BAND EDGES OF PERIODIC SCHRÖDINGER OPERATORS ARE GENERICALLY ISOLATED AND NONDEGENERATE

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Abstract. For periodic Schrödinger operators $H_V = -\Delta + V$ with bounded real-valued potentials on $\mathbb{R}^d$ with $d \geq 2$, we show that for generic potentials, each endpoint of every spectral gap is attained by a single Bloch band at only finitely many quasimomenta, and has a nondegenerate Hessian at every attaining point. This proves the Spectral Edge Conjecture for periodic Schrödinger operators.

1. Introduction

Let $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, be a lattice of full rank, with the dual lattice and the Brillouin zone given by

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^d : \gamma^* \cdot \gamma \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Gamma\}, \qquad \mathbb{T}^d_* = \mathbb{R}^d/\Gamma^*. \tag{1.1}$$

We consider Schrödinger operators $H_V = -\Delta + V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with bounded real-valued Γ-periodic potentials in the space

$$\mathcal{X} = L^\infty(\mathbb{R}^d/\Gamma; \mathbb{R}). \tag{1.2}$$

For $V \in \mathcal{X}$, Floquet–Bloch theory gives the Floquet–Bloch transformed operators

$$H_V(k) = (-i\nabla + k)^2 + V : \; L^2(\mathbb{R}^d/\Gamma) \to L^2(\mathbb{R}^d/\Gamma) \qquad k \in \mathbb{T}^d_*, \tag{1.3}$$

with domain $H^2(\mathbb{R}^d/\Gamma)$ and eigenvalues (counted with multiplicity)

$$E_1(V,k) \leq E_2(V,k) \leq \cdots, \qquad k \in \mathbb{T}^d_*. \tag{1.4}$$

Then by Floquet–Bloch theory, we have the decomposition of the continuous spectrum

$$\operatorname{Spec} H_V = \bigcup_{n \geq 1} E_n(V, \mathbb{T}^d_*). \tag{1.5}$$

A subset of $\mathcal{X}$ is called *residual* if it contains a countable intersection of open dense subsets of $\mathcal{X}$. The endpoints $a, b \in \operatorname{Spec} H_V$ of a bounded connected component (a,b) of $\mathbb{R} \setminus \operatorname{Spec} H_V$ are called *band edges* (or *spectral edges*), and the interval (a,b) is called a *spectral gap*.[1] Our main result is the following

Theorem 1.1. *There exists a residual set $\mathcal{G} \subset \mathcal{X}$ such that the following holds. For every $V \in \mathcal{G}$ and spectral gap $(a,b) \subset \mathbb{R} \setminus \operatorname{Spec} H_V$,*

(i) *there exist unique indices $n_-, n_+ \geq 1$ such that*

$$a \in E_{n_-}(V, \mathbb{T}^d_*), \qquad b \in E_{n_+}(V, \mathbb{T}^d_*);$$

for every k with $E_{n_-}(V,k) = a$, the number a is a simple eigenvalue of $H_V(k)$, and for every k with $E_{n_+}(V,k) = b$, the number b is a simple eigenvalue of $H_V(k)$;

(ii) *the sets*

$$\{k : E_{n_-}(V,k) = a\}, \qquad \{k : E_{n_+}(V,k) = b\} \tag{1.6}$$

are finite;

[1] The bottom of the spectrum is well understood in the literature (cf. [14]), thus we exclude it from the definition of the band edges.

(iii) *for every k_- in the first set and k_+ in the second set in* (1.6)*, we have*

$$d_k^2 E_{n_-}(V, k_-) < 0, \qquad d_k^2 E_{n_+}(V, k_+) > 0.$$

The theorem proves the Spectral Edge Conjecture (cf. [18, Conjecture 8.5]) for periodic Schrödinger operators, which predicts that, for generic potentials, any band edge belongs to a single band function $E_n(k)$, is attained at isolated quasimomenta k, and has a nondegenerate Hessian there. The structure of band edges of periodic Schrödinger operators is a fundamental open question in mathematical physics. Filonov–Kachkovskiy [12] proved that global band extrema are isolated for a broad class of periodic elliptic operators when $d = 2$. However, their argument does not establish nondegeneracy of the Hessian and is specific to dimension two. The generic nondegeneracy of spectral edges has remained open even for periodic Schrödinger operators. Theorem 1.1 establishes both isolation and Hessian nondegeneracy for generic potentials in every dimension $d \geq 2$.

The Spectral Edge Conjecture has several important consequences. It enters the definition of effective masses in solid-state physics [1, Chapter 12, equation (12.29)] and is also a fundamental hypothesis in results on Liouville theorems and Green function asymptotics [19, 20], homogenization [24], impurity eigenvalues created by localized perturbations [5, 26], and Anderson localization near internal band edges [15, 25].

Part (i) follows from the theorem of Klopp–Ralston [16]. We prove parts (ii)–(iii) by showing that, for a fixed band $E_n(V, k)$, a generic perturbation can make the Hessian nondegenerate at its global extrema. Without loss of generality, we only give the proof for global minima. Global maxima can be treated in a similar fashion.

Proof idea and structure of the paper. We briefly describe the main mechanism of the proof. Fix a band $E_n(V, k)$ and let k_0 be a global minimum at which $E_n(V, k_0)$ is a simple eigenvalue. Locally near k_0, we write the corresponding eigenvalue and normalized Bloch eigenfunction as $E(k)$ and $u(k)$, and set

$$K = \ker d_k^2 E(k_0), \qquad Y = K^\perp.$$

Since k_0 is a minimum, the Hessian is positive definite on Y, so the only obstruction to nondegeneracy lies in the kernel K. The implicit function theorem allows us to eliminate the nondegenerate Y directions and reduce the problem to the behavior of the band function in the K directions. By the Feynman–Hellmann formula,

$$\partial_t|_{t=0} E(V + tq, k) = F_q(k), \quad F_q(k) = \int_{\mathbb{R}^d/\Gamma} q(x)|u(k, x)|^2 dx,$$

and by differentiation,

$$dF_q(k_0) \cdot \eta = \int_{\mathbb{R}^d/\Gamma} q(x)\rho_\eta(x) dx, \qquad \rho_\eta = (\eta \cdot \partial_k)|u(k_0, \cdot)|^2.$$

The proof now splits according to whether these density variations vanish on K.

(1) In Section 3, we study the case when

$$\rho_\eta \not\equiv 0 \qquad \text{for some } \eta \in K. \tag{1.7}$$

In this case, we may choose $q = \rho_\eta$ to make $dF_q(k_0) \cdot \eta > 0$. Thus the potential perturbation changes, to first order, the gradient of the band function in a direction belonging to the Hessian kernel K. After eliminating the Y variables, we obtain a one-parameter family of reduced band functions on K. Then nonvanishing $dF_q(k_0) \cdot \eta > 0$ gives a transversality direction for this family. Sard's theorem then implies that, for generic values of the perturbation parameter, the reduced Hessian has positive rank at every nearby critical point which is a local minimum. A Schur complement argument identifies the corank of the reduced Hessian with that of the full Hessian. Hence the Hessian corank decreases by at least one.

