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FLAT TRACE ESTIMATES FOR ANOSOV FLOWS

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ABSTRACT. We prove a high energy flat trace estimate for the modified resolvent of the generator of an Anosov flow. This fills a gap in the proof of the local trace formula in [JiZw17] and is a by-product of the authors' ongoing project of its generalization to Axiom A flows.

1. INTRODUCTION

This note is a by-product of the authors' ongoing project on the local trace formula for Axiom A flows, which leads to the discovery of some issues in [JiZw17]. Since the situation for Anosov flows is simpler than the one for Axiom A flows, we give here a separate presentation to fix the issues in [JiZw17].

Let X be a smooth compact manifold, $\varphi_t : X \to X$ be an Anosov flow generated by a smooth vector field V, and $P = -iV$, Jin–Zworski [JiZw17] proved the following local trace formula relating the Pollicott–Ruelle resonances $\mathrm{Res}(P)$ to the lengths of closed geodesics.

Theorem 1. *For any $A > 0$ there exists a distribution $F_A \in \mathcal{S}'(\mathbb{R})$ supported in $[0, \infty)$ such that*

$$\sum_{\mu \in \mathrm{Res}(P), \mathrm{Im}(\mu) > -A} e^{-i\mu t} + F_A(t) = \sum_{\gamma} \frac{T_\gamma^\# \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}, \quad t > 0$$

in $\mathcal{D}'((0, \infty))$, where the sum on the right hand side is taken over all closed geodesics, $\mathcal{P}_\gamma$ is the Poincaré map, and

$$|\widehat{F}_A(\lambda)| = \mathcal{O}_{A,\varepsilon}(\langle \lambda \rangle^{2n+1+\varepsilon}), \quad \mathrm{Im}\, \lambda < A - \varepsilon \tag{1.1}$$

for any $\varepsilon > 0$.

The last estimate (1.1) has been modified comparing to [JiZw17, (1.5)]. The additional loss of ε in the exponent in (1.1) comes from the following mistake in [JiZw17]: rescaling from [JiZw17, (4.20)] back to [JiZw17, (4.1)], we should gain an additional h from the derivative changing from $\frac{d}{dz}$ to $\frac{d}{d\lambda}$, but also have $|z| = h|\lambda| \sim h^{1/2}$ and thus the result should be $h^{-2n} \sim \lambda^{4n}$. However, we can go back to the setting of [DyZw16, Proposition 3.4] and replace $h^{1/2}$ by any h^ε with $\varepsilon \in (0, 1)$ arbitrarily small. This way we also replace the bound in [JiZw17, (4.19)] and [JiZw17, (4.20)] by $h^{-(2-\varepsilon)n-2}$ and thus we obtain the bound in (1.1). In section 3, we will give a simpler proof for a weaker high energy flat trace estimate, comparing to [JiZw17, Proposition 3.1], see Theorem 2. From this, the bound in [JiZw17, (4.20)] becomes h^{-2n-2}, but still gives the same bound in (1.1). The advantage is that we can avoid the complicated construction for complex absorbing potential Q as in [JiZw17, §2.5].

In [JiZw17], the proof for the high energy flat trace estimate [JiZw17, Proposition 3.1] was incomplete as it relied on the following flawed statement ([JiZw17, (2.14)]) about the semiclassical wavefront set for the resolvent $R_h(z) = (hP - z)^{-1}$:

$$\mathrm{WF}'_h(R_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}),$$

which was used to deduce the same statement [JiZw17, (2.19)] for the modified resolvent $\widetilde{R}_h(z) = (hP - iQ - z)^{-1}$. However, $R_h(z)$ has poles which are exactly the Pollicott–Ruelle resonances. Even in the set where it is well-defined, it is not clear that the kernel is h-tempered uniformly in z, and thus $\mathrm{WF}'_h(R_h(z))$ may not be defined. To remedy this issue, we analyze the modified resolvent $\widetilde{R}_h(z)$ directly to give the statement [JiZw17, (2.19)], which is the correct statement eventually used in the proof of Theorem 1 in [JiZw17]. This will be done in Proposition 2.1 in Section 2.

For more details on the notations we refer to [JiZw17]. For preliminaries on semiclassical analysis we refer to Zworski [Zw12] and Dyatlov–Zworski [DyZw19, Appendix E]. For other recent developments concerning trace formulas for Pollicott-Ruelle resonances, see [Je20], [Je21].

