FLAT TRACE ESTIMATES FOR ANOSOV FLOWS

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Abstract. We prove a high energy flat trace estimate for the modified resolvent of the generator of an Anosov flow. This fills a gap in the proof of the local trace formula in [JiZw17] and is a by-product of the authors’ ongoing project of its generalization to Axiom A flows.

1. Introduction

This note is a by-product of the authors’ ongoing project on the local trace formula for Axiom A flows, which leads to the discovery of some issues in [JiZw17]. Since the situation for Anosov flows is simpler than the one for Axiom A flows, we give here a separate presentation to fix the issues in [JiZw17].

Let $X$ be a smooth compact manifold, $\varphi_t : X \to X$ be an Anosov flow generated by a smooth vector field $V$, and $P = -iV$, Jin–Zworski [JiZw17] proved the following local trace formula relating the Pollicott–Ruelle resonances $\text{Res}(P)$ to the lengths of closed geodesics.

**Theorem 1.** For any $A > 0$ there exists a distribution $F_A \in \mathcal{S}'(\mathbb{R})$ supported in $[0, \infty)$ such that

$$\sum_{\mu \in \text{Res}(P), \text{Im}(\mu) > -A} e^{-it\mu} + F_A(t) = \sum_{\gamma} \frac{T_\gamma \delta(t - T_\gamma)}{|\det(I - P_{\gamma})|}, \quad t > 0$$

in $\mathcal{D}'((0, \infty))$, where the sum on the right hand side is taken over all closed geodesics, $P_{\gamma}$ is the Poincaré map, and

$$|\hat{F}_A(\lambda)| = O_{A, \varepsilon}(\langle \lambda \rangle^{2n+1+\varepsilon}), \quad \text{Im } \lambda < A - \varepsilon \tag{1.1}$$

for any $\varepsilon > 0$.

The last estimate (1.1) has been modified comparing to [JiZw17, (1.5)]. The additional loss of $\varepsilon$ in the exponent in (1.1) comes from the following mistake in [JiZw17]: rescaling from [JiZw17, (4.20)] back to [JiZw17, (4.1)], we should gain an additional $h$ from the derivative changing from $\frac{d}{dz}$ to $\frac{d}{d\lambda}$, but also have $|z| = h|\lambda| \sim h^{1/2}$ and thus the result should be $h^{-2n} \sim \lambda^{4n}$. However, we can go back to the setting of [DyZw16, Proposition 3.4] and replace $h^{1/2}$ by any $h^\varepsilon$ with $\varepsilon \in (0, 1)$ arbitrarily small. This way we also replace
the bound in [JiZw17, (4.19)] and [JiZw17, (4.20)] by $h^{-(2-\varepsilon)n-2}$ and thus we obtain the bound in (1.1). In section 3, we will give a simpler proof for a weaker high energy flat trace estimate, comparing to [JiZw17, Proposition 3.1], see Theorem 2. From this, the bound in [JiZw17, (4.20)] becomes $h^{-2n-2}$, but still gives the same bound in (1.1). The advantage is that we can avoid the complicated construction for complex absorbing potential $Q$ as in [JiZw17, §2.5].

In [JiZw17], the proof for the high energy flat trace estimate [JiZw17, Proposition 3.1] was incomplete as it relied on the following flawed statement ([JiZw17, (2.14)]) about the semiclassical wavefront set for the resolvent $R_h(z) = (hP - z)^{-1}$:

$$WF'_h(R_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}),$$

which was used to deduce the same statement [JiZw17, (2.19)] for the modified resolvent $\tilde{R}_h(z) = (hP - iQ - z)^{-1}$. However, $R_h(z)$ has poles which are exactly the Pollicott–Ruelle resonances. Even in the set where it is well-defined, it is not clear that the kernel is $h$-tempered uniformly in $z$, and thus $WF'_h(R_h(z))$ may not be defined. To remedy this issue, we analyze the modified resolvent $\tilde{R}_h(z)$ directly to give the statement [JiZw17, (2.19)], which is the correct statement eventually used in the proof of Theorem 1 in [JiZw17]. This will be done in Proposition 2.1 in Section 2.

For more details on the notations we refer to [JiZw17]. For preliminaries on semiclassical analysis we refer to Zworski [Zw12] and Dyatlov–Zworski [DyZw19, Appendix E]. For other recent developments concerning trace formulas for Pollicott-Ruelle resonances, see [Je20], [Je21].

