e_0 - e_{WKB}\| = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Suppose we have

$$(P - z)e_0 = v,$$

then $\|v\|^2 = \langle Qv, v\rangle = t_0^2 = \mathcal{O}(e^{-\frac{1}{Ch}})$, and

$$\begin{aligned} e_0(x) &= c_0(z,h)h^{-\frac{1}{4}}a(h)e^{\frac{i}{h}\phi_+(x)} + \frac{1}{h}\int_{x_+(z)}^{x} e^{\frac{i}{h}(\phi_+(x)-\phi_+(y))}v(y)dy \\ &= c_0(z,h)h^{-\frac{1}{4}}a(h)e^{\frac{i}{h}\phi_+(x)} + Kv. \end{aligned}$$

Since $|e^{\frac{i}{h}(\phi_+(x)-\phi_+(y))}| \sim e^{-\frac{|x-y|}{h}}$ away from $x_\pm(z)$ and $|e^{\frac{i}{h}(\phi_+(x)-\phi_+(y))}| \gtrsim e^{-\frac{|x-y|^2}{h}}$ near $x_\pm(z)$, we have

$$\int |K(x,y)|dx, \int |K(x,y)|dy \lesssim \frac{1}{h}\int_0^1 e^{-\frac{t^2}{h}}dt \sim h^{-\frac{1}{2}}.$$

By Schur's lemma we know $\|k\| = \mathcal{O}(h^{-\frac{1}{2}})$. Therefore $Kv = \mathcal{O}(e^{-\frac{1}{Ch}})$ and

$$e_0 = e_{WKB} + \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Step 2 We need to prove for $u \perp e_0$,

$$\langle Qu,, u\rangle \geq \frac{h}{C}\|u\|^2.$$

Recall

$$u = c_0 h^{-\frac{1}{4}}a(z,h)e^{\frac{i}{h}\phi_+(x)} + Kv,$$

and

$$\begin{aligned} 0 &= \langle u, e_0\rangle \\ &= c_0\langle h^{-\frac{1}{4}}a(z,h)e^{\frac{i}{h}\phi_+(x)}, e_{WKB}\rangle + \mathcal{O}(e^{-\frac{1}{Ch}})\|u\| + \mathcal{O}(h^{-\frac{1}{2}})\|v\|. \end{aligned}$$

This implies

$$|c_0| = \mathcal{O}(e^{-\frac{1}{Ch}})\|u\| + \mathcal{O}(h^{-\frac{1}{2}})\|v\|$$

and then

$$\|u\| = \mathcal{O}(h^{-\frac{1}{2}})\|v\|.$$

Thus

$$\|(P-z)u\| \geq \frac{\sqrt{h}}{C}\|u\|$$

and

$$\langle Qu, u\rangle \geq \frac{h}{C}\|u\|^2,$$

□

There is a conjecture by Zelditch.

Conjecture 1. *Let $\phi \in \mathbb{R}[x_1, x_2, \cdots, x_n]$, if for $\Omega \subset \mathbb{R}^n$ and any $a \in C_0^\infty(\Omega)$ we have*

$$\int a(x)e^{i\phi(x)/h}dx = \mathcal{O}(h^\infty),$$

then $\nabla\phi \neq 0$ in Ω.

Now suppose $(P-z)e_j = \alpha_j f_j$, then $(P^* - \bar{z})f_j = \beta_j e_j$. Moreover, we have

$$\alpha_j\beta_j = t_j^2 \quad \alpha_j = \overline{\beta_j},$$

so without loss of generality we can assume $\alpha_j = \beta_j = t_j$.

Now we can construct a Grushin problem.

Theorem 12. *Suppose $R_+ : H^1(S^1) \to \mathbb{C}$ and $R_- : \mathbb{C} \to L^2(S^1)$ are defined as follows*

$$R_+u = \langle u, e_0\rangle, \quad R_-u_- = u_-f_0,$$

then

$$\mathcal{P}(z) = \begin{pmatrix} P-z & R_- \\ R_+ & 0 \end{pmatrix} : H_h^1(S^1) \times \mathbb{C} \to L^2(S^1) \times \mathbb{C}$$

is invertible with

$$\mathcal{P}(z)^{-1} = \mathcal{E}(z) := \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

where

$$\|E\|_{L^2 \to H_h^1} = \mathcal{O}(\frac{1}{\sqrt{h}}),\ \|E_\pm\| = \mathcal{O}(1),\ \|E_{-+}\| = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Here

$$\|u\|_{H_h^1}^2 = \|u\|_{L^2}^2 + \|hD_xu\|_{L^2}^2.$$

Moreover, we have the following explicit formula.

$$E_+v_+ = v_+e_0, \quad E_-v = \langle v, f_0\rangle.$$

Proof. The proof is simple. For any $(v, v_+) \in L^2 \times \mathbb{C}$, we want to find $(u, u_-) \in H^1_h \times \mathbb{C}$ such that

$$\begin{cases} (P-z)u + R_- u_- = v, \\ R_+ u = v_+. \end{cases}$$

Suppose $v = \sum v_j f_j$, $u = \sum u_j e_j$, then

$$u_0 = v_+,\ u = v_+ e_0 + \sum_{j \geq 1} \frac{v_j}{t_j} e_j,\ u_- = v_0 - t_j v_+.$$

