LECTURE NOTES ON BOGOVSKII TYPE OPERATORS AND FLEXIBILITY OF THE EINSTEIN CONSTRAINT EQUATION

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1. The Vacuum Einstein constraint equations

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1.1. **The vacuum constraints.** The vacuum Einstein equation states

$$\mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}\mathbf{g}\,\mathrm{tr}_{\mathbf{g}}\,\mathrm{Ric}_{\mathbf{g}} = 0, \tag{1.1}$$

where $\mathbf{g}$ is a Lorentzian metric on a $d+1$-dimensional manifold, and $\mathrm{Ric}_{\mathbf{g}}$ is the Ricci curvature of $\mathbf{g}$.

Let (M^d, g) be a spacelike hypersurface of dimension $d \geq 2$ in a vacuum spacetime, and let k be its second fundamental form. The Gauss and Codazzi equations imply

$$2\mathcal{H}(g,k) := R_g + (\mathrm{tr}_g\, k)^2 - |k|_g^2 = 0, \tag{1.2}$$
$$\mathcal{M}(g,k) := \mathrm{div}_g\, k - \mathrm{d}(\mathrm{tr}_g\, k) = 0. \tag{1.3}$$

Indeed, suppose $\mathbf{R}_{\mathbf{g}}$ is the Riemann curvature tensor of $\mathbf{g}$, and $\mathbf{R}_g$ is the Riemann curvature tensor of g. The Gauss equation states that

$$\mathbf{g}(\mathbf{R}_{\mathbf{g}}(e_i,e_j)e_j,e_i) = g(\mathbf{R}_g(e_i,e_j)e_j,e_i) - k(e_i,e_j)^2 + k(e_i,e_i)k(e_j,e_j).$$

Summing in i,j, the left hand side becomes

$$\sum_{i,j}\mathbf{g}(\mathbf{R}_{\mathbf{g}}(e_i,e_j)e_j,e_i) = \sum_j \mathrm{Ric}_{\mathbf{g}}(e_j,e_j) - \mathrm{Ric}_{\mathbf{g}}(n,n) = 0,$$

while the right hand side becomes $R_g + (\mathrm{tr}_g\, k)^2 - |k|_g^2$. The Codazzi equation states that

$$\mathbf{g}(\mathbf{R}_{\mathbf{g}}(e_i,e_j)e_j,n) = \nabla_{e_i}k(e_j,e_j) - \nabla_{e_j}k(e_i,e_j).$$

Summing in j, the left hand side becomes

$$\sum_j \mathbf{g}(\mathbf{R}_{\mathbf{g}}(e_i,e_j)e_j,n) = \mathrm{Ric}_{\mathbf{g}}(e_i,n) = 0,$$

while the right hand side becomes $\mathrm{d}(\mathrm{tr}_g\, k) - \mathrm{div}_g\, k$. The fundamental theorem of Choquet-Bruhat [8] states that any (sufficiently regular) triple (M,g,k) solving (1.2)–(1.3) corresponds to a geometrically unique spacetime with a metric solving (1.1) (see [3, 13] for the uniqueness).

At each point, (g,k) contains $d(d+1)$ scalar components, whereas (1.2)–(1.3) give only $d+1$ scalar equations. The system is therefore underdetermined. The basic questions are:

- Can one build nontrivial vacuum data supported in a prescribed region?
- Can two solutions be glued in a prescribed region? What is the obstruction to glue two solutions?
- How fast may asymptotically flat vacuum data decay?

1.2. **Linearization at flat data.** Let us linearize around the flat data $(\delta, 0)$ on $\mathbb{R}^d$. Write

$$g = \delta + \dot{g}, \qquad k = \dot{k}.$$

The first variation of scalar curvature at the Euclidean metric is

$$DR_\delta(u) = \partial_i \partial_j \dot{g}^{ij} - \Delta \operatorname{tr}_\delta \dot{g}. \tag{1.4}$$

One quick derivation starts from

$$D\Gamma^k_{ij}(\dot{g}) = \frac{1}{2}\big(\partial_i \dot{g}_j{}^k + \partial_j \dot{g}_i{}^k - \partial^k \dot{g}_{ij}\big),$$

then contracts $D\mathrm{Ric}_{ij} = \partial_k D\Gamma^k_{ij} - \partial_j D\Gamma^k_{ik}$. The terms involving $\dot{k}$ in the Hamiltonian constraint are quadratic, so they do not appear in the linearization at $k = 0$.

For the momentum constraint,

$$D\mathcal{M}_{(\delta,0)}(\dot{g}, \dot{k})_j = \partial^i \dot{k}_{ij} - \partial_j \operatorname{tr}_\delta \dot{k}. \tag{1.5}$$

The metric perturbation $\dot{g}$ does not occur at linear order because the background second fundamental form vanishes.

We make the following change of variables to make the equations (1.4)–(1.5) in a simpler form:

$$h_{ij} = g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta(g_{ij} - \delta_{ij}), \qquad \pi_{ij} = k_{ij} - \delta_{ij} \operatorname{tr}_\delta k. \tag{1.6}$$

Since $\operatorname{tr}_\delta h = -(d-1)\operatorname{tr}_\delta(g - \delta)$ and $\operatorname{tr}_\delta \pi = -(d-1)\operatorname{tr}_\delta k$, the inverse transformation is

$$g_{ij} = \delta_{ij} + h_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta h, \qquad k_{ij} = \pi_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta \pi. \tag{1.7}$$

Thus the linearized constraint operator becomes

$$P(h, \pi) = \big(P_H h, P_M \pi\big) := \big(\partial_i \partial_j h^{ij},\, \partial_i \pi^{ij}\big). \tag{1.8}$$

The two components are called the double divergence and the symmetric divergence.

1.3. **The nonlinear equations as a quadratic perturbation.** In the variables (h, π), the constraint equations (1.2)–(1.3) can be written as

$$P(h, \pi) = \Phi(h, \pi). \tag{1.9}$$

Near $(0, 0)$, the nonlinear map has the schematic form

$$\Phi_H(h, \pi) = M^{(0)}(h)\big(h, \partial^2 h\big) + M^{(1)}(h)\big(\partial h, \partial h\big) + M^{(2)}(h)(\pi, \pi), \tag{1.10}$$

$$\Phi_M(h, \pi) = N^{(0)}(h)\big(h, \partial \pi\big) + N^{(1)}(h)\big(\partial h, \pi\big), \tag{1.11}$$

where $M^{(0)}, M^{(1)}, M^{(2)}, N^{(0)}, N^{(1)}$ are smooth tensor-valued coefficient maps as long as g remains positive definite. The notation means finite linear combinations of contractions with coefficients depending smoothly on h. The important facts are

$$\Phi(0, 0) = 0, \qquad D\Phi(0, 0) = 0. \tag{1.12}$$

Consequently, in reasonable spaces we expect

$$\|\Phi(h, \pi)\|_Y \le C\|(h, \pi)\|_X^2, \tag{1.13}$$

$$\|\Phi(h_1, \pi_1) - \Phi(h_2, \pi_2)\|_Y \le C\big(\|(h_1, \pi_1)\|_X + \|(h_2, \pi_2)\|_X\big)\|(h_1 - h_2, \pi_1 - \pi_2)\|_X. \tag{1.14}$$

1.4. **Formal adjoints and finite-dimensional obstructions.** On a bounded domain, compactly supported solvability is constrained by the kernels of the formal adjoints.

For a scalar test function φ,

$$\int (\partial_i \partial_j h^{ij})\varphi \,\mathrm{d}x = \int h^{ij} \partial_i \partial_j \varphi \,\mathrm{d}x,$$

so

$$P_H^* \varphi = \nabla^2 \varphi. \tag{1.15}$$

On a connected Euclidean domain, $\nabla^2 \varphi = 0$ exactly when φ is affine:

$$\varphi(x) = a + b_i x^i.$$

Therefore a compactly supported solution of $P_H h = f$ must satisfy

$$\int f \,\mathrm{d}x = 0, \qquad \int x^a f \,\mathrm{d}x = 0, \quad a = 1, \ldots, d. \tag{1.16}$$

For a 1-form $\boldsymbol{\omega}$,

$$\begin{aligned}\int (\partial_i \pi^{ij})\boldsymbol{\omega}_j \,\mathrm{d}x &= -\int \pi^{ij}\partial_i \boldsymbol{\omega}_j \,\mathrm{d}x \\ &= -\frac{1}{2}\int \pi^{ij}(\partial_i \boldsymbol{\omega}_j + \partial_j \boldsymbol{\omega}_i)\,\mathrm{d}x.\end{aligned}$$

Thus

$$P_M^* \boldsymbol{\omega} = -\frac{1}{2}(\partial_i \boldsymbol{\omega}_j + \partial_j \boldsymbol{\omega}_i). \tag{1.17}$$

Its kernel is the space of Euclidean Killing fields. In dimension three it is spanned by the translations

$$e_a = \partial_a, \qquad a = 1, 2, 3,$$

and rotations

$$Y_a = \epsilon_{aij} x^i \partial_j, \qquad a = 1, 2, 3.$$

Hence a compactly supported solution of $P_M \pi = F$ must satisfy

$$\int F \cdot e_a \,\mathrm{d}x = 0, \qquad \int F \cdot Y_a \,\mathrm{d}x = 0, \quad a = 1, 2, 3. \tag{1.18}$$

Together, (1.16) and (1.18) give ten conditions in $d = 3$. These are the linear ancestors of energy, linear momentum, center of mass, and angular momentum.

1.5. **Notations.** Unless stated otherwise, indices are raised and lowered with the Euclidean metric δ. Repeated indices are summed. The construction of Bogovskii-type operators work in dimension $d \geq 1$ while the discussion of constraint equations works in dimension $d \geq 3$. We specialize to $d = 3$ when discussing the ten charges and annular gluing (although the same works in general dimensions).

For a symmetric tensor u, $\operatorname{tr}_\delta u = \delta^{ij} u_{ij}$. Let $B_r(\xi) := \{x \in \mathbb{R}^d : |x - \xi| < r\}$ and $B_r := B_r(0)$. We write

$$\langle x \rangle = (1 + |x|^2)^{1/2}, \qquad A_r = B_{2r} \setminus \overline{B_r}, \qquad \widetilde{A}_r = B_{4r} \setminus \overline{B_{r/2}}.$$

A solid cone with vertex y, axis $\omega \in \mathbb{S}^{d-1}$, and opening angle θ is

$$C_{y,\theta}(\omega) = \{x \in \mathbb{R}^d : \angle(x - y, \omega) \leq \theta\}.$$

When $y = 0$ we write $C_\theta(\omega)$. For $0 < \theta < \pi/2$ this is a convex cone.

2. Conic solution operators

In this section, we construct conic solution operators for the linearized constraint operator (1.8), which are right inverses with controlled support of the Schwartz kernel in a conic region.

2.1. **A right inverse for the divergence equation.** Consider

$$\partial_i u^i = f. \tag{2.1}$$

Fix $\omega \in \mathbb{S}^{d-1}$. The vector-valued distribution supported on the ray $\mathbb{R}_+\omega$ and defined by

$$\langle T_\omega, \psi \rangle := \int_0^\infty \omega\, \psi(t\omega)\,\mathrm{d}t, \qquad \psi \in C_c^\infty(\mathbb{R}^d), \tag{2.2}$$

satisfies $\partial_i (T_\omega)^i = \delta_0$. Indeed,

$$\left\langle \partial_i (T_\omega)^i, \varphi \right\rangle = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t\omega)\,\mathrm{d}t = \varphi(0).$$

The singular ray can be smoothed in the angular variables without losing its one-sided support. Choose $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with

$$\int_{\mathbb{S}^{d-1}} \kappa(\omega)\,\mathrm{d}\omega = 1. \tag{2.3}$$

Averaging (2.2) gives

$$K^i_\kappa(x) = \kappa(\widehat{x})\frac{x^i}{|x|^d}, \quad \widehat{x} := \frac{x}{|x|}, \qquad \partial_i K^i_\kappa = \delta_0. \tag{2.4}$$

The kernel is homogeneous of degree $1-d$ and is supported in the closed cone generated by $\operatorname{supp}\kappa$.

Exercise 2.1. Check (2.4) directly.

This elementary construction contains three features:

(i) the right inverse is obtained by integrating along rays;
(ii) angular averaging gives a kernel smooth away from the origin;
(iii) choosing the angular density localizes the kernel in a prescribed cone.

2.2. **The double divergence equation.** Now we study the double divergence equation

$$\partial_i\partial_j h^{ij} = f. \tag{2.5}$$

For κ satisfying (2.3), set

$$K^{ij}_\kappa(x) := \kappa(\widehat{x})\frac{x^i x^j}{|x|^d}. \tag{2.6}$$

Notice that K_κ is homogeneous of degree $2-d$.

Proposition 2.2 (Conic fundamental solution for P_H)**.** *The distribution K_κ is a symmetric tensor, smooth on $\mathbb{R}^d \setminus \{0\}$, and*

$$\partial_i\partial_j K^{ij}_\kappa = \delta_0. \tag{2.7}$$

Its support is contained in the closed cone generated by $\operatorname{supp}\kappa$.

Proof. For $\varphi \in C^\infty_c(\mathbb{R}^d)$, polar coordinates give

$$\begin{aligned}\langle \partial_i\partial_j K^{ij}_\kappa, \varphi\rangle &= \int_{\mathbb{S}^{d-1}} \kappa(\omega)\int_0^\infty r\,\omega^i\omega^j(\partial_i\partial_j\varphi)(r\omega)\,\mathrm{d}r\,\mathrm{d}\omega\\ &= \int_{\mathbb{S}^{d-1}} \kappa(\omega)\int_0^\infty r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\varphi(r\omega)\,\mathrm{d}r\,\mathrm{d}\omega.\end{aligned}$$

Integration by parts twice, with compact support at infinity, yields

$$\int_0^\infty r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\varphi(r\omega)\,\mathrm{d}r = \varphi(0).$$

The angular integral is one by (2.3). The support and homogeneity statements follow immediately from (2.6). ∎

The factor r in the preceding proof explains the order of the operator: solving a double-divergence equation gains two derivatives and produces a kernel of order -2.

2.3. **The symmetric divergence equation.** We next solve

$$\partial_j \pi^{jk} = F^k, \qquad \pi^{jk} = \pi^{kj}. \tag{2.8}$$

For a fixed $v \in \mathbb{R}^d$, define

$$L^{jk}_{\kappa,v}(x) := \partial_\ell\left[\kappa(\widehat{x})\frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d}\right]. \tag{2.9}$$

Proposition 2.3 (Conic fundamental solution for P_M)**.** *The tensor $L_{\kappa,v}$ is symmetric in j,k, is homogeneous of degree $1-d$, is smooth away from the origin, and satisfies*

$$\partial_j L^{jk}_{\kappa,v} = \delta_0 v^k. \tag{2.10}$$

Its support is contained in the closed cone generated by $\operatorname{supp}\kappa$.

Proof. For a single direction ω, let ϕ_ω be the distribution

$$\langle \phi_\omega, \psi\rangle = \int_0^\infty t\psi(t\omega)\,\mathrm{d}t.$$

Then $\omega^j\omega^\ell\partial_j\partial_\ell\phi_\omega = \delta_0$. Put

$$\pi^{jk}_\omega = \partial_\ell\phi_\omega\big(\omega^j\omega^\ell v^k + \omega^\ell\omega^k v^j - v^\ell\omega^k\omega^j\big).$$

The last two terms cancel after applying ∂_j, while the first gives $\delta_0 v^k$. Averaging this identity over ω against κ and changing to polar coordinates gives precisely (2.9). Symmetry and the remaining properties follow from the formula. □

For a vector $F = (F_1,\dots,F_d)$, we use the standard basis e_a and sum the corresponding kernels.

Definition 2.4 (Conic solution operator)**.** Let Ω be a convex cone and choose κ satisfying (2.3) with $\operatorname{supp}\kappa \subset \Omega\cap\mathbb{S}^{d-1}$. Define

$$S_c(f,F) := \left(K_\kappa * f,\ \sum_{a=1}^{d} L_{\kappa,e_a} * F_a\right). \tag{2.11}$$

It follows from (2.7) and (2.10) that

$$PS_c(f,F) = (f,F). \tag{2.12}$$

2.4. **Support preservation and the outgoing property.** Note that

$$\operatorname{supp}(u*v) \subset \operatorname{supp} u + \operatorname{supp} v := \{a+b : a\in\operatorname{supp} u,\ b\in\operatorname{supp} v\}.$$

A convex cone is closed under addition. We therefore obtain the following:

Corollary 2.5 (Conic support)**.** *If* $\operatorname{supp}(f,F)\subset\Omega$ *and Ω is the convex cone used in Definition 2.4, then*

$$\operatorname{supp} S_c(f,F)\subset\Omega.$$

The operator is outgoing in the following sense.

Definition 2.6 (Outgoing kernel)**.** An integral kernel $K(x,y)$ on a cone Ω is outgoing if there is $C_\Omega > 0$ such that

$$K(x,y)\neq 0 \quad\Longrightarrow\quad |y|\le C_\Omega(1+|x|). \tag{2.13}$$

Exercise 2.7. Define the incoming integral kernel similar as Definition 2.6. If an operator is both outgoing and incoming, we say it is diagonal.

Lemma 2.8 (Conic convolution is outgoing)**.** *Let $\Omega = C_\theta(\omega)$ with $0<\theta<\pi/2$. The kernel $K(x-y)$ is outgoing on Ω.*

Proof. If $x,y\in\Omega$ and $x-y\in\Omega_{\omega,\theta'}$, then $(x-y)\cdot\omega\ge 0$. If $|y|/|x|$ were arbitrarily large, then $(x-y)/|x-y|$ would approach $-y/|y|$, whose scalar product with ω is negative, contradicting $x-y\in\Omega_{\omega,\theta'}$. Compactness gives a uniform constant C_Ω. □

Exercise 2.9. Prove Lemma 2.8 with explicit C_Ω.

2.5. **Weighted Sobolev spaces.** For an integer $m \geq 0$, define
$$\|u\|_{H_{\mathrm{b}}^m}^2 := \sum_{|\alpha|\leq m} \|\langle x\rangle^{|\alpha|}\partial^\alpha u\|_{L^2(\mathbb{R}^d)}^2. \tag{2.14}$$
For $s \in \mathbb{R}$, we define H_{b}^s using complex interpolation and duality.

For $\delta \in \mathbb{R}$ set
$$H_{\mathrm{b}}^{s,\delta} = \langle x\rangle^{-\delta} H_{\mathrm{b}}^s, \qquad \|u\|_{H_{\mathrm{b}}^{s,\delta}} := \|\langle x\rangle^\delta u\|_{H_{\mathrm{b}}^s}. \tag{2.15}$$
For integer m, the norm is equivalent to
$$\sum_{|\alpha|\leq m} \|\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u\|_{L^2}. \tag{2.16}$$
Each derivative is accompanied by one extra power of $\langle x\rangle$.

For an open subset $\Omega \subset \mathbb{R}^d$, let $H_{\mathrm{b}}^{s,\delta}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in the $H_{\mathrm{b}}^{s,\delta}$ norm. When Ω is the closure of an open set with Lipschitz boundary, we use the same notation.

For $u = \langle r\rangle^{-\gamma}$, we have
$$\langle r\rangle^{-\gamma} \in H_{\mathrm{b}}^{m,\delta} \quad \iff \quad \int_1^\infty r^{2\delta-2\gamma+d-1}\,\mathrm{d}r < \infty \quad \iff \gamma > \frac{d}{2} + \delta. \tag{2.17}$$
In $d = 3$, the weights $\delta = -1/2$ and $\delta = 1/2$ correspond respectively to the decay rate r^{-1} and r^{-2}.

Exercise 2.10. Let $\chi \in C^\infty(\mathbb{R}^d)$ vanish on B_1 and equal one outside B_2. For $\gamma, q \in \mathbb{R}$, set
$$u(x) = \chi(x)|x|^{-\gamma}\big(\log(2+|x|)\big)^{-q}.$$
For an integer $m \geq 0$ and $\delta \in \mathbb{R}$: determine when $u \in H_{\mathrm{b}}^{m,\delta}$.

Let $\{\chi_j\}_{j\geq 0}$ be a dyadic partition of unity, with χ_j supported where $r \simeq 2^j$, and let $\varphi_j(x) = 2^j x$. Then we have the dyadic decomposition:
$$\|u\|_{H_{\mathrm{b}}^{s,\delta}}^2 \simeq \sum_{j\geq 0} 2^{2j(\delta+d/2)}\|\varphi_j^*(\chi_j u)\|_{H^s(\mathbb{R}^d)}^2. \tag{2.18}$$
This reduces weighted estimates to ordinary Sobolev estimates on a dyadic annuli.