## 2. Wavefront set estimates

In this section, we fix the issue in [JiZw17] by proving the following semiclassical wavefront set estimate for the modified resolvent $\tilde{R}_h(z)$. We briefly recall the notations from [JiZw17]: Let $Q$ be the absorbing potential as in [JiZw17], to be more precise, we require

- $WF_h(Q) \subset \{ |\xi| < 1 \}$;
- $\sigma_h(Q) > 0$ on $\{ |\xi| \leq 1/2 \}$;
- and $\sigma_h(Q) \geq 0$ everywhere.

The additional requirement in [JiZw17, §2.5] is used to improve the power in the flat trace estimate (3.1) and we will give a simpler argument in Section 3 to avoid the complications. In [DyZw16, Proposition 3.4], it is shown that for fixed $C_1, C_2, \varepsilon > 0$, $\tilde{P}_h(z) = hP - iQ - z$ is invertible for $z \in [-C_1 h^{\varepsilon}, C_1 h^{\varepsilon}] + i[-C_2 h, 1]$ and its inverse satisfies the following estimate

$$\|\tilde{R}_h(z)\|_{\mathcal{H}^1_h \to \mathcal{H}^1_h} \leq C h^{-1}.$$
Here $\mathcal{H}_h^s = H_{sG(h)}$ is the semiclassical anisotropic Sobolev space defined in [DyZw16, §3.3] and $s > 0$ is a parameter chosen large enough depending on $C_1$ and $C_2$. The weight function $G(h)$ is constructed in a way that $\tilde{P}_h(z) : D_h^s := \{u \in \mathcal{H}_h^s : \tilde{P}_h(z)u \in \mathcal{H}_h^s\} \rightarrow \mathcal{H}_h^s$ is invertible. In the following we will only use the fact that 

$$H_h^1 \subset \mathcal{H}_h^s \subset H_h^{-s},$$

where $H_h^s$ is the usual semiclassical Sobolev spaces on $X$.

**Proposition 2.1.** We have

$$\text{WF}^s_h(\tilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_u^*) \setminus \{0\})$$

(2.1)

where $\Omega_+$ is the flowout 

$$\Omega_+ = \{(e^{ihp}(y, \eta), y, \eta) : p(y, \eta) = 0\} \subset T^*(X \times X) \simeq T^*X \times T^*X,$$

and $\kappa : T^*(X \times X) \setminus \{0\} \rightarrow S^*(X \times X)$ is the natural projection map.

**Remark 2.2.** Note that $S^*(X \times X) \neq S^*X \times S^*X$, hence there are difficulties to deal with the fiber infinity directly. In fact, unlike the finite part of the wavefront set $T^*(X \times X) \simeq T^*X \times T^*X$, there is no natural way to identify the element in $S^*X \times S^*X$ where $S^*X = \kappa(T^*X \setminus \{0\})$ with the element in $S^*(X \times X) = \kappa(T^*(X \times X) \setminus \{0\})$. However, we do have the natural identification of the diagonal elements $\Delta(S^*X) = \kappa(\Delta(T^*X) \setminus \{0\})$.

The rest of this section will be devoted to the proof of Proposition 2.1. We will follow the strategy of [DyZw16, Proposition 3.4], where the authors prove the estimate for the finite part of $\text{WF}^s_h(\tilde{R}_h(z))$. To deal with the wavefront set at fiber infinity we introduce another small parameter $h > 0$ (which will play the role of $|\langle \xi, \eta \rangle|^{-1}$).

**Step 1:** Let $p^{-1}(0) = \{(x, \xi) \in T^*X : p(x, \xi) = 0\} \supset E_u^* \cup E_u^*$, we first show a weaker statement:

$$\text{WF}^s_h(\tilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times p^{-1}(0)) \setminus \{0\}).$$

(2.2)

Suppose $(x_0, \xi_0, y_0, \eta_0) \in \{(\langle \xi, \eta \rangle) = 1\} \setminus (\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times p^{-1}(0)))$, then as in [DyZw16, Proposition 3.4], using the propagation estimate ([DyZw16, Proposition 2.5]) and the radial source estimate ([DyZw16, Proposition 2.6]), we can find a sufficiently large $\rho > 0$, neighbourhoods $U$ of $(x_0, \rho \xi_0)$ and $W$ of $(y_0, \rho \eta_0)$, and $A, B \in \Psi_h^0(X)$ such that