2. Wavefront set estimates

In this section, we fix the issue in [JiZw17] by proving the following semiclassical wavefront set estimate for the modified resolvent $\widetilde{R}_h(z)$. We briefly recall the notations from [JiZw17]: Let Q be the absorbing potential as in [JiZw17], to be more precise, we require

- $\mathrm{WF}_h(Q) \subset \{|\xi| < 1\}$;
- $\sigma_h(Q) > 0$ on $\{|\xi| \le 1/2\}$;
- and $\sigma_h(Q) \ge 0$ everywhere.

The additional requirement in [JiZw17, §2.5] is used to improve the power in the flat trace estimate (3.1) and we will give a simpler argument in Secion 3 to avoid the complications. In [DyZw16, Proposition 3.4], it is shown that for fixed $C_1, C_2, \varepsilon > 0$, $\widetilde{P}_h(z) = hP - iQ - z$ is invertible for $z \in [-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]$ and its inverse satisfied the following estimate

$$\|\widetilde{R}_h(z)\|_{\mathcal{H}_h^s \to \mathcal{H}_h^s} \le Ch^{-1}.$$

Here $\mathcal{H}_h^s = H_{sG(h)}$ is the semiclassical anisotropic Sobolev space defined in [DyZw16, §3.3] and $s > 0$ is a parameter chosen large enough depending on C_1 and C_2. The weight function $G(h)$ is constructed in a way that $\widetilde{P}_h(z) : D_h^s := \{u \in \mathcal{H}_h^s : \widetilde{P}_h(z)u \in \mathcal{H}_h^s\} \to \mathcal{H}_h^s$ is invertible. In the following we will only use the fact that

$$H_h^s \subset \mathcal{H}_h^s \subset H_h^{-s},$$

where H_h^s is the usual semiclassical Sobolev spaces on X.

Proposition 2.1. *We have*

$$\mathrm{WF}_h'(\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}) \tag{2.1}$$

where Ω_+ is the flowout

$$\Omega_+ = \{(e^{tH_p}(y,\eta), y, \eta) : p(y,\eta) = 0\} \subset T^*(X \times X) \simeq T^*X \times T^*X,$$

and $\kappa : T^*(X \times X) \setminus \{0\} \to S^*(X \times X)$ is the natural projection map.

Remark 2.2. *Note that $S^*(X \times X) \neq S^*X \times S^*X$, hence there are difficulties to deal with the fiber infinity directly. In fact, unlike the finite part of the wavefront set $T^*(X \times X) \simeq T^*X \times T^*X$, there is no natural way to identify the element in $S^*X \times S^*X$ where $S^*X = \kappa(T^*X \setminus \{0\})$ with the element in $S^*(X \times X) = \kappa(T^*(X \times X) \setminus \{0\})$. However, we do have the natural identification of the diagonal elements $\Delta(S^*X) = \kappa(\Delta(T^*X) \setminus \{0\})$.*

The rest of this section will be devoted to the proof of Proposition 2.1. We will follow the strategy of [DyZw16, Proposition 3.4], where the authors prove the estimate for the finite part of $\mathrm{WF}_h'(\widetilde{R}_h(z))$. To deal with the wavefront set at fiber infinity we introduce another small parameter $\tilde{h} > 0$ (which will play the role of $|(\xi,\eta)|^{-1}$).

Step 1: Let $p^{-1}(0) = \{(x,\xi) \in T^*X : p(x,\xi) = 0\} \supset E_u^* \cup E_s^*$, we first show a weaker statement:

$$\mathrm{WF}_h'(\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times p^{-1}(0)) \setminus \{0\}). \tag{2.2}$$

Suppose $(x_0, \xi_0, y_0, \eta_0) \in \{|(\xi,\eta)| = 1\} \setminus (\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times p^{-1}(0)))$, then as in [DyZw16, Proposition 3.4] there exist $\rho > 0$ and neighbourhoods U of $(x_0, \rho\xi_0)$ and W of $(y_0, \rho\eta_0)$, and $A, B \in \Psi_h^0(X)$ such that

$$\begin{gathered}\|Au\|_{\mathcal{H}_h^s} \le Ch^{-1}\|B\widetilde{P}_h(z)u\|_{\mathcal{H}_h^s} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}, \\ U \subset \mathrm{ell}_h(A), \quad (\{|\xi| \le 1\} \cup W) \cap \mathrm{WF}_h(B) = \varnothing.\end{gathered} \tag{2.3}$$