Exercise 2.11. Use (2.18) to show that

(i) If $s > d/2$, then
$$\|\langle x\rangle^{d/2+\delta}u\|_{L^\infty} \lesssim \|u\|_{H_{\mathrm{b}}^{s,\delta}}. \tag{2.19}$$
(ii) For $s_1 + s_2 > 0$, we have
$$\|uv\|_{H_{\mathrm{b}}^{s_0,\delta_1+\delta_2+d/2}} \lesssim \|u\|_{H_{\mathrm{b}}^{s_1,\delta_1}}\|v\|_{H_{\mathrm{b}}^{s_2,\delta_2}}, \quad s_0 = \begin{cases} \min(s_1,s_2), & \max(s_1,s_2) > d/2, \\ s_1+s_2-d/2, & \max(s_1,s_2) < d/2. \end{cases} \tag{2.20}$$

The natural solution space for the constraints is
$$X^{s,\delta} := H_{\mathrm{b}}^{s,\delta}(\mathbb{R}^d;\mathrm{Sym}^2) \times H_{\mathrm{b}}^{s-1,\delta+1}(\mathbb{R}^d;\mathrm{Sym}^2). \tag{2.21}$$
The second fundamental form is assigned one extra power of decay and one fewer derivative.

2.6. **Weighted estimates for homogeneous kernels.** The conic kernels have nice mapping properties between weighted Sobolev spaces.

Theorem 2.12 (Homogeneous convolution estimate). *Let $K \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ be homogeneous of degree $k-d$, with $0 < k < d$. If*
$$-\frac{d}{2} < \delta < \frac{d}{2} - k, \tag{2.22}$$
then convolution extends to a bounded map
$$K* : H_{\mathrm{b}}^{s-k,\delta+k} \longrightarrow H_{\mathrm{b}}^{s,\delta}. \tag{2.23}$$

If the kernel is outgoing on a cone, the lower bound in (2.22) is not needed for the outgoing part; if it is incoming, the upper bound is not needed for the incoming part.

Proof. Split the Schwartz kernel $K(x-y)$ into three regions:

$$K_{\text{diag}} + K_{\text{out}} + K_{\text{in}},$$

where $|x| \simeq |y|$, $|x| > (1+c)|y|$, and $|x| < (1-c)|y|$, respectively.

On the diagonal region, rescale each dyadic annulus to unit size. Homogeneity gives a factor $2^{j(k-d)}$, and standard pseudodifferential estimates show that the localized operator has order $-k$.

On the outgoing region, the off-diagonal kernel is smooth and satisfies

$$|\partial_{x,y}^{\alpha} K_{\text{out}}(x,y)| \lesssim_{\alpha} (1+|x|+|y|)^{k-d-|\alpha|}.$$

This implies that

$$\|K_{\text{out},j,j'}\|_{H_{\mathrm{b}}^{s-k,\delta+k}(A_{2^{j'}}) \to H_{\mathrm{b}}^{N,\delta}(A_{2^j})} \lesssim 2^{j(k-d)} 2^{(\delta+d/2)j-(\delta-d/2+k)j'} = 2^{(\delta-d/2+k)(j-j')}. \tag{2.24}$$

In the outgoing region we have $j > j'$, this sum converges when $\delta < d/2 - k$. On the incoming region we have

$$\|K_{\text{in},j,j'}\|_{H_{\mathrm{b}}^{s-k,\delta+k}(A_{2^{j'}}) \to H_{\mathrm{b}}^{N,\delta}(A_{2^j})} \lesssim 2^{j'(k-d)} 2^{(\delta+d/2)j-(\delta-d/2+k)j'} = 2^{(\delta+d/2)(j-j')}, \quad j < j'. \tag{2.25}$$

The convergence requires $\delta > -d/2$. Combining the three pieces gives (2.23). ☐

For K_κ we take $k=2$; for $L_{\kappa,v}$ we take $k=1$. Both components of the constraint source can be placed in the same space:

Corollary 2.13 (Mapping property of S_c)**.** *For every $\delta < \frac{d}{2} - 2$ and $s \in \mathbb{R}$,*

$$S_c : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega; \mathbb{R} \oplus \mathbb{R}^d) \longrightarrow H_{\mathrm{b}}^{s,\delta}(\Omega; \mathrm{Sym}^2) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega; \mathrm{Sym}^2) \tag{2.26}$$

is bounded.

The endpoint is sharp: for generic compactly supported f, $K_\kappa * f$ has a leading term of size r^{2-d}, and this power lies just outside $H_{\mathrm{b}}^{s,d/2-2}$.

Exercise 2.14. A function is called polyhomogeneous if it has the asymptotic expansion (as well as all the derivatives)

$$u(x) \sim \sum_{j,\ell} u_j(\widehat{x}) r^{-j} (\log r)^{\ell}, \quad r \to \infty,$$

where j lies in a discrete index set and $\ell \in \mathbb{N}$. Prove that S_c maps polyhomogeneous functions to polyhomogeneous functions.

2.7. **Linearized solutions.** To construct solutions to the constraint equation (1.9) we need linearized solutions $(h_0, \pi_0) \in \ker P$.

Lemma 2.15. *For $\phi \in C_c^\infty(\mathbb{R}^d)$ define*

$$\mathcal{T}(\phi)_{ij} = \begin{pmatrix} \partial_2\partial_2\phi & -\partial_1\partial_2\phi & 0 \\ -\partial_1\partial_2\phi & \partial_1\partial_1\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.27}$$

Then

$$\partial_i \mathcal{T}(\phi)^{ij} = 0. \tag{2.28}$$

Consequently $(\mathcal{T}(\phi), 0)$ and $(0, \mathcal{T}(\phi))$ both lie in $\ker P$.

Exercise 2.16. Check that this linearized solution has nontrivial linearized Riemannian curvature when $d \geq 3$. Thus it will lead to a nontrivial metric instead of just a gauge change:

$$(D\mathbf{R})_{ijk\ell} = \frac{1}{2}(\partial_i\partial_k \dot{g}_{j\ell} + \partial_j\partial_\ell \dot{g}_{ik} - \partial_i\partial_\ell \dot{g}_{jk} - \partial_j\partial_k \dot{g}_{i\ell}) \neq 0.$$

3. Bogovskii type operators

For the gluing problem, we need to solve (1.8) in a compact region. We adapt the Bogovskii operators from fluid mechanics to more general settings.

3.1. The annular correction problem.

Suppose two initial data are prescribed near an inner and an outer annulus. A cutoff interpolation agrees with the desired data at both ends, but the derivatives of the cutoff create error in the transition annulus. The correction problem is therefore:

> Given (f, F) supported in an annulus, find symmetric tensors (h, π) supported in the same annulus such that
> $$\partial_i \partial_j h^{ij} = f, \qquad \partial_i \pi^{ij} = F^j.$$

By (1.16)–(1.18), this is possible only when (f, F) is orthogonal to the ten-dimensional cokernel. Unlike the conic case, the cokernel is unavoidable because both the source and the output are required to have compact support.

3.2. The scalar Bogovskii-type operator on a star-shaped set.

Let $\Omega \subset \mathbb{R}^3$ be star-shaped with respect to a ball $B \Subset \Omega$: for every $x \in \Omega$ and $y \in B$, the segment $[x, y]$ lies in Ω. Choose $\eta \in C_c^\infty(B)$ with $\int \eta = 1$. For $z = x - y$, define

$$\Psi_\eta^{ij}(z + y, y) = \frac{z^i z^j}{|z|^3} \int_{|z|}^{\infty} \eta\left(r \frac{z}{|z|} + y\right) r^2 \, \mathrm{d}r, \tag{3.1}$$

and

$$(S_\eta f)^{ij}(x) = \int_{\mathbb{R}^3} \Psi_\eta^{ij}(x, y) f(y) \, \mathrm{d}y. \tag{3.2}$$

The radial integral searches along the ray beginning at y and passing through x. It is nonzero only when that ray eventually meets B.

Lemma 3.1. *In the z variable,*

$$\partial_{z^i} \partial_{z^j} \Psi_\eta^{ij}(z + y, y) = \delta_0(z) - 4\eta(z + y) - z^a \partial_a \eta(z + y). \tag{3.3}$$

Integrating (3.3) with $f(y)$ gives

$$\partial_i \partial_j (S_\eta f)^{ij}(x) = f(x) - (\eta(x) + \partial_a(x^a \eta(x))) \int f(y) \, \mathrm{d}y + \partial_a \eta(x) \int y^a f(y) \, \mathrm{d}y. \tag{3.4}$$

Proposition 3.2. *If $f \in C_c^\infty(\Omega)$ satisfies*

$$\int f = 0, \qquad \int y^a f(y) \, \mathrm{d}y = 0, \quad a = 1, 2, 3, \tag{3.5}$$

then

$$\partial_i \partial_j (S_\eta f)^{ij} = f, \qquad \operatorname{supp} S_\eta f \subset \Omega.$$

Moreover, S_η is a classical pseudodifferential operator of order -2. In particular, for every $s \in \mathbb{R}$, let $H_0^s(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $H^s(\mathbb{R}^3)$,

$$\|S_\eta f\|_{H_0^s} \lesssim \|f\|_{H_0^{s-2}}, \qquad \|[S_\eta, \partial_a] f\|_{H_0^s} \lesssim \|f\|_{H_0^{s-2}}. \tag{3.6}$$

Proof. If the integrand in (3.1) is nonzero, there is an $r \geq |x - y|$ such that

$$y_1 = y + r \frac{x - y}{|x - y|} \in B.$$

The point x lies on the segment from y to y_1. Since $y \in \Omega$, $y_1 \in B$, and Ω is star-shaped with respect to B, it follows that $x \in \Omega$. To show that S_η is a pseudodifferential operator, we define

$$a(y, \xi) := \int \chi(z) \Psi_\eta(z + y, y) e^{-iz \cdot \xi} dz \tag{3.7}$$

so that

$$\Psi_\eta(x,y)\chi(x-y) = \frac{1}{(2\pi)^d}\int a(y,\xi)e^{i(x-y)\cdot\xi}d\xi. \tag{3.8}$$

We divide $a(y,\xi) := \sum a_j(y,\xi)$ where the integral is on the dyadic annulus $|z| \simeq 2^{-j}$. Since

$$|\partial_y^\alpha \partial_z^\beta \Psi_\eta(z+y,y)| \lesssim |z|^{-1-|\beta|}. \tag{3.9}$$

We conclude that

$$|\partial_y^\alpha \partial_\xi^\beta a_j(y,\xi)| \lesssim \min(2^{-j(|\beta|+2)}, 2^{j|\beta'|}|\xi|^{-(|\beta|+2+|\beta'|)}). \tag{3.10}$$

Thus we conclude that

$$|\partial_y^\alpha \partial_\xi^\beta a(y,\xi)| \lesssim |\xi|^{-(|\beta|+2)}. \tag{3.11}$$

$\square$

3.3. **The symmetric-divergence Bogovskii operator.** Define

$$\begin{aligned}(\Psi_\eta)^{ij}{}_k(z+y,y) = &\ \frac{1}{2}\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r\right)|z|^{-3}(z^i\delta_k^j + \delta_k^i z^j)\\ &+ \frac{1}{2}\partial_{z^m}\left[\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r\right)|z|^{-3}z^m(z^i\delta_k^j + \delta_k^i z^j)\right]\\ &- \partial_{z^k}\left[\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^2\,\mathrm{d}r\right)|z|^{-3}z^i z^j\right],\end{aligned} \tag{3.12}$$

and

$$(T_\eta F)^{ij}(x) = \int (\Psi_\eta)^{ij}{}_k(x,y)F^k(y)\,\mathrm{d}y. \tag{3.13}$$

The output is symmetric in i,j.

Lemma 3.3. *We have*

$$\partial_{z^i}(\Psi_\eta)^{ij}{}_k(z+y,y) = \delta_0(z)\delta_k^j - 2\eta(z+y)\delta_k^j - \frac{1}{2}z^\ell\partial_{z^\ell}\eta(z+y)\delta_k^j + \frac{1}{2}z^j\partial_{z^k}\eta(z+y). \tag{3.14}$$

Proposition 3.4. *If $F \in C_c^\infty(\Omega;\mathbb{R}^3)$ obeys*

$$\int F\cdot e_a\,\mathrm{d}x = 0, \qquad \int F\cdot Y_a\,\mathrm{d}x = 0, \quad a=1,2,3, \tag{3.15}$$

then

$$\partial_i(T_\eta F)^{ij} = F^j, \qquad \operatorname{supp} T_\eta F \subset \Omega.$$

The operator has order -1 and for all $s\in\mathbb{R}$,

$$\|T_\eta F\|_{H_0^s(\Omega)} \lesssim \|F\|_{H_0^{s-1}(\Omega)}, \qquad \|[T_\eta,\partial_a]F\|_{H_0^s(\Omega)} \lesssim \|F\|_{H_0^{s-1}(\Omega)}. \tag{3.16}$$

Exercise 3.5. Prove Proposition 3.4 following the proof of Proposition 3.2.

3.4. **From star-shaped sets to an annulus.** An annulus is not star-shaped, but it is a finite union of sets that are star-shaped with respect to balls.

Suppose $U = U_1\cup U_2$ and that $U_1\cap U_2$ contains a ball that is star-shaped in U_1 and U_2, so that we can define the solution operators S_1 and S_2 in U_1 and U_2, respectively. Let $\chi_1+\chi_2 = 1$ be subordinate to the cover. For the solution operator in Proposition 3.2, write

$$g_0 = 1, \qquad g_a = x^a, \quad a=1,2,3.$$

Choose $\rho\in C_c^\infty(U_1\cap U_2)$, $\rho\not\equiv 0$, and form the Gram matrix

$$G_{\mu\nu} = \int g_\mu g_\nu \rho^2\,\mathrm{d}x, \qquad 0\le\mu,\nu\le 3.$$

It is positive definite. Set

$$\vartheta_\mu = (G^{-1})_{\mu\nu}g_\nu\rho^2, \qquad \int \vartheta_\mu g_{\mu'}\,\mathrm{d}x = \delta_{\mu\mu'}. \tag{3.17}$$

For $f \in C_c^\infty(U)$, define

$$f_1 = \chi_1 f - \sum_{\mu=0}^{3} \left(\int \chi_1 f g_\mu \, \mathrm{d}x \right) \vartheta_\mu, \tag{3.18}$$

$$f_2 = \chi_2 f - \sum_{\mu=0}^{3} \left(\int \chi_2 f g_\mu \, \mathrm{d}x \right) \vartheta_\mu. \tag{3.19}$$

Then each f_a is supported in U_a and $\int f_1 g_\mu = \int f_2 g_\mu = 0$. Hence

$$Sf = S_1 f_1 + S_2 f_2$$

satisfies

$$PSf = f_1 + f_2 = f - \sum_{\mu=0}^{3} \left(\int f g_\mu \, \mathrm{d}x \right) \vartheta_\mu.$$

Iterating over a finite cover yields the annular operators.

The same construction works for the solution operator in Proposition 3.4. We have proved the following.

Theorem 3.6 (Bogovskii operators on A_1)**.** *For $P = (P_H, P_M)$, there are linear operators*

$$S_B : C_c^\infty(A_1) \to C_c^\infty(A_1; \mathrm{Sym}^2) \times C_c^\infty(A_1; \mathrm{Sym}^2)$$

with the following properties:

(i) $PS_B(f, F) = (f, F)$ *for* (f, F) *satisfying* (1.16) *and* (1.18)*;*
(ii) $S_B : H_0^{s-2}(A_1) \to H_0^s(A_1) \times H_0^{s-1}(A_1)$*;*
(iii) $\mathrm{supp}(f, F) \subset A_1$ *implies that* $\mathrm{supp}\, S_B(f, F) \subset A_1$.

3.5. **The ten surface charges.** For data (g, k) on a neighborhood of $\partial B_r \subset \mathbb{R}^3$, define

$$\mathbf{E}[(g,k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, \mathrm{d}S, \tag{3.20}$$

$$\mathbf{P}_a[(g,k); \partial B_r] = \int_{\partial B_r} (k_{aj} - \delta_{aj} \operatorname{tr}_\delta k) \nu^j \, \mathrm{d}S, \tag{3.21}$$

$$\mathbf{C}_a[(g,k); \partial B_r] = \frac{1}{2} \int_{\partial B_r} \Big[x^a (\partial_i g_{ij} - \partial_j g_{ii}) - \delta_{ia}(g_{ij} - \delta_{ij}) + \delta_{ja}(g_{ii} - \delta_{ii}) \Big] \nu^j \, \mathrm{d}S, \tag{3.22}$$

$$\mathbf{J}_a[(g,k); \partial B_r] = \int_{\partial B_r} (k_{ij} - \delta_{ij} \operatorname{tr}_\delta k) Y_a^i \nu^j \, \mathrm{d}S. \tag{3.23}$$

Write

$$\mathbf{Q} = (\mathbf{E}, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3).$$

To avoid traces on rough Sobolev data, one may average the surface charges: for $\eta \in C_c^\infty((1,2))$ such that $\int \eta = 1$ and $\eta_r(r') = r^{-1}\eta(r^{-1}r')$, define

$$\mathbf{Q}[(g,k); A_r] = \int \eta_r(r') \mathbf{Q}[(g,k); \partial_{B_{r'}}] dr'. \tag{3.24}$$

The link between charges and cokernels is given by the following. For $0 < r_0 < r_1$,

$$\int_{B_{r_1} \setminus B_{r_0}} \tfrac{1}{2} \partial_i \partial_j h^{ij} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrm{d}x = \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g,k); \partial B_{r_1}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g,k); \partial B_{r_0}], \tag{3.25}$$

$$\int_{B_{r_1}\setminus B_{r_0}} \sum_j \partial_i \pi^{ij} \begin{pmatrix} \mathbf{e}_1^j \\ \mathbf{e}_2^j \\ \mathbf{e}_3^j \\ \mathbf{Y}_1^j \\ \mathbf{Y}_2^j \\ \mathbf{Y}_3^j \end{pmatrix} \mathrm{d}x = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g,k); \partial B_{r_1}] - \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g,k); \partial B_{r_0}], \tag{3.26}$$

and in terms of averaged charges, if

$$\chi_{r_0,r_1}(r) = \int_{-\infty}^{r} \left(\eta_{r_0}(r') - \eta_{r_1}(r')\right) \mathrm{d}r', \tag{3.27}$$

we have

$$\int \chi_{r_0,r_1}(r) \tfrac{1}{2} \partial_i \partial_j h^{ij} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrm{d}x = \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g,k); A_{r_1}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g,k); A_{r_0}], \tag{3.28}$$

$$\int \sum_j \chi_{r_0,r_1}(r) \partial_i \pi^{ij} \begin{pmatrix} \mathbf{e}_1^j \\ \mathbf{e}_2^j \\ \mathbf{e}_3^j \\ \mathbf{Y}_1^j \\ \mathbf{Y}_2^j \\ \mathbf{Y}_3^j \end{pmatrix} \mathrm{d}x = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g,k); A_{r_1}] - \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{pmatrix} [(g,k); A_{r_0}]. \tag{3.29}$$

Under the invariant rescaling

$$g^{(R)}(x) = g(Rx), \qquad k^{(R)}(x) = Rk(Rx), \tag{3.30}$$

the charges transform as

$$(\mathbf{E},\mathbf{P})[g^{(R)}, k^{(R)}; A_1] = R^{-1}(\mathbf{E},\mathbf{P})[g,k;A_R], \qquad (\mathbf{C},\mathbf{J})[g^{(R)}, k^{(R)}; A_1] = R^{-2}(\mathbf{C},\mathbf{J})[g,k;A_R]. \tag{3.31}$$

4. Conic gluing and obstruction-free gluing

We recall the constraint equations:

$$\begin{aligned} R_g + (\mathrm{tr}_g\, k)^2 - |k|_g^2 &= 0, \\ \mathrm{div}_g\, k - \mathrm{d}(\mathrm{tr}_g\, k) &= 0. \end{aligned} \tag{4.1}$$

4.1. The Picard iteration.

Proposition 4.1. *Let $P : X \to Y$ and $S : Y \to X$ be bounded operators. Suppose $\Phi : X \to Y$ is smooth near 0, satisfies (1.12), and obeys (1.13)–(1.14). If $u_0 \in X$ is sufficiently small, then there is a unique u_1 in a ball of radius comparable to $\|u_0\|_X^2$ such that*

$$u_1 = S\Phi(u_0 + u_1). \tag{4.2}$$

Consequently, if $u_0 \in \ker P$ and $PS = \mathrm{Id}_Y$, then $u := u_0 + u_1$ solves $Pu = \Phi(u)$ and

$$\|u - u_0\|_X \lesssim \|u_0\|_X^2.$$

Any support property shared by u_0, S, and Φ is inherited by u.