This tells us $\mathcal{P}(z)$ is invertible, and

$$E_+ v_+ = v_+ e_0, \quad E_- v = v_0 = \langle v, f_0 \rangle.$$

The bounds follows from the spectral estimates. □

Remark 3. *The operator $Q(z) = (P-z)^*(P-z)$ is not holomorphic, so the Grushin problem is also not holomorphic. To overcome this difficulty, we need the following technique.*

Proposition 40. *Let $f_+ = (\partial_{\bar{z}} R_+) E_+$ and $f_- = E_- \partial_{\bar{z}} R_-$, then*

$$\partial_{\bar{z}} E_{-+}(z) + f(z) E_{-+}(z) = 0.$$

Proof. This follows from the formula

$$\partial_{\bar{z}} \mathcal{E}(z) = -\mathcal{E}(z) \partial_{\bar{z}} \mathcal{P}(z) \mathcal{E}(z).$$

□

To compute $f(z)$, we need to use the approximate solution e_{WKB}. So we need the following lemma.

Lemma 41.

$$e_0 = e_{WKB} + \mathcal{O}(e^{-\frac{1}{Ch}})$$

holds with all derivatives $\partial_z, \partial_{\bar{z}}$.

Proof. Let $\Pi(z) : L^2(S^1) \to \mathbb{C} e_0$ be the orthogonal projection, so that $e_0 = \alpha(z) \Pi(z) e_{WKB}$ with $\alpha(z) = 1 + \mathcal{O}(e^{-\frac{1}{Ch}})$. We claim

$$\|\partial_z^\alpha \partial_{\bar{z}}^\beta \Pi(z)\|_{L^2 \to L^2} = \mathcal{O}(h^{-N_{\alpha,\beta}}). \tag{3.1}$$

This follows form the projection formula

$$\Pi(z) = \frac{1}{2\pi i} \int_\gamma (w - Q(z))^{-1} dw$$

and the spectral gap tells us

$$\|(w - Q(z))^{-1}\|_{L^2} = \mathcal{O}(h^{-1}).$$

We claim we actually have

$$\|(w-Q(z))^{-1}\|_{L^2\to H^1_h}=\mathcal{O}(h^{-1}).$$

Recall $Q=(hD_x)^2-2(\mathrm{Re}z)hD_x+a(x)$ for some real smooth function $a(x)$, we get

$$\begin{aligned}\langle Qu,u\rangle &= \|hD_xu\|^2-2\mathrm{Re}z\langle hD_xu,u\rangle+\langle au,u\rangle\\ &\geq \frac{1}{2}\|hD_xu\|^2-C\|u\|^2.\end{aligned}$$

Thus,

$$|\langle (Q-w)u,u\rangle|+\|u\|^2\geq \frac{1}{C}\|hD_xu\|^2.$$

This justifies

$$\|(w-Q(z))^{-1}\|_{L^2\to H^1_h}=\mathcal{O}(h^{-1}).$$

Now

$$\begin{aligned}\partial_{\bar z}(w-Q(z))^{-1}&=(w-Q)^{-1}\partial_{\bar z}Q(w-Q)^{-1}\\ &=(w-Q)^{-1}(-hD_x+\partial_{\bar z}a)(w-Q)^{-1}\end{aligned}$$

and

$$\|\partial_{\bar z}(w-Q(z))^{-1}\|\leq \|(w-Q)^{-1}\|\|(-hD_x+\partial_{\bar z}a)(w-Q)^{-1}\|=\mathcal{O}(h^{-2}).$$

We can proceed similarly to justify (3.1).

Now our lemma follows easily: First $\partial_z^\alpha\partial_{\bar z}^\beta e_{WKB}$ is of tempered growth, then by our estimate of $\partial_z^\alpha\partial_{\bar z}^\beta\Pi(z)$, e_0 is also of tempered growth. Then since e_0-e_{WKB} is small, we get $\partial_z^\alpha\partial_{\bar z}^\beta(e_0-e_{WKB})=\mathcal{O}(e^{-\frac{1}{Ch}})$ be interpolation

$$|f'(0)|\leq C_\varepsilon(\|f\|^{\frac12}_{L^\infty(-\varepsilon,\varepsilon)}\|f''\|^{\frac12}_{L^\infty(-\varepsilon,\varepsilon)}+\|f\|_{L^\infty(-\varepsilon,\varepsilon)}).$$

□

Lemma 42.

$$\mathrm{Re}\Delta F=4\mathrm{Re}\partial_z f=\frac{2}{h}\left(\frac{1}{\frac{1}{i}\{p,\bar p\}(\rho_+)}-\frac{1}{\frac{1}{i}\{p,\bar p\}(\rho_-)}\right)+\mathcal{O}(1).$$

Proof. Recall $f_+=(e_0,\partial_z e_0)=(e_{WKB},\partial_z e_{WKB})+\mathcal{O}(e^{-\frac{1}{Ch}})$. A direct calculation shows that

$$\begin{aligned}(e_{WKB},\partial_z e_{WKB})&=-\frac{i}{h}\overline{\partial_z\phi_+(x_+(z),z)}+\mathcal{O}(1)\\ &=\frac{i}{h}\xi_+(z)\partial_{\bar z}x_+(z)+\mathcal{O}(1).\end{aligned}$$

So

$$\mathrm{Re}\partial_z f_+=\mathrm{Re}\frac{i}{2h}\partial_{\bar z}x_+(z)+\mathcal{O}(1).$$

A similar computation for f_- proves the lemma. □

Corollary 43.

$$\mathrm{Re}\Delta F dy \wedge dx = \frac{1}{h}(d\xi_+ \wedge dx_+ - d\xi_- \wedge dx_-).$$

3.3.1. *The Grushin problem.* To prove Hager's theorem, we set up the following Grushin problem.

$$\mathcal{P}^\delta(z) = \begin{pmatrix} P - z + \delta Q & R_- \\ R_+ & 0 \end{pmatrix}.$$

The following lemma is similar to the one we proved before.