(2) In Section 4, we study the case when

$$\rho_\eta \equiv 0 \qquad \text{for every } \eta \in K. \tag{1.8}$$

By (1.8), $dF_q(k_0)|_K = 0$ for every $q \in \mathcal{X}$, so we study the variation of the reduced Hessian itself. Let g_t denote the band function after eliminating the Y variables for the perturbed potential $V + tq$. Section 4 shows that

$$\frac{d}{dt}|_{t=0} d_z^2 g_t(0) = \mathcal{R}(q),$$

where $\mathcal{R}(q)$ is a quadratic form on K depending linearly on q. To remove the degeneracy in the K directions, the main point is to prove that there exists $q \in \mathcal{X}$ for which

$$\mathcal{R}(q) > 0 \quad \text{on } K. \tag{1.9}$$

Thus the two mechanisms are complementary: in the first case a first-order perturbation lowers the Hessian corank, while in the second case a second-order perturbation removes the remaining kernel. The nondegeneracy of the Hessian follows by induction, and the isolation of the minima/maxima follows as a corollary.

The key mechanism to establish (1.9) is Lemma 2.2, which says that under the assumption (1.8), $(\eta \cdot \partial_k) u(k_0, \cdot)$ cannot vanish on the nodal set of u for $\eta \neq 0$. The key Lemma 2.2 for the Bloch density is proved in Section 2 using a current identity and unique continuation. Section 5 applies the local arguments in Section 3 and 4 to show that nondegeneracy is open and dense whenever the band extrema are simple. Section 6 combines this with Klopp–Ralston [16] to prove the main Theorem 1.1.

Related work. The Spectral Edge Conjecture is a long-standing problem in mathematical physics. An early formulation appears in the work of Colin de Verdière [7], who established the Morse property for finitely many bands of small generic potentials in dimension two. The conjecture was later formulated for periodic elliptic operators by Kuchment–Pinchover [19, Conjecture 20]. See also [17, Conjecture 5.25] and the recent survey [18, Conjecture 8.5]. Klopp–Ralston [16] proved generic simplicity of the endpoints at a spectral gap. Filonov–Kachkovskiy [12] proved that global band extrema are isolated for a wide class of two-dimensional periodic elliptic operators, while Kirsch–Simon [14] proved uniqueness and a quantitative nondegeneracy at the bottom of the spectrum. Further perturbative results include [7, 22, 24].

There is a parallel algebraic theory for discrete and graph models. Do–Kuchment–Sottile [9] established an algebraic genericity dichotomy for degenerate extrema and verified the Spectral Edge Conjecture for a specific maximal two-atomic $\mathbb{Z}^2$-periodic graph. Liu [21] proved, using irreducibility of the Fermi variety, that for discrete periodic Schrödinger operators an extremal level set is finite in dimension two and has dimension at most $d-2$ in dimensions $d \geq 3$. See also [13] for a different proof for non-divisible lattices. Faust–Liu–Luo [10] subsequently obtained an improved quantitative two-dimensional cardinality bound. Berkolaiko–Canzani–Cox–Marzuola [4] gave local-to-global criteria for extrema of dispersion bands on periodic graphs, and Faust–Sottile [11] verified the Spectral Edge Conjecture for two infinite families of periodic graphs.

The isolated, single-band, nondegenerate edge hypothesis also has important consequences: Kuchment–Pinchover [19] relate it to Liouville theorems, and Kuchment–Raich [20] derive Green function asymptotics at internal spectral edges under this hypothesis. Parnovski–Shterenberg enlarge the period lattice in their perturbation argument and state in [22, Remark 1.2] that nondegeneracy in two dimensions under the same lattice perturbation was not known, and Theorem 1.1 also answers this question. Stable degenerate edges occur for periodic quantum graphs [3] and for discrete periodic operators [12, Section 7]; these examples reflect structural constraints of the graph or discrete models and do not contradict the continuum fixed-lattice result proved here.

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2. Analytic perturbation theory and a current identity

We first fix notation for a simple eigenvalue under perturbation. Let $V_0 \in \mathcal{X}$, $k_0 \in \mathbb{T}^d_*$, and let E_0 be a simple eigenvalue of $H_{V_0}(k_0)$. Choose a positively oriented circle $\gamma \subset \mathbb{C}$ which encloses E_0 and no other point of $\operatorname{Spec} H_{V_0}(k_0)$. After shrinking a neighborhood $\mathcal{V}$ of V_0 in $\mathcal{X}$ and a coordinate neighborhood U of k_0, the circle γ lies in the resolvent set of $H_W(k)$ for all $(W,k) \in \mathcal{V} \times U$, and the spectral projection

$$\Pi(W,k) = \frac{1}{2\pi i}\oint_\gamma (z - H_W(k))^{-1}\,dz \tag{2.1}$$

has rank one. Define

$$E(W,k) = \operatorname{tr}\left(\frac{1}{2\pi i}\oint_\gamma z(z - H_W(k))^{-1}\,dz\right). \tag{2.2}$$

Thus $E(W,k)$ is the unique eigenvalue of $H_W(k)$ inside γ. It is continuous in (W,k), analytic in k, and jointly analytic when W is restricted to a finite-dimensional affine subspace of $\mathcal{X}$. If U and the finite-dimensional parameter set are contractible, one can choose a normalized eigenfunction $u(W,k)$ which is analytic in the finite-dimensional variables and satisfies

$$H_W(k)u(W,k) = E(W,k)u(W,k), \qquad \|u(W,k)\|_{L^2(\mathbb{R}^d/\Gamma)} = 1. \tag{2.3}$$

The following lemma is a consequence of standard elliptic estimates.

Lemma 2.1. *Let $U \subset \mathbb{R}^d$ be open. Assume that $E : U \to \mathbb{R}$ and $u : U \to H^2(\mathbb{R}^d/\Gamma)$ are analytic and satisfy*

$$H_V(k)u(k) = E(k)u(k), \qquad \|u(k)\|_{L^2(\mathbb{R}^d/\Gamma)} = 1.$$

For every multi-index β, every $1 < p < \infty$ and every $0 < \sigma < 1$,

$$\partial_k^\beta u(k,\cdot) \in W^{2,p}(\mathbb{R}^d/\Gamma) \quad \text{and} \quad \partial_k^\beta u(k,\cdot) \in C^{1,\sigma}(\mathbb{R}^d/\Gamma)$$

locally uniformly in $k \in U$.

Let $E(k)$ and $u(k)$ satisfy the assumptions of Lemma 2.1, and let k_0 be a critical point of E. For $\xi, \zeta \in \mathbb{R}^d$, write

$$\rho_\xi := (\xi \cdot \partial_k)|u(k_0,\cdot)|^2, \qquad \rho_{\xi\zeta} := (\xi \cdot \partial_k)(\zeta \cdot \partial_k)|u(k_0,\cdot)|^2. \tag{2.4}$$

These functions are real-valued, continuous, bounded, and Γ-periodic.

Lemma 2.2. *Assume that $\eta \in \ker d_k^2 E(k_0)$. Take*

$$u = u(k_0,\cdot), \qquad v = (\eta \cdot \partial_k)u(k_0,\cdot), \qquad Z = \{x \in \mathbb{R}^d/\Gamma : u(x) = 0\}.$$

If

$$(\eta \cdot \partial_k)|u(k_0,\cdot)|^2 = 0 \quad \text{on } \mathbb{R}^d/\Gamma, \qquad v = 0 \quad \text{on } Z, \tag{2.5}$$

then we have $\eta = 0$.