Proof. Let $\epsilon = \|u_0\|_X$ and consider the ball $B_{C\epsilon^2}(0) \subset X$. For u_1 in this ball,

$$\|S\Phi(u_0 + u_1)\|_X \leq \|S\|\, C_0(\epsilon + C\epsilon^2)^2 \leq C\epsilon^2$$

when C is chosen large and ϵ small. Moreover,

$$\|S\Phi(u_0 + u_1) - S\Phi(u_0 + v_1)\|_X \leq C_1 \|S\| (\epsilon + C\epsilon^2) \|u_1 - v_1\|_X.$$

For ϵ small, the last coefficient is less than $1/2$. Banach's fixed point theorem implies a unique u_1 of norm $\leq C\epsilon^2$ satisfying (4.2). Finally, if $u_0 \in \ker P$ and $PS = \mathrm{Id}_Y$, then

$$P(u_0 + u_1) = Pu_1 = PS\Phi(u_0 + u_1) = \Phi(u_0 + u_1). \qquad \square$$

The difficult work is therefore concentrated in two tasks: construct a right inverse S with the correct support, and choose spaces in which S and Φ satisfy the required estimates.

4.2. **Solutions localized in a cone.** We now show how to construct solutions supported in a cone. Let $X^{s,\delta}$ be as in (2.21) and let

$$Y^{s,\delta} = H_{\mathrm{b}}^{s-2,\delta+2}(\mathbb{R}^d; \mathbb{R} \oplus \mathbb{R}^d).$$

For $s > d/2$ and $\delta \geq -d/2$, the product estimate (2.20) implies that the nonlinear map (1.10)–(1.11) is smooth

$$\Phi : X^{s,\delta} \longrightarrow Y^{s,\delta} \tag{4.3}$$

near the origin. Combining this with 2.13, we have the following interval

$$-\frac{d}{2} \leq \delta < \frac{d}{2} - 2. \tag{4.4}$$

Theorem 4.2 (Conic vacuum data)**.** *Let $d \geq 2$, let $C_\theta(\omega)$ be a solid cone with $0 < \theta < \pi/2$, let $s > d/2$, and choose δ as in (4.4). If $u_0 = (h_0, \pi_0) \in X^{s,\delta}$ is a sufficiently small solution of $Pu_0 = 0$ supported in $C_\theta(\omega)$, then there is a vacuum solution $u = (h, \pi)$ supported in $C_\theta(\omega)$ such that*

$$\|u - u_0\|_{X^{s,\delta}} \lesssim \|u_0\|_{X^{s,\delta}}^2. \tag{4.5}$$

If u_0 is smooth, then the solution u is also smooth.

Proof. Set the fixed point problem

$$u_1 = S_c\Phi(u_0 + u_1). \tag{4.6}$$

The mapping statements above and the quadratic estimates for Φ verify the hypotheses of Proposition 4.1. The local nature of Φ and Corollary 2.5 preserve support. Smoothness follows by differentiating the fixed point equation, commuting derivatives through the diagonal part of S_c, and using that the outgoing pieces are smoothing away from the diagonal. $\square$

4.3. **The tail.** A compactly supported u_0 does not in general lead to compactly supported nonlinear data. This is because the solution operator S_c generates tails. We note that for $|x| > 2|y| + 1$,

$$K_\kappa(x - y) = K_\kappa(x) + O(|x|^{1-d}|y|), \tag{4.7}$$

$$L_{\kappa,v}(x - y) = L_{\kappa,v}(x) + O(|x|^{-d}|y|). \tag{4.8}$$

Consequently, for (f, F) decaying fast enough,

$$K_\kappa * f(x) = K_\kappa(x) \int f(y)\,\mathrm{d}y + O(|x|^{1-d}), \tag{4.9}$$

$$L_{\kappa,v} * F(x) = L_{\kappa,v}(x) \int F(y)\,\mathrm{d}y + O(|x|^{-d}). \tag{4.10}$$

A careful iteration on derivatives for the fixed point equation (4.6) implies

$$|\partial^\ell h(x)| \lesssim_\ell \langle x\rangle^{2-d-\ell}, \qquad |\partial^\ell \pi(x)| \lesssim_\ell \langle x\rangle^{1-d-\ell}. \tag{4.11}$$

Theorem 4.3 (Carlotto–Schoen [2], Aretakis–Czimek–Rodnianski [1], Mao–Tao [11])**.** *For every $d \geq 3$ and every cone $C_\theta(\omega)$ with $0 < \theta < \pi/2$, there exist nontrivial smooth vacuum data on $\mathbb{R}^d$ satisfying*

$$(g, k) = (\delta, 0) \quad \text{on } \mathbb{R}^d \setminus C_\theta(\omega)$$

and the estimates (4.11) for all $\ell \geq 0$.

This decay rate is sharp because if it decays faster, the ADM energy vanishes, and by the positive mass theorem, the solution must come from a spacelike hypersurface in the Minkowski spacetime.

4.4. **Corvino–Schoen type gluing.** Now we study the gluing problem on an annulus, à la Corvino–Schoen [6]. The proof we state below is from Mao–Oh–Tao [10].

Definition 4.4 ($\mathcal{Q}$-admissible initial data sets)**.** Let a bounded open subset $\mathcal{Q} \subseteq \mathbb{R}^{10}$ and $s \in \mathbb{R}$ be given. We say that a 10-parameter family $\{(g_Q, k_Q)\}_{\mathcal{Q}}$ is a *$\mathcal{Q}$-admissible family of annular initial data sets* on $\widetilde{A}_r$ with Sobolev regularity s if:

(1) for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in (H^1 \cap C^0) \times L^2(\widetilde{A}_r)$ and solves (4.1) in $\widetilde{A}_r$;
(2) for each $Q \in \mathcal{Q}$, $(g_Q, k_Q) \in H^s \times H^{s-1}(\widetilde{A}_r)$;
(3) the map $\mathcal{Q} \to H^s \times H^{s-1}(\widetilde{A}_r)$ defined by $Q \mapsto (g_Q, k_Q)$ is Lipschitz;
(4) for each $Q \in \mathcal{Q} \subseteq \mathbb{R}^{10}$, we have $Q = \mathbf{Q}[(g_Q, k_Q); A_r]$.

Theorem 4.5 (Gluing up to linear obstructions at unit scale)**.** *For each $s > \frac{3}{2}$, there exist $\epsilon_c = \epsilon_c(s) > 0$ and $M_c = M_c(s) > 0$ such that the following holds. Let $(\mathring{g}, \mathring{k})$ be a solution to (4.1) in $H^s \times H^{s-1}(\widetilde{A}_1)$ such that*

$$\|(\mathring{g} - \delta, \mathring{k})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq \epsilon. \tag{4.12}$$

Define $\mathring{Q} \in \mathbb{R}^{10}$ by $\mathring{Q} = \mathbf{Q}[(\mathring{g}, \mathring{k}); A_1]$. Let $\mathcal{Q} \subseteq \mathbb{R}^{10}$ be a bounded open set such that

$$\mathring{Q} \in \mathcal{Q}, \quad B_{M_c \epsilon^2}(\mathring{Q}) \subseteq \mathcal{Q}. \tag{4.13}$$

and consider an $\mathcal{Q}$-admissible family of annular initial data sets $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ on $\widetilde{A}_1$ with Sobolev regularity s such that, for all $Q, Q' \in \mathcal{Q}$,

$$\|(g_Q - \delta, k_Q)\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq \epsilon, \tag{4.14}$$

$$\|(g_Q - g_{Q'}, k_Q - k_{Q'})\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \leq K|Q - Q'|. \tag{4.15}$$

Then the following holds:

(1) **Exterior gluing.** *If $\epsilon < \epsilon_c$ and $K\epsilon < \epsilon_c$, then there exists a solution $(g, k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ to (4.1) and $Q \in \mathcal{Q}$ such that $\|(g - \delta, k)\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$, $|Q - \mathring{Q}| \leq M_c \epsilon^2$, and*

$$(g, k) = \begin{cases} (\mathring{g}, \mathring{k}) & \text{in } A_{\frac{1}{2}}, \\ (g_Q, k_Q) & \text{in } A_2. \end{cases}$$

(2) **Interior gluing.** *If $\epsilon < \epsilon_c$ and $K\epsilon < \epsilon_c$, then there exists a solution $(g, k) \in H^s \times H^{s-1}(\widetilde{A}_1)$ to (4.1) and $Q \in \mathcal{Q}$ such that $\|(g - \delta, k)\|_{H^s \times H^{s-1}(\widetilde{A}_1)} \lesssim \epsilon$, $|Q - \mathring{Q}| \leq M_c \epsilon^2$, and*

$$(g, k) = \begin{cases} (g_Q, k_Q) & \text{in } A_{\frac{1}{2}}, \\ (\mathring{g}, \mathring{k}) & \text{in } A_2. \end{cases}$$

In both cases, the following additional statements hold as well:

- *(Lipschitz continuity) The map $(\mathring{g}, \mathring{k}) \mapsto (g, k, Q)$ is Lipschitz as a map from the subset of $H^s \times H^{s-1}$ restricted by (4.12)–(4.13) into $H^s \times H^{s-1} \times \mathbb{R}^{10}$.*
- *(Persistence of regularity) If $(\mathring{g}, \mathring{k}) \in H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)$ and the family $\{(g_Q, k_Q)\}_{Q \in \mathcal{Q}}$ is of Sobolev regularity $s + m$ for $m \in \mathbb{N}$, then $(g, k) \in H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)$ and*

$$\begin{aligned} \|(g - \delta, k)\|_{H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)} \leq{} & C\|(\mathring{g} - \delta, \mathring{k})\|_{H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)} \\ & + C\|(g_Q - \delta, k_Q)\|_{H^{s+m} \times H^{s+m-1}(\widetilde{A}_1)}, \end{aligned}$$

where $C = C(s, m, \|(\mathring{g} - \delta, \mathring{k})\|_{H^{s+1} \times H^s(\widetilde{A}_1)}, \|(g_Q - \delta, k_Q)\|_{H^{s+1} \times H^s(\widetilde{A}_1)})$.

Proof. **Step 1.** We begin by forming the first trial for the desired initial data set. Let $\chi(x)$ be a smooth radial function which equals 0 for $|x| < 1$ and 1 for $|x| > 2$. We introduce

$$(g_{in;Q}, k_{in;Q}, g_{out;Q}, k_{out;Q}) = \begin{cases} (\mathring{g}, \mathring{k}, g_Q, k_Q) & \text{for Statement (1)}, \\ (g_Q, k_Q, \mathring{g}, \mathring{k}) & \text{for Statement (2)}. \end{cases}$$

We introduce the first guess for the glued initial data, namely,

$$(\overline{h}_Q, \overline{\pi}_Q) = (1-\chi)(h_{in;Q}, \pi_{in;Q}) + \chi(h_{out;Q}, \pi_{out;Q}). \tag{4.16}$$

Of course, $(\overline{h}_Q, \overline{\pi}_Q)$ would not solve the constraint equations (4.1); our aim is to find a correction $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ supported in A_1 and a choice of $Q \in \mathcal{Q}$ such that $(h, \pi) = (\overline{h}_Q + \widetilde{h}_Q, \overline{\pi}_Q + \widetilde{\pi}_Q)$ solves (4.1).

A quick algebraic computation shows that we want $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ to satisfy

$$\partial_i\partial_j \widetilde{h}_Q^{ij} = \widetilde{M}_Q, \quad \partial_i \widetilde{\pi}_Q^{ij} = \widetilde{N}_Q, \tag{4.17}$$

where $\widetilde{M}_Q, \widetilde{N}_Q$ are the errors.

Let S_B be the solution operator defined in Theorem 3.6. For each $Q \in \mathcal{Q}$, we look for $(\widetilde{h}_Q, \widetilde{\pi}_Q)$ solving the fixed point problems

$$(\widetilde{h}_Q, \widetilde{\pi}_Q) = S_B(\widetilde{M}_Q, \widetilde{N}_Q). \tag{4.18}$$

By the first part of Proposition 4.1, there exists such solution $(\widetilde{h}_Q, \widetilde{\pi}_Q) \in H_0^s \times H_0^{s-1}(A_1)$.

Step 2: Finding Q. To conclude the proof of existence, it only remains to find $Q \in \mathcal{Q}$ such that

$$\int \tfrac{1}{2}\widetilde{M}_Q(1, x_1, x_2, x_3)^\dagger \,\mathrm{d}x = 0, \tag{4.19}$$

$$\int \widetilde{N}_Q \cdot (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)^\dagger \,\mathrm{d}x = 0, \tag{4.20}$$

Introduce $\chi_{\frac{1}{2},2}$ as in (3.27). By (3.28), the LHS of (4.19) equals

$$\begin{aligned}
&\int -\chi_{\frac{1}{2},2}(r)\tfrac{1}{2}\partial_i\partial_j \overline{h}_Q^{ij} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrm{d}x + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},2}(r)\tfrac{1}{2}\widetilde{M}_Q^{nonlin} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrm{d}x \\
&= \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] - \begin{pmatrix} \mathbf{E} \\ \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_2] + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},2}(r)\tfrac{1}{2}\widetilde{M}_Q^{nonlin} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mathrm{d}x,
\end{aligned}$$

To compute further the first two terms on the last line, we use (3.28) again:

$$\begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_{\frac{1}{2}}] = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] + \int_{\widetilde{A}_1} \chi_{\frac{1}{2},1}(r)\tfrac{1}{2}M_{in;Q}^{nonlin} \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} \mathrm{d}x,$$

and similarly for $[(g_{out;Q}, k_{out;Q}); A_2]$. We arrive at

$$(\text{LHS of } (4.19)) = \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{E} \\ \vdots \\ \mathbf{C}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] \quad + \begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix} \tag{4.21}$$

where

$$\begin{pmatrix} n[Q]_{\mathbf{E}} \\ \vdots \\ n[Q]_{\mathbf{C}_3} \end{pmatrix} = \int_{\widetilde{A}_1} \tfrac{1}{2}\left(\chi_{\frac{1}{2},2}(r)\widetilde{M}_Q^{nonlin} + \chi_{\frac{1}{2},1}(r)M_{in;Q}^{nonlin} + \chi_{1,2}(r)M_{out;Q}^{nonlin}\right) \begin{pmatrix} 1 \\ \vdots \\ x_3 \end{pmatrix} \mathrm{d}x. \tag{4.22}$$

Similarly, we may show that

$$(\text{LHS of } (4.20)) = \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{in;Q}, k_{in;Q}); A_1] - \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{J}_3 \end{pmatrix} [(g_{out;Q}, k_{out;Q}); A_1] + \begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix} \tag{4.23}$$

with

$$\begin{pmatrix} n[Q]_{\mathbf{P}_1} \\ \vdots \\ n[Q]_{\mathbf{J}_3} \end{pmatrix} = \int_{\widetilde{A}_1} \left(\chi_{\frac{1}{2},2}(r) \widetilde{N}_Q^{nonlin} + \chi_{\frac{1}{2},1}(r) N_{in;Q}^{nonlin} + \chi_{1,2}(r) N_{out;Q}^{nonlin} \right) \cdot \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{Y}_3 \end{pmatrix} \, \mathrm{d}x. \tag{4.24}$$

For the nonlinear contribution $n[Q]$, we note that since $n[Q]$ involves quadratic terms,

$$|n[Q]| \lesssim \epsilon^2, \quad \|n[\cdot]\|_{Lip} \lesssim K\epsilon.$$

For the sake of concreteness, we focus on the case of exterior gluing from this point on; the case of interior gluing is similar. Then $\mathbf{Q}[(g_{in;Q}, k_{in;Q}); A_1] = \mathring{Q}$, whereas $\mathbf{Q}[(g_{out;Q}, k_{out;Q}); A_1] = Q$. Hence, (4.21)–(4.23) reduce to a fixed point problem

$$Q = \mathring{Q} + n[Q]. \tag{4.25}$$

Moreover, we have shown that $|n[Q]| \lesssim \epsilon^2$ and $\|n[Q]\|_{Lip} \lesssim K\epsilon$. **We take M_c to be the implicit constant in the first inequality** so that $|n[Q]| \leq M_c \epsilon^2$. By hypothesis, $\mathrm{dist}(\mathring{Q}, \partial \mathcal{Q}) > M_c \epsilon^2$. Hence, for ϵ_c sufficiently small (recall that $\epsilon < \epsilon_c$), the RHS of (4.25) is thus a contraction on $\{Q : |Q - \mathring{Q}| \leq M_c \epsilon^2\} \subseteq \mathcal{Q}$. It follows that a unique $Q \in \mathcal{Q}$ satisfying (4.25) and $|Q - \mathring{Q}| \leq M_c \epsilon^2$ exists by the Banach fixed point theorem.

□

Exercise 4.6. Show the Lipschitz dependence property and the persistence of regularity in the Theorem. Hint: Differentiate on the equation (4.18). It suffices to control the highest derivative on the right. Then use the smallness of the solution.

4.5. **Obstruction-free gluing.** At the linear level, a compactly supported annular solution requires equality of all ten charges. The nonlinear constraint equations contain quadratic terms that can create charge. We will construct *bump solutions* that generate charges, which leads to the obstruction-free gluing à la Czimek–Rodnianski [7]. The proof here follows Mao–Oh–Tao [10].

Theorem 4.7 (Obstruction-free gluing for almost flat data on annuli). *Given $s > \frac{3}{2}$ and $\Gamma > 1$, there exist $\epsilon_o = \epsilon_o(s, \Gamma) > 0$, $\mu_o = \mu_o(s, \Gamma) > 0$ and $C_o = C_o(s, \Gamma) > 0$ such that the following holds. Let $(g_{in}, k_{in}) \in H^s \times H^{s-1}(A_1)$, $(g_{out}, k_{out}) \in H^s \times H^{s-1}(A_{32})$ be pairs solving (4.1). Define $\Delta Q = (\Delta \mathbf{E}, \ldots, \Delta \mathbf{J}_3) \in \mathbb{R}^{10}$ by*

$$\Delta Q = \mathbf{Q}[(g_{out}, k_{out}); A_{32}] - \mathbf{Q}[(g_{in}, k_{in}); A_1], \tag{4.26}$$

and assume that

$$\Delta \mathbf{E} > |\Delta \mathbf{P}|, \tag{4.27}$$

$$\frac{\Delta \mathbf{E}}{\sqrt{(\Delta \mathbf{E})^2 - |\Delta \mathbf{P}|^2}} < \Gamma, \tag{4.28}$$

$$\Delta \mathbf{E} < \epsilon_o^2, \tag{4.29}$$

$$|\Delta \mathbf{C}| + |\Delta \mathbf{J}| < \mu_o \Delta \mathbf{E}, \tag{4.30}$$

$$\|(g_{in} - \delta, k_{in})\|^2_{H^s \times H^{s-1}(A_1)} + \|(g_{out} - \delta, k_{out})\|^2_{H^s \times H^{s-1}(A_{32})} < \mu_o \Delta \mathbf{E}. \tag{4.31}$$

Then there exists $(g, k) \in H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})$ solving (4.1) such that

$$(g, k) = (g_{in}, k_{in}) \quad \textit{in } A_1, \quad (g, k) = (g_{out}, k_{out}) \quad \textit{in } A_{32}, \tag{4.32}$$

$$\|(g - \delta, k)\|^2_{H^s \times H^{s-1}(B_{64} \setminus \overline{B_1})} < C_o \Delta \mathbf{E}. \tag{4.33}$$

The condition (4.27) is a reflection of the positive mass theorem. (4.28) is a quantitative version of (4.27). (4.29)–(4.31) are smallness conditions.

We also state the asymptotically flat version.