Lemma 44.

$$\|Q\|_{HS} \le \frac{C}{h}$$

with probability $\ge 1 - \mathcal{O}(e^{-\frac{1}{Ch^2}})$.

Now we know $\|\mathcal{P}(z)\| = \mathcal{O}(h^{-1/2})$, so for $\|\delta Q\| \ll \sqrt{h}$ we have $\mathcal{P}^\delta(z)$ is invertible. A direct calculation shows that

$$E^\delta = E + \mathcal{O}\left(\frac{\delta}{h^2}\right)$$
$$E_+^\delta = E_+ + \mathcal{O}\left(\frac{\delta}{h^{\frac{3}{2}}}\right)$$
$$E_-^\delta = E_- + \mathcal{O}\left(\frac{\delta}{h^{\frac{3}{2}}}\right)$$
$$E_{-+}^\delta = E_{-+} - \delta E_- Q E_+ + \mathcal{O}\left(\frac{\delta^2}{h^{\frac{5}{2}}}\right).$$

Lemma 45.

$$|\widehat{e_{WKB}}(k)| = \mathcal{O}\left(\left(\frac{h}{|k|}\right)^\infty\right).$$

Proof. The crucail thing is

$$e_{WKB} \approx h^{-\frac{1}{4}} e^{-\frac{x^2}{h}}.$$

A direct calculation shows that

$$\partial_x^n e_{WKB}(x) \lesssim h^{-\frac{1}{4}} \left(\left(\frac{x}{h}\right)^n + h^{-\frac{n}{2}}\right) e^{-\frac{x^2}{h}}$$

and

$$\begin{aligned}\int e_{WKB}(x)e^{-ikx}dx &= \frac{1}{k^n}\int D_x^n e_{WKB}(x)e^{-ikx}dx\\ &\lesssim h^{-\frac{1}{4}}k^{-n}h^{-\frac{n}{2}}\\ &\lesssim h^{-\frac{1}{4}}h^{\frac{n}{4}}|k|^{-\frac{n}{4}}.\end{aligned}$$

□

Corollary 46.

$$E_-QE_+ \sim \mathcal{N}_{\mathbb{C}}(0, 1-\mathcal{O}(h^\infty)).$$

Proof. This is because

$$\begin{aligned}E_-QE_+ &= \langle f_0, Qe_0\rangle\\ &= \sum_{|k|,|j|\le\frac{C}{h}} \alpha_{jk}(\omega)\hat{f}_0(j)\overline{\hat{e}_0(k)}\\ &\sim \mathcal{N}_{\mathbb{C}}(0, \sum_{|k|,|j|\le\frac{C}{h}} |\hat{e}_0(k)|^2|\hat{f}_0(j)|^2).\end{aligned}$$

□

Now we have

Proposition 47. *For* $0<t\ll 1$, $0<\delta\ll h^{\frac{3}{2}}$, $\delta t \gg e^{-\frac{1}{Ch}}$, $t\gg\frac{\delta}{h^{\frac{5}{2}}}$, *we have*

- *"*$\forall z\in\Omega, |E^\delta_{-+}(z)|\le e^{-\frac{1}{Ch}}+\frac{C\delta}{h}$*", with probability* $\ge 1-\mathcal{O}(e^{-\frac{1}{Ch}})$.
- $\forall z\in\Omega$, *"*$|E^\delta_{-+}(z)|\ge\frac{t\delta}{C}$*, with probability* $\ge 1-\mathcal{O}(t^2)-\mathcal{O}(e^{-\frac{1}{Ch}})$*".*

Proof. This follows from

$$E^\delta_{-+} = E_{-+} - \delta E_-QE_+ + \mathcal{O}\left(\frac{\delta^2}{h^{\frac{5}{2}}}\right).$$

□

3.3.2. *Counting zeros of holomorphic functions.* Now we can estimate the zeros of $E^\delta_{-+}(z)$ by the following lemma due to Hager-Sjöstrand.

Theorem 13. *Let* $\Omega\Subset\tilde{\Omega}\Subset\mathbb{C}$, $\partial\Omega$ *is smooth.* $\varphi\in C^2(\tilde{\Omega})$, $z\mapsto u(z,h)$ *is a holomorphic function in* $\tilde{\Omega}$, $0<\varepsilon\ll 1$. *Suppose*

- $|u(z,h)|\le\exp(\frac{1}{h}(\varphi(z)+\varepsilon))$, *for* $z\in\text{nbhd}(\partial\Omega)$.
- $z_1, z_2, \cdots, z_n\in\partial\Omega$, $z_j=z_j(h)$, $N\sim\frac{1}{\sqrt{\varepsilon}}$, *and* $\partial\Omega\subset\cup_j D(Z_j,\sqrt{\varepsilon})$, *such that*

$$|u(z_j,h)|\ge\exp(\frac{1}{h}(\varphi(z)-\varepsilon)).$$

Then

$$\sharp u^{-1}(0) \cap \Omega = \frac{1}{2\pi h} \int_{\Omega} \Delta \varphi dm(z) + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}).$$

This theorem follow from the local version of Hadamard's factorization theorem.