Moreover, A is microlocally supported near $(x_0, \rho\xi_0)$ and B microlocally supported in a neighbourhood of $\{e^{-tH_p}(x_0, \rho\xi_0) : t \ge 0\}$. The condition that $(x_0, \xi_0, y_0, \eta_0) \notin E_u^* \times p^{-1}(0)$

guarantees that $\mathrm{WF}_h(B)\cap\{|\xi|\le 1\}=\varnothing$ for some large number $\rho>0$. We can also assume that

$$A=\mathrm{Op}_h(a),\quad B=\mathrm{Op}_h(b),\quad Q=\mathrm{Op}_h(q)$$

with symbols $b\in S^0$ and $a,q\in C_0^\infty$ independent of h, and $\operatorname{supp} q\subset\{|\xi|\le 1\}$ so that $\operatorname{supp} q\cap\operatorname{supp} b=\varnothing$. Here Op_h denotes a semiclassical quantization on a compact manifold, see [DyZw19, Appendix E].

Replacing h by $h\tilde{h}$ in the estimate (2.3), we get

$$A_{\tilde{h}}=\mathrm{Op}_{h\tilde{h}}(a),\quad B_{\tilde{h}}=\mathrm{Op}_{h\tilde{h}}(b),\quad Q_{\tilde{h}}=\mathrm{Op}_{h\tilde{h}}(q)\in\Psi^0_{h\tilde{h}}(X)$$

such that

$$\begin{aligned}\|A_{\tilde{h}}u\|_{\mathcal{H}^s_{h\tilde{h}}}\le & C(h\tilde{h})^{-1}\|B_{\tilde{h}}(h\tilde{h}P-\tilde{h}z-iQ_{\tilde{h}})u\|_{\mathcal{H}^s_{h\tilde{h}}}+\mathcal{O}((h\tilde{h})^\infty)\|u\|_{H^{-N}_{h\tilde{h}}},\\ & U\subset\mathrm{ell}_{h\tilde{h}}(A_{\tilde{h}}),\quad(\{|\xi|\le 1\}\cup W)\cap\mathrm{WF}_{h\tilde{h}}(B_{\tilde{h}})=\varnothing.\end{aligned}\tag{2.4}$$

Note $z\in[-C_1h^\varepsilon,C_1h^\varepsilon]+i[-C_2h,1]$ implies $\tilde{h}z\in[-C_1(\tilde{h}h)^\varepsilon,C_1(\tilde{h}h)^\varepsilon]+i[-C_2\tilde{h}h,1]$. However we wish to recover $\widetilde{P}_h$ in estimate (2.4), and this require us to replace $Q_{\tilde{h}}$ by $\tilde{h}Q$ and to deal with the Q term. We need the following lemma:

Lemma 2.3. *For every $N\in\mathbb{N}$,*

$$\|B_{\tilde{h}}Qu\|_{H^N_{h\tilde{h}}}=\mathcal{O}(h^\infty\tilde{h}^\infty)\|u\|_{H^{-N}_{h\tilde{h}}}.$$

Proof. Using a partition of unity argument we may assume that we are on $\mathbb{R}^n$ and all the symbols are compactly supported in $\mathbb{R}^n$. Recall (e.g. [Zw12, Theorem 4.23]) for a sufficiently large constant $M>0$,

$$\|\mathrm{Op}_h(a)\|_{L^2\to L^2}\lesssim\|a\|_{S^{0,M}},\quad\|a\|_{S^{k,M}}:=\sum_{|\alpha|+|\beta|\le M}\left\|\langle\xi\rangle^{|\alpha|-k}\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\right\|_{L^\infty}.$$

Since $\{\xi=0\}\cap\operatorname{supp} b=\varnothing$, for $m\gg 1$,

$$\begin{aligned}\|B_{\tilde{h}}Q\|_{H^{-N}_{h\tilde{h}}\to H^N_{h\tilde{h}}}&=\|\langle h\tilde{h}D\rangle^N B_{\tilde{h}}Q\langle h\tilde{h}D\rangle^N\|_{L^2\to L^2}\\&\lesssim\|\langle\tilde{h}\xi\rangle^N\#b(x,\tilde{h}\xi)\#q(x,\xi)\#\langle\tilde{h}\xi\rangle^N\|_{S^{0,M}}\\&\lesssim h^m\|\langle\tilde{h}\xi\rangle^N\|^2_{S^{N,M'}}\|b(x,\tilde{h}\xi)\|_{S^{m,M'}}\|q(x,\xi)\|_{S^{-m-2N,M'}}\\&\lesssim\mathcal{O}(h^m\tilde{h}^m).\end{aligned}$$