Definition 4.8. Let $(g, k) \in H^s_{\mathrm{loc}} \times H^{s-1}_{\mathrm{loc}}(B^c_{R_0})$ for $s > \frac{3}{2}$ and $R_0 \in 2^{\mathbb{N}}$. We say that (g, k) is *α-asymptotically flat* if, for some $\alpha \in \mathbb{R}$ and $D_0 > 0$, we have

$$\|(g - \delta, k)\|_{\dot{\mathcal{H}}^s \times \dot{\mathcal{H}}^{s-1}(\widetilde{A}_r)} \leq D_0 r^{-s + \frac{3}{2} - \alpha} \qquad \text{for all } r \in 2^{\mathbb{N}} \cap B^c_{R_0}, \tag{4.34}$$

where[1] $\|u\|_{\dot{\mathcal{H}}^s(\widetilde{A}_R)} := R^{-s+\frac{3}{2}}\|u(Rx)\|_{H^s(\widetilde{A}_1)}$.

Theorem 4.9 (Obstruction-free gluing for asymptotically flat data)**.** *Let $s > \frac{3}{2}$ and $\alpha > \frac{1}{2}$. Let (g_{in}, k_{in}), $(g_{out}, k_{out}) \in H^s_{\mathrm{loc}} \times H^{s-1}_{\mathrm{loc}}(B^c_{R_0})$ be α-asymptotically flat pairs solving (4.1). Define, for those components that are well-defined,*

$$\Delta Q := \mathbf{Q}^{ADM}[(g_{out}, k_{out})] - \mathbf{Q}^{ADM}[(g_{in}, k_{in})],$$

and assume that

$$\Delta\mathbf{E} > |\Delta\mathbf{P}|.$$

Then there exists an asymptotically flat pair $(g, k) \in H^s_{\mathrm{loc}} \times H^{s-1}_{\mathrm{loc}}(B^c_{R_0})$ solving (4.1) and $r \geq R_0$ such that $(g, k) = (g_{in}, k_{in})$ on B_{2r} and $(g, k) = (g_{out}, k_{out})$ on B^c_{32r}.

Proof of Theorem 4.7. By Theorem 4.5, it suffices to construct a family of solutions $(g_{\Delta Q}, k_{\Delta Q})$ in the annulus A_{16} such that it extends (g_{in}, k_{in}) and satisfies the conditions in Definition 4.4. For this we construct bump solutions with the following properties:

(1) Each bump solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$ solves (4.1).
(2) Each bump solution is the sum of a linearized solution of size t, supported in a small ball $B_b(\xi)$, and a nonlinear correction of size t^2, supported in an outgoing cone.
(3) The charges are given by
$$(\mathbf{E}, \mathbf{P}, \mathbf{C}, \mathbf{J}) = \gamma(\ell)t^2(1, \ell_i, \xi_i, \epsilon_i^{\ jk}\xi_j\ell_k), \quad \gamma(\ell) = (1 - |\ell|^2)^{-\frac{1}{2}}$$
up to small error terms.

We can then use those bump solutions to build a family of solutions $(g_{\Delta Q}, k_{\Delta Q})$ with the desired properties.

□

4.6. **A localized bump solution.** In this section, we show how to construct a bump solution to (4.1) "localized" in a small ball $B_b(\xi)$ with radius b and center ξ, and given charges. This bump solution is used in the proof of Theorem 4.7.

Proposition 4.10 (Bump solution)**.** *Fix any $-1 < \delta < -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2})$ and $\omega \in \mathbb{S}^2$. For any $s > \frac{3}{2}$, $b > 0$ and $\Gamma > 1$, there exists $\epsilon_{b,0} = \epsilon_{b,0}(b, \Gamma) > 0$ such that the following holds. Given $t > 0$, $\ell \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ obeying*

$$t < \epsilon_{b,0}, \quad 1 \leq \gamma(\ell) := (1 - |\ell|^2)^{-\frac{1}{2}} < \Gamma, \quad \overline{B_b(\xi)} \subseteq C_\theta(\omega) \cap A_8,$$

there exists a solution $(g_{t,\ell,\xi}, k_{t,\ell,\xi}) \in C^\infty(\mathbb{R}^3)$ to (4.1) satisfying

$$\mathrm{supp}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi}) \subseteq C_\theta(\omega) \cap \{|x| > 8\}, \tag{4.35}$$

and, for $n = 0, 1$,

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}} \lesssim_{s,b,\Gamma} t, \tag{4.36}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{E}[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\right] = O_{b,\Gamma}(t^3), \tag{4.37}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{P}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\ell_i\right] = O_{b,\Gamma}(t^3), \tag{4.38}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{C}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\xi_i\right] = O(bt^2) + O_{b,\Gamma}(t^3), \tag{4.39}$$

$$(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}\left[\mathbf{J}_i[(g_{t,\ell,\xi}, k_{t,\ell,\xi}); A_{16}] - \gamma(\ell)t^2\epsilon_i^{\ jk}\xi_j\ell_k\right] = O(bt^2) + O_{b,\Gamma}(t^3). \tag{4.40}$$

Moreover, the following improved bound holds in $\{|x| > 16\}$ (for $n = 0, 1$):

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^{(n)}(g_{t,\ell,\xi} - \delta, k_{t,\ell,\xi})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}(\{|x|>16\})} \lesssim_{s,b,\Gamma} t^2. \tag{4.41}$$

We remark that $B_b(\xi)$ contains the support of the seed bump, or more precisely the $O(t)$ part of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$; the reason behind the improvement (4.41) is that $\{|x| > 16\}$ is disjoint from $B_b(\xi)$.

The following lemma shows how to compute the charges for a specific type of solutions.

[1]When $s \in \mathbb{N}$, observe that $\|u\|_{\dot{\mathcal{H}}^s(\widetilde{A}_R)} \simeq \|\partial_x^{(s)}u\|_{L^2(\widetilde{A}_R)} + R^{-s}\|u\|_{L^2(\widetilde{A}_R)}$.

Lemma 4.11 (Charge computation for special initial data)**.** *Given $r_0 \in 2^{\mathbb{N}_0}$, $\xi_0 \in \mathbb{R}^3$, $t, b_0, A > 0$ and $\tau \in (-\delta_0, \delta_0)$ for some $\delta_0 > 0$, let $(g,k) = (g(\tau), k(\tau)) \in C^2 \times C^1(B_{2r_0}(\xi_0))$ be a solution to (4.1) satisfying, for $n = 0, 1$,*

$$\|\partial_\tau^n (g_{ij} - \delta_{ij} - \mathring{\alpha}_{ij}, k_{ij})\|_{C^2 \times C^1(B_{2r_0}(\xi_0))} \leq At^2,$$

where $\mathring{\alpha}_{ij} = \mathring{\alpha}(\tau)_{ij}$ obeys the special condition

$$\partial^i (\mathring{\alpha}_{ij} - \delta_{ij} \operatorname{tr}_\delta \mathring{\alpha}) = 0, \tag{4.42}$$

as well as the size and support properties (for $n = 0, 1$ and $n' = 0, 1, 2$)

$$|\partial_\tau^n (b_0 \partial_x)^{(n')} \mathring{\alpha}| \leq A b_0^{-\frac{1}{2}} t, \quad \operatorname{supp} \mathring{\alpha} \subseteq B_{b_0}(0). \tag{4.43}$$

Provided that $\overline{B_{b_0}(0)} \subseteq B_{r_0}(\xi_0)$, we have, for $n = 0, 1$,

$$\begin{aligned}
\partial_\tau^n \Big[\mathbf{E}[(g,k); A_{r_0}(\xi_0)] - \tfrac{1}{16} \int \sum_{j,k,\ell} (\partial_k \mathring{\alpha}_{j\ell} - \partial_\ell \mathring{\alpha}_{jk})^2 \, \mathrm{d}x \Big] &= O_{A,b_0}(t^3), \\
\partial_\tau^n \mathbf{P}_i[(g,k); A_{r_0}(\xi_0)] &= O_{A,b_0}(t^3), \\
\partial_\tau^n \mathbf{C}_i[(g,k); A_{r_0}(\xi_0)] &= O(A^2 b_0 t^2) + O_{A,b_0}(t^3), \\
\partial_\tau^n \mathbf{J}_i[(g,k); A_{r_0}(\xi_0)] &= O_{A,b_0}(t^3).
\end{aligned}$$

Proof. For simplicity, we drop the τ-dependence; we leave the similar proof of the C^1-dependence on τ to the reader. Define $(h, \pi) := ((g - \delta) - \delta \operatorname{tr}_\delta (g - \delta), k - \delta \operatorname{tr}_\delta k)$ and $(h_1, \pi_1) := (\mathring{\alpha} - \delta \operatorname{tr}_\delta \mathring{\alpha}, 0)$. The charges are given by (where $\chi_{r_0}(r) = \int_r^\infty \eta_{r_0}(r') \, \mathrm{d}r'$ as in (3.27))

$$\begin{aligned}
&\begin{pmatrix} \mathbf{E} \\ \mathbf{C}_k \end{pmatrix} [(g,k); A_{r_0}(\xi_0)] = \frac{1}{2} \int \chi_{r_0}(x - \xi_0) \partial_i \partial_j h^{ij} \begin{pmatrix} 1 \\ x_k \end{pmatrix} \mathrm{d}x = \frac{1}{2} \int \chi_{r_0}(x - \xi_0) M(h, \pi) \begin{pmatrix} 1 \\ x_k \end{pmatrix} \mathrm{d}x \\
&= \frac{1}{2} \int M(h_1, \pi_1) \begin{pmatrix} 1 \\ x_k \end{pmatrix} \mathrm{d}x + \mathcal{O}_{A,b_0}(t^3) = -\frac{1}{2} \int R[\delta + \mathring{\alpha}] \begin{pmatrix} 1 \\ x_k \end{pmatrix} \mathrm{d}x + \mathcal{O}_{A,b_0}(t^3),
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} \mathbf{P}_k \\ \mathbf{J}_k \end{pmatrix} [(g,k); A_{r_0}(\xi_0)] = \int \sum_j \chi_{r_0}(x - \xi_0) \partial_i \pi^{ij} \begin{pmatrix} \mathbf{e}_k^j \\ \mathbf{Y}_k^j \end{pmatrix} \mathrm{d}x = \int \sum_j \chi_{r_0}(x - \xi_0) N^j(h, \pi) \begin{pmatrix} \mathbf{e}_k^j \\ \mathbf{Y}_k^j \end{pmatrix} \mathrm{d}x \\
&= \int \sum_j N^j(h_1, \pi_1) \begin{pmatrix} \mathbf{e}_k^j \\ \mathbf{Y}_k^j \end{pmatrix} \mathrm{d}x + \mathcal{O}_{A,b_0}(t^3) = \mathcal{O}_{A,b_0}(t^3),
\end{aligned}$$

where we used the fact that $\partial_i \partial_j h_1^{ij} = 0$, $\pi_1 = 0$ and $\operatorname{supp}(h_1, \pi_1) \subseteq B_{r_0}(\xi_0)$.

The above already implies the desired assertions for $\mathbf{P}_k$ and $\mathbf{J}_k$. For $\mathbf{E}$ and $\mathbf{C}_k$, recall the general formulae

$$R[g] = (g^{-1})^{ij} (\partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{i\ell}^k \Gamma_{jk}^\ell), \quad \Gamma_{ij}^k = \frac{1}{2} (g^{-1})^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Let $(g_1)_{ij} := \delta_{ij} + \mathring{\alpha}_{ij}$ and write $(g_1^{-1})^{ij} = \delta^{ij} + \widetilde{\alpha}^{ij}$, where we observe that $\widetilde{\alpha}^{ij} = -\mathring{\alpha}_{ij} + O_{A,b_0}(t^2)$. We will abbreviate terms that are linear in $\mathring{\alpha}$ by lin – note that they will all vanish since $(\mathring{\alpha}, 0)$ is chosen to solve the linearized constraint equation. We have

$$\begin{aligned}
R[g_1] &= \frac{1}{2} (g_1^{-1})^{ij} \partial_k \widetilde{\alpha}^{k\ell} (\partial_i \mathring{\alpha}_{j\ell} + \partial_j \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ij}) + \frac{1}{2} (g_1^{-1})^{ij} (g_1^{-1})^{k\ell} \partial_k (\partial_i \mathring{\alpha}_{j\ell} + \partial_j \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ij}) \\
&\quad - \frac{1}{2} (g_1^{-1})^{ij} \partial_j \widetilde{\alpha}^{k\ell} \partial_i \mathring{\alpha}_{k\ell} - \frac{1}{2} (g_1^{-1})^{ij} (g_1^{-1})^{k\ell} \partial_j \partial_i \mathring{\alpha}_{k\ell} + (g_1^{-1})^{ij} (\Gamma_{k\ell}^k \Gamma_{ij}^\ell - \Gamma_{i\ell}^k \Gamma_{jk}^\ell) \\
&= \frac{1}{2} \partial_k (\widetilde{\alpha}^{k\ell} (2 \partial_i \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ii})) + \frac{1}{2} \widetilde{\alpha}^{ij} \partial_k (2 \partial_i \mathring{\alpha}_{jk} - \partial_k \mathring{\alpha}_{ij}) - \frac{1}{2} \partial_i (\widetilde{\alpha}^{k\ell} \partial_i \mathring{\alpha}_{k\ell}) - \frac{1}{2} \widetilde{\alpha}^{ij} \partial_i \partial_j \mathring{\alpha}_{kk} \\
&\quad + \frac{1}{4} \partial_\ell \mathring{\alpha}_{kk} (2 \partial_i \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ii}) - \frac{1}{4} (\partial_i \mathring{\alpha}_{k\ell})^2 + \frac{1}{4} (\partial_k \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ik})^2 + \mathrm{lin} + O_{A,b_0}(t^3).
\end{aligned}$$

To simplify the nonlinear terms, we use the special condition (4.42), which implies

$$\partial^i \mathring{\alpha}_{ij} = \partial_j \operatorname{tr}_\delta \mathring{\alpha}, \quad \partial^i \partial^j \mathring{\alpha}_{ij} = \Delta \operatorname{tr}_\delta \mathring{\alpha}.$$

Recall also that $\mathrm{lin} = \mathrm{D}_\delta R[\mathring{\alpha}] = 0$, $g_1 = \delta + \mathring{\alpha}$ and $\widetilde{\alpha}^{ij} = -\alpha_{ij} + O_{A,b_0}(t^2)$. Thus

$$\begin{aligned}R[\delta+\mathring{\alpha}] &= \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{2}\right)(\partial_j \operatorname{tr}_\delta \mathring{\alpha})^2 + \left(-\frac{1}{2} - \frac{1}{4} + \frac{1}{2}\right)(\partial_k \mathring{\alpha}_{ij})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + O_{A,b_0}(t^3)\\ &= \frac{1}{4}(\partial_j \operatorname{tr}_\delta \mathring{\alpha})^2 - \frac{1}{4}(\partial_k \mathring{\alpha}_{ij})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + O_{A,b_0}(t^3)\\ &= -\frac{1}{8}(\partial_k \mathring{\alpha}_{i\ell} - \partial_\ell \mathring{\alpha}_{ik})^2 + \partial(\mathring{\alpha}, \partial\mathring{\alpha}) + O_{A,b_0}(t^3).\end{aligned}$$

Plugging in this formula into the above formulae for $\mathbf{E}$ and $\mathbf{C}_k$, and using the fact that $|x_k| \le b_0$ in $\operatorname{supp} \mathring{\alpha} \subseteq B_{b_0}(0)$, the desired assertions for $\mathbf{E}$ and $\mathbf{C}_k$ follow. $\square$

Proof of Proposition 4.10. As a first step, we choose a localized linearized solution and apply the solution operator to make it a solution to (4.1).

Step 1. Choice of $\mathring{\alpha}$. We begin by choosing $\mathring{\alpha}_{ij} \in C_c^\infty(\mathbb{R}^3)$, to which we shall apply Lemma 4.11 in the end (in particular, for our motivation behind the choices $b_0 = \frac{1}{2}\Gamma^{-2}b$ and (4.44), see Steps 2 and 4 below, respectively). Fix a function $\chi \in C_c^\infty(B_1)$ such that

$$\frac{1}{16}\int |\partial(\partial_{11}^2 + \partial_{22}^2)\chi|^2 \,\mathrm{d}x = 1. \tag{4.44}$$

Following §2.7, we set

$$\mathring{\alpha} = t\begin{pmatrix} \frac{1}{2}(\partial_{22}^2\chi_{b,\Gamma} - \partial_{11}^2\chi_{b,\Gamma}) & -\partial_{12}^2\chi_{b,\Gamma} & 0\\ -\partial_{12}^2\chi_{b,\Gamma} & \frac{1}{2}(\partial_{11}^2\chi_{b,\Gamma} - \partial_{22}^2\chi_{b,\Gamma}) & 0\\ 0 & 0 & -\frac{1}{2}(\partial_{11}^2 + \partial_{22}^2)\chi_{b,\Gamma}\end{pmatrix}$$

where $\chi_{b,\Gamma}(x) = (\frac{1}{2}\Gamma^{-2}b)^{\frac{3}{2}}\chi(2\Gamma^2 b^{-1}x)$ so that $\|\partial_{ijk}^3\chi_{b,\Gamma}\|_{L^2}^2 = \|\partial_{ijk}^3\chi\|_{L^2}^2$. From the definition, it is straightforward to verify that $\partial^i\mathring{\alpha}_{ij} = \partial_j \operatorname{tr}_\delta \mathring{\alpha}$ and (4.43) hold with $b_0 = \frac{1}{2}\Gamma^{-2}b$.

It is, in principle, possible to choose more general linearized solutions to get bump solutions with more general charges. Instead, we will only use the linearized *Lorentz boost* to get solutions with more general charges.

Step 2. Construction of $\boldsymbol{\alpha}$ and $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$.

The Lorentz boost is the following process. Consider a Lorentzian metric $\mathbf{g} \in C^2(\boldsymbol{\Omega})$, where $\boldsymbol{\Omega} \subseteq \mathbb{R}^{1+3}$. Given $(\Lambda, \boldsymbol{\xi}) \in SO^+(1,3) \times \mathbb{R}^{1+3}$, let $y^\mu = \Lambda^\mu{}_{\mu'}x^{\mu'} + \boldsymbol{\xi}^\nu$. Then, with respect to the coordinates $(y^0, \dots, y^3)$, define

$$g_{ij}^{(\Lambda,\boldsymbol{\xi})} = \mathbf{g}_{ij}\Big|_{\{y^0=0\}\cap\boldsymbol{\Omega}}, \quad k_{ij}^{(\Lambda,\boldsymbol{\xi})} = \frac{-(\mathbf{g}^{-1})^{0\mu}}{2\sqrt{-(\mathbf{g}^{-1})^{00}}}\left(\partial_\mu \mathbf{g}_{ij} - \partial_i \mathbf{g}_{j\mu} - \partial_j \mathbf{g}_{i\mu}\right)\Big|_{\{y^0=0\}\cap\boldsymbol{\Omega}}. \tag{4.45}$$

Consider the Einstein tensor $\mathbf{G}[\mathbf{g}] = \operatorname{Ric}[\mathbf{g}] - \frac{1}{2}\mathbf{g}\operatorname{tr}_{\mathbf{g}}\operatorname{Ric}[\mathbf{g}]$. Its linearization around the flat metric $\boldsymbol{\eta}$, which trivially satisfies $\mathbf{G}[\boldsymbol{\eta}] = 0$, takes the form

$$\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta} = \frac{1}{2}\left(-\nabla^\gamma\nabla_\gamma \mathbf{H}_{\alpha\beta} + \nabla_\alpha\nabla^\gamma\mathbf{H}_{\gamma\beta} + \nabla_\beta\nabla^\gamma\mathbf{H}_{\gamma\alpha} - \boldsymbol{\eta}_{\alpha\beta}\nabla^\gamma\nabla^\delta\mathbf{H}_{\gamma\delta}\right), \tag{4.46}$$

where $\mathbf{H} = \dot{\mathbf{g}} - \frac{1}{2}\boldsymbol{\eta}\operatorname{tr}_{\boldsymbol{\eta}}\dot{\mathbf{g}}$.