Theorem 14. *Suppose $f(z)$ is a holomorphic function in $|z| \le 2R$ and $|f(z)| \le M$ for $|z| \le 2R$. Also, $|f(0)| \ge M^{-1}$. Then there exists $C > 0$ idependent of R such that*

$$f(z) = e^{i\theta} e^{g(z)} \prod_{j=1}^{N} (z - z_j), \quad |z| \le R,$$

where z_j are zeros of f in $|z| \le \frac{3R}{2}$, and

$$N \le C \log M, \quad |g(z)| \le C \log M (1 + \log \langle R \rangle).$$

Proof. We will use three steps to prove this theorem.

Step 1: Jensen's formula.

$$\log |f(0)| + \int_0^r \frac{N(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

Suppose $f(z)$ does not no zeros in $|z| \le r$, then it follows directly from the fact that $\operatorname{Re} \log f(z)$ is a harmonic function.

If $f(z)$ has no zero on the circle $|z| = r$, then we can apply the formula to

$$\tilde{f}(z) = \prod_{j=1}^{N} \frac{r^2 - z\bar{z}_j}{r(z - z_j)} f(z)$$

and get the desired formula. Finally, the case when there are zeros on the circle $|z| = r$ follows by continuity.

The estiamte for the number of zeros $N \le C \log M$ follows directly from Jensen's formula. But to find a bound for $g(z)$, we need a lower bound for the polynomial $\prod_{j=1}^{N} (z - z_j)$, which is obtained by the following Cartan's lemma.

Step 2: Cartan's lemma.

Lemma 48. *Let μ be a finite Radon measure on $\mathbb{C}$ and consider the logrithmic potential of μ:*

$$u(z) = \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta).$$

Then for any $0 < \eta < 1$, there exists a set of discs C_j of radii r_j, s.t.

- $\sum_j r_j < 5\eta$
- *For $z \notin \cup C_j$, $|u(z)| \ge \mu(\mathbb{C}) \log \frac{\eta}{e}$.*

For polynomials, the constant 5 can be replaced by 2.

Proof. We only prove for the polynomial case, since this is the case we will be using. Let $Z = \{z_j\}$ with multiplicity, and set

$$\mathcal{C} = \{D(z, \lambda\frac{\eta}{N})\} : \sharp Z \cap D(z, \lambda\frac{\eta}{N}) = \lambda\}.$$

If we take discs near the boundary of the convex hall of Z, it is easy to see $\mathcal{C}$ is not empty. Now let $\lambda_1 = \max\{\lambda : D(z, \lambda\frac{\eta}{N}) \in \mathcal{C}\}$. Then we observe

$$\lambda > \lambda_1 \Rightarrow \sharp Z \cap D(z, \lambda\frac{\eta}{N}) < \lambda.$$

Now let C_1 be a disc of radius $\lambda\frac{\eta}{N}$ such that $\sharp Z \cap C_1 = \lambda_1$ (we call the points of rank λ_1), and let $Z_1 = Z \setminus C_1$. For this new Z_1, we can repeat the procedure and get smaller and smaller discs $C_2, C_3, \cdots, C_k$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, $\sum \lambda_i = N$. Now let $\tilde{C}_j$ be the concentric discs with C_j with twice radii. We have

$$\begin{aligned} z \notin \bigcup_1^p \tilde{C}_j \Rightarrow D(z, \lambda\frac{\eta}{N}) \bigcap \bigcup_{\lambda \leq \lambda_j} C_j = \emptyset \\ \Rightarrow \text{ rank of points in } D(z, \lambda\frac{\eta}{N}) < \lambda \\ \Rightarrow \sharp Z \cap D(z, \lambda\frac{\eta}{N}) \leq \lambda - 1. \end{aligned}$$

Suppose

$$|z - z_1| \leq |z - z_2| \leq \cdots \leq |z - z_N|,$$

then

$$\sharp Z \cap D(z, \lambda\frac{\eta}{N}) \leq \lambda - 1 \Rightarrow |z - z_j| \geq \frac{j\eta}{N}.$$

Thus

$$\prod_j |z - z_j| \geq \prod_j \frac{j\eta}{N} \geq \left(\frac{\eta}{N}\right)^N N! \geq \left(\frac{\eta}{e}\right)^N.$$

□

Step 3: Borel-Carathéodory inequality.

For a holomorphic function $g(z)$ in $|z| \leq R$, and $|z| = r < R$, we have the following Borel-Carathéodory inequality.

$$|g(z)| \leq \frac{2r}{R - r} \max_{|z| \leq R} \mathrm{Re} g(z) + \frac{R + r}{R - r} |g(0)|.$$

To prove the lemma, we can first assume $g(0) = 0$ without loss of generality, then let

$$u(z) = \frac{g(z)}{2 \max_{|z| \leq R} \mathrm{Re} g(z) - g(z)},$$

we have

$$u(0)=0 \text{ and } |u(z)|^2=\frac{|g(z)|^2}{(2\max\limits_{|z|\le R}\mathrm{Re}g(z)-\mathrm{Re}g(z))^2+(\mathrm{Im}g(z))^2}\le 1.$$