Proof. Lemma 2.1 gives $u, v \in W^{2,p} \cap C^{1,\sigma}(\mathbb{R}^d/\Gamma)$ for every $0 < \sigma < 1$ and $1 < p < \infty$. The function $e^{ik_0\cdot x}u(x)$ solves

$$(-\Delta + V - E(k_0))\Big(e^{ik_0\cdot x}u(x)\Big) = 0$$

on $\mathbb{R}^d$. A nontrivial solution of this equation cannot vanish on a set of positive Lebesgue measure: at a density point it vanishes to infinite order by the argument of [8], and strong unique continuation [2, 6] then implies that it vanishes identically. Hence

$$|Z| = 0. \tag{2.6}$$

Set $D = (\mathbb{R}^d/\Gamma) \setminus Z$. The first condition in (2.5) is $2\operatorname{Re}(\overline{u}v) = 0$. Therefore

$$v = i\alpha u \quad \text{on } D, \qquad \alpha = -i\frac{v}{u} \in C^1(D;\mathbb{R}). \tag{2.7}$$

The function α is periodic on D, which need not be connected.

Define

$$\begin{aligned} J(k,x) &= \operatorname{Im}\Big(\overline{u(k,x)}\nabla_x u(k,x)\Big) + k|u(k,x)|^2, \\ j = (\eta\cdot\partial_k)J(k_0,\cdot) &= \operatorname{Im}\Big(\overline{v}\nabla_x u + \overline{u}\nabla_x v\Big) + \eta|u|^2 + 2k_0\operatorname{Re}(\overline{u}v). \end{aligned} \tag{2.8}$$

Multiplying (2.3) by $\overline{u(k)}$ and taking imaginary parts gives

$$\nabla_x \cdot J(k,\cdot) = 0.$$

The Feynman–Hellmann identity in k-variable is given by

$$\nabla_k E(k) = 2\int_{\mathbb{R}^d/\Gamma} J(k,x)\,dx.$$

Differentiating this identity and $\nabla_x \cdot J(k,\cdot) = 0$ in the direction η at k_0 gives

$$\nabla_x \cdot j = 0, \qquad \int_{\mathbb{R}^d/\Gamma} j\,dx = \frac{1}{2}d_k^2E(k_0)\eta = 0. \tag{2.9}$$

Differentiating (2.8) and using (2.7) gives

$$j = |u|^2(\eta + \nabla\alpha) \quad \text{on } D. \tag{2.10}$$

The second condition in (2.5) implies $j = 0$ on Z. Since u and v are Lipschitz and vanish on Z, while their gradients are bounded,

$$|j(x)| \le C\operatorname{dist}(x,Z), \qquad x \in \mathbb{R}^d/\Gamma. \tag{2.11}$$

We next justify integration by parts without assumptions on the geometry of Z. If $Z = \varnothing$, take $\chi_\varepsilon = 1$. Otherwise choose periodic Lipschitz functions $\chi_\varepsilon : \mathbb{R}^d/\Gamma \to [0,1]$ such that

$$\chi_\varepsilon = 0 \text{ on } \{\operatorname{dist}(x,Z) \le \varepsilon\}, \qquad \chi_\varepsilon = 1 \text{ on } \{\operatorname{dist}(x,Z) \ge 2\varepsilon\}, \qquad |\nabla\chi_\varepsilon| \le C\varepsilon^{-1}. \tag{2.12}$$

We choose the cutoffs so that $\chi_\varepsilon(x) \to 1$ monotonically for every $x \in D$ as $\varepsilon \to 0$. Choose bounded smooth functions $T_N : \mathbb{R} \to \mathbb{R}$ satisfying

$$T_N(0) = 0, \qquad 0 \le T_N' \le 1, \qquad T_N'(s) \uparrow 1 \quad \text{for every } s \in \mathbb{R}. \tag{2.13}$$

The truncation is needed because $\alpha = v/u$ need not be bounded near Z. Testing the first identity in (2.9) against $\chi_\varepsilon T_N(\alpha)$ gives

$$\int_D \chi_\varepsilon T_N'(\alpha) j\cdot\nabla\alpha\,dx = -\int_D T_N(\alpha) j\cdot\nabla\chi_\varepsilon\,dx. \tag{2.14}$$

For fixed N, the right-hand side is bounded in absolute value by

$$C_N\Big|\{x : \operatorname{dist}(x,Z) < 2\varepsilon\}\Big|,$$

which tends to zero by (2.6). From (2.10),

$$j \cdot \nabla\alpha = |u|^2|\eta + \nabla\alpha|^2 - \eta \cdot j.$$

Letting first $\varepsilon \to 0$ and then $N \to \infty$ in (2.14), using monotone convergence for the first term and dominated convergence for the second, gives

$$\int_D |u|^2|\eta + \nabla\alpha|^2\, dx = \eta \cdot \int_{\mathbb{R}^d/\Gamma} j\, dx = 0. \tag{2.15}$$

Thus

$$\nabla\alpha = -\eta \quad \text{on } D. \tag{2.16}$$

Lift all periodic functions to $\mathbb{R}^d$. On a nonempty connected component D_* of the lifted set $\{u \neq 0\}$,

$$\alpha(x) = C_0 - \eta \cdot x. \tag{2.17}$$

Take $D_0 = -i\nabla + k_0$ and $L = D_0^2 + V - E(k_0)$. Differentiating the eigenvalue equation in the direction η and using $dE(k_0) \cdot \eta = 0$ gives

$$Lv = -2\eta \cdot D_0 u. \tag{2.18}$$

For $g(x) = i(C_0 - \eta \cdot x)$, the commutator identity $[D_{0,j}, g] = -\eta_j$ gives

$$L(gu) = -2\eta \cdot D_0 u. \tag{2.19}$$

The function $v - gu$ solves the homogeneous equation and vanishes on the open set D_*. Unique continuation [2] gives $v = gu$ on $\mathbb{R}^d$. For every $\gamma \in \Gamma$, periodicity of u and v gives

$$(\eta \cdot \gamma)u(x) = 0 \qquad \text{for all } x \in \mathbb{R}^d.$$

Since $u \not\equiv 0$ and Γ spans $\mathbb{R}^d$, we obtain $\eta = 0$. □

3. First-order perturbations

In this section, we discuss the first-order perturbation, which works when $\rho_\eta \not\equiv 0$. For $V \in \mathcal{X}$, assume that

$$k_0 \in \mathcal{M}_n(V) := \operatorname{argmin}_{k \in \mathbb{T}^d_*} E_n(V, k), \tag{3.1}$$

and $E_n(V, k_0)$ is a simple eigenvalue of $H_V(k_0)$. Let $E(W, k)$ be the eigenvalue defined by (2.2) on $\mathcal{V} \times U$, after choosing γ, $\mathcal{V}$, and U as in Section 2. We use the notation

$$E(k) = E(V, k), \qquad u(k) = u(V, k).$$

Shrink U so that $E(k) = E_n(V, k)$ for $k \in U$. Define

$$K := \ker d_k^2 E(k_0), \qquad Y := K^\perp. \tag{3.2}$$

Since k_0 is a minimum,

$$H_Y = d_k^2 E(k_0)|_{Y \times Y} > 0. \tag{3.3}$$

By the Feynman–Hellmann formula, for $q \in \mathcal{X}$

$$\partial_t|_{t=0} E(V + tq, k) = F_q(k), \qquad F_q(k) := \int_{\mathbb{R}^d/\Gamma} q(x)|u(k, x)|^2\, dx. \tag{3.4}$$