- $U \subset \text{ell}_h(A)$ and $A$ is microlocally supported near $(x_0, \rho \xi_0)$;
- $B$ is microlocally supported in a neighbourhood of $\{e^{-tH_p}(x_0, \rho \xi_0) : t \geq 0\}$ and $$(\{|\xi| \leq 1\} \cup W) \cap \text{WF}_h(B) = \emptyset.$$
there exists a constant $C > 0$, for any $h$-tempered $u \in \mathcal{S}(X)$,
\[
\|A u\|_{\mathcal{H}^s_{h \hbar}} \leq C h^{-1} \|B \overline{P}_h(z) u\|_{\mathcal{H}^s_{h \hbar}} + \mathcal{O}(h^\infty) \|u\|_{H^N_{h \hbar}}.
\] (2.4)

Here we use the condition $(x_0, \xi_0, y_0, \eta_0) \notin E^*_p \times p^{-1}(0)$ to guarantee $\text{WF}_{h,B}(B) \cap \{\|\xi\| \leq 1\} = \emptyset$ in (2.3) when $\rho > 0$ is large enough. We can also assume that
\[
A = \text{Op}_h(a), \quad B = \text{Op}_h(b), \quad Q = \text{Op}_h(q)
\]
with symbols $b \in S^0$ and $a, q \in C^\infty_0$ independent of $h$, and $\text{supp} \, q \subset \{\|\xi\| \leq 1\}$ so that $\text{supp} \, q \cap \text{supp} \, b = \emptyset$. Here $\text{Op}_h$ denotes a semiclassical quantization on a compact manifold, see [DyZw19, Appendix E].

Now we introduce another small parameter $\hbar \to 0^+$ independent of $h$ to describe the behaviour of the semiclassical Fourier transform as $(\hbar, \eta) \to \infty$ in a conic neighborhood of $(\xi_0, \eta_0)$. Replacing $h$ by $\hbar h$ in the estimate (2.4), we get
\[
A_{\hbar} = \text{Op}_{h \hbar}(a), \quad B_{\hbar} = \text{Op}_{h \hbar}(b), \quad Q_{\hbar} = \text{Op}_{h \hbar}(q) \in \Psi^0_{h \hbar}(X)
\]
such that
\[
\|A_{\hbar} u\|_{\mathcal{H}^s_{h \hbar}} \leq C(h \hbar)^{-1} \|B_{\hbar}(h \hbar P - \hbar h z - iQ_{\hbar}) u\|_{\mathcal{H}^s_{h \hbar}} + \mathcal{O}(h \hbar^\infty) \|u\|_{H^N_{h \hbar}},
\]
\[
U \subset \text{ell}_{h \hbar}(A_{\hbar}), \quad (\{(\xi) | \xi| \leq 1\} \cup W) \cap \text{WF}_{h \hbar}(B_{\hbar}) = \emptyset.
\] (2.5)

Note $z \in [-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]$ implies $\hbar h z \in [-C_1(h \hbar)^\varepsilon, C_1(h \hbar)^\varepsilon] + i[-C_2 \hbar h, 1]$. However we wish to recover $\overline{P}_h$ in estimate (2.5), and this require us to replace $Q_{\hbar}$ by $\hbar Q$ and to deal with the $Q$ term. We need the following lemma:

**Lemma 2.3.** For every $N \in \mathbb{N}$,
\[
\|B_{\hbar} Q u\|_{H^N_{h \hbar}} = \mathcal{O}(h^\infty \hbar^\infty) \|u\|_{H^N_{h \hbar}}.
\]

**Proof.** Using a partition of unity argument we can reduce to the case $M = \mathbb{R}^n$ and assume that all the symbols are compactly supported in $\mathbb{R}^{2n}$. Recall that (e.g. [Zw12, Theorem 4.23]) for a sufficiently large constant $M > 0$ only depending on $n = \dim M$,
\[
\|\text{Op}_h(a)\|_{L^2 \to L^2} \lesssim \|a\|_{S^0,M}, \quad \|a\|_{S^k,M} := \sum_{|\alpha| + |\beta| \leq M} \|\langle \xi \rangle^{\alpha-k} \partial_x^\alpha \partial_\xi^\beta a(x, \xi)\|_{L^\infty}.
\]