We solve the following linearized Einstein equation to define $\boldsymbol{\alpha} \in C^\infty(\mathbb{R}^{1+3})$ to be the unique solution to the problem

$$\begin{gathered}\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha\beta} = 0, \quad \nabla^\alpha\mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = 0,\\ (\boldsymbol{\alpha}_{ij}, \partial_0\boldsymbol{\alpha}_{ij})|_{\{x^0=0\}} = (\mathring{\alpha}_{ij}, 0), \quad \boldsymbol{\alpha}_{00}|_{\{x^0=0\}} = 0, \quad \boldsymbol{\alpha}_{0j}|_{\{x^0=0\}} = 0,\end{gathered} \tag{4.47}$$

where $\partial_0\boldsymbol{\alpha}_{00}|_{\{x^0=0\}}$, $\partial_0\boldsymbol{\alpha}_{0j}|_{\{x^0=0\}}$ are induced by $\nabla^\alpha\mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta} = 0$. It is direct to check that this is equivalent to a wave equation

$$\begin{gathered}\Box\boldsymbol{\alpha}_{\alpha\beta} = 0,\\ (\boldsymbol{\alpha}_{ij}, \partial_0\boldsymbol{\alpha}_{ij})|_{\Sigma_0} = (\mathring{\alpha}_{ij}, 0), \quad \boldsymbol{\alpha}_{00}|_{\Sigma_0} = 0, \quad \boldsymbol{\alpha}_{0j}|_{\Sigma_0} = 0,\\ (\nabla^\alpha\mathbf{H}[\boldsymbol{\alpha}]_{\alpha\beta})|_{\Sigma_0} = 0, \quad (\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\boldsymbol{\alpha}]_{\alpha 0})|_{\Sigma_0} = 0.\end{gathered}$$

Recall that $\operatorname{supp}(\boldsymbol{\alpha}_{\mu\nu}, \partial_t \boldsymbol{\alpha}_{\mu\nu})|_{\{x^0=0\}} \subseteq \{x^0 = 0,\, |x| < \frac{1}{2}\Gamma^{-2}b\}$ for each μ, ν. Hence, by finite speed of propagation,

$$\operatorname{supp} \boldsymbol{\alpha} \subseteq \boldsymbol{\Omega}_{\boldsymbol{\alpha}} := \{|x| < |x^0| + \tfrac{1}{2}\Gamma^{-2}b\}. \tag{4.48}$$

Using the parameters ℓ and ξ, define $\Lambda_\ell \in SO^+(1,3)$ (Lorentz boost by velocity ℓ) and $\boldsymbol{\xi} \in \mathbb{R}^{1+3}$ by

$$(\Lambda_\ell x)^0 = \gamma(\ell)(x^0 - \ell \cdot P_\ell x), \quad (\Lambda_\ell x)^j = \gamma(\ell)(-\ell^j x^0 + (P_\ell x)^j) + (P_\ell^\perp x)^j, \quad \boldsymbol{\xi}^0 = 0, \quad \boldsymbol{\xi}^j = \xi^j,$$

where $\gamma(\ell) = (1 - |\ell|^2)^{-\frac{1}{2}}$ (Lorentz contraction factor), and $P_\ell, P_\ell^\perp : \mathbb{R}^3 \to \mathbb{R}^3$ are orthogonal projections to (ℓ) and $(\ell)^\perp$, respectively. Define

$$y^\mu = (\Lambda_\ell)^\mu{}_\nu x^\nu + \boldsymbol{\xi}^\mu,$$

which constitute another set of canonical coordinates. With respect to the coordinates $(y^0, \dots, y^3)$, we define $(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})$ on $\{y^0 = 0\}$ as

$$(\alpha_{t,\ell,\xi})_{ij} = \boldsymbol{\alpha}_{ij}\big|_{\{y^0=0\}}, \quad (\beta_{t,\ell,\xi})_{ij} = \tfrac{1}{2}\left(\partial_0 \boldsymbol{\alpha}_{ij} - \partial_i \boldsymbol{\alpha}_{j0} - \partial_j \boldsymbol{\alpha}_{i0}\right)\big|_{\{y^0=0\}}. \tag{4.49}$$

Note that this pair solves the linearized constraint equation, in the sense that $(h_{t,\ell,\xi}, \pi_{t,\ell\xi}) = (\alpha_{t,\ell,\xi} - \delta \operatorname{tr}_\delta \alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi} - \delta \operatorname{tr}_\delta \beta_{t,\ell,\xi})$ solves $P(h_{t,\ell,\xi}, \pi_{t,\ell,\xi}) = 0$. We claim that

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi})\|_{H^s \times H^{s-1}(\mathbb{R}^3)} + \|\nabla (t\partial_t, \partial_\ell, \partial_\xi)^n \boldsymbol{\alpha}_{0\mu}\|_{H^{s-1}(\mathbb{R}^3)} \lesssim_{n,s,b,\Gamma} t, \tag{4.50}$$

$$\operatorname{supp}(\alpha_{t,\ell,\xi}, \beta_{t,\ell,\xi}) \subseteq B_b(\xi). \tag{4.51}$$

The $H^s \times H^{s-1}$ bound (4.50) follows from the standard energy estimate applied to $\Box(t\partial_t, \partial_\ell, \partial_\xi)^n \boldsymbol{\alpha}_{\mu\nu} = 0$ in the region between the two planes $\{x^0 = 0\}$ and $\{y^0 = 0\}$. To verify (4.51), observe first that, since $|x^0| \le |\ell||x|$ on $\{y^0 = 0\}$, we have

$$\{y^0 = 0\} \cap \{|x| > \tfrac{1}{2}(1 - |\ell|)^{-1}\Gamma^{-2}b\} \subseteq \{|x| > |x^0| + \tfrac{1}{2}\Gamma^{-2}b\} = \mathbb{R}^{1+3} \setminus \overline{\boldsymbol{\Omega}_{\boldsymbol{\alpha}}}.$$

Note also that, on $\{y^0 = 0\}$, the coordinates (y^1, y^2, y^3) see length contraction by $\gamma(\ell)^{-1}$ in the direction of ℓ relative to (x^1, x^2, x^3), i.e.,

$$(y^j - \xi^j)\big|_{\{y^0=0\}} = \gamma(\ell)^{-1}(P_\ell x)^j + (P_\ell^\perp x)^j\big|_{\{y^0=0\}}. \tag{4.52}$$

In particular, $|x| \ge |y - \xi|$. Putting these facts together, and using $\frac{1}{2}(1 - |\ell|)^{-1} \le \gamma^2 < \Gamma^2$, we have

$$\{y^0 = 0,\, |y - \xi| \ge b\} \subseteq \{y^0 = 0,\, |x| \ge b\} \subseteq \{y^0 = 0,\, |x| > \tfrac{1}{2}(1 - |\ell|)^{-1}\Gamma^{-2}b\} \subseteq \mathbb{R}^{1+3} \setminus \overline{\boldsymbol{\Omega}_{\boldsymbol{\alpha}}}.$$

In view of (4.48), $\boldsymbol{\alpha} = 0$ in an open neighborhood of $\{y^0 = 0,\, |y - \xi| \ge b\}$, which implies the desired support statement.

Step 3. Construction of $(g_{t,\ell,\xi}, k_{t,\ell,\xi})$. We work on the hyperplane $\{y^0 = 0\}$ using the coordinates (y^1, y^2, y^3). By (4.50), (4.51) and hypothesis, note that

$$\operatorname{supp}(h_{t,\ell,\xi}, \pi_{t,\ell,\xi}) \subseteq C_\theta(\omega) \cap A_8, \quad \|(t\partial_t, \partial_\ell, \partial_\xi)(h_{t,\ell,\xi}, \pi_{t,\ell,\xi})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} = O_{s,b,\Gamma}(t),$$

where the weights do not play any role in the second assertion by the support property. Let S_c be the conic solution operators in Definition 2.4 adapted to $C_\theta(\boldsymbol{\omega})$. In order to construct a solution to (4.1) of the form $(g_{t,\ell,\xi}, k_{t,\ell,\xi}) = (\delta + \alpha_{t,\ell,\xi} + \widetilde{\alpha}, \beta_{t,\ell,\xi} + \widetilde{\beta})$, it suffices to solve

$$(\widetilde{h}, \widetilde{\pi}) = S_c N(h_{t,\ell,\xi} + \widetilde{h}, \pi_{t,\ell,\xi} + \widetilde{\pi}),$$

where $(\widetilde{h}, \widetilde{\pi}) = (\widetilde{\alpha} - \delta \operatorname{tr}_\delta \widetilde{\alpha}, \widetilde{\beta} - \delta \operatorname{tr}_\delta \widetilde{\beta})$. By standard Picard iteration, there is a unique solution $(\widetilde{h}, \widetilde{\pi}) \in (H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1})_0(C_\theta(\omega))$ as long as t is sufficiently small depending on s, b, Γ, such that

$$\|(t\partial_t, \partial_\ell, \partial_\xi)^n (\widetilde{h}, \widetilde{\pi})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim_{n,s,b,\Gamma} t^2.$$

Step 4. Computation of charges. By (4.44), we have

$$\begin{aligned}\frac{1}{16} \int \sum_{j,k,\ell} (\partial_k \mathring{\alpha}_{j\ell} - \partial_\ell \mathring{\alpha}_{jk})^2 \, \mathrm{d}x &= \frac{1}{8} \int \left[\sum_{j,k,\ell} (\partial_j \mathring{\alpha}_{k,\ell})^2 - \sum_j (\partial_j \operatorname{tr}_\delta \mathring{\alpha})^2\right] \mathrm{d}x \\ &= t^2 \int \left[\frac{1}{4} |\partial \partial_{12}^2 \chi_{b,\Gamma}|^2 + \frac{1}{16} |\partial(\partial_{11}^2 - \partial_{22}^2)\chi_{b,\Gamma}|^2\right] \mathrm{d}x\end{aligned}$$

$$= \frac{t^2}{16}\int |\partial(\partial_{11}^2+\partial_{22}^2)\chi|^2\,\mathrm{d}x = t^2.$$

Since $\{x^0=0,\,16\le|x|\le 32\}$ and $\{y^0=0,\,16\le|y|\le 32\}$ are contained in $\mathbf{\Omega}$ yet disjoint from $\overline{\mathbf{\Omega}_{\boldsymbol{\alpha}}}$, for each σ we may find a hypersurface U_σ such that $\partial U_\sigma=\{x^0=0,\,|x|=16(1+\sigma)\}\cup\{y^0=0,\,|y|=16(1+\sigma)\}$, $\overline{U_\sigma}\cap\operatorname{supp}\boldsymbol{\alpha}=\emptyset$ while $\mathrm{Vol}(U_\sigma)=O(1)$. By Lemma 4.11, we have

$$(\mathbf{E},\mathbf{P},\mathbf{C},\mathbf{J})[(g,k);A_{16}] = (t^2,0,O(\Gamma^{-2}bt^2),0)+O_{b,\Gamma}(t^3),$$

and similarly after applying of $(t\partial_t,\partial_\ell,\partial_\xi)$.

Exercise 4.12. Introduce the notation

$$(\mathbb{P}_0,\mathbb{P}_i,\mathbb{J}_{i0},\mathbb{J}_{jk})[(g,k);A_r] = (\mathbf{E},\mathbf{P}_i,\mathbf{C}_i,\epsilon^i{}_{jk}\mathbf{J}_i)[(g,k);A_r]. \tag{4.53}$$

Show that after the Lorentz boost,

$$\mathbb{P}_\mu\mapsto \Lambda_\mu{}^{\mu'}\mathbb{P}_{\mu'},\quad \mathbb{J}_{\mu\nu}\mapsto\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\mathbb{J}_{\mu'\nu'}+\boldsymbol{\xi}_\mu\Lambda_\nu{}^{\nu'}\mathbb{P}_{\nu'}-\boldsymbol{\xi}_\nu\Lambda_\mu{}^{\mu'}\mathbb{P}_{\mu'} \tag{4.54}$$

up to $O(t^4)$ errors. Thus we deduce (4.37)–(4.38), (4.39)–(4.40).

The proof goes along the following steps.

Let $\mathbf{X}$ be a Killing vector field with respect to $\boldsymbol{\eta}$, i.e., $(\mathcal{L}_{\mathbf{X}}\boldsymbol{\eta})_{\alpha\beta}=\nabla_\alpha\mathbf{X}_\beta+\nabla_\beta\mathbf{X}_\alpha=0$. By the linearized second Bianchi identity $\nabla^\beta \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}=0$, observe that $\nabla^\beta(\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha)=0$; hence, the 1-form $\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha$ is co-closed. In fact, there exists a 2-form whose co-differential is this 1-form; following [5, Appendix E] (see also [4]), we introduce

$${}^{(\mathbf{X})}\mathbb{U}_{\alpha\beta}[\dot{\mathbf{g}}]=\frac{1}{2}\left[\left(-\nabla_\alpha\mathbf{H}_{\gamma\beta}+\nabla_\beta\mathbf{H}_{\gamma\alpha}+\boldsymbol{\eta}_{\gamma\alpha}\nabla^\delta\mathbf{H}_{\beta\delta}-\boldsymbol{\eta}_{\gamma\beta}\nabla^\delta\mathbf{H}_{\alpha\delta}\right)\mathbf{X}^\gamma+\mathbf{H}_{\gamma\alpha}\nabla^\gamma\mathbf{X}_\beta-\mathbf{H}_{\gamma\beta}\nabla^\gamma\mathbf{X}_\alpha\right]. \tag{4.55}$$

By a straightforward computation, it may be verified that

$$\nabla^\alpha({}^{(\mathbf{X})}\mathbb{U}_{\alpha\beta}[\dot{\mathbf{g}}])=\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{\alpha\beta}\mathbf{X}^\alpha. \tag{4.56}$$

Equivalently, $\mathrm{d}(\star{}^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}])=-\star\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}](\mathbf{X},\cdot)$ using the Hodge star operator $\star$ associated to $\boldsymbol{\eta}$. By the Stokes theorem,

$$\int_{\partial U}\star{}^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}]=\int_U\mathrm{d}(\star{}^{(\mathbf{X})}\mathbb{U}[\dot{\mathbf{g}}])=-\int_U\star\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}](\mathbf{X},\cdot). \tag{4.57}$$

Consider a system of coordinates $\{x^0,\dots,x^3\}$ with respect to which $\boldsymbol{\eta}=-(\mathrm{d}x^0)^2+(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2$ and $\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$ has positive orientation. On $\Sigma_0=\{x^0=0\}$, define

$$\dot g_{ij}=\dot{\mathbf{g}}_{ij}\Big|_{\{x^0=0\}},\quad \dot k_{ij}=\frac{1}{2}\left(\partial_0\dot{\mathbf{g}}_{ij}-\partial_i\dot{\mathbf{g}}_{j0}-\partial_j\dot{\mathbf{g}}_{i0}\right)\Big|_{\{x^0=0\}}. \tag{4.58}$$

It may be checked that

$$\mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{00}=\tfrac12\partial_j\partial_k(\dot g^{jk}-\delta^{jk}\operatorname{tr}_\delta\dot g),\quad \mathrm{D}_{\boldsymbol{\eta}}\mathbf{G}[\dot{\mathbf{g}}]_{0j}=\partial^\ell(\dot k_{j\ell}-\delta_{j\ell}\operatorname{tr}_\delta\dot k). \tag{4.59}$$

The space of Killing vector fields with respect to $\boldsymbol{\eta}$ are spanned by ∂_{x^μ} and $x_\mu\partial_{x^\nu}-x_\nu\partial_{x^\mu}$ $(\mu,\nu=0,1,2,3)$. For $r>0$, we define the associated charges by

$$\begin{aligned}\mathbb{P}_\mu[\dot{\mathbf{g}};\{x^0=0,\,|x|=r\}]&=\int_{\{x^0=0,\,|x|=r\}}\star^{(\partial_{x^\mu})}\mathbb{U}[\dot{\mathbf{g}}],\\ \mathbb{J}_{\mu\nu}[\dot{\mathbf{g}};\{x^0=0,\,|x|=r\}]&=\int_{\{x^0=0,\,|x|=r\}}\star^{(x_\mu\partial_{x^\nu}-x_\nu\partial_{x^\mu})}\mathbb{U}[\dot{\mathbf{g}}],\end{aligned} \tag{4.60}$$

where $\{x^0=0,\,|x|=r\}$ is oriented with $\star(\mathrm{d}x^0\wedge\mathrm{d}|x|)$. Check that the $(\mathbb{P},\mathbb{J})$ defined in (4.53) is the space average of (4.60). Finally, (4.54) follows from the transformation of the Killing vector fields and (4.57).

□

Exercise 4.13. Use a linear combination of the bumps solutions to build a family satisfying the properties in Definition 4.4.

5. Slowly decaying data and rapidly decaying data

We now discuss some variants.

5.1. Initial data with slow decay. In Theorem 4.3, if we take the linearized solution to have slow decay, we will get initial data with slow decay. This is investigated in Shen–Wan [14].

More precisely, we want to have solutions (g, k) to (4.1) such that for some $\alpha \in [0, 1)$,

$$|\partial^\ell (g - \delta)| \lesssim \epsilon_0 \langle x\rangle^{-\alpha-\ell}, \quad 0 \le \ell \le q + 1, \tag{5.1}$$

$$|\partial^\ell k| \lesssim \epsilon_0 \langle x\rangle^{-\alpha-1-\ell}, \quad 0 \le \ell \le q, \tag{5.2}$$

and such that (5.1) and (5.2) do not hold for any $\alpha' > \alpha$.

The corresponding weighted Sobolev space is

$$H_{\mathrm{b}}^{s,\alpha-3/2} \times H_{\mathrm{b}}^{s-1,\alpha-1/2}. \tag{5.3}$$

For $\alpha = 0$, the metric perturbation is allowed to tend to zero more slowly than every positive power of r, while k is only just beyond the r^{-1} scale.

We take $L : [0, \infty) \to (0, \infty)$ to be nondecreasing, and such that

$$\int_1^\infty \frac{\mathrm{d}r}{rL(r)^2} < \infty, \qquad \int_1^\infty \frac{r^{\eta-1}}{L(r)^2}\,\mathrm{d}r = \infty \quad \text{for every } \eta > 0, \tag{5.4}$$

$$|(L^{-1})^{(m)}(r)| \le \frac{C_m}{r^m L(r)}, \quad 0 \le m \le q + 3, \tag{5.5}$$

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0. \tag{5.6}$$

The model example is

$$L(r) = \big(\log(2 + r)\big)^\beta, \qquad \beta > \frac{1}{2}. \tag{5.7}$$

Choose an angular cutoff χ supported in a cone and equal to one on a smaller cone, and a radial cutoff η that equals one for large r. Define

$$\phi_h(x) = \epsilon_0\eta(r)\chi(\widehat{x})\frac{r^{2-\alpha}}{L(r)}, \qquad \phi_\pi(x) = \epsilon_0\eta(r)\chi(\widehat{x})\frac{r^{1-\alpha}}{L(r)}. \tag{5.8}$$

Set (h_0, π_0) as in Lemma 2.15.

The derivative conditions on L imply

$$|\partial^\ell h_0(x)| \lesssim_\ell \epsilon_0 \frac{r^{-\alpha-\ell}}{L(r)}, \tag{5.9}$$

$$|\partial^\ell \pi_0(x)| \lesssim_\ell \epsilon_0 \frac{r^{-\alpha-1-\ell}}{L(r)}. \tag{5.10}$$

Thus (h_0, π_0) belongs to (5.3).

Exercise 5.1. Show that

$$|h_0(x)| \gtrsim \epsilon_0 \frac{r^{-\alpha}}{L(r)}, \qquad |\pi_0(x)| \gtrsim \epsilon_0 \frac{r^{-\alpha-1}}{L(r)}. \tag{5.11}$$

Recall that the conic solution operator (2.11) has the specialized mapping property

$$S_c : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega; \mathbb{R} \oplus \mathbb{R}^3) \longrightarrow H_{\mathrm{b}}^{s,\delta}(\Omega; \mathrm{Sym}^2) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega; \mathrm{Sym}^2). \tag{5.12}$$

Running the Picard iteration as in §4.1 with $s = q+3$ gives a slowly decaying solution $(h, \pi) = (h_0+h_1, \pi_0+\pi_1)$ with

$$\|(h_1, \pi_1)\|_{H_{\mathrm{b}}^{s,\alpha-3/2}\times H_{\mathrm{b}}^{s-1,\alpha-1/2}} \lesssim \epsilon_0^2. \tag{5.13}$$

Theorem 5.2 (Shen–Wan [14]). *For every $\alpha \in [0,1)$, $q \in \mathbb{N}$, and sufficiently small ϵ_0, there are nontrivial smooth initial data supported in a cone satisfying (4.1) and (5.1)–(5.2). Moreover, for every $\alpha' > \alpha$, the data fail to lie in $H_{\mathrm{b}}^{q+3,\alpha'-3/2} \times H_{\mathrm{b}}^{q+2,\alpha'-1/2}$.*

Exercise 5.3. Verify the last claim of Theorem 5.2 using (5.11).

Exercise 5.4. Taking $\alpha = 0$ in Theorem 5.2 gives a solution with very slow decay. It is possible to modify the construction to construct a solution with no decay at all, in the following way.