By Schwarz lemma we have

$$|u(z)|\le\frac{|z|}{R}$$

and then

$$|g(z)|\le\frac{|z|}{R}\left|2\max_{|z|\le R}\mathrm{Re}g(z)-g(z)\right|\Rightarrow|g(z)|\le\frac{2r}{R-r}\max_{|z|\le R}\mathrm{Re}g(z).$$

The final step to to apply the Borel-Carathéodory inequality to $g(z)$ given by the decomposition

$$f(z)=e^{g(z)}\prod_j(z-z_j)$$

Since

$$\begin{aligned}\mathrm{Re}g(z)&\le\log|f(z)|-\log|\prod_j(z-z_j)|\\&\le C\log M-C\log\left(\frac{\eta}{e}\right)^N\\&\le C(1+\log\left(\frac{\eta}{e}\right))\log M\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}g(0)&\ge\log|f(0)|-\sum_{j=1}^N\log|z_j|\\&\ge -C(1+\log\langle R\rangle)\log M.\end{aligned}$$

Proof of Theorem 13. Let $i\varphi_j(z)=\varphi(z_j)+2\partial_z\varphi(z_j)(z-z_j)$, then

$$\varphi(z)=\mathrm{Re}\,(i\varphi_j(z))+\mathcal{O}((z-z_j)^2)$$

and

$$\partial_z\varphi_j(z)=\frac{2}{i}\partial_z\varphi(z)+\mathcal{O}((z-z_j)).$$

Let

$$v_j(z)=u(z)e^{-i\varphi_j(z)/h},$$

then

$$e^{-\frac{C\varepsilon}{h}}\le|v_j(z)|\le e^{\frac{C\varepsilon}{h}}$$

in the disc $D(z_j, C\sqrt{\varepsilon})$. Let $f(z) = v_j(z_j + \sqrt{\varepsilon}(z - z_j))$, by our previous lemma we get

$$f(z) = e^{i\theta} e^{g(z)} \prod_{j=1}^{N} (z - z_j)$$

for $N \lesssim \frac{\varepsilon}{h}$ and $|g(z)| \lesssim \frac{\varepsilon}{h}$. Now the number of zeros of $u(z)$ in Ω is

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u'(z)}{u(z)} dz &= \frac{1}{2\pi i} \sum_j \int_{\gamma_j} \left(\frac{i}{h} \varphi_j'(z) + \frac{v_j'(z)}{v_j(z)} \right) dz \\ &= \frac{1}{2\pi h} \int_{\partial\Omega} \frac{2}{i} \partial_z \varphi(z) dz + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}) \\ &= \frac{1}{2\pi h} \int_{\Omega} \Delta\varphi(z) dm(z) + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}). \end{aligned}$$

□

□

Lemma 49. *Let $u(z) = e^{F^\delta(z)} E_{-+}^\delta(z)$, then the zeros of $u(z)$ coincides with eigenvalues of P^δ with multiplicity.*

Proof. By Lemma 13, we have

$$\begin{aligned} \lim_{\gamma \to z_0} \operatorname{tr} \int_\gamma P^\delta(z)^{-1} dP^\delta(z) &= \lim_{\gamma \to z_0} \int_\gamma E_{-+}^\delta(z)^{-1} dE_{-+}^\delta(z) \\ &= \lim_{\gamma \to z_0} \int_\gamma (e^{F^\delta(z)} E_{-+}^\delta(z))^{-1} d(e^{F^\delta(z)} E_{-+}^\delta(z)). \end{aligned}$$

□

Proof of Hager's theorem. Use Theorem 13 for $\varphi(z) = hF(z)$ and $\varepsilon = h \log(\frac{1}{\delta})$, then

$$\begin{aligned} \sharp u^{-1}(0) \cap \Omega &= \frac{1}{2\pi h} \int_\Omega \Delta\varphi dm(z) + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}) \\ &= \frac{1}{2\pi h} \int_\Omega (d\xi_+ \wedge dx_+ - d\xi_- \wedge dx_-) + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}) \\ &= \frac{1}{2\pi h} \int_{p^{-1}(\Omega)} d\xi \wedge dx + \mathcal{O}(\frac{\sqrt{\varepsilon}}{h}). \end{aligned}$$

□

4. Higher oder generalizations

4.0.1. *Examples.* Consider the operator

$$P = \partial_x (\sin x) \partial_x + \partial_x.$$

The spectrum is discrete on the imaginary axis.

4.1. **Basic constructions.** Let

$$P(x, hD_x, h) = \sum_{h \leq m} b_k(x, h)(hD_x)^k,$$

we want to use WKB method to find an approximate eigenvalue.

Lemma 50.

$$p\sharp q = \sum_l \frac{1}{l!} \partial_\xi^l p(x, \xi, h)(hD_x)^l q(x, \xi, h).$$

Proof.

$$\begin{aligned} p\sharp q(x, \xi, h) &= PQ1 \\ &= e^{-\frac{-ix\xi}{h}} P e^{\frac{ix\xi}{h}} e^{-\frac{ix\xi}{h}} Q e^{\frac{ix\xi}{h}} 1 \\ &= P(x, \xi + hD_x, h) q(x, \xi, h) \\ &= \sum_l \frac{1}{l!} \partial_\xi^l p(x, \xi, h)(hD_x)^l q(x, \xi, h). \end{aligned}$$

□

Now consider

$$P_\varphi = e^{-\frac{i\varphi}{h}} P e^{i\frac{\varphi(x)}{h}},$$

if

$$p(x, \varphi'(x)) = 0 \quad \partial_\xi p(x, \varphi'(x)) \neq 0,$$

then

$$P_\varphi = Q_0 + Q_1 h + Q_2 h^2 + \cdots,$$

where

$$Q_0 = p(x, \varphi'(x)) = 0, \quad Q_1 = \partial_\xi p(x, \varphi'(x)) D_x + P_{\varphi,1}(x, 0).$$

Thus, we can inductively solve

$$P_\varphi a = 0$$

for $a \sim a_0 + a_1 h + a_2 h^2 + \cdots$. By Borel's lemma we get a (local) WKB solution $P_\varphi a = \mathcal{O}(h^\infty)_{C^\infty}$.