Therefore for any $N \geq 0$, we can estimate
\[
\|B_{\hbar} Q\|_{H^N_{h \hbar} \to H^N_{h \hbar}} = \|\langle h \hbar D \rangle^N B_{\hbar} Q \langle h \hbar D \rangle^N\|_{L^2 \to L^2} \lesssim \|\langle \hbar \xi \rangle^N \# b(x, \hbar \xi) \# q(x, \xi) \# \langle \hbar \xi \rangle^N\|_{S^0,M}.
\]
Recall that
\[
(a, b) \mapsto a \# b \text{ is a continuous bilinear map from } S^k \times S^\ell \text{ to } S^{k+\ell} \text{ for any } k, \ell \in \mathbb{R},
\]
Therefore we have
\[ \text{supp } a \cap \text{supp } b = \emptyset, \quad a \# b = O_{S^{k+\ell}}(h^\infty) , \]
and when sup \( a \cap \text{supp } b = \emptyset \), a \# b = O_{S^{k+\ell}}(h^\infty), or more precisely, for any \( m \geq 0 \), any seminorm of \( a \# b \) in \( S^{k+\ell} \) is bounded by \( h^m \) times the product of a seminorm of \( a \) in \( S^k \) and a seminorm of \( b \) in \( S^\ell \).

Now sup \( q \subset \{ |\xi| \leq 1 \} \) and the wavefront set condition (2.3) shows that for any \( \tilde{h} \in (0, 1) \), we have sup \( b(x, \tilde{h}\xi) \cap \text{supp } q(x, \xi) = \emptyset \), therefore for any \( m \geq 1 \), there exists \( M' \) depending only on \( N, M \) and \( m \), such that uniformly for sufficiently small \( h, \tilde{h} > 0 \)
\[ \| B_{\tilde{h}} Q \|_{H^{-N}_{h\tilde{h}} H^N_{h\tilde{h}}} \lesssim h^m \| \langle \tilde{h}\xi \rangle^N \|_{S^{N, M'}}^2 \| b(x, \tilde{h}\xi) \|_{S^{m, M'}} \| q(x, \xi) \|_{S^{-m-2N, M'}}. \]

Finally, we note that \( \| \langle \tilde{h}\xi \rangle^N \|_{S^{N, M'}}^2 \) is uniformly bounded in \( \tilde{h} > 0 \) and since \( \{ \xi = 0 \} \cap \text{supp } b = \emptyset \),
\[ \| b(x, \tilde{h}\xi) \|_{S^{m, M'}} = O(\tilde{h}^m). \]

We conclude that uniformly for sufficiently small \( h, \tilde{h} > 0 \),
\[ \| B_{\tilde{h}} Q \|_{H^{-N}_{h\tilde{h}} H^N_{h\tilde{h}}} = O(h^m \tilde{h}^m), \]
and since \( m \) can be chosen arbitrarily large, this concludes the proof. \( \square \)

Now we go back to (2.5) and taking \( u(x) = \tilde{R}_h(z)(\psi(x)e^{ix\cdot\rho h_0/h\tilde{h}}) \) (here we choose a local coordinates and identity a neighborhood of \( x_0 \) to subset of \( \mathbb{R}^n \)) where sup \( \psi \times \{ \rho h_0 \} \subset W \), the wavefront set condition (2.3) for \( B \) gives
\[ \| B_{\tilde{h}} Q \|_{H^{-N}_{h\tilde{h}} H^N_{h\tilde{h}}} = O(h^\infty \tilde{h}^\infty), \quad \| B_{\tilde{h}} (\psi(x)e^{ix\cdot\rho h_0/h\tilde{h}}) \|_{H^N_{h\tilde{h}}} = O(h^\infty \tilde{h}^\infty). \]

Therefore we have
\[ \| A_{\tilde{h}} u \|_{H^s_{h\tilde{h}}} \leq C h^{-1} \| B_{\tilde{h}} \tilde{P}_h u \|_{H^s_{h\tilde{h}}} + C (h\tilde{h})^{-1} \| B_{\tilde{h}} Q \tilde{P}_h u \|_{H^s_{h\tilde{h}}} + C h^{-1} \| B_{\tilde{h}} Q u \|_{H^s_{h\tilde{h}}} + O((h\tilde{h})^\infty) \| u \|_{H^{-N}_{h\tilde{h}}} \]
\[ = O(h^{-1}) \| B_{\tilde{h}} (\psi(x)e^{ix\cdot\rho h_0/h\tilde{h}}) \|_{H^s_{h\tilde{h}}} + O((h\tilde{h})^\infty) \| u \|_{H^{-N}_{h\tilde{h}}} \]
\[ = O(h^\infty \tilde{h}^\infty). \]