For $\eta \in K$, define ρ_η by (2.4). Then

$$dF_q(k_0) \cdot \eta = \int_{\mathbb{R}^d/\Gamma} q\rho_\eta\, dx. \tag{3.5}$$

In particular, if $\rho_\eta \not\equiv 0$, then we may take $q = \rho_\eta \in \mathcal{X}$ such that

$$dF_q(k_0) \cdot \eta = \|\rho_\eta\|^2_{L^2(\mathbb{R}^d/\Gamma)} > 0.$$

Proposition 3.1. *If $r = \dim K > 0$ and $q \in \mathcal{X}$ satisfy*

$$dF_q(k_0)|_K \neq 0, \tag{3.6}$$

then there exist a coordinate neighborhood $U_0 \Subset U$ of k_0, a neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of V, and $\varepsilon_0 > 0$ such that for every $W \in \mathcal{V}_0$, there exists an open dense set $\mathcal{G}_{W,\varepsilon_0} \subset (-\varepsilon_0, \varepsilon_0)$ of full Lebesgue measure such that for any $a \in \mathcal{G}_{W,\varepsilon_0}$

$$\dim\ker d_k^2 E(W + aq, k) \leq r - 1 \tag{3.7}$$

at critical points $k \in U_0$ satisfying

$$d_k E(W + aq, k) = 0, \qquad d_k^2 E(W + aq, k) \geq 0. \tag{3.8}$$

Moreover, for each $a \in \mathcal{G}_{W,\varepsilon_0}$, there exists $\delta > 0$ such that (3.7) remains valid after replacing $W + aq$ by any $W' \in \mathcal{V}$ with

$$\|W' - (W + aq)\|_{L^\infty} < \delta.$$

Proof. Use a local coordinate chart at k_0 and write

$$k = k_0 + y + z, \qquad y \in Y, \quad z \in K.$$

Taking some neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of V and $\varepsilon_0 > 0$, for any fixed $W \in \mathcal{V}_0$ and $|a| < \varepsilon_0$, we define

$$f(W, a, y, z) = E(W + aq, k_0 + y + z).$$

By (3.3), we may choose closed small balls $\overline{B}_Y \subset Y$ and $\overline{B}_K \subset K$ centered at zero and shrinking $\mathcal{V}_0$, and ε_0 if needed, such that $\partial_y^2 f$ is positive definite on $\mathcal{V}_0 \times (-\varepsilon_0, \varepsilon_0) \times \overline{B}_Y \times \overline{B}_K$. By the implicit function theorem, there exists a function $\psi(W, a, z) \in B_Y$ such that

$$\partial_y f(W, a, \psi(W, a, z), z) = 0, \qquad z \in \overline{B}_K. \tag{3.9}$$

The positivity of $\partial_y^2 f$ implies the uniqueness of $\psi(W, a, z)$ as a solution of (3.9) in B_Y. Hence, for $k = k_0 + y + z$ with $(y, z) \in B_Y \times B_K$ such that $d_k E(W + aq, k) = 0$, we have $y = \psi(W, a, z)$.

By (3.6), we may choose $e \in K$ such that $dF_q(k_0) \cdot e \neq 0$. For simplicity, we define

$$g(W, a, z) = f(W, a, \psi(W, a, z), z), \qquad \Phi(W, a, z) = d_z g(W, a, z) \cdot e. \tag{3.10}$$

Differentiating (3.9) in z at $(V, 0, 0)$ and taking $\zeta \in K$ gives

$$H_Y(D_z\psi(V, 0, 0)\zeta, \cdot) + d_k^2 E(k_0)(\zeta, \cdot)\Big|_Y = 0.$$

By (3.3) and that K is the kernel of $d_k^2 E(k_0)$, we have

$$D_z\psi(V, 0, 0) = 0. \tag{3.11}$$

Differentiating the identity

$$\partial_a g(V, a, z)|_{a=0} = F_q(k_0 + \psi(V, 0, z) + z)$$

in z and using (3.4), (3.11) and the choice of $e \in K$, we obtain

$$\partial_a \Phi(V, 0, 0) = dF_q(k_0) \cdot (D_z\psi(V, 0, 0)e + e) = dF_q(k_0) \cdot e \neq 0. \tag{3.12}$$

After shrinking the neighborhood $\mathcal{V}_0 \times (-\varepsilon_0, \varepsilon_0) \times B_Y \times B_K$ of $(V, 0, 0)$, $\partial_a \Phi$ does not vanish on it. Hence, for $W \in \mathcal{V}_0$, the map $(a, z) \mapsto \Phi(W, a, z)$ is a submersion, and the set

$$\Sigma_W = \{(a, z) : \Phi(W, a, z) = 0\}$$

is a smooth submanifold of $\mathbb{R} \times K$.

Let $\pi : \Sigma_W \to \mathbb{R}$ be the projection $\pi(a, z) = a$. A point $(a, z) \in \Sigma_W$ is critical for π if and only if

$$D_z\Phi(W, a, z) = 0.$$

Indeed, the tangent space of Σ_W is the kernel of $\partial_a\Phi\, da + D_z\Phi$, and $\partial_a\Phi \neq 0$. Thus the critical parameters are exactly the critical values of the projection π. Using Sard's theorem on the projection π and by the previous equivalence, for almost every $a \in (-\varepsilon_0, \varepsilon_0)$, we have

$$D_z\Phi(W,a,z) \neq 0 \quad \text{whenever } z \in \overline{B}_K \text{ and } \Phi(W,a,z) = 0, \tag{3.13}$$

which is an open condition. Hence, the corresponding set $\mathcal{G}_{W,\varepsilon_0} \subset (-\varepsilon_0, \varepsilon_0)$ is open, dense, and of full Lebesgue measure.

For $a \in \mathcal{G}_{W,\varepsilon_0}$ and $k = k_0 + \psi(W,a,z) + z$ satisfying (3.8), we have $d_z g(W,a,z) = 0$. Hence $\Phi(W,a,z) = 0$. Put

$$H = d_z^2 g(W,a,z) \geq 0.$$

Since $\Phi = d_z g \cdot e$,

$$D_z\Phi(W,a,z)v = H(v,e), \qquad v \in K. \tag{3.14}$$

By (3.13), $D_z\Phi(W,a,z) \neq 0$. Hence, we have $H(\,\cdot\,,e) \neq 0$ and H has rank at least one. Since $\dim K = r$, we have

$$\dim\ker H \leq r - 1.$$

To compare H with the full Hessian, write the latter acting on $Y \oplus K$ as

$$\begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}, \qquad A > 0.$$

The Hessian of g is the Schur complement

$$H = C - B^{\mathsf{T}} A^{-1} B.$$

The factorization

$$\begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^{\mathsf{T}} A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

shows that their kernels have the same dimension. This proves (3.7).