Under strong conditions, we can prove a global WKB method.

Theorem 15. *Suppose $p(x_0, \xi_0) = 0$, $\frac{1}{i}\{p, \bar{p}\}(x_0, \xi_0) > 0$, then we can find an approximate solution $u \in C^\infty$ with $\|u\|_{L^2} = 1$ and*

$$\|Pu\|_{L^2} = \mathcal{O}(h^\infty).$$

Proof. We consider the function φ such that

$$p(x,\varphi'(x)) = 0, \quad \varphi'(x_0) = \xi_0$$

(such function exists by implicit function theorem), then

$$p'_x(x_0,\xi_0) + p'_\xi(x_0,\xi_0)\varphi''(x_0) = 0$$

and

$$\begin{aligned}\mathrm{Im}\varphi''(x_0) &= -\mathrm{Im}\frac{p'_x(x_0,\xi_0)}{p'_\xi(x_0,\xi_0)} \\ &= -\mathrm{Im}\frac{p'_x(x_0,\xi_0)\bar{p'_\xi}(x_0,\xi_0)}{|p'_\xi(x_0,\xi_0)|^2} \\ &= \frac{1}{2|p'_\xi|^2}\frac{1}{i}\{p,\bar{p}\}(x_0,\xi_0) > 0.\end{aligned}$$

Now we can define

$$f(x,h) = h^{-\frac{1}{4}}a(x,h)e^{i\frac{\varphi}{h}}$$

near x_0, and we alraedy proved that

$$Pf = re^{i\frac{\varphi}{h}}, \quad r = \mathcal{O}(h^\infty).$$

By stationary phase we know

$$\|f\|^2_{L^2} = \frac{|a(0)|\sqrt{2\pi}}{\sqrt{2\mathrm{Im}\varphi''(x_0)}} + o(h).$$

Moreover, we have

$$\int_{\delta<|x-x_0|<\frac{1}{C}} |f|^2 = \mathcal{O}(e^{-\frac{1}{Ch}})$$

since $e^{i\frac{\varphi}{h}}$ is localized (exponentially) near x_0. Let

$$u = \frac{\chi f}{\|\chi f\|},$$

we have $\|u\|_{L^2} = 1$ and

$$Pu = \frac{\chi Pf + [P,\chi]f}{\|\chi f\|} = \mathcal{O}(h^\infty).$$

$\square$

Definition 51. *Let*

$$\Sigma = p(S^1 \times \mathbb{R}) \subset \mathbb{C}$$
$$\Sigma_+ = \left\{ z : \exists (x,\xi) \text{ such that } p(x,\xi) = z, \frac{1}{i}\{p,\bar{p}\} > 0 \right\}$$
$$\Sigma_- = \left\{ z : \exists (x,\xi) \text{ such that } p(x,\xi) = z, \frac{1}{i}\{p,\bar{p}\} < 0 \right\}.$$

The global WKB method proves that for any $K \Subset \Sigma_+$ we have

$$K \subset \sigma_{H^\infty}(P).$$

Here we also recall two trivial bounds for approximate solutions.

Proposition 52.
- *For $z \in p(S^1 \times \mathbb{R})$, there exists $u \in C^\infty$, $\|u\|_{L^2} = 1$ such that $\|(P-z)u\| = \mathcal{O}(h^{\frac{1}{2}})$.*
- *If p is real-valued, $z = p(x_0, \xi_0)$ and $dp(x_0,\xi_0) \neq 0$, then there exists $u \in C_0^\infty$ with $\|u\|_{L^2} = 1$ and $\|(P-z)u\|_{L^2} = \mathcal{O}(h)$.*

Proof.
- For the first one, let us try

$$u(x) = e^{\frac{i(x-x_0)\xi_0}{h}} \chi(h^{-\gamma}(x-x_0)) h^{-\frac{\gamma}{2}},$$

then

$$\begin{aligned} Pu &= e^{\frac{i(x-x_0)\xi_0}{h}} P(x, \xi_0 + hD_x, h)\chi(h^{-\gamma}(x-x_0))h^{-\frac{\gamma}{2}} \\ &= e^{\frac{i(x-x_0)\xi_0}{h}} h^{-\frac{\gamma}{2}} p(x,\xi_0,h)\chi(h^{-\gamma}(x-x_0)) \\ &= e^{\frac{i(x-x_0)\xi_0}{h}} h^{-\frac{\gamma}{2}} \left(p(x,\xi_0,h)\chi(h^{-\gamma}(x-x_0)) + \sum_{k>0} \frac{h^k}{k!}\partial_\xi^k p(x,\xi_0,h) D_x^k \left(\chi\left(\frac{x-x_0}{h^\gamma} \right)\right)\right) \\ &= \mathcal{O}(h^{\frac{\gamma}{2}}) \mathbb{1}_{|x-x_0| \le h^\gamma} + \mathcal{O}(h^{1-\frac{3\gamma}{2}}) \mathbb{1}_{|x-x_0|\le h^\gamma}. \end{aligned}$$