(1) Choose initial data without decay;
(2) Consider the space with the norm
$$\|u\|_{H_{\mathrm{b}}^{s,\delta,\infty}} := \|u\|_{H^s(B_{10})} + \sup_{j\in\mathbb{N}} \|u\|_{H_{\mathrm{b}}^{s,\delta}(\widetilde{A}_{2^j})}.$$
Show that the solution operator is bounded
$$S_c : H_{\mathrm{b}}^{s-2,\delta+2,\infty}(\Omega; \mathbb{R}\oplus\mathbb{R}^3) \longrightarrow H_{\mathrm{b}}^{s,\delta,\infty}(\Omega; \mathrm{Sym}^2) \times H_{\mathrm{b}}^{s-1,\delta+1,\infty}(\Omega; \mathrm{Sym}^2),$$
and the Picard iteration in §4.1 works on
$$X^{s,\delta} := H_{\mathrm{b}}^{s,\delta,\infty}(\Omega; \mathrm{Sym}^2) \times H_{\mathrm{b}}^{s-1,\delta+1,\infty}(\Omega; \mathrm{Sym}^2), \quad Y^{s,\delta} := H_{\mathrm{b}}^{s-2,\delta+2,\infty}(\Omega; \mathbb{R}\oplus\mathbb{R}^3), \quad s > 3/2,\ \delta = -3/2.$$
(3) Show that the solution we get has a uniform lower bound along a ray going to infinity.

5.2. **Incoming solution operator.** There is also an incoming construction of solution operators.

First we make the observation that the construction in §3 gives an operator also on $\mathbb{R}^d$. For example, the formula
$$\Psi_\eta^{ij}(z+y, y) = \frac{z^i z^j}{|z|^3} \int_{|z|}^\infty \eta\left(r\frac{z}{|z|} + y\right) r^2 \,\mathrm{d}r \tag{5.14}$$
makes sense on $\mathbb{R}^d$. However, this operator does not have a very good regularizing property on weighted Sobolev spaces. The reason is, after rescaling to the unit scale, the conic region we are averaging becomes very thin.

Therefore we consider the following modification. Let $\eta(y, y_1) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be homogeneous in the sense that
$$\eta(\lambda y, \lambda y_1) = \lambda^{-d}\eta(y, y_1), \quad \lambda > 1,\ |y| > 1.$$
Moreover, we assume for any $y \in \mathbb{R}^d$, we have $\operatorname{supp}\eta(y,\cdot) \subset \{|y|/100 < |y_1| < |y|/10 + 1/10\}$ and
$$\int \eta(y, y_1) dy_1 = 1, \quad \forall y \in \mathbb{R}^d. \tag{5.15}$$

Now we consider the operator with the Schwartz kernel
$$\Psi_\eta^{ij}(z+y, y) = \frac{z^i z^j}{|z|^3} \int_{|z|}^\infty \eta\left(y, r\frac{z}{|z|} + y\right) r^2 \,\mathrm{d}r \tag{5.16}$$
After the process as in §3.4, this operator maps a dyadic annulus to essentially the same dyadic annulus, and thus has nice properties on the weighted Sobolev spaces. Then we can use the Bogovskii type operator (5.14) on $\mathbb{R}^d$ to correct the smooth errors in each dyadic annulus.

Using (5.16) we have the following lemma.

Lemma 5.5. *Let $d \ge 2$. Then there exist linear operators S_{hom} on $\mathbb{R}^d$ such that for $s \in \mathbb{R}$ and $\delta > -d/2$,*

(1) $S_{\mathrm{hom}} : H_{\mathrm{b}}^{s-2,\delta+2} \to H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ and $[S_{\mathrm{hom}}, \partial] : H_{\mathrm{b}}^{s-2,\delta+2} \to H_{\mathrm{b}}^{s,\delta+1} \times H_{\mathrm{b}}^{s-1,\delta+2}$ are bounded;
(2) There exist $\zeta(x,y) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that
$$PS_{\mathrm{hom}}f = f - \int f(y)\zeta(x,y)dy,$$
where ζ is homogeneous in the sense that
$$\zeta(\lambda x, \lambda y) = \lambda^{-d}\zeta(x,y), \quad \lambda > 1,\ |x| > 1, |y| > 1.$$

(3) S_{hom} *is homogeneous in the sense that*

$$S_{\mathrm{hom}}(\lambda x, \lambda y) = (\lambda^{2-d} S^{(1)}_{\mathrm{hom}}(x,y), \lambda^{1-d} S^{(2)}_{\mathrm{hom}}(x,y)), \quad \lambda > 1,\ |x| > 1, |y| > 1.$$

(4) S_{hom}*is incoming in the sense that*

$$\operatorname{supp} S_{\mathrm{hom}}(x,y) \subset \{|x| \le |y|\} \cup B_1 \times B_1. \tag{5.17}$$

Moreover, S_{hom} *is diagonal: there exists* $C > 0$ *such that the Schwartz kernel* $K(x,y)$ *satisfies*

$$\operatorname{supp} K(x,y) \subset \{C^{-1}|y| - C \le |x| \le |y| + C\}.$$

Exercise 5.6. Check that S_{hom} maps polyhomogeneous functions to polyhomogeneous functions.

We can use incoming solution operators to construct solutions that decay fast with respect to a given solution.

We recall the following notation for the error of (4.1):

$$\mathcal{E}_0[g,k] := R[g] + (\operatorname{tr}_g k)^2 - |k|^2_g, \quad \mathcal{E}_j[g,k] := (\operatorname{div}_g k)_j - \nabla_j \operatorname{tr}_g k,$$

Theorem 5.7. *Let* $s > \frac{3}{2}$, $-\frac{3}{2} < \delta_0 < \delta_1$ *and* $\delta_0 \le \delta < \delta_1$ *such that* $\delta_1 > 1/2$ *and* $\delta_0 + \delta > -1$. *For* $\epsilon_0, \epsilon_1, \epsilon > 0$ *sufficiently small (depending on* s, δ_0, δ_1 *and* δ*), the following holds. Let* $(g_0, k_0) \in H^{s,\delta_0}_{\mathrm{b}} \times H^{s-1,\delta_0+1}_{\mathrm{b}}(\mathbb{R}^3 \setminus \overline{B_1})$ *satisfy*

$$\|g_0 - \delta\|_{H^{s,\delta_0}_{\mathrm{b}}} + \|k_0\|_{H^{s-1,\delta_0+1}_{\mathrm{b}}} \le \epsilon_0,$$

and solve (4.1) *approximately in the sense that*

$$\|(\mathcal{E}_0, \dots, \mathcal{E}_3)[g_0, k_0]\|_{H^{s-2,\delta_1+2}_{\mathrm{b}}} \le \epsilon_1.$$

Moreover, for δ *satisfying* $\delta_0 \le \delta < \delta_1$, *let* $(\dot{g}, \dot{k}) \in H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}(\mathbb{R}^3 \setminus \overline{B_1})$ *be a solution to the linearized system*

$$P(h, \pi) = (\partial_i \partial_j h^{ij}, \partial_i \pi^{ij}) = 0 \tag{5.18}$$

satisfying

$$\|(\dot{g}, \dot{k})\|_{H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}} \le \epsilon.$$

Then there exists $(g,k) \in H^{s,\delta_0}_{\mathrm{b}} \times H^{s-1,\delta_0+1}_{\mathrm{b}}(\mathbb{R}^3 \setminus \overline{B_1})$ *solving* (4.1) *such that*

$$\|(g - g_0 - \dot{g}, k - k_0 - \dot{k})\|_{H^{s,\min\{\delta_1, \delta_0+\delta+3/2\}}_{\mathrm{b}} \times H^{s-1,\min\{\delta_1,\delta_0+\delta\}+5/2}_{\mathrm{b}}} \lesssim \epsilon_1 + \epsilon^2.$$

Proof. We will apply the homogeneous solution operator S_{hom} in Lemma 5.5 and the Bogovskii operator S_η in the whole space as in (5.14).

Let $\eta \in C^\infty_c(B_{1/10})$ with $\int \eta(y_1) dy_1 = 1$ and

$$Sf = S_{\mathrm{hom}} f + S_\eta (f - P S_{\mathrm{hom}} f). \tag{5.19}$$

The incoming property of S_{hom} and S_η implies the incoming property of S:

$$\operatorname{supp} S(x,y) \subset \{|x| \le |y|\} \cup B_1 \times B_1$$

and thus S is well defined on $\mathbb{R}^3 \setminus \overline{B_1}$.

We consider the fixed point problem

$$(h_1, \pi_1) = S(\Phi(h_0 + \dot{h} + h_1, \pi_0 + \dot{\pi} + \pi_1) - P(h_0 + \dot{h}, \pi_0 + \dot{\pi})).$$

Note that the right hand side is in $H^{s-2,\delta_1+2}_{\mathrm{b}}$ with $\delta_1 > 1/2$ so S_η can be applied to it. Banach fixed point theorem gives a unique solution

$$\|(h_1, \pi_1)\|_{H^{s,\delta_1}_{\mathrm{b}} \times H^{s-1,\delta_1+1}_{\mathrm{b}}} \lesssim \epsilon_1.$$

Moreover, $(h_0 + \dot{h} + h_1, \pi_0 + \dot{\pi} + \pi_1)$ solves (4.1) in $\mathbb{R}^d \setminus \overline{B_1}$ due to the support of η. □

Exercise 5.8. Modify the proof above to construct exponentially decreasing perturbations.

6. Recovery on curves and the finite-dimensional cokernel condition

The explicit conic and Bogovskii operators we have discussed are instances of a much broader construction developed by Isett–Mao–Oh–Tao [9], which works for a general underdetermined PDE system in the sense of Definition 6.3 below.

6.1. Derivation of Bogovskii type operators.

6.1.1. *Double divergence equation.* Suppose $\mathcal{P}h = \partial_i\partial_j h^{ij}$ for a symmetric two tensor h^{ij}. We would like a construct solution operators to $\mathcal{P}h = f$. We start with the adjoint problem

$$\langle h, \mathcal{P}^*\varphi\rangle = \langle f, \varphi\rangle, \quad (\mathcal{P}^*\varphi)_{jk} = \partial_i\partial_j\varphi.$$

Taking $f = \delta_y$, we would like to recover $\varphi(y)$ from $\mathcal{P}^*\varphi$. We will make the construction on a family of curves and then average among them.

We have the following ODE system.

$$\begin{aligned} \nabla_i\varphi &= \boldsymbol{\omega}_i, \\ \nabla_i\boldsymbol{\omega}_j &= (\mathcal{P}^*\varphi)_{ij}. \end{aligned} \tag{6.1}$$

Suppose we have a curve $\gamma(t) : [0,1] \to \mathbb{R}^d$ such that $\gamma(0) = y$, $\gamma(1) = y_1$. We have

$$\boldsymbol{\omega}_i(\gamma(t)) = \boldsymbol{\omega}_i(y_1) - \int_t^1 (\mathcal{P}^*\varphi)_{ij}(\gamma(s))\gamma'(s)^j ds$$

and

$$\begin{aligned} \varphi(y) &= \varphi(y_1) - \int_0^1 \boldsymbol{\omega}_i(\gamma(s))\gamma'(s)^i ds \\ &= \varphi(y_1) + \boldsymbol{\omega}_i(y_1)(y - y_1)^i + \int_0^1 (\mathcal{P}^*\varphi)_{ij}(\gamma(s))(\gamma(s)^i - \gamma(0)^i)\gamma'(s)^j ds. \end{aligned}$$

From this, we conclude a distribution $G_{y_1}(x, y)$ supported on the curve γ defined by

$$\langle G_{y_1}^{ij}(x,y), f_{ij}(x)\rangle = \int_0^1 f_{ij}(\gamma(s))(\gamma(s)^i - \gamma(0)^i)\gamma'(s)^j ds$$

such that it solves $\mathcal{P}h = 0$ up to a finite dimensional cokernel

$$\partial_{x^i}\partial_{x^j} G_{y_1}^{ij}(x,y) = \delta(x-y) - \delta(x-y_1) + (y-y_1)^i\partial_{x^i}\delta(x-y_1).$$

Averaging over a family of curves $\gamma(t, y, y_1)$ gives us the solution operator considered before.

6.1.2. *Symmetric divergence equation.* Suppose $\mathcal{P}\pi = \partial_i\pi^{ij}$ for a symmetric two tensor π^{ij}. We would like to construct solution operators to $(\mathcal{P}\pi)^j = f^j$. We start with the adjoint problem

$$\langle \pi, \mathcal{P}^*\boldsymbol{\omega}\rangle = \langle f, \boldsymbol{\omega}\rangle, \quad (\mathcal{P}^*\boldsymbol{\omega})_{jk} = -\frac{1}{2}(\partial_j\boldsymbol{\omega}_k + \partial_k\boldsymbol{\omega}_j).$$

Taking $f^j = \delta_y \mathbf{e}^j$, we would like to recover $\boldsymbol{\omega}(y)$ from $\mathcal{P}^*\boldsymbol{\omega}$. We will make the construction on a family of curves and then average among them.

Introduce $\boldsymbol{\eta}_{jk} = \frac{1}{2}(\partial_j\boldsymbol{\omega}_k - \partial_k\boldsymbol{\omega}_j)$. We have the following ODE system.

$$\begin{aligned} \nabla_i\boldsymbol{\omega}_j &= \boldsymbol{\eta}_{ij} - (\mathcal{P}^*\boldsymbol{\omega})_{ij}, \\ \nabla_i\boldsymbol{\eta}_{jk} &= \partial_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \partial_j(\mathcal{P}^*\boldsymbol{\omega})_{ik}. \end{aligned} \tag{6.2}$$

Suppose we have a curve $\gamma(t) : [0,1] \to \mathbb{R}^d$. We have

$$\boldsymbol{\eta}_{jk}(\gamma(t)) = \boldsymbol{\eta}_{jk}(y_1) - \int_t^1 \partial_k(\mathcal{P}^*\boldsymbol{\omega})_{ij}(\gamma(s))\gamma'(s)^i - \partial_j(\mathcal{P}^*\boldsymbol{\omega})_{ik}(\gamma(s))\gamma'(s)^i ds$$

and

$$\begin{aligned}\boldsymbol{\omega}_j(y) &= \boldsymbol{\omega}_j(y_1) - \int_0^1 \boldsymbol{\eta}_{ij}(\gamma(t))\gamma'(t)^i dt + \int_0^1 (\mathcal{P}^*\boldsymbol{\omega})_{ij}\gamma'(t)^i dt \\ &= \boldsymbol{\omega}_j(y_1) + \boldsymbol{\eta}_{ij}(y_1)(y-y_1)^i + \int_0^1 (\partial_j((\mathcal{P}^*\boldsymbol{\omega})_{\ell i}(\gamma(s))\gamma'(s)^\ell - \partial_i(\mathcal{P}^*\boldsymbol{\omega})_{\ell j}(\gamma(s))\gamma'(s)^\ell)(\gamma(s)-\gamma(0))^i ds \\ &\quad + \int_0^1 (\mathcal{P}^*\boldsymbol{\omega})_{ij}\gamma'(t)^i dt.\end{aligned}$$

From this, one concludes a solution operator supported on the curve γ defined by

$$\langle K^{ij}_{y_1,k}(x,y), f_{ij}(x)\rangle = \int_0^1 (\partial_k(f_{\ell i}(\gamma(s))\gamma'(s)^\ell - \partial_i(f_{\ell k}(\gamma(s))\gamma'(s)^\ell)(\gamma(s)-\gamma(0))^i ds + \int_0^1 f_{ik}\gamma'(t)^i dt$$

such that it solves $\mathcal{P}\pi = 0$ up to a finite dimensional cokernel:

$$\partial_{x^i} K^{ij}_{y_1,k}(x,y) = \delta(x-y)\delta^j_k - \delta(x-y_1)\delta^j_k + \frac{1}{2}(y-y_1)^\ell \partial_\ell \delta(x-y_1)\delta^j_k - \frac{1}{2}(y-y_1)^j \partial_k \delta(x-y_1).$$

Averaging over a family of curves $\gamma(t,y,y_1)$ gives us the solution operator considered before.

6.2. **The general operator and its formal cokernel.** Let $U \subset \mathbb{R}^d$ be open, and let

$$\mathcal{P} : C^\infty(U;\mathbb{C}^{s_0}) \longrightarrow C^\infty(U;\mathbb{C}^{r_0}), \qquad r_0 \le s_0, \tag{6.3}$$

be an $r_0 \times s_0$ matrix of linear differential operators with smooth coefficients. We write $u = (u^K)_{K=1}^{s_0}$ and $f = (f^J)_{J=1}^{r_0}$. Its formal L^2 adjoint is

$$\mathcal{P}^* : C^\infty(U;\mathbb{C}^{r_0}) \longrightarrow C^\infty(U;\mathbb{C}^{s_0}).$$

The unknown side of (6.3) has at least as many components as the equation side, so $\mathcal{P}$ is underdetermined, while $\mathcal{P}^*$ is overdetermined.

Definition 6.1 (Formal cokernel)**.** The formal cokernel of $\mathcal{P}$ on a connected open set U is

$$\mathcal{Z}(U) := \ker \mathcal{P}^*(U) = \{Z \in C^\infty(U;\mathbb{C}^{r_0}) : \mathcal{P}^* Z = 0\}. \tag{6.4}$$

No boundary condition is imposed on Z.

The terminology comes from compactly supported solvability. If $u \in C_c^\infty(U;\mathbb{C}^{s_0})$ and $\mathcal{P}u = f$, then every $Z \in \mathcal{Z}(U)$ satisfies

$$\langle Z, f\rangle_U = \langle Z, \mathcal{P}u\rangle_U = \langle \mathcal{P}^* Z, u\rangle_U = 0. \tag{6.5}$$

Thus compactly supported right inverses can exist only after projecting away the formal cokernel. The affine functions for double divergence and the Euclidean Killing fields for symmetric divergence are precisely the two blocks of this space for the flat linearized constraint operator.

From the above computation, it is clear that the Bogovskii type operator follows from the following graded augmented system.

Definition 6.2 (Graded augmented system)**.** Let $\mathcal{P}$ be an $r_0 \times s_0$-matrix-valued differential operator on an open subset $U \subseteq \mathbb{R}^d$, with m_K denoting the order of $(\mathcal{P}^*\varphi)_K$ for each $K \in \{1,\dots,s_0\}$. Given $m'_K \in \mathbb{Z}_{\ge 0}$ for each $K \in \{1,\dots,s_0\}$ and an $\mathbb{C}^{r_0}$-valued function $(\varphi_J)_{J\in\{1,\dots,r_0\}}$ on U, consider $(\Phi_{\mathbf{A}} = \Phi_{\mathbf{A}}(y))_{\mathbf{A}\in\mathcal{A}}$ (called *augmented variables*) satisfying the following properties:

- (Φ-1) $(\Phi_{\mathbf{A}})$ **is an augmentation of** (φ_J)**.** The index set $\mathcal{A}$ contains $\{1,\dots,r_0\}$; moreover, $\Phi_J = \varphi_J$ for $J \in \{1,\dots,r_0\} \subseteq \mathcal{A}$.
- (Φ-2) $(\varphi_J) \mapsto (\Phi_{\mathbf{A}})$ **is a differential operator.** There exist functions $c[\Phi_{\mathbf{A}}]^{(\alpha,J)}$ on U, where α is a multi-index and $J \in \{1,\dots,r_0\}$, such that

$$\Phi_{\mathbf{A}}(y) = c[\Phi_{\mathbf{A}}]^{(\alpha,J)}(y)\partial^\alpha \varphi_J(y),$$

  for all $\mathbf{A} \in \mathcal{A}$ and $y \in U$.

(Φ-3) **Φ and $\mathcal{P}^*\varphi$ solve a first-order PDE (augmented PDE system).** There exist matrix-valued 1-forms $(\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}$ and $(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}$ on U, where γ is a multi-index and $K \in \{1,\ldots,s_0\}$, such that

$$\partial_i \Phi_{\mathbf{A}}(y) = (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}(y)\Phi_{\mathbf{A}'}(y) + (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)}(y)\partial^\gamma(\mathcal{P}^*\varphi)_K(y), \tag{6.6}$$

for all $\mathbf{A} \in \mathcal{A}$, $i \in \{1,\ldots,d\}$ and $y \in U$. Furthermore,

$$(\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} = 0 \quad \text{ if } |\gamma| > m'_K.$$

(Φ-4) **Graded structure.** Define the *degree* $d_{\mathbf{A}}$ of $\Phi_{\mathbf{A}}$ by

$$d_{\mathbf{A}} := -\max\{|\alpha| : c[\Phi_{\mathbf{A}}]^{(\alpha,J)} \neq 0 \text{ for some } J\}.$$

In particular, φ_J has degree 0, i.e., $d_J = 0$ for $J \in \{1,\ldots,r_0\}$. We assume that:

- **Graded structure for the augmented system.**

$$\begin{aligned} (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'} &= 0 \qquad \text{if } d_{\mathbf{A}} > d_{\mathbf{A}'} + 1,\\ (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)} &= 0 \qquad \text{if } d_{\mathbf{A}} > -m_K - |\gamma| + 1. \end{aligned}$$

We write $N_0 := \max\{|d_{\mathbf{A}}|\}_{\mathbf{A}\in\mathcal{A}} + 1$ for the maximal degree[2] that occurs in (6.6).