Since m can be chosen arbitrarily large, this concludes the proof. $\square$

Now we go back to (2.4) and taking $u(x) = \widetilde{R}_h(z)(\psi(x)e^{ix\cdot\rho\eta_0/h\tilde{h}})$ (here we choose a local coordinates and identify a neighborhood of x_0 to subset of $\mathbb{R}^n$) where $\operatorname{supp}\psi \times \{\rho\eta_0\} \subset W$, the wavefront set condition (2.3) for B gives

$$\|B_{\tilde{h}}Q_{\tilde{h}}\|_{H^{-N}_{h\tilde{h}} \to H^N_{h\tilde{h}}} = \mathcal{O}(h^\infty \tilde{h}^\infty), \quad \|B_{\tilde{h}}(\psi(x)e^{ix\cdot\rho\eta_0/h\tilde{h}})\|_{H^N_{h\tilde{h}}} = \mathcal{O}(h^\infty \tilde{h}^\infty).$$

Therefore we have

$$\begin{aligned}\|A_{\tilde{h}}u\|_{\mathcal{H}^r_{h\tilde{h}}} &\le Ch^{-1}\|B_{\tilde{h}}\widetilde{P}_h u\|_{\mathcal{H}^r_{h\tilde{h}}} + C(h\tilde{h})^{-1}\|B_{\tilde{h}}Q_{\tilde{h}}u\|_{\mathcal{H}^r_{h\tilde{h}}} \\ &\qquad + Ch^{-1}\|B_{\tilde{h}}Qu\|_{\mathcal{H}^r_{h\tilde{h}}} + \mathcal{O}((h\tilde{h})^\infty)\|u\|_{H^{-N}_{h\tilde{h}}} \\ &= \mathcal{O}(h^{-1})\|B_{\tilde{h}}(\psi(x)e^{ix\cdot r\eta_0/h\tilde{h}})\|_{\mathcal{H}^r_{h\tilde{h}}} + \mathcal{O}(h^\infty\tilde{h}^\infty)\|u\|_{H^{-N}_{h\tilde{h}}} \\ &= \mathcal{O}(h^\infty\tilde{h}^\infty).\end{aligned}$$

This means $\mathrm{WF}_{h\tilde{h}}(u) \cap U = \varnothing$, and thus if $\chi \in C^\infty(X)$ and $\operatorname{supp}\chi \times \{\rho\xi_0\} \subset U$, then

$$\int \chi(x)e^{-ix\cdot\rho\xi_0/h\tilde{h}}\widetilde{R}_h(z)(\psi(x)e^{ix\cdot\rho\eta_0/h\tilde{h}})dx = \mathcal{O}(h^\infty\tilde{h}^\infty)$$

Moreover, by construction it is easy to see the estimate is locally uniform in $(x_0, \xi_0, y_0, \eta_0)$. Therefore by the equivalent definition of semiclassical wavefront sets using the semiclassical Fourier transform (see [Al08, Definition 3.2]), $\kappa(x_0, \xi_0, y_0, \eta_0) = \kappa(x_0, \rho\xi_0, y_0, \rho\eta_0) \notin \mathrm{WF}'_h(\widetilde{R}_h(z)) \cap S^*(X \times X)$ and we have (2.2).

Step 2: The previous method does not work for $(x_0, \xi_0, y_0, \eta_0) \in E_u^* \times p^{-1}(0)$ since $\mathrm{WF}_h(B)$ has to intersect the zero section $\{\xi = 0\}$. Here we argue by duality. Suppose $(x_0, \xi_0, y_0, \eta_0) \in \{|(\xi, \eta)| = 1\} \setminus (\Delta \cup \Omega_+ \cup (p^{-1}(0) \times E_s^*))$, we consider the following operator

$$-\widetilde{P}_h(z)^* := -hP - iQ - (-\bar{z}),$$

acting on $\mathcal{H}_h^{-s}$. We see that this corresponds to the reversed Anosov flow φ_{-t} generated by $-V$ and $z \in [-C_h h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]$ also gives $-\bar{z}$ in the same region. We can repeat the same argument with the opposite propagation direction we get $\widetilde{P}_h(z)^*$ is invertible, with inverse $\widetilde{R}_h(z)^* : \mathcal{H}_h^{-s} \to \mathcal{H}_h^{-s}$ satisfying

$$\|\widetilde{R}_h(z)^*\|_{\mathcal{H}_h^{-s} \to \mathcal{H}_h^{-s}} \le Ch^{-1}.$$