For a fixed regular value a, compactness of the z-ball gives

$$|\Phi(W,a,z)| + |D_z\Phi(W,a,z)| \geq c > 0 \tag{3.15}$$

for all $z \in \overline{B}_K$. The inequality (3.15), the positivity of $\partial_y^2 f$, and the rank-one property of (2.1) persist under a sufficiently small perturbation of the potential. This proves the last assertion. □

4. Second-order perturbations

In this section, we follow the notation of Section 3 and focus on the case

$$\rho_\eta = 0 \qquad \text{for every } \eta \in K. \tag{4.1}$$

For $q \in \mathcal{X}$, define

$$f_t(y,z) = E(V + tq, k_0 + y + z).$$

As in (3.9), let $\psi(t,z)$ be the function such that

$$\partial_y f_t(\psi(t,z), z) = 0, \tag{4.2}$$

and, for simplicity, we write

$$g_t(z) = f_t(\psi(t,z), z). \tag{4.3}$$

If $Y = \{0\}$, all terms involving Y are omitted. We introduce

$$\kappa(q) = -H_Y^{-1}\Big(dF_q(k_0)|_Y\Big), \qquad \mathcal{R}(q) = d^2F_q(k_0)|_{K\times K} + d^3E(k_0)(\kappa(q),\cdot,\cdot)|_{K\times K}. \tag{4.4}$$

The following lemma gives the infinitesimal change of the Hessian in the K direction.

Lemma 4.1. *For every $q \in \mathcal{X}$,*

$$\partial_t|_{t=0}\, d_z^2 g_t(0) = \mathcal{R}(q), \tag{4.5}$$

where for $\eta, \zeta \in K$,

$$\mathcal{R}(q)(\eta,\zeta) = \int_{\mathbb{R}^d/\Gamma} q(x)\Theta_{\eta,\zeta}(x)\, dx, \tag{4.6}$$

with

$$\Theta_{\eta,\zeta} = \rho_{\eta\zeta} + \rho_{\chi(\eta,\zeta)}, \quad \chi(\eta,\zeta) = -H_Y^{-1}C(\eta,\zeta), \quad C(\eta,\zeta)[y] = d^3E(k_0)(y,\eta,\zeta). \tag{4.7}$$

Proof. Differentiate (4.2) in t at $(t,z) = (0,0)$. (3.4) gives

$$\partial_t\psi(0,0) = \kappa(q). \tag{4.8}$$

As in (3.11),

$$D_z\psi(0,0) = 0.$$

Differentiating (4.2) twice in z and pairing with $\eta, \zeta \in K$ gives

$$D_z^2\psi(0,0)(\eta,\zeta) = \chi(\eta,\zeta). \tag{4.9}$$

Differentiating with respect to t gives

$$\begin{aligned} \partial_t g_t(z) &= \partial_t E_t(k_0 + \psi(t,z) + z) + dE_t(k_0 + \psi(t,z) + z) \cdot \partial_t\psi(t,z) \\ &= \partial_t E_t(k_0 + \psi(t,z) + z). \end{aligned} \tag{4.10}$$

where the second term in (4.10) vanishes by (4.2) and $\partial_t\psi(t,z) \in Y$. Evaluating at $t = 0$,

$$\partial_t|_{t=0}\, g_t(z) = F_q(k_0 + \psi(0,z) + z). \tag{4.11}$$

Taking two derivatives in z at zero and using (4.9) gives

$$\partial_t|_{t=0}\, d_z^2 g_t(0)(\eta,\zeta) = d^2F_q(k_0)(\eta,\zeta) + dF_q(k_0) \cdot \chi(\eta,\zeta).$$

By symmetry of H_Y^{-1}, the last term is $d^3E(k_0)(\kappa(q),\eta,\zeta)$. This proves (4.5). $\square$

Lemma 4.2. *Assume that $K \neq \{0\}$ and (4.1) holds. Then there exists $q \in \mathcal{X}$ such that*

$$\mathcal{R}(q) > 0 \qquad \textit{on } K. \tag{4.12}$$

Proof. As $\mathcal{R}(\mathcal{X}) \subset \mathrm{Sym}^2(K^*)$, if $\mathcal{R}(\mathcal{X})$ did not intersect the cone of positive definite forms

$$\mathcal{C} = \{A \in \mathrm{Sym}^2(K^*) : A > 0\},$$

finite-dimensional convex separation (cf. [23, Theorem 11.2]) would give a nonzero linear functional vanishing on $\mathcal{R}(\mathcal{X})$ and nonnegative on $\mathcal{C}$. By duality, this gives a nonzero positive semidefinite $M \in \mathrm{Sym}^2(K)$ such that

$$\langle M, \mathcal{R}(q)\rangle = 0 \qquad \text{for every } q \in \mathcal{X}. \tag{4.13}$$

By (4.6), the continuous periodic function $x \mapsto \langle M, \Theta(x)\rangle$ has zero integral against every $q \in \mathcal{X}$. Taking $q = \langle M, \Theta\rangle$ gives

$$\langle M, \Theta(x)\rangle = 0 \qquad \text{for every } x \in \mathbb{R}^d/\Gamma. \tag{4.14}$$

As $M \in \mathrm{Sym}^2(K)$, we may write

$$M = \sum_{j=1}^m \mu_j \eta_j \otimes \eta_j, \qquad \mu_j > 0, \quad 0 \neq \eta_j \in K.$$

Let $Z = \{u(k_0,\cdot) = 0\}$. If $Z = \varnothing$, then (4.1) and Lemma 2.2, with the condition on Z vacuous, yields $\eta_j = 0$, which is a contradiction. Thus $Z \neq \varnothing$.

For $x \in Z$, since $u(k_0,x) = 0$, the terms containing a factor u vanish and

$$\rho_{\eta_j\eta_j}(x) = 2|\eta_j \cdot \partial_k u(k_0,x)|^2, \qquad \rho_{\chi(\eta_j,\eta_j)}(x) = 0.$$

Consequently,
$$\Theta_{\eta_j,\eta_j}(x) = 2|\eta_j \cdot \partial_k u(k_0,x)|^2, \qquad x \in Z. \tag{4.15}$$
Substituting (4.15) into (4.14) yields
$$\eta_j \cdot \partial_k u(k_0,x) = 0 \qquad \text{for } x \in Z$$
for every j. Together with (4.1) and $\eta_j \in \ker d_k^2 E(k_0)$, Lemma 2.2 implies $\eta_j = 0$. This gives a contradiction. □

With Lemma 4.2, we can state the main second-order perturbation proposition.

Proposition 4.3. *Assume that $K \neq \{0\}$, and let $q \in \mathcal{X}$ satisfy*
$$dF_q(k_0)|_K = 0, \qquad \mathcal{R}(q) > 0 \quad \text{on } K. \tag{4.16}$$
There is a neighborhood U_0 of k_0 and $t_0 > 0$ such that for every $0 < t < t_0$ the function
$$k \longmapsto E(V + tq, k)$$
attains its minimum over $\overline{U_0}$ at exactly one point $k_t \in U_0$ with
$$d_k^2 E(V + tq, k_t) > 0. \tag{4.17}$$
For each $t \in (0, t_0)$, there exists $\delta_t > 0$ such that, if $W \in \mathcal{V}$ and
$$\|W - (V + tq)\|_{L^\infty} < \delta_t,$$
then $k \mapsto E(W,k)$ attains its minimum over $\overline{U_0}$ at exactly one point $k_W \in U_0$, and $d_k^2 E(W, k_W) > 0$.