We call a collection $(\mathcal{A}, (\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}, (\mathbf{B}_i)_{\mathbf{A}}{}^{\mathbf{A}'}, (\mathbf{C}_i)_{\mathbf{A}}{}^{(\gamma,K)})$ satisfying (Φ-1)–(Φ-4) a (*graded*) *augmented system* for $\mathcal{P}$.

6.3. **The complex elliptic condition.** Suppose that the Kth component of $\mathcal{P}^*\varphi$ has order m_K. Its principal symbol defines a linear map

$$p^*(x,\xi) : \mathbb{C}^{r_0} \longrightarrow \mathbb{C}^{s_0}, \qquad \xi \in \mathbb{C}^d. \tag{6.7}$$

The corresponding symbol $p(x,\xi)$ of $\mathcal{P}$ is the transpose-adjoint map in the formal algebraic sense.

Definition 6.3 (complex elliptic condition)**.** The operator $\mathcal{P}$ satisfies *(CE)* on U if

$$p^*(x,\xi) \text{ is injective for every } x \in U \text{ and every } \xi \in \mathbb{C}^d \setminus \{0\}. \tag{6.8}$$

Equivalently, $p(x,\xi)$ is surjective for every nonzero complex covector.

The use of complex covectors is essential. Ordinary underdetermined ellipticity asks for surjectivity only when $\xi \in \mathbb{R}^d \setminus \{0\}$. Condition (CE) excludes complex characteristics as well.

It turns out that the complex elliptic condition implies the existence of a graded augmented system.

Theorem 6.4. *Let U be an open subset of $\mathbb{R}^d$. Suppose that $\mathcal{P}$ has smooth coefficients. Assume that the $s_0 \times r_0$-matrix-valued principal symbol $p^*(x,\xi)$ of $\mathcal{P}^*$ (where $s_0 \geq r_0$) satisfies*

(CE) $p^*(x,\xi)$ is injective for all $x \in U$ and $\xi \in \mathbb{C}^d \setminus \{0\}$.

Then there exist augmented variables $(\Phi_{\mathbf{A}})_{\mathbf{A}\in\mathcal{A}}$ on U satisfying (Φ-1)–(Φ-4) with index set $\mathcal{A} = \{(\alpha,J) : 1 \leq J \leq r_0,\ |\alpha| \leq N_0 - 1\}$, degree $d_{(\alpha,J)} = -|\alpha|$, and coefficients $\mathbf{B}_i, \mathbf{C}_i \in C^\infty(U)$ with quantitative estimates.

For the proof we recall

Proposition 6.5 (Hilbert's Nullstellensatz)**.** *Suppose $f_1(x_1,\cdots,x_d),\cdots,f_N(x_1,\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$ are polynomials with the set of common zeros*

$$Z = \{(z_1,\cdots,z_d) \in \mathbb{C}^d : f_j(z_1,\cdots,z_d) = 0,\ j = 1,2,\cdots,N\}.$$

Then for any polynomial $h(x_1.\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$, if h vanishes on Z, then there exist $M > 0$ and polynomials $g_j(x_1,\cdots,x_d) \in \mathbb{C}[x_1,\cdots,x_d]$, $j = 1,2,\cdots,N$ such that

$$h(x_1,\cdots,x_d)^M = \sum_j g_j(x_1,\cdots,x_d)f_j(x_1,\cdots,x_d).$$

[2] Here, $+1$ accounts for the derivative ∂_i on the LHS of (6.6).

Proof of Theorem 6.4. **Step 1:** For any $x_0 \in U$, there exist $N_0 > 0$ and polynomial valued $r_0 \times s_0$ matrices $g_\alpha(\xi)$ for $|\alpha| = N_0$ such that

$$\xi^\alpha I_{r_0} = g_\alpha(\xi) p^*(x_0, \xi), \quad |\alpha| = N_0. \tag{6.9}$$

Moreover, ${(p^*)_K}^J(\xi)$ is homogeneous of degree m_K and ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

The claim follows from applying Hilbert's Nullstellensatz (Proposition 6.5) to $\det(M_j(\xi))$ where $M_j(\xi)$ goes through all the $r_0 \times r_0$ minors of $p^*(x_0, \xi)$. By assumption, the common zeros of $\det(M_j(\xi))$ in $\mathbb{C}^d$ is contained in $\{0\}$. Since ξ_ℓ vanishes at 0, there exist $N_1, \cdots, N_d$ and polynomials $h^j_\ell(\xi)$ such that

$$\xi_\ell^{N_\ell} = \sum_\ell h^j_\ell(\xi) \det(M_j(\xi)).$$

We notice that $\det(M_j(\xi)) I_{r_0}$ can be written as the product of its adjugate matrix $\mathrm{adj}(M_j(\xi))$ and itself:

$$\det(M_j(\xi)) I_{r_0} = \mathrm{adj}(M_j(\xi)) M_j(\xi).$$

Moreover, $M_j(\xi)$ is the product of a constant $0-1$ matrix and $p^*(x_0, \xi)$. Therefore we conclude that there are polynomial-valued $r_0 \times s_0$ matrices $\widetilde{g}^\ell(\xi)$ so that

$$\xi_\ell^{N_\ell} I_{r_0} = \widetilde{g}^\ell(\xi) p^*(x_0, \xi).$$

Taking $N_0 > N_1 + \cdots + N_d$, we conclude (6.9). We may assume ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$ since terms of other degrees do not contribute.

Step 2: There exist ξ-polynomial-valued $r_0 \times s_0$ matrices $g_\alpha(x, \xi)$ for $|\alpha| = N_0$ such that

$$\xi^\alpha I_{r_0} = g_\alpha(x, \xi) p^*(x, \xi), \quad |\alpha| = N_0. \tag{6.10}$$

The matrix $g_\alpha(x, \xi)$ depends smoothly on the coefficients of $p^*(x, \xi)$ and some smooth cutoff functions. Moreover, ${(p^*)_K}^J(\xi)$ is homogeneous of degree m_K and ${(g_\alpha)_J}^K(\xi)$ is homogeneous of degree $N_0 - m_K$.

When $\mathcal{P}$ has constant coefficients, then it suffices to take $g_\alpha(x, \xi) = g_\alpha(\xi)$ (from Step 1); hence, it suffices to only consider the case when $\mathcal{P}$ does not have constant coefficients. After a partition of unity, it suffices to prove the claim in a small neighborhood of a given point x_0. By Step 1, we have

$$\xi^\alpha I_{r_0} = g_\alpha(x_0, \xi) p^*(x_0, \xi), \quad |\alpha| = N_0. \tag{6.11}$$

Thus

$$\xi^\alpha I_{r_0} = g_\alpha(x_0, \xi) p^*(x, \xi) + g_\alpha(x_0, \xi)(p^*(x_0, \xi) - p^*(x, \xi)).$$

Since each entry of $g_\alpha(x_0, \xi)(p^*(x_0, \xi) - p^*(x, \xi))$ is a homogeneous polynomial of degree N_0 in ξ, we apply (6.11) again:

$$\xi^\alpha I_{r_0} = g_\alpha(x_0, \xi) p^*(x, \xi) + \sum_\beta \epsilon^\alpha_\beta(x) g^\beta(x_0, \xi) p^*(x_0, \xi)$$

where $\epsilon^\alpha_\beta(x)$ are $r_0 \times r_0$ matrices depending smoothly on coefficients of $p^*(x, \xi)$ such that $\epsilon(x, \xi) \to 0$ as $x \to x_0$. This can be done repeatedly and we get

$$\begin{aligned}
\xi^\alpha I_{r_0} &= g_\alpha(x_0, \xi) p^*(x, \xi) + \sum_\beta \epsilon^\beta_\alpha(x) g_\beta(x_0, \xi) p^*(x, \xi) + \sum_{\beta, \gamma} \epsilon^\beta_\alpha(x) \epsilon^\gamma_\beta(x) g_\gamma(x_0, \xi) p^*(x, \xi) + \cdots \\
&= \left(g_\alpha(x_0, \xi) + \sum_\beta \epsilon^\beta_\alpha(x) g_\beta(x_0, \xi) + \sum_{\beta, \gamma} \epsilon^\beta_\alpha(x) \epsilon^\gamma_\beta(x) g_\gamma(x_0, \xi) + \cdots \right) p^*(x, \xi).
\end{aligned}$$

For x in a small neighborhood of x_0, the Neumann series converges and we conclude (6.10).

Step 3. We claim that taking $\Phi_{\mathbf{A}} = \partial^\alpha \varphi_J$ to be the jet of φ_J up to order $N_0 - 1$ indexed by $\mathbf{A} = (\alpha, J)$ in the set $\mathcal{A} := \{(\alpha, J) : 1 \le J \le r_0, |\alpha| \le N_0 - 1\}$ gives an augmented system satisfying (Φ-1)-(Φ-4). First, (Φ-1) and (Φ-2) are obvious from the definition. For (Φ-3), we take the trivial relation $\partial_i \partial^\alpha \varphi_J(y) = \partial^{\alpha + \mathbf{e}_i} \varphi_J(y)$ to be the equation if $|\alpha| < N_0 - 1$, i.e.,

$${(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')} = \begin{cases} 1 & \text{if } J = J',\ \alpha' = \alpha + \mathbf{e}_i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)} = 0 \quad \text{for } |\alpha| < N_0 - 1.$$

It remains to check that, for $|\alpha| = N_0 - 1$, there exist ${(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')}(y)$ and ${(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y)$ such that

$$\partial_i\partial^\alpha\varphi_J(y) = {(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')}(y)\partial^{\alpha'}\varphi_{J'}(y) + {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y)\partial^\gamma(\mathcal{P}^*\varphi)_K(y). \tag{6.12}$$

This follows from (6.10) and putting $(\mathcal{P}^*_{\text{prin}} - \mathcal{P}^*)\varphi$ into the first term on the right-hand side. Indeed, $\mathbf{C}_i$ is determined by rewriting (6.10) in the form

$$\xi_i\xi^\alpha{(I_{r_0})_J}^{J'} = {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y)\xi^\gamma{(p^*)_K}^{J'}(y,\xi) \quad \text{for } |\alpha| = N_0 - 1, \tag{6.13}$$

(i.e., ${(g_{\alpha+\mathbf{e}_i})_J}^K(y,\xi) = {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y)\xi^\gamma$) and $\mathbf{B}_i$ is determined by

$${(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')}(y)\partial^{\alpha'}\varphi_{J'}(y) = {(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y){(\mathcal{P}^*_{\text{prin}}\partial^\gamma - \partial^\gamma\mathcal{P}^*)_K}^{J'}\varphi_{J'}(y) \quad \text{for } |\alpha| = N_0 - 1. \tag{6.14}$$

It remains to check (Φ-4). We define the degree of $\partial^\alpha\varphi_J$ to be $d_{(\alpha,J)} := -|\alpha|$. The condition ($\Phi$-4) for terms with degree $d_{(\alpha,J)} > -N_0 + 1$ is obvious. It suffices to check ${(\mathbf{B}_i)_{(\alpha,J)}}^{(\alpha',J')}$ and ${(\mathbf{C}_i)_{(\alpha,J)}}^{(\gamma,K)}(y)$ which corresponds to $d_{(\alpha,J)} = -N_0 + 1$ in (Φ-4). Since $d_{(\alpha',J')} \geq -N_0 + 1$, the condition for $\mathbf{B}$ is automatic. Moreover, $m_K + |\gamma| \leq N_0$, so $d_{(\alpha,J)} \leq -m_K - |\gamma| + 1$ and the condition for $\mathbf{C}$ follows. □

In the constant-coefficient homogeneous case, we have the following equivalence.

Proposition 6.6. *Let U be an connected open subset of $\mathbb{R}^d$, and $\mathcal{P}$ an $r_0 \times s_0$-matrix-valued operator on U that is homogeneous with constant coefficients. Then the following are equivalent:*

1. *$p^*(\xi)$ satisfies (CE).*
2. *There exists a maximal graded augmented system $\{\Phi_\mathbf{A}\}_{\mathbf{A}\in\mathcal{A}}$ for $\mathcal{P}$ on U.*
3. *$\dim\ker\mathcal{P}^*(U) < +\infty$.*

Proof. The implication (1) ⇒ (2) is exactly Theorem 6.4. Moreover, (2) ⇒ (3) follows from integrating along curves as above. It remains to prove (3) ⇒ (1), or equivalently, its contrapositive. Indeed, if p^* does *not* satisfy (CE), then there exists a constant vector $\{\varphi_J\}_{J\in\{1,\ldots,r_0\}} \neq 0$ and $\xi \in \mathbb{C}^d \setminus \{0\}$ such that $p^*(\xi)\varphi = 0$, which means that $e^{i\lambda\xi x}\varphi \in \ker\mathcal{P}^*$ for every $\lambda \in \mathbb{C}$. Thus $\ker\mathcal{P}^*$ is infinite dimensional. □

6.4. **Preliminaries for explicit computations in the constant curvature case.** In the following sections, we collect some explicit computations for the double divergence operator and symmetric divergence operator in the constant curvature case. Here we recall some preliminaries.

Under the convention ${\mathbf{R}_{ij}}^k{}_\ell u^\ell = (\nabla_i\nabla_j - \nabla_j\nabla_i)u^k$ for the Riemann curvature tensor, recall that the Riemann, Ricci, and scalar curvature tensors on a Riemannian manifold $(\mathcal{M},\mathbf{g})$ with constant sectional curvature κ are given by

$$\mathbf{R}_{ijk\ell} = \kappa(\mathbf{g}_{ik}\mathbf{g}_{j\ell} - \mathbf{g}_{i\ell}\mathbf{g}_{jk}), \quad \mathrm{Ric} = (d-1)\kappa\mathbf{g}, \quad R = d(d-1)\kappa.$$

We also introduce the *generalized sine function* that is defined for $\kappa \in \mathbb{R}$ as

$$s_\kappa(t) = \begin{cases} t & \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} & \kappa < 0, \\ \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}} & \kappa > 0. \end{cases}$$

The generalized cosine function is defined as

$$c_\kappa(t) = s'_\kappa(t) = \begin{cases} 1 & \kappa = 0, \\ \cosh(\sqrt{-\kappa}t) & \kappa < 0, \\ \cos(\sqrt{\kappa}t) & \kappa > 0. \end{cases}$$

Fix a point $y \in \mathcal{M}$, and consider the polar coordinate system (r,ω) (where $\omega = (\omega^1,\ldots\omega^{d-1})$ is a parametrization of $\mathbb{S}^{d-1}$) centered at y (i.e., r is the geodesic distance from y). The metric can be written as

$$\mathbf{g} = \mathrm{d}r^2 + s_\kappa(r)^2\mathbf{g}_{\mathbb{S}^{d-1}}.$$

Using ordinary capital latin letters $A, B, \ldots$ to denote the angular variables ω^A (so $A, B, \ldots \in \{1, \ldots, d-1\}$), the Christoffel symbols take the form

$$\Gamma^A_{rB} = \Gamma^A_{Br} = \frac{s'_\kappa(r)}{s_\kappa(r)}\delta^A_B, \quad \Gamma^r_{AB} = -s'_\kappa(r)s_\kappa(r)(\mathbf{g}_{\mathbb{S}^{d-1}})_{AB}, \quad \Gamma^A_{BC} = {}^{(\mathbf{g}_{\mathbb{S}^{d-1}})}\Gamma^A_{BC},$$

where all the other components vanish. The normalized angular forms and vector fields are defined as $d\widetilde{\omega}^A = s_\kappa(r)d\omega^A$ and $\partial^A_{\widetilde{\omega}} = s_\kappa(r)^{-1}\partial_{\omega^A}$.

6.5. **Double divergence operator (or linearized scalar curvature operator).** We consider

$$\mathcal{P}\mathbf{h} = \nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj}, \tag{6.15}$$

for $d \geq 1$. The operator $\mathcal{P}$ is obtained by making a simple change of variables to the linearization of the scalar curvature operator. In fact, the *linearized scalar curvature operator* is directly given by

$$\mathrm{D}R(\mathbf{g})\dot{\mathbf{g}} = -\nabla^\ell\nabla_\ell \operatorname{tr}_{\mathbf{g}}\dot{\mathbf{g}} + \nabla^j\nabla^k\dot{\mathbf{g}}_{jk} - \mathrm{Ric}^{jk}\dot{\mathbf{g}}_{jk},$$

where $\operatorname{tr}_{\mathbf{g}}\dot{\mathbf{g}} = \mathbf{g}^{jk}\dot{\mathbf{g}}_{jk}$. We introduce[3]

$$\mathbf{h}^{jk} = \dot{\mathbf{g}}^{jk} - \mathbf{g}^{jk}\operatorname{tr}_{\mathbf{g}}\dot{\mathbf{g}}.$$

Then since $\operatorname{tr}_{\mathbf{g}}\mathbf{h} = (1-d)\operatorname{tr}_{\mathbf{g}}\dot{\mathbf{g}}$, this change of variables is invertible and we have

$$\dot{\mathbf{g}}^{jk} = \mathbf{h}^{jk} - \frac{1}{d-1}\mathbf{g}^{jk}\operatorname{tr}_{\mathbf{g}}\mathbf{h}.$$

Under this change of variables, we have

$$\mathrm{D}R(\mathbf{g})\dot{\mathbf{g}} = \nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\mathbf{h}^{jk}$$

where the RHS is equal to $\mathcal{P}(\mathbf{h})$ defined above.

We first compute the formal adjoint operator $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported symmetric 2-tensor $\mathbf{h}$ on U and $\varphi \in C^\infty_c(U)$, we have

$$\begin{aligned}\int_U \mathcal{P}(\mathbf{h})\varphi\,\mathrm{d}x &= \int_U \left(\nabla_j\nabla_k\mathbf{h}^{jk} - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\mathbf{h}^{jk}\right)\right)\varphi\,\mathrm{d}x \\ &= \int_U \mathbf{h}^{jk}\left(\nabla_j\nabla_k - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\right)\varphi\,\mathrm{d}x.\end{aligned}$$

So the formal adjoint is

$$(\mathcal{P}^*\varphi)_{jk} = \nabla_j\partial_k\varphi - \left(\mathrm{Ric}_{jk} - \frac{1}{d-1}R\mathbf{g}_{jk}\right)\varphi. \tag{6.16}$$

Its principal symbol is

$$(p^*)_{jk}(\xi) = -\xi_j\xi_k, \tag{6.17}$$

which clearly satisfies (CE) for all $d \geq 1$.

6.5.1. *Covariant graded augmented system and* $\ker\mathcal{P}^*$. Given a function φ, define

$$\boldsymbol{\omega}_j = \partial_j\varphi. \tag{6.18}$$

Then

$$\begin{cases}\nabla_i\varphi = \boldsymbol{\omega}_i, \\ \nabla_i\boldsymbol{\omega}_j = \left(\mathrm{Ric}_{jk} - \dfrac{1}{d-1}R\mathbf{g}_{jk}\right)\varphi + (\mathcal{P}^*\varphi)_{ij}.\end{cases} \tag{6.19}$$

As we will see in Section 6.5.2, this system of covariant first-order PDEs (6.19) is useful for performing explicit computations on space forms.