Moreover, there exist $\rho > 0$, $U = \operatorname{nbd}(x_0, \rho\xi_0)$ and $W = \operatorname{nbd}(y_0, \rho\eta_0)$ such that for $\operatorname{supp}\psi \times \{\rho\eta_0\} \subset W$ and $\operatorname{supp}\chi \times \{\rho\xi_0\} \subset U$ we have

$$\int \psi(x)e^{-ix\cdot\rho\eta_0/h\tilde{h}}\widetilde{R}_h(z)^*(\chi(x)e^{ix\cdot\rho\xi_0/h\tilde{h}})dx = \mathcal{O}(h^\infty\tilde{h}^\infty),$$

and the estimate is locally uniform in $(x_0, \xi_0, y_0, \eta_0)$. Therefore $\kappa(y_0, \eta_0, x_0, \xi_0) \notin \mathrm{WF}'_h(\widetilde{R}_h(z)^*) \cap S^*(X \times X)$. Since the Schwartz kernel of $\widetilde{R}_h(z)^*$ is $\overline{K(y,x)}$ if the $K(x,y)$ is the Schwartz kernel of $\widetilde{R}_h(z)$, we have $\kappa(x_0, \xi_0, y_0, \eta_0) \notin \mathrm{WF}'_h(\widetilde{R}_h(z)) \cap S^*(X \times X)$ and thus

$$\mathrm{WF}'_h(\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (p^{-1}(0) \times E_s^*)) \setminus \{0\}).$$

Combining this with (2.2) we get the desired estimate (2.1) and finish the proof of Proposition 2.1.

3. Flat trace estimates

In this section, we present a simpler argument than the one in [JiZw17] to give the following flat trace estimate (see [JiZw17, Proposition 3.1]). The result is slightly weaker than the original one in [JiZw17], but avoid using [NoZw15, Proposition 10.3] and thus the assumption [JiZw17, (2.7)] for the complex absorbing potential Q.

Theorem 2. *The flat trace*

$$T(z) = \mathrm{tr}^\flat(e^{-it_0 h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z))$$

is well-defined and holomorphic for z in $[-C_1 h^\varepsilon, C_1 h^\varepsilon] + i[-C_2 h, 1]$. Moreover, we have

$$T(z) = \mathcal{O}(h^{-2n-2}). \tag{3.1}$$

To prove it we need a wavefront set estimate for the Schwartz kernel of $e^{-it_0 h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z)$:

Lemma 3.1.

$$\begin{aligned}&\mathrm{WF}'_h(e^{-it_0 h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \\ &\quad \kappa(\{(x,\xi,y,\eta) : (e^{-t_0 H_p}(x,\xi), y, \eta) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\} \text{ or } \xi = 0, \eta \neq 0\}).\end{aligned}$$

Proof. Proposition 2.1 gives

$$\mathrm{WF}'_h(\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \kappa(\Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*) \setminus \{0\}).$$

Thus

$$\begin{aligned}&\mathrm{WF}'_h(e^{-t_0 V}\widetilde{R}_h(z)) \cap S^*(X \times X) \subset \\ &\kappa(\{(x,\xi,y,\eta) : (e^{-t_0 H_p}(x,\xi), y, \eta) \in \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*)\} \setminus \{0\}).\end{aligned}$$

We have

$$e^{-it_0 P} - e^{-it_0 h^{-1}(hP - iQ)} = h^{-1}\int_0^{t_0} e^{-i(t_0 - t)P} Q e^{-ith^{-1}(hP - iQ)} dt,$$

and using $\mathrm{WF}'_h(Q)\cap S^*(X\times X)=\emptyset$ and [Al08, Lemma 3.7(iii)], we can compute

$$\mathrm{WF}'_h(e^{-i(t_0-t)P}Qe^{-ith^{-1}(hP-iQ)}\widetilde{R}_h(z))\cap S^*(X\times X)\subset (X\times\{0\})\times S^*X.$$

Therefore

$$\begin{aligned}&\mathrm{WF}'_h(e^{-it_0h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z))\cap S^*(X\times X)\\ &\subset \Big(\mathrm{WF}'_h(e^{-it_0P}\widetilde{R}_h(z))\bigcup\cup_{t=0}^{t_0}\mathrm{WF}'_h(e^{-i(t_0-t)P}Qe^{-ith^{-1}(hP-iQ)}\widetilde{R}_h(z))\Big)\bigcap S^*(X\times X)\\ &\subset \kappa(\{(x,\xi,y,\eta)\,:\,(e^{-t_0H_p}(x,\xi),y,\eta)\in\Delta(T^*X)\cup\Omega_+\cup(E_u^*\times E_s^*)\setminus\{0\}\text{ or }\xi=0,\eta\neq 0\}).\end{aligned}$$

□

Theorem 2 then follows from the following general lemma.