Taking $\gamma = \frac{1}{2}$ we get

$$\|Pu\|_{L^2} = \mathcal{O}(h^{\frac{\gamma}{2}} + h^{1-\frac{\gamma}{2}}) = \mathcal{O}(h^{\frac{1}{2}}).$$

- To prove the second one, recall in local WKB method we get

$$P(e^{\frac{i\varphi}{h}} a) = e^{\frac{i\varphi}{h}} r, \quad r = \mathcal{O}(h^\infty).$$

The crucial point is that when p is real-valued then potential φ is also real-valued. So let

$$\tilde{u} = \chi e^{\frac{i\varphi}{h}} a$$

we have

$$P\tilde{u} = \chi e^{\frac{i\varphi}{h}} r + [P,\chi] e^{\frac{i\varphi}{h}} a = \mathcal{O}(h^\infty) + \mathcal{O}(h).$$

□

We offer an easy case of Morse-Sard theorem.

Theorem 16. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map, then the singular values of f has zero (Lebesgue) measure.*

Proof. Suppose $K \Subset \mathbb{R}^2$ is a set of singular values, then there is a covering

$$K = \bigcup I_j$$

with disjoint cubes I_j of diameter $\varepsilon > 0$ such that

$$\sum m(I_j) \leq C.$$

Moreover, if $z_j \in K \cap I_j$, then

$$f(z) = f(z_j) + \partial f(z_j)(z - z_j) + o(z - z_j)$$

and

$$m(f(I_j)) = o(\varepsilon^2) = o(1)m(I_j).$$

Then

$$m(f(K)) \leq \sum m(f(I_j)) = o(1) \sum m(I_j) = o(1).$$

□

Another important step in Hager-Sjöstrand theorem is that $\Sigma_+ = \Sigma_-$, we provide a statement that holds in general.

Theorem 17. *Suppose $p(x, \xi) = \sum_{k \leq m} \xi^k b_k(x)$ and there exists z_0 such that*

$$|p(x, \xi) - z_0| \geq \frac{1}{C} \langle \xi \rangle^m,$$

(i.e. p is elliptic of order m), then $\Sigma_+ = \Sigma_-$. Moreover, if $\Omega \subset \mathbb{C}$ is simply-connected, and $\{p, \bar{p}\} \neq 0$ on $p^{-1}(\Omega)$, then for any $z \in \Omega$,

$$p^{-1}(z) = \{\rho_1^+, \cdots, \rho_N^+, \rho_1^-, \cdots, \rho_N^-\},$$

where

$$\rho_j^\pm = (x_j^\pm(z), \xi_j^\pm(z)), \quad \pm \frac{1}{i}\{p, \bar{p}\}(\rho_j^\pm) > 0.$$

Using all the previous ingradients (WKB method and topological properties for general elliptic differential operators), we can proceed as before and get Hager-Sjöstrand's theorem.

Theorem 18. *Let* $p(x,\xi) = \sum_{k\le m} \xi^k b_k(x)$ *and* $P = \sum_{k\le m} b_k(x)(hD)^k$*. Assume there exists* $z_0 \in \mathbb{C}$ *such that*

$$|p(x,\xi) - z_0| \ge \frac{1}{C}\langle \xi \rangle^m,$$

$\Omega \Subset p(S^1 \times \mathbb{R})$ *simply connected,* $\partial\Omega \in C^\infty$*, and* $\{p, \bar{p}\}(x,\xi) \neq 0$ *for any* $(x,\xi) \in \Omega$*. Let*

$$Q = \sum_{i,j\le \frac{C}{h}} \alpha_{ij}(w) e^i \otimes e_j^*$$

with α_{ij} *i.i.d standard Gaussian distributions, then*

$$\sharp\sigma(P + \delta Q) \cap \Omega = \frac{\mathrm{vol}(p^{-1}(\Omega))}{2\pi h} + o(h^{-1})$$

with probability $\ge 1 - o(h^\eta)$ *for some* $\eta > 0$*.*

There is an even finer description by Vogel-Nonnenmacher in the case $p(x,\xi) = p(x,-\xi)$ which even hold for pertubation by potentials

$$Q = \sum_{j\le \frac{C}{h^2}} v_j e_j.$$

$$\mathcal{L}_{h,z_0} \to \mathcal{L}_{G(z_0)}, \quad h \to 0$$

where

$$\mathcal{L}_{h,z_0} = \sum_{z\in\sigma(P+\delta Q)} \delta_{\frac{z-z_0}{\sqrt{h}}}$$

is the distribution for the spectrum and $\mathcal{L}_{G(z_0)}$ is the distribution of zeros of Gaussian analytic functions defined as follows.

Let

$$g_\sigma(w) = \sum_n \alpha_n \frac{\sigma^{\frac{n}{2}} \omega^n}{\sqrt{n!}}, \quad \alpha_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

be a Gaussian analytic function, we define its distribution of zeros as $\mathcal{L}_{g_\sigma} = \sum_{z\in g_\sigma^{-1}(0)} \delta_z$.