This means \( \text{WF}_{h\tilde{h}}(u) \cap U = \emptyset \), and thus if \( \chi \in C^\infty(X) \) and sup \( \chi \times \{ \rho \xi_0 \} \subset U \), then
\[ \int \chi(x)e^{-ix\cdot\rho \xi_0/h\tilde{h}} \tilde{R}_h(z)(\psi(x)e^{ix\cdot\rho h_0/h\tilde{h}})dx = O(h^\infty \tilde{h}^\infty). \]

Moreover, by construction it is easy to see the estimate is locally uniform in \( (x_0, \xi_0, y_0, \eta_0) \). Therefore by the equivalent definition of semiclassical wavefront sets using the semiclassical Fourier transform (see [Al08, Definition 3.2]), \( \kappa(x_0, \xi_0, y_0, \eta_0) = \kappa(x_0, \rho \xi_0, y_0, \rho \eta_0) \not\in \text{WF}'(\tilde{R}_h(z)) \cap S^*(X \times X) \) and we have (2.2).
Step 2: The previous method does not work for \((x_0, \xi_0, y_0, \eta_0)\) \(\in E_0^* \times p^{-1}(0)\) since \(WF_h(B)\) has to intersect the zero section \(\{\xi = 0\}\). Here we argue by duality. Suppose \((x_0, \xi_0, y_0, \eta_0)\) \(\in \{(\xi, \eta)| = 1\} \setminus (\Delta \cup \Omega_+ \cup (p^{-1}(0) \times E_0^*))\), we consider the following operator

\[ -\tilde{P}_h(z)^* := -hP^* - iQ - (z), \]

acting on \(\mathcal{H}_h^{-s}\). We see that this corresponds to the reversed Anosov flow \(\varphi_{-t}\) generated by \(-V\) and \(z \in [-C_1h^\varepsilon, C_1h^\varepsilon] + i[-C_2h, 1]\) also gives \(-z\) in the same region. We can repeat the same argument with the opposite propagation direction we get \(\tilde{P}_h(z)^*\) is invertible, with inverse \(\tilde{R}_h(z)^* : \mathcal{H}_h^{-s} \rightarrow \mathcal{H}_h^{-s}\) satisfying

\[ \|\tilde{R}_h(z)^*\|_{\mathcal{H}_h^{-s} \rightarrow \mathcal{H}_h^{-s}} \leq Ch^{-1}. \]

Moreover, there exist \(\rho > 0, U = nbhd(x_0, \rho \xi_0)\) and \(W = nbhd(y_0, \rho \eta_0)\) such that for \(\text{supp } \psi \times \{(\rho \eta_0)\} \subset U\) we have

\[ \int \psi(x)e^{-ix.\rho \xi_0/hh}\tilde{R}_h(z)^*(\chi(x)e^{ix.\rho \xi_0/hh})dx = O(h^\infty), \]

and the estimate is locally uniform in \((x_0, \xi_0, y_0, \eta_0)\). Therefore

\[ \kappa(y_0, \eta_0, x_0, \xi_0) \not\in WF_h(\tilde{R}_h(z)^*) \cap S^*(X \times X). \]

Since the Schwartz kernel of \(\tilde{R}_h(z)^*\) is \(\overline{K(y, x)}\) if the \(K(x, y)\) is the Schwartz kernel of \(\tilde{R}_h(z)\), we have \(\kappa(x_0, \xi_0, y_0, \eta_0) \not\in WF_h'(\tilde{R}_h(z)) \cap S^*(X \times X)\) and thus

\[ WF_h'(\tilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (p^{-1}(0) \times E_0^*)) \setminus \{0\}. \]

Combining this with (2.2) we get the desired estimate (2.1) and finish the proof of Proposition 2.1.