We note that (6.19) immediately leads to graded augmented variables in the sense of Definition 6.2. Indeed, if we specialize (6.19) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}^{*\mathrm{d}x}_{\mathrm{prin}}$),

[3] We carefully note that $\dot{\mathbf{g}}^{jk}$ is the metric dual of $\dot{\mathbf{g}}_{jk}$, *not* the first order variation of $(\mathbf{g}^{-1})^{jk}$. These two objects differ by a sign.

then we see that $\Phi_\varphi := \varphi$, $\Phi_{\boldsymbol{\omega}_j} := \boldsymbol{\omega}_j$ define augmented variables for $\mathcal{P}_{\text{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\varphi, \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_d\}$ (in particular, $\#\mathcal{A} = d+1$), $d_\varphi = 0$, $d_{\boldsymbol{\omega}_j} = -1$, $m_{ij} = 2$, and $m'_{ij} = 0$ for all $i, j \in \{1, \ldots, d\}$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\text{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

In view of (6.19), we see that $\dim\ker\mathcal{P}^* \leq \#\mathcal{A} = d+1$. This bound is optimal, and the maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\mathcal{P}^*\varphi = \partial_j\partial_k\varphi$ and $\ker\mathcal{P}^* = \operatorname{span}\{1, x^1, \ldots, x^d\}$.
- For the sphere $\mathbb{S}^d$, $\mathcal{P}^*\varphi = \nabla_j\partial_k\varphi + \mathbf{g}_{jk}\varphi$. If we embed the sphere into $\mathbb{R}^{d+1}$ by $\{(x^0)^2 + \cdots + (x^d)^2 = 1\} \subseteq \mathbb{R}^d$, then $\ker\mathcal{P}^*$ is spanned by the restrictions on $\mathbb{S}^d$ of the ambient coordinates $x^0, x^1, \cdots, x^d$, which are linearly independent. Indeed, for each $\mu \in \{0, 1, \ldots, d\}$, that $\mathcal{P}^*x^\mu = 0$ can be seen by a direct computation using the fact that the second fundamental form $\Pi(X, Y)$ of the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$, which proceeds as follows:
$$(\mathcal{P}^*x^\mu)(X,Y) = X(Yx^\mu) - (\overline{\nabla}_X Y)^\parallel x^\mu + \mathbf{g}(X,Y)x^\mu = (\Pi(X,Y) + \mathbf{g}(X,Y))x^\mu = 0,$$
where $\overline{\nabla}$ is the covariant derivative in the ambient coordinates, $X, Y \in T\mathbb{S}^d$, and $(\overline{\nabla}_X Y)^\parallel$ is the tangential part of $\overline{\nabla}_X Y$. The expressions $X(Yx^\mu)$ and $(\overline{\nabla}_X Y)^\parallel$ are computed using any extension of Y as a smooth vector field; as is well-known, the identity holds independently of this choice. The linear independence claim is clear. Since $\dim\ker\mathcal{P}^* \leq d+1$, $\{x^0, x^1, \ldots, x^d\}$ indeed also spans $\mathcal{P}^*$.
- For the hyperbolic space $\mathbb{H}^d$, $\mathcal{P}^*\varphi = \nabla_j\partial_k\varphi - \mathbf{g}_{jk}\varphi$. We embed the hyperbolic space into the Minkowski space $\mathbb{R}^{1,d}$ with metric $-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^d)^2$ by $\{-(x^0)^2 + (x^1)^2 + \cdots + (x^d)^2 = -1\}$. Then $\ker\mathcal{P}^*$ is again spanned by the restrictions to $\mathbb{H}^d$ of the ambient coordinates $x^0, x^1, \cdots, x^d$. This statement can be verified in a similar manner to the case of $\mathbb{S}^d$. In particular, for $\mu \in \{0, 1, \ldots, d\}$ and any $X, Y \in T\mathbb{H}^d$, we have
$$(\mathcal{P}^*x^\mu)(X,Y) = X(Yx^\mu) - (\overline{\nabla}_X Y)^\parallel x^\mu - \mathbf{g}(X,Y)x^\mu = (\Pi(X,Y) - \mathbf{g}(X,Y))x^\mu = 0,$$
where we used that the second fundamental form $\Pi(X,Y)$ of the embedding $\mathbb{H}^d \hookrightarrow \mathbb{R}^{1,d}$ equals $\mathbf{g}$.

6.5.2. *Explicit computations in the constant curvature case.* Let κ be the constant sectional curvature of $(\mathcal{M}, \mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from y to y_1. We may write $\mathbf{x}(t) = (t, \omega_0)$ in polar coordinates at y for some $\omega_0 \in \mathbb{S}^{d-1}$. From (6.19) and the formulas in Section 6.4, we have
$$\partial_r\varphi = \boldsymbol{\omega}_r, \quad \partial_r\boldsymbol{\omega}_r = -\kappa\varphi + (\mathcal{P}^*\varphi)_{rr}.$$
So
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(\mathbf{x}(t)) = \left(-\kappa\varphi + (\mathcal{P}^*\varphi)_{rr}\right).$$
The solution is given by
$$\varphi(\mathbf{x}(0)) = \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\mathcal{P}^*\varphi)_{ij}\circ\mathbf{x}(s)\,\mathrm{d}s + \varphi\circ\mathbf{x}(d(y_1,y))c_\kappa(d(y_1,y)) - \dot{\mathbf{x}}^j(\partial_j\varphi)\circ\mathbf{x}(d(y_1,y))s_\kappa(d(y_1,y)).$$
Thus,
$$\begin{aligned}\langle K_{y_1}(\cdot,y), \psi\rangle &= \int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij}\circ\mathbf{x}(s)\,\mathrm{d}s,\\ \langle b_{y_1}(\cdot,y), \varphi\rangle &= \varphi(y_1)c_\kappa(d(y_1,y)) - \dot{\mathbf{x}}^j(y, y_1, d(y_1,y))(\partial_j\varphi)(y_1)s_\kappa(d(y_1,y)),\end{aligned}$$
which are tensor-valued distributions.

6.5.3. *Explicit formulas on flat spaces.* For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y, y_1, s) = y + s(y_1 - y)$ for the Bogovski-type solution operator and over straight rays $\mathbf{x}(y, \omega, s) = y + s\omega$ for the conic solution operator.

(1) Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$K_\eta^{ij}(z+y, y) = \left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|} + y \right) r^{d-1} \mathrm{d}r \right) \frac{z^i z^j}{|z|^d}$$

with

$$b_\eta(x,y) = (d+1)\eta(x) + (x-y)^i \partial_i \eta(x).$$

(2) Let $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \kappa = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j \nabla_k \mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$K_\kappa^{ij}(z+y, y) = \frac{z^i z^j}{|z|^d} \kappa\left(\frac{z}{|z|} \right).$$

6.6. **Symmetric divergence operator (or adjoint Killing operator).** We consider the *symmetric divergence operator*

$$\mathcal{P}\mathbf{h} = \nabla_j \mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj}, \tag{6.20}$$

for $d \geq 1$.

We first compute the formal adjoint operator $\mathcal{P}^*$ (with respect to $\mathrm{d}V$). For a smooth compactly supported symmetric 2-tensor $\mathbf{h}$ and 1-form $\boldsymbol{\omega}$ on U, we have

$$\int_U (\mathcal{P}\mathbf{h})^k \boldsymbol{\omega}_k \,\mathrm{d}V = \int_U (\nabla_j \mathbf{h}^{jk}) \boldsymbol{\omega}_k \,\mathrm{d}V = -\int_U \mathbf{h}^{jk} \nabla_j \boldsymbol{\omega}_k \,\mathrm{d}V = -\int_U \mathbf{h}^{jk} \frac{1}{2} \left(\nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j \right) \mathrm{d}V.$$

The formal L^2-adjoint is

$$(\mathcal{P}^* \boldsymbol{\omega})_{jk} = -\frac{1}{2} \left(\nabla_j \boldsymbol{\omega}_k + \nabla_k \boldsymbol{\omega}_j \right). \tag{6.21}$$

Observe that $\mathcal{P}^* \boldsymbol{\omega} = 0$ is precisely the condition that the vector field $\boldsymbol{\omega}_\sharp$ is a Killing vector field of $(\mathcal{M}, \mathbf{g})$; for this reason, we will call $\mathcal{P}^*$ the *Killing operator*. Its principal symbol is

$$(p^*)_{jk}{}^\ell(\xi) = -\frac{i}{2}(\xi_j \delta_k^\ell + \xi_k \delta_j^\ell). \tag{6.22}$$

6.6.1. *Covariant graded augmented system and* $\ker \mathcal{P}^*$. Given a 1-form $\boldsymbol{\omega}_j$, define

$$\boldsymbol{\eta}_{jk} = \tfrac{1}{2} (\mathrm{d}\boldsymbol{\omega})_{jk} = \tfrac{1}{2} (\nabla_j \boldsymbol{\omega}_k - \nabla_k \boldsymbol{\omega}_j). \tag{6.23}$$

Then

$$\begin{cases} \nabla_i \boldsymbol{\omega}_j = \boldsymbol{\eta}_{ij} - (\mathcal{P}^* \boldsymbol{\omega})_{ij}, \\ \nabla_i \boldsymbol{\eta}_{jk} = -\mathbf{R}_{jki}{}^\ell \boldsymbol{\omega}_\ell + \nabla_k (\mathcal{P}^* \boldsymbol{\omega})_{ij} - \nabla_j (\mathcal{P}^* \boldsymbol{\omega})_{ki}. \end{cases} \tag{6.24}$$

We postpone the proof of (6.24) and discuss its consequences first.

We note that (6.24) immediately leads to graded augmented variables in the sense of Definition 6.2. Indeed, if we specialize (6.24) to the Euclidean space in rectangular coordinates (so that $\mathcal{P} = \mathcal{P}_{\mathrm{prin}}$ and $\mathcal{P}^* = \mathcal{P}_{\mathrm{prin}}^{*\mathrm{d}x}$), then we see that $\Phi_{\boldsymbol{\omega}_j} := \boldsymbol{\omega}_j$, $\Phi_{\boldsymbol{\eta}_{jk}} := \boldsymbol{\eta}_{jk}$ define augmented variables for $\mathcal{P}_{\mathrm{prin}}$ that satisfy (Φ-1)–(Φ-4), where $\mathcal{A} = \{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_d, \boldsymbol{\eta}_{12}, \ldots, \boldsymbol{\eta}_{(d-1)d}\}$ (in particular, $\#\mathcal{A} = d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$), $d_{\boldsymbol{\omega}_{jk}} = 0$, $d_{\boldsymbol{\eta}_j} = -1$, $m_{ij} = 1$, and $m'_{ij} = 1$ for all $i, j \in \{1, \ldots, d\}$. These augmented variables also satisfy (Φ-1)–(Φ-4) for any lower order perturbations of $\mathcal{P}_{\mathrm{prin}}$ (viewed as an operator on an open subset of $\mathbb{R}^d$, the Euclidean space in rectangular coordinates).

As is well-known, $\mathcal{P}^*$ has a finite dimensional kernel with $\dim \ker \mathcal{P}^* \leq \#\mathcal{A} = \frac{d(d+1)}{2}$. The maximal dimension is reached (i.e., the augmented system is completely integrable) on space forms:

- For $\mathbb{R}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of the Killing vector fields, which are

  $$\mathrm{span}\left(\{\mathbf{e}_J\}_{J=1,\ldots,d} \cup \{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{1 \leq J < K \leq d} \right).$$

  Geometrically, these correspond to translation and rotation vector fields on $\mathbb{R}^d$.

- For the sphere $\mathbb{S}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of
$$\ker \mathcal{P}^* = \operatorname{span}\left(\{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{0\le J<K\le d}\right).$$
Geometrically, these correspond to the rotation vector fields on the ambient Euclidean space $\mathbb{R}^{d+1}$.
- For the hyperbolic space $\mathbb{H}^d$, $\ker \mathcal{P}^*$ consists of the metric duals of
$$\ker \mathcal{P}^* = \operatorname{span}\left(\{x^0 \mathbf{e}_J + x^J \mathbf{e}_0\}_{1\le J\le d} \cup \{x_K \mathbf{e}_J - x_J \mathbf{e}_K\}_{1\le J<K\le d}\right).$$
Geometrically, these correspond to the Lorentz boost and rotation vector fields on the ambient Minkowski space $\mathbb{R}^{1,d}$.

Finally, we give a proof of (6.24), which is a covariant generalization of [12]. Indeed, the first identity is obvious. To prove the second identity, we begin by computing $\nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{jk}$ and cycling the indices i,j,k:

$$\begin{aligned}
-2\nabla_i(\mathcal{P}^*\boldsymbol{\omega})_{jk} &= \nabla_i\nabla_j\boldsymbol{\omega}_k + \nabla_i\nabla_k\boldsymbol{\omega}_j,\\
-2\nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki} &= \nabla_j\nabla_k\boldsymbol{\omega}_i + \nabla_j\nabla_i\boldsymbol{\omega}_k = \nabla_k\nabla_j\boldsymbol{\omega}_i + \nabla_i\nabla_j\boldsymbol{\omega}_k - \mathbf{R}_{jk}{}^{\ell}{}_{i}\boldsymbol{\omega}_\ell + \mathbf{R}_{ij}{}^{\ell}{}_{k}\boldsymbol{\omega}_\ell,\\
-2\nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} &= \nabla_k\nabla_i\boldsymbol{\omega}_j + \nabla_k\nabla_j\boldsymbol{\omega}_i = \nabla_i\nabla_k\boldsymbol{\omega}_j + \nabla_k\nabla_j\boldsymbol{\omega}_i - \mathbf{R}_{ki}{}^{\ell}{}_{j}\boldsymbol{\omega}_\ell.
\end{aligned}$$

Then we subtract the second equation from the third equation. We obtain

$$\begin{aligned}
\nabla_i\boldsymbol{\eta}_{jk} &= \tfrac{1}{2}(\mathbf{R}_{jk}{}^{\ell}{}_{i} - \mathbf{R}_{ij}{}^{\ell}{}_{k} - \mathbf{R}_{ki}{}^{\ell}{}_{j})\boldsymbol{\omega}_\ell + \nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki}\\
&= \mathbf{R}_{jk}{}^{\ell}{}_{i}\boldsymbol{\omega}_\ell + \nabla_k(\mathcal{P}^*\boldsymbol{\omega})_{ij} - \nabla_j(\mathcal{P}^*\boldsymbol{\omega})_{ki},
\end{aligned}$$

where we used the first Bianchi identity for the last equality.

6.6.2. *Explicit computations in the constant curvature case.* Let κ be the constant sectional curvature of $(\mathcal{M},\mathbf{g})$. Fix $y, y_1 \in \mathcal{M}$ and a geodesic segment $\mathbf{x}$ from y to y_1. We may write $\mathbf{x}(t) = (t,\omega_0)$ in polar coordinates at y for some $\omega_0 \in \mathbb{S}^{d-1}$. From (6.24) and the formulas in Section 6.4, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_r(\mathbf{x}(t)) &= -(\mathcal{P}^*\boldsymbol{\omega})_{rr},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}_A(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\omega}_B + \boldsymbol{\eta}_{rA} - (\mathcal{P}^*\boldsymbol{\omega})_{rA},\\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_{rA}(\mathbf{x}(t)) &= \Gamma^B_{rA}\boldsymbol{\eta}_{rB} - \kappa\boldsymbol{\omega}_A + \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - \nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA}.
\end{aligned}$$

For the first equation we can solve directly:

$$\boldsymbol{\omega}_r(\mathbf{x}(t)) = \boldsymbol{\omega}_r(y_1) + \int_t^{d(y_1,y)} (\mathcal{P}^*\boldsymbol{\omega})_{rr}(\mathbf{x}(s))\,\mathrm{d}s.$$

The equation for $\boldsymbol{\omega}_A$ can be reduced:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t)) = -\kappa s_\kappa(t)^{-1}\boldsymbol{\omega}_A + s_\kappa(t)^{-1}\nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr} - 2s_\kappa(t)^{-1}\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA}.$$

Therefore,

$$\begin{aligned}
s_\kappa(t)^{-1}\boldsymbol{\omega}_A(\mathbf{x}(t)) = &\int_t^{d(y_1,y)} s_\kappa(t-s)s_\kappa(s)^{-1}(\nabla_r(\mathcal{P}^*\boldsymbol{\omega})_{rA} - \nabla_A(\mathcal{P}^*\boldsymbol{\omega})_{rr})(\mathbf{x}(s))\mathrm{d}s\\
&+ \int_t^{d(y_1,y)} c_\kappa(t-s)s_\kappa(s)^{-1}(\mathcal{P}^*\boldsymbol{\omega})_{rA}\mathrm{d}s\\
&+ \boldsymbol{\omega}_A(y_1)c_\kappa(t-d(y_1,y))/s_\kappa(d(y_1,y)) + \boldsymbol{\eta}_{rA}(y_1)s_\kappa(t-d(y_1,y))/s_\kappa(d(y_1,y)).
\end{aligned}$$

Thus

$$\begin{aligned}
\langle K_{y_1}(\cdot,y),\psi\rangle = &\left(\int_0^{d(y_1,y)} \dot{\mathbf{x}}^i\dot{\mathbf{x}}^j\psi_{ij}(\mathbf{x}(s))\,\mathrm{d}s\right)\mathrm{d}r + \left(\int_0^{d(y_1,y)} c_\kappa(s)\dot{\mathbf{x}}^j\psi_{j\widetilde{A}}(\mathbf{x}(s))\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A\\
&- \left(\int_0^{d(y_1,y)} s_\kappa(s)\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j(\nabla_i\psi_{j\widetilde{A}} - \nabla_{\widetilde{A}}\psi_{ij})(\mathbf{x}(s))\,\mathrm{d}s\right)\mathrm{d}\widetilde{\omega}^A
\end{aligned}$$

(we recall that $\mathrm{d}\widetilde{\omega}^A = s_\kappa(r)\,\mathrm{d}\omega^A$ is the normalized angular 1-form and $\psi_{j\widetilde{A}} = s_\kappa(r)^{-1}\psi_{jA}$ since $\partial_{\widetilde{\omega}^A} = s_\kappa^{-1}\partial_{\omega^A}$) and

$$\langle b_{y_1}(\cdot,y),\varphi\rangle = \varphi_r(y_1)\,\mathrm{d}r + \left(\varphi_{\widetilde{A}}(y_1)c_\kappa(d(y_1,y)) - \frac{1}{2}\dot{\mathbf{x}}^j(d(y_1,y))(\nabla_j\varphi_{\widetilde{A}} - \nabla_{\widetilde{A}}\varphi_j)(y_1)s_\kappa(d(y_1,y))\right)\mathrm{d}\widetilde{\omega}^A.$$

6.6.3. *Explicit formulas on flat spaces.* For the readers' convenience, we record the explicit formulas for the Bogovskii-type and conic solution operators on $\mathbb{R}^d$. We average over straight line segments $\mathbf{x}(y,y_1,s) = y + s(y_1-y)$ for the Bogovski-type solution operator and over straight half-lines $\mathbf{x}(y,\omega,s) = y + s\omega$ for the conic solution operator.

(1) Let $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\eta = 1$. The Bogovskii-type solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$\begin{aligned}(K_\eta)^{ij}_k(z+y,y) =& \frac{1}{2}\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k}{|z|^d}\\ &+ \frac{1}{2}\partial_{z^m}\left(\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^m(z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k)}{|z|^d}\right)\\ &- \partial_{z^k}\left(\left(\int_{|z|}^\infty \eta\left(r\frac{z}{|z|}+y\right)r^{d-1}\,\mathrm{d}r\right)\frac{z^iz^j}{|z|^d}\right)\end{aligned}$$

with

$$(b_\eta)^j_k(x,y) = \frac{d+1}{2}\eta(x)\boldsymbol{\delta}^j_k + \frac{1}{2}(x-y)^\ell\partial_\ell\eta(x)\boldsymbol{\delta}^j_k - \frac{1}{2}(x-y)^j\partial_k\eta(x).$$

(2) Let $\kappa \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\kappa = 1$. The conic solution operator for $\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk}$ where $\mathbf{h}^{ij} = \mathbf{h}^{ji}$ on $\mathbb{R}^d$ with flat metric is given by

$$(K_\kappa)^{ij}_k(z+y,y) = \frac{1}{2}\frac{z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k}{|z|^d}\kappa\left(\frac{z}{|z|}\right) + \frac{1}{2}\partial_{z^m}\left(\frac{z^m(z^i\boldsymbol{\delta}^j_k + z^j\boldsymbol{\delta}^i_k)}{|z|^d}\kappa\left(\frac{z}{|z|}\right)\right) - \partial_{z^k}\left(\frac{z^iz^j}{|z|^d}\kappa\left(\frac{z}{|z|}\right)\right).$$

Exercise 6.7. Do the computation for the *trace-free symmetric divergence operator*

$$\mathcal{P}\mathbf{h} = \nabla_j\mathbf{h}^{jk} \qquad \text{where } \mathbf{h}^{jk} = \mathbf{h}^{kj} \text{ and } \operatorname{tr}_{\mathbf{g}}\mathbf{h} = 0, \tag{6.25}$$

for $d \geq 3$.

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