Proof. Let
$$G_q(z) = \partial_t|_{t=0}\, g_t(z). \tag{4.18}$$
The first condition in (4.16), (4.11) and (4.5) give
$$dG_q(0) = 0, \qquad d^2 G_q(0) = \mathcal{R}(q) > 0. \tag{4.19}$$
Choose a closed ball $\overline{B} \subset K$ centered at zero such that
$$g_0(z) \geq g_0(0), \qquad G_q(z) - G_q(0) \geq c|z|^2, \qquad z \in \overline{B}, \tag{4.20}$$
for some $c > 0$. We may shrink the Y-neighborhood so that $\partial_y^2 f_t$ is uniformly positive definite. Then, for each $z \in \overline{B}$ and small t, the point $y = \psi(t,z)$ is the unique minimum of $y \mapsto f_t(y,z)$ in that neighborhood. After shrinking again, minima of $E(V + tq, \cdot)$ over a fixed product neighborhood U_0 correspond exactly to minima of g_t over $\overline{B}$.

Joint analyticity gives
$$g_t(z) = g_0(z) + tG_q(z) + t^2 S(t,z) \tag{4.21}$$
uniformly in $C^2(\overline{B})$. By (4.20), the value of g_t on ∂B is larger than $g_t(0)$ for all sufficiently small $t > 0$. Thus every minimizer lies in B. Let z_t be a minimizer. Since S is uniformly Lipschitz in z,
$$|S(t,z) - S(t,0)| \leq C_S |z|.$$
Comparing $g_t(z_t)$ with $g_t(0)$ in (4.21) gives
$$0 \geq g_t(z_t) - g_t(0) \geq ct|z_t|^2 - C_S t^2 |z_t|.$$
Therefore
$$|z_t| \leq C_1 t \tag{4.22}$$
for a constant C_1 independent of t. Since $K = \ker d_k^2 E(k_0)$, we have $d_z^2 g_0(0) = 0$. For each $v \in K$, the function $s \mapsto g_0(sv)$ has a minimum at zero and has vanishing second derivative there. Its third derivative at zero is therefore zero. Polarization gives
$$d_z^3 g_0(0) = 0,$$

and hence

$$d_z^2 g_0(z) = O(|z|^2). \tag{4.23}$$

Using (4.19), (4.21), and (4.23), we obtain uniformly for $|z| \le C_1 t$,

$$d_z^2 g_t(z) = t\mathcal{R}(q) + O(t^2) > 0 \tag{4.24}$$

for t sufficiently small. The ball $\{|z| \le C_1 t\}$ is convex and contains every minimizer by (4.22); thus (4.24) implies uniqueness of z_t. The Schur complement argument similar to Proposition 3.1 gives (4.17).

For fixed $t > 0$, the rank-one spectral projection (2.1), the strict boundary inequality $g_t|_{\partial B} > g_t(0)$, uniqueness of the minimum, and the positive lower bound in (4.17) persist under a sufficiently small perturbation of the potential. This proves the last assertion. □

5. Generic nondegeneracy for a single band

For $n \ge 1$, define

$$\mathcal{S}_n^{\min} = \Big\{V \in \mathcal{X} : E_n(V,k) \text{ is a simple eigenvalue of } H_V(k) \text{ for every } k \in \mathcal{M}_n(V)\Big\}. \tag{5.1}$$

Lemma 5.1. *The set $\mathcal{S}_n^{\min}$ is open in $\mathcal{X}$.*

Proof. Fix $V \in \mathcal{S}_n^{\min}$ and define

$$m_n(V) = \min_{k \in \mathbb{T}_*^d} E_n(V,k).$$

For each $k \in \mathcal{M}_n(V)$, choose a coordinate neighborhood U_k and a circle γ_k as in (2.1). Compactness of $\mathcal{M}_n(V)$ gives a finite subcover $U_{k_1}, \dots, U_{k_N}$. Their union U contains $\mathcal{M}_n(V)$, and there exists $\delta > 0$ such that

$$E_n(V,k) \ge m_n(V) + 3\delta, \qquad k \in \mathbb{T}_*^d \setminus U. \tag{5.2}$$

The min–max principle gives

$$\sup_{k \in \mathbb{T}_*^d} |E_n(W,k) - E_n(V,k)| \le \|W - V\|_{L^\infty}. \tag{5.3}$$

If $\|W - V\|_{L^\infty} < \delta$, then (5.2)–(5.3) imply that every point of $\mathcal{M}_n(W)$ lies in U. The contours γ_{k_j} remain in the resolvent sets for W close to V, so the eigenvalue $E_n(W,k)$ is simple at every $k \in \mathcal{M}_n(W)$ by analytic perturbation theory. □

For $V \in \mathcal{S}_n^{\min}$ define

$$R_n(V) := \max_{k \in \mathcal{M}_n(V)} \dim\ker d_k^2 E_n(V,k). \tag{5.4}$$

We use the following notation at a pair (W,k) with $W \in \mathcal{S}_n^{\min}$ and $k \in \mathcal{M}_n(W)$. Let $E_W(\kappa)$ and $u_W(\kappa)$ be defined as in (2.2) and (2.3) near $\kappa = k$. Write

$$K(W,k) = \ker d_\kappa^2 E_W(k), \qquad \rho_{W,k,\eta} = (\eta \cdot \partial_\kappa)|u_W(k,\cdot)|^2, \quad \eta \in K(W,k). \tag{5.5}$$

For $q \in \mathcal{X}$, let $F_{W,k,q}$ be the function in (3.4) formed with u_W, and let

$$\mathcal{R}_{W,k}(q) \in \mathrm{Sym}^2(K(W,k)^*) \tag{5.6}$$

be the form defined by (4.4). These quantities do not depend on the phase of u_W.

Lemma 5.2. *Let $V \in \mathcal{S}_n^{\min}$, and let $r = R_n(V) > 0$ as in (5.4), and let $k_0 \in \mathcal{M}_n(V)$ satisfy*

$$\dim\ker d_k^2 E_n(V,k_0) = r.$$

There exist a compact coordinate neighborhood Q of k_0, with $k_0 \in \operatorname{int} Q$, and an open neighborhood $\mathcal{V}_0 \subset \mathcal{S}_n^{\min}$ of V such that the set

$$\mathcal{G}_Q = \Big\{W \in \mathcal{V}_0 : \dim\ker d_k^2 E_n(W,k) \le r - 1 \textit{ for every } k \in \mathcal{M}_n(W) \cap Q\Big\} \tag{5.7}$$

is open and dense in $\mathcal{V}_0$.

Proof. Write $K_0 = K(V, k_0)$ defined in (5.5). Shrink a coordinate neighborhood U of k_0 and a neighborhood $\mathcal{V}_0$ of V so that $\mathcal{V}_0 \subset \mathcal{S}_n^{\min}$ and