3. Flat trace estimates

In this section, we present a simpler argument than the one in [JiZw17] to give the following flat trace estimate (see [JiZw17, Proposition 3.1]). The result is slightly weaker than the original one in [JiZw17], but avoid using [NoZw15, Proposition 10.3] and thus the assumption [JiZw17, (2.7)] for the complex absorbing potential \(Q\).

Theorem 2. Fix any \(\varepsilon > 0\), the flat trace

\[ T(z) = tr^b(e^{-it_0h^{-1}\tilde{P}_h(z)}\tilde{R}_h(z)) \]

is well-defined and holomorphic for \(z\) in \([-C_1h^\varepsilon, C_1h^\varepsilon] + i[-C_2h, 1]\). Moreover, we have

\[ T(z) = O(h^{-2n-2}). \]

(3.1)

To prove it we need a wavefront set estimate for the Schwartz kernel of \(e^{-it_0h^{-1}\tilde{P}_h(z)}\tilde{R}_h(z)\):
Lemma 3.1.

\[ \text{WF}' \left( e^{-it_0h^{-1}} \tilde{R}_h(z) \right) \cap S^*(X \times X) \supset \kappa(\{(x, \xi, y, \eta) : (e^{-it_0H_p}(x, \xi), y, \eta) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\} \text{ or } \xi = 0, \eta \neq 0\}). \]

**Proof.** Proposition 2.1 gives

\[ \text{WF}' \left( \tilde{R}_h(z) \right) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}). \]

Thus

\[ \text{WF}' \left( e^{-it_0V} \tilde{R}_h(z) \right) \cap S^*(X \times X) \subset \kappa(\{(x, \xi, y, \eta) : (e^{-it_0H_p}(x, \xi), y, \eta) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}). \]

We have

\[ e^{-it_0P} - e^{-it_0h^{-1}(hP-iQ)} = h^{-1} \int_{t_0}^{t_0} e^{-i(t_0-t)P} Q e^{-ith^{-1}(hP-iQ)} \, dt, \]

and using \( \text{WF}' \left( Q \right) \cap S^*(X \times X) = \emptyset \) and [A108, Lemma 3.7(iii)], we can compute

\[ \text{WF}' \left( e^{-i(t_0-t)P} Q e^{-ith^{-1}(hP-iQ)} \tilde{R}_h(z) \right) \cap S^*(X \times X) \subset (X \times \{0\}) \times S^*X. \]

Therefore

\[ \begin{align*}
\text{WF}' \left( e^{-it_0h^{-1}} \tilde{R}_h(z) \right) \cap S^*(X \times X) \\
\subset \left( \text{WF}' \left( e^{-it_0P} \tilde{R}_h(z) \right) \bigcup \bigcup_{t=0}^{t_0} \text{WF}' \left( e^{-i(t_0-t)P} Q e^{-ith^{-1}(hP-iQ)} \tilde{R}_h(z) \right) \right) \cap S^*(X \times X) \\
\subset \kappa(\{(x, \xi, y, \eta) : (e^{-it_0H_p}(x, \xi), y, \eta) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\} \text{ or } \xi = 0, \eta \neq 0\}).
\end{align*} \]

\[ \square \]

Theorem 2 then follows from the following general lemma.

**Lemma 3.2.** Let \( X \) be an \( n \)-dimensional smooth manifold and \( m \in \mathbb{R} \). If \( P(h) : C^\infty(X) \to \mathcal{D}'(X) \) is \( h \)-tempered and satisfies

- \( \text{WF}'(P(h)) \cap \Delta(S^*X) = \emptyset; \)
- \( \|AP(h)B\|_{L^2 \to L^2} = \mathcal{O}(h^{-m}) \) for any \( A, B \in \Psi_h^{\text{comp}}(X) \);

then \( \text{tr}^b(P(h)) \) is well-defined with

\[ \text{tr}^b(P(h)) = \mathcal{O}(h^{-2n-m}). \]