The function $G(z_0)$ is defined as $\det(g_{z_0}^{ij})$ where

$$g_{z_0}^{ij} = g_{\sigma_{z_0}^{ij}}, \quad \sigma_{z_0}^{ij} = \sum_{\pm} \frac{i}{\{p,\bar{p}\}(\rho_j^\pm(z_0))}.$$

5. WKB methods for analytic PDEs

If we have a PDE with analytic coefficients, we can find some special phenomenon. The material comes from [Sj19, Chapter 7].

Example 17. *Let*

$$P_t = (hD_x)^2 + (1-t+ti)\sin x : H^2(S^1) \to L^2(S^1),$$

we want to study the spectrum of P_t. It turns out that there exists a holomorphic family

$$E \mapsto I(E,h) = I_0(E) + h^2 I_2(E) + \cdots$$

such that $\operatorname{Spec} P_t$ *are given by solutions to $I(E,h) = 2\pi h(n+\frac{1}{2})$ (the Bohr-Sommerfeld quantization condition). Moreover, $I_0(E)$ is given by*

$$I_0(E) = \int_\gamma \xi dx$$

where $\gamma \in H_1(p^{-1}(E))$ is the generator of the homology.

To study the general case, we need to first look at the equation

$$(h\partial_x - A(x))u = 0, \quad u(x_0) = u_0, \quad A(x) \in C^\infty(I, M_{2\times 2}).$$

Proposition 53. *There exists a unique solution operator $E(x,y)$ such that*

$$u(x) = E(x, x_0)u_0$$

solves the equation. Moreover, we have an estimate

$$\|E(x,y)\| \le \begin{cases} \exp\left(\int_y^x \mu_+(A(t))\frac{dt}{h}\right), & x \ge y, \\ \exp\left(\int_y^x \mu_-(A(t))\frac{dt}{h}\right), & x \le y. \end{cases}$$

where

$$\mu_+(A(x)) = \sup_{\|v\|=1} \operatorname{Re}\langle A(x)v, v\rangle, \quad \mu_-(A(x)) = \inf_{\|v\|=1} \operatorname{Re}\langle A(x)v, v\rangle.$$

- Now let us assume $A(x)$ has two distinct eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$.
- $\operatorname{Re}\lambda_1(x) \ge \operatorname{Re}\lambda_2(x)$.

Example 18. *Consider the Schrödinger operator $P = (hD_x)^2 + V(x)$, if $V(x) \neq 0$, then we can consider the following equation.*

$$\left(h\partial_x - \begin{pmatrix} 0 & 1 \\ V(x) & 0 \end{pmatrix}\right) u(x) = 0.$$

The matrix $A(x) = \begin{pmatrix} 0 & 1 \\ V(x) & 0 \end{pmatrix}$ has eigenvalues $\lambda_1(x) = -\sqrt{V(x)}$ and $\lambda_2(x) = \sqrt{V(x)}$.

Proposition 54. *There exists a smooth family of operators*

$$U(x,h) \sim U_0(x) + hU_1(x) + \cdots$$

such that

$$U^{-1}(h\partial_x - A(x))U = h\partial_x - \Lambda(x,h)$$

where

$$\Lambda(x,h) = \Lambda_0(x) + h\Lambda_1(x) + \cdots$$

is diagonal.

Corollary 55. *Let $\varphi_j'(z) = \lambda_j(z)$. There exists*

$$a \sim a_0(z) + ha_1(z) + \cdots$$

with $a_0 \neq 0$, $A(z)a_0 = \lambda_j a_0$ such that

$$(h\partial_z - A(z))(a(z,h)e^{\varphi_j(z)/h}) = r(z,h)e^{\varphi_j(z)/h}, \quad r(z,h) = \mathcal{O}(h^\infty).$$

Theorem 19. *If $\operatorname{Re}(\dot{\gamma\lambda_1}) \geq \operatorname{Re}(\dot{\gamma\lambda_2})$. Let $u^j_{WKB}(z,h) = e^{\varphi_j(z)/h}a_j(z,h)$, suppose u solves $(h\partial_z - A(z))u = 0$ in Ω, and*

- *$u(\gamma(a)) = u_{WKB}(\gamma(a))$, $j = 1$,*
- *or $u(\gamma(b)) = u_{WKB}(\gamma(b))$, $j = 2$,*

then

$$|u - u_{WKB}| = \mathcal{O}(h^\infty)e^{\varphi_j(z)/h} \qquad \text{on } \gamma([a,b]).$$

If $\operatorname{Re}(\dot{\gamma\lambda_1}) > \operatorname{Re}(\dot{\gamma\lambda_2})$ on γ, then

$$|u - u_{WKB}| = \mathcal{O}(h^\infty)e^{\varphi_j(z)/h} \qquad \text{on } \begin{cases} \text{nbhd}(\gamma((a,b])), & j = 1, \\ \text{nbhd}(\gamma([a,b))), & j = 2. \end{cases}$$

Definition 56. *Suppose we have a phase function $\varphi(z)$, the Stokes line is defined as $\operatorname{Re}\varphi = 0$ and the anti-Stokes lines is defined as $\operatorname{Im}\varphi = 0$.*

Example 19. *A standard example is given by $V(z) = z$, $\varphi'(z) = \sqrt{z}$ and $\varphi(z) = \frac{2}{3}z^{\frac{3}{2}}$.*

We have the relations

$$(\varphi_j')^{1/2} = i^{\nu_{j,k}}(\varphi_k')^{1/2}$$

with

$$\nu_{j,k} = -\nu_{k,j}, \quad \nu_{i,j} + \nu_{j,k} + \nu_{k,i} = 1.$$

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