(a) for $(W, k) \in \mathcal{V}_0 \times U$, the ordered eigenvalue $E_n(W, k)$ is the unique eigenvalue inside one fixed contour γ;
(b) if $k \in \mathcal{M}_n(W) \cap U$, then

$$\dim K(W, k) \leq r = R_n(V).$$

Note that the second property follows from upper semicontinuity of the Hessian corank. We first prove the density of $\mathcal{G}_Q$ in $\mathcal{V}_0$.
Case 1. If $\rho_{V,k_0,\eta} \not\equiv 0$ for some $\eta \in K_0$, we choose $q = \rho_{V,k_0,\eta}$. Then

$$dF_{V,q}(k_0) \cdot \eta > 0.$$

After shrinking U and $\mathcal{V}_0$, Proposition 3.1 applies to every potential $W \in \mathcal{V}_0$, with the same direction q and the same coordinate splitting at (V, k_0). Let $Q \Subset U$ contain k_0 in its interior. For every $W \in \mathcal{V}_0$ and every neighborhood $\mathcal{W}$ of W, Proposition 3.1 gives a arbitrarily close to zero such that $W + aq \in \mathcal{W} \cap \mathcal{G}_Q \cap \mathcal{V}_0$. Thus $\mathcal{G}_Q$ is dense.
Case 2. If

$$\rho_{V,k_0,\eta} = 0 \qquad \text{for every } \eta \in K_0. \tag{5.8}$$

By Lemma 4.2, we may choose $q \in \mathcal{X}$ such that

$$\mathcal{R}_{V,k_0}(q) > 0 \qquad \text{on } K_0. \tag{5.9}$$

We may shrink U and $\mathcal{V}_0$ so that

$$\left.\begin{array}{c} W \in \mathcal{V}_0, \quad k \in \mathcal{M}_n(W) \cap U, \\ \dim K(W,k) = r, \quad dF_{W,q}(k)|_{K(W,k)} = 0 \end{array}\right\} \Longrightarrow \mathcal{R}_{W,k}(q) > 0 \text{ on } K(W,k). \tag{5.10}$$

To see (5.10), choose $c > 0$ smaller than every positive eigenvalue of $d_k^2 E_n(V, k_0)$. For (W, k) close to (V, k_0), the spectral projection of the Hessian onto $[0, c)$ is continuous and has rank r whenever the Hessian corank is r; its range is then $K(W, k)$. The energy $E(W, k)$ in (2.2), its first three k-derivatives, and the first two k-derivatives of $F_{W,q}$ depend continuously on (W, k). By equation (4.6) and under the identification by these spectral projections, $\mathcal{R}_{W,k}(q)$ converges to $\mathcal{R}_{V,k_0}(q)$. Hence, (5.9) implies (5.10).

Choose $Q \Subset U$ with $k_0 \in \operatorname{int} Q$. Fix $W \in \mathcal{V}_0$ and set

$$B_W = \{k \in \mathcal{M}_n(W) \cap Q : \dim K(W, k) = r\}. \tag{5.11}$$

The set B_W is compact. If $B_W = \varnothing$, then $W \in \mathcal{G}_Q$. Suppose that $B_W \neq \varnothing$.

For each $k \in B_W$, take the fixed direction q and the base point k. If

$$dF_{W,q}(k)|_{K(W,k)} \neq 0, \tag{5.12}$$

then Proposition 3.1 gives a neighborhood U_k of k and an open dense subset $G_k \subset (0, \varepsilon_k)$ such that, for $s \in G_k$, every point $\kappa \in U_k$ satisfying

$$d_\kappa E_n(W + sq, \kappa) = 0, \qquad d_\kappa^2 E_n(W + sq, \kappa) \geq 0$$

has Hessian corank at most $r - 1$. If (5.12) fails, then (5.10) and Proposition 4.3 give a neighborhood U_k and $\varepsilon_k > 0$ such that every minimum in U_k has Hessian corank at most $r - 1$ for every $s \in (0, \varepsilon_k)$.

Choose $k_1, \ldots, k_m \in B_W$ such that

$$B_W \subset \bigcup_{j=1}^{m} U_{k_j}. \tag{5.13}$$

After replacing all ε_{k_j} by their minimum, the set

$$G = \bigcap_{j=1}^{m} G_{k_j} \tag{5.14}$$

is open and dense in an interval $(0,\varepsilon)$; in the second-order case we take $G_{k_j} = (0,\varepsilon)$, where we may shrink ε so that $W + sq \in \mathcal{V}_0$ for every $s \in (0,\varepsilon)$.

For all sufficiently small $s > 0$, every point of

$$\{\kappa \in \mathcal{M}_n(W+sq) \cap Q : \dim K(W+sq,\kappa) = r\}$$

belongs to the union in (5.13). Otherwise there exist $s_j \to 0$ and $\kappa_j \in Q \setminus \bigcup_i U_{k_i}$ such that $\kappa_j \in \mathcal{M}_n(W+s_jq)$ and $\dim K(W+s_jq,\kappa_j) = r$. Passing to a subsequence gives $\kappa_j \to \kappa \in Q$. The estimate (5.3) gives $\kappa \in \mathcal{M}_n(W)$. The local eigenvalues converge in C^2, so upper semicontinuity gives $\dim K(W,\kappa) \geq r$. Property (b) gives equality; hence $\kappa \in B_W$, contradicting (5.13).

Choose $s \in G$ arbitrarily small. Then $W + sq \in \mathcal{G}_Q$, which proves the density in Case 2.

It remains to show that $\mathcal{G}_Q$ is open. Suppose $W_j \to W_*$ in $\mathcal{V}_0$, $W_* \in \mathcal{G}_Q$, and $W_j \notin \mathcal{G}_Q$. Choose

$$k_j \in \mathcal{M}_n(W_j) \cap Q, \qquad \dim K(W_j,k_j) \geq r.$$

After passing to a subsequence, $k_j \to k_* \in Q$. The estimate (5.3) gives $k_* \in \mathcal{M}_n(W_*)$, and C^2 convergence of the local eigenvalues gives

$$\dim K(W_*,k_*) \geq r,$$

contradicting $W_* \in \mathcal{G}_Q$. Thus $\mathcal{G}_Q$ is open. $\square$

Proposition 5.3. *For every $n \geq 1$, the set*

$$\mathcal{N}_n^{\min} = \Big\{V \in \mathcal{S}_n^{\min} : d_k^2 E_n(V,k) > 0 \text{ for every } k \in \mathcal{M}_n(V)\Big\} \tag{5.15}$$

is open and dense in $\mathcal{S}_n^{\min}$.

Proof. We first show openness. Fix $V \in \mathcal{N}_n^{\min}$. $\mathcal{M}_n(V)$ is finite and each point of $\mathcal{M}_n(V)$ is an isolated minimum. Choose pairwise disjoint coordinate neighborhoods $U_1,\dots,U_N$ of its points such that the local eigenvalues have Hessian bounded below by a positive constant on each U_j. The openness follows from the continuity of the Hessian and the eigenvalues $E_n(W,\cdot)$ with respect to the perturbations in W.

We now prove density. Let $\mathcal{U} \subset \mathcal{S}_n^{\min}$ be nonempty and open. Choose $V_0 \in \mathcal{U}$ and put $r_0 = R_n(V_0)$. If $r_0 = 0$, then $V_0 \in \mathcal{N}_n^{\min}$. Assume $r_0 > 0$, and define

$$B_{r_0}(V_0) = \{k \in \mathcal{M}_n(V_0) : \dim\ker d_k^2 E_n(V_0,k) = r_0\}. \tag{5.16}$$

This set is compact. Apply Lemma 5.2 at each point of $B_{r_0}(V_0)$. Choose finitely many compact neighborhoods $Q_1,\dots,Q_N$ whose interiors cover $B_{r_0}(V_0)$, and choose an open neighborhood $\mathcal{V}_0 \subset \mathcal{U}$ of V_0 on which Lemma 5.2 holds.

After shrinking $\mathcal{V}_0$, every $W \in \mathcal{V}_0$ and every

$$k \in \mathcal{M}_n(W) \setminus \bigcup_{j=1}^{N} \operatorname{int} Q_j$$

satisfy

$$\dim\ker d_k^2 E_n(W,k) \leq r_0 - 1. \tag{5.17}$$

Indeed, otherwise there exist $W_j \to V_0$ and k_j outside the union with Hessian corank at least r_0. A subsequence converges to $k \in \mathcal{M}_n(V_0)$ with corank at least r_0, so $k \in B_{r_0}(V_0)$, contradicting the choice of the interiors.