**Proof.** Since \( \text{WF}'(P(h)) \cap \Delta(S^*X) = \emptyset \), we have \( \text{WF}'(P(h)) \cap \Delta(T^*X) = \emptyset \), it is then a classical theorem (see e.g. [Hö83, Theorem 8.2.4]) that the flat trace is well-defined as long as the wavefront set does not intersect the diagonal.
Let \( u = K_h \) be the Schwartz kernel of \( P(h), \iota : X \to X \times X \) be the diagonal embedding, then for \( \chi \in C^\infty(X), \varphi(x,y) = \psi(x)\psi(y) \in C^\infty(X \times X) \) supported near the diagonal,

\[
\langle \iota^* (\varphi u), \chi \rangle = \langle \varphi u, \iota_* \chi \rangle = \frac{1}{(2\pi h)^{2n}} \int \mathcal{F}_h(\varphi u) I_{X,h}(\xi, \eta) d\xi d\eta
\]

where

\[
I_{X,h}(\xi, \eta) = \int \chi(x) e^{ix(\xi + \eta)/h} dx.
\]

If \(|\xi + \eta| > |\xi|/C\), then

\[
I_{X,h}(\xi, \eta) = \mathcal{O}(h^\infty(|\xi| + |\eta|)^{-\infty}).
\]

Thus we only need to consider the case when \((\xi, \eta)\) lies in a small conical neighbourhood of \(\{\xi + \eta = 0\}\) or in a neighbourhood of \(\{\xi = \eta = 0\}\).

(i) When \(|\xi| + |\eta| \leq C\) is bounded, we have for some \(A, B \in \Psi_h^\text{comp}(X)\)

\[
|\mathcal{F}_h(\varphi u)| = |\langle P(h)B(\psi(y)e^{-ix\eta/h}), A(\psi(x)e^{-ix\xi/h}) \rangle| + \mathcal{O}(h^\infty)
\]

\[
\lesssim \|AP(h)B\|_{L^2 \to L^2} + \mathcal{O}(h^\infty)
\]

\[
= \mathcal{O}(h^{-m}).
\]

(ii) When \((\xi, \eta)\) is near fiber infinity and in a small conic neighbourhood of \(\{\xi + \eta = 0\}\) which does not intersect \(WF_h'(P(h))\), we have

\[
\mathcal{F}_h(\varphi u) = \mathcal{O}(h^\infty(|\xi| + |\eta|)^{-\infty})
\]

thanks to the wavefront condition \(WF_h'(P(h)) \cap \Delta(S^*X) = \emptyset\).

Now (3.2) gives us

\[
|\langle \iota^* (\varphi u), \chi \rangle| = h^{-2n} \mathcal{O}(h^{-m}) = \mathcal{O}(h^{-2n-m})
\]

and a partition of unity argument finishes the proof. \(\square\)

Proof of Theorem 2. The operator \( \tilde{R}_h(z) : \mathcal{H}^s_h \to \mathcal{H}^s_h \) is bounded and thus \( h \)-tempered. Lemma 3.1 gives

\[
WF_h'(e^{-it_0 h^{-1}\tilde{R}_h(z)} \tilde{R}_h(z)) \cap \Delta(S^*X) = \emptyset
\]

if we choose \( t_0 > 0 \) smaller than the least length of the closed orbits. For any \( A, B \in \Psi_h^\text{comp}(X) \) recall

\[
e^{-it_0 P} - e^{-it_0 h^{-1}(P - iQ)} = h^{-1} \int_0^{t_0} e^{-ith^{-1}(P - iQ)} Q e^{-i(t_0-t)P} dt,
\]
we have
\[
\| A e^{-it_0 h^{-1}} \tilde{R}(z) B \|_{L^2 \to L^2} \\
\lesssim \| A e^{-it_0 P} \tilde{R}(z) B \|_{L^2 \to L^2} + h^{-1} \int_0^{t_0} \| A e^{-ith^{-1}(hP-iQ)} Q e^{-i(t_0-t)P} \tilde{R}(z) B \|_{L^2 \to L^2} dt \\
\lesssim \| A e^{-it_0 P} \tilde{R}(z) B \|_{\mathcal{H}_h^s \to \mathcal{H}_h^s} + h^{-1} \int_0^{t_0} \| Q e^{-i(t_0-t)P} \tilde{R}(z) B \|_{L^2 \to L^2} dt \\
= O(h^{-1}) + h^{-1} \int_0^{t_0} \| Q e^{-i(t_0-t)P} \tilde{R}(z) B \|_{\mathcal{H}_h^s \to \mathcal{H}_h^s} dt \\
= O(h^{-2}).
\]

Here we use the fact that on compact sets in the phase space $L^2$ norm is equivalent to any $\mathcal{H}^s$ norm. Now the claim follows from Lemma 3.2. 

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