For each j, Lemma 5.2 gives an open dense subset of $\mathcal{V}_0$ on which every global minimum in Q_j has Hessian corank at most $r_0 - 1$, whose finite intersection is again open and dense. Together with (5.17), it contains a nonempty open set $\mathcal{U}_1 \subset \mathcal{U}$ such that

$$R_n(W) \leq r_0 - 1, \qquad W \in \mathcal{U}_1. \tag{5.18}$$

Choose $V_1 \in \mathcal{U}_1$. If $R_n(V_1) = 0$, then $V_1 \in \mathcal{N}_n^{\min}$. Otherwise, repeat the construction inside $\mathcal{U}_1$. At each induction, R_n decreases by at least one. The induction ends after at most d steps, yielding a potential in $\mathcal{U} \cap \mathcal{N}_n^{\min}$. This proves density. □

6. Proof of Theorem 1.1

We take intervals $I = [r, s]$ with rational endpoints $r, s \in \mathbb{Q}$. Define $\mathcal{V}_I \subset \mathcal{X}$ to be the set of V for which I is contained in a bounded connected component of $\mathbb{R} \setminus \operatorname{Spec} H_V$. For $V \in \mathcal{V}_I$, denote this component by $(a_I(V), b_I(V))$. Define $\mathcal{D}_I \subset \mathcal{V}_I$ by the following condition: $V \in \mathcal{D}_I$ if there exist unique indices $n_-(V), n_+(V)$ such that

$$a_I(V) \in E_{n_-(V)}(V, \mathbb{T}^d_*), \qquad b_I(V) \in E_{n_+(V)}(V, \mathbb{T}^d_*). \tag{6.1}$$

Note that the condition (6.1) implies that each endpoint is a simple eigenvalue of $H_V(k)$ at every attaining quasimomentum k.

Lemma 6.1. *The set $\mathcal{V}_I$ is open in $\mathcal{X}$, and $\mathcal{D}_I$ is open and dense relative to $\mathcal{V}_I$.*

Proof. The min–max principle gives

$$\operatorname{dist}_H(\operatorname{Spec} H_V, \operatorname{Spec} H_W) \leq \|V - W\|_{L^\infty}. \tag{6.2}$$

Thus $\mathcal{V}_I$ is open. Applying [16, Theorem 1.1] at the two endpoints yields the density of $\mathcal{D}_I$ in $\mathcal{V}_I$. The relative openness of $\mathcal{D}_I$ in $\mathcal{V}_I$ again follows from (6.2) and the existence of a uniform band gap near band extrema. □

Lemma 6.2. *Let (a, b) be a connected component of $\mathbb{R} \setminus \operatorname{Spec} H_V$.*

(1) If $E_n(V, k_0) = a$ for some k_0, then $a = \max_{k \in \mathbb{T}^d_*} E_n(V, k)$.
(2) If $E_m(V, k_0) = b$ for some k_0, then $b = \min_{k \in \mathbb{T}^d_*} E_m(V, k)$.

We omit the proof as it is trivial. For $V \in \mathcal{V}_I$, let $\mathcal{N}_I$ be the set of potentials such that (1) $V \in \mathcal{D}_I$; (2) for every k attaining $a_I(V)$, $d_k^2 E_{n_-(V)}(V, k) < 0$; (3) for every k attaining $b_I(V)$, $d_k^2 E_{n_+(V)}(V, k) > 0$.

Proposition 6.3. *The set $\mathcal{N}_I$ is open and dense relative to $\mathcal{V}_I$.*

Proof. Fix $V \in \mathcal{N}_I$. Lemma 6.2 identifies the lower endpoint as a global maximum of $E_{n_-(V)}$ and the upper endpoint as a global minimum of $E_{n_+(V)}$. The openness of (5.15) together with Lemma 6.1 shows that $\mathcal{N}_I$ is relatively open.

Now we show density. Let $\mathcal{U} \subset \mathcal{V}_I$ be nonempty and relatively open. By Lemma 6.1, choose $V_0 \in \mathcal{U} \cap \mathcal{D}_I$. After shrinking, there exists a nonempty open set

$$\mathcal{U}_0 \subset \mathcal{U} \cap \mathcal{D}_I \tag{6.3}$$

and fixed indices n_-, n_+ such that (6.1) holds with these indices for every $V \in \mathcal{U}_0$. Lemma 6.2 and the definition (5.1) give

$$\mathcal{U}_0 \subset \mathcal{S}_{n_-}^{\max} \cap \mathcal{S}_{n_+}^{\min},$$

where $\mathcal{S}_{n_-}^{\max}$ is defined analogously for band maxima. By the density of $\mathcal{N}_{n_+}^{\min}$, there exists a nonempty open set

$$\mathcal{U}_1 \subset \mathcal{U}_0 \cap \mathcal{N}_{n_+}^{\min}. \tag{6.4}$$

By the density of $\mathcal{N}_{n_-}^{\max}$, there exists a nonempty open set

$$\mathcal{U}_2 \subset \mathcal{U}_1 \cap \mathcal{N}_{n_-}^{\max}, \tag{6.5}$$

where $\mathcal{N}_{n_-}^{\max}$ is defined analogously for band maxima. Thus, by (6.3), (6.4) and (6.5), we have a nonempty open set $\mathcal{U}_2 \subset \mathcal{N}_I$. This proves the density of $\mathcal{N}_I$ in $\mathcal{V}_I$. □

Proof of Theorem 1.1. Define

$$\mathcal{O}_I = \mathcal{N}_I \cup \left(\mathcal{X} \setminus \overline{\mathcal{V}_I}\right). \tag{6.6}$$

The set $\mathcal{O}_I$ is open. It is dense as follows. Let $\mathcal{W} \subset \mathcal{X}$ be nonempty and open. If $\mathcal{W}$ meets $\mathcal{X} \setminus \overline{\mathcal{V}_I}$, then it meets $\mathcal{O}_I$. Otherwise $\mathcal{W} \subset \overline{\mathcal{V}_I}$. Since $\mathcal{V}_I$ is dense in its closure, $\mathcal{W} \cap \mathcal{V}_I$ is nonempty; Proposition 6.3 then gives $\mathcal{W} \cap \mathcal{N}_I \neq \varnothing$.

Set

$$\mathcal{G} = \bigcap_{\substack{r,s\in\mathbb{Q}\\ r<s}} \mathcal{O}_{[r,s]}. \tag{6.7}$$

By definition, $\mathcal{G}$ is residual. Take $V \in \mathcal{G}$. For any bounded spectral gap (a, b) of $\operatorname{Spec} H_V$. Choose $r, s \in \mathbb{Q}$ with

$$a < r < s < b.$$

Then $V \in \mathcal{V}_{[r,s]}$, and (6.6)–(6.7) imply $V \in \mathcal{N}_{[r,s]}$. This proves Parts (i) and (iii) of Theorem 1.1. A nondegenerate critical point is isolated, and the set of attaining points is closed. Therefore, there are only finitely many of them as $\mathbb{T}^d_*$ is compact. This proves part (ii) of Theorem 1.1. □

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