Lemma 3.2. *Let X be an n-dimensional smooth manifold and $m\in\mathbb{R}$. If $P(h):C^\infty(X)\to\mathcal{D}'(X)$ is h-tempered and satisfies*

- $\mathrm{WF}'_h(P(h))\cap\Delta(S^*X)=\varnothing$;
- $\|AP(h)B\|_{L^2\to L^2}=\mathcal{O}(h^{-m})$ *for* $A,B\in\Psi_h^{\mathrm{comp}}(X)$;

then $\mathrm{tr}^\flat(P(h))$ *is well-defined with*

$$\mathrm{tr}^\flat(P(h))=\mathcal{O}(h^{-2n-m}).$$

Proof. Since $\mathrm{WF}'_h(P(h))\cap\Delta(S^*X)=\varnothing$, we have $\mathrm{WF}'(P(h))\cap\Delta(T^*X)=\varnothing$, it is then a classical theorem (see e.g. [Hö83, Theorem 8.2.4]) that the flat trace is well-defined as long as the wavefront set does not intersect the diagonal.

Let $u=K_h$ be the Schwartz kernel of $P(h)$, $\iota:X\to X\times X$ be the diagonal embedding, then for $\chi\in C^\infty(X)$, $\varphi(x,y)=\psi(x)\psi(y)\in C^\infty(X\times X)$ supported near the diagonal,

$$\langle\iota^*(\varphi u),\chi\rangle=\langle\varphi u,\iota_*\chi\rangle=\frac{1}{(2\pi h)^{2n}}\int\mathcal{F}_h(\varphi u)I_{\chi,h}(\xi,\eta)d\xi d\eta \tag{3.2}$$

where

$$I_{\chi,h}(\xi,\eta)=\int\chi(x)e^{ix\cdot(\xi+\eta)/h}dx.$$

If $|\xi+\eta|>|\xi|/C$, then

$$I_{\chi,h}(\xi,\eta)=\mathcal{O}(h^\infty(|\xi|+|\eta|)^{-\infty}).$$

Thus we only need to consider the case when (ξ,η) lies in a small conical neighbourhood of $\{\xi+\eta=0\}$ or in a neighbourhood of $\{\xi=\eta=0\}$.

(i) When $|\xi| + |\eta| \leq C$ is bounded, we have for some $A, B \in \Psi_h^{\mathrm{comp}}(X)$

$$\begin{aligned}|\mathcal{F}_h(\varphi u)| &= |\langle P(h)B(\psi(y)e^{-iy\cdot\eta/h}), A(\psi(x)e^{-ix\cdot\xi/h})\rangle| + \mathcal{O}(h^\infty)\\ &\lesssim \|AP(h)B\|_{L^2\to L^2} + \mathcal{O}(h^\infty)\\ &= \mathcal{O}(h^{-m}).\end{aligned}$$

(ii) When (ξ, η) is near fiber infinity and in a small conic neighbourhood of $\{\xi + \eta = 0\}$ which does not intersect $\mathrm{WF}'_h(P(h))$, we have

$$\mathcal{F}_h(\varphi u) = \mathcal{O}(h^\infty \langle |\xi| + |\eta|\rangle^{-\infty})$$

thanks to the wavefront condition $\mathrm{WF}'_h(P(h)) \cap \Delta(S^*X) = \varnothing$.

Now (3.2) gives us

$$|\langle \iota^*(\varphi u), \chi\rangle| = h^{-2n}\mathcal{O}(h^{-m}) = \mathcal{O}(h^{-2n-m})$$

and a partition of unity argument finishes the proof. $\square$

Proof of Theorem 2. The operator $\widetilde{R}_h(z) : \mathcal{H}_h^s \to \mathcal{H}_h^s$ is bounded and thus h-tempered. Lemma 3.1 gives

$$\mathrm{WF}'_h(e^{-it_0h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z)) \cap \Delta(S^*X) = \varnothing$$

if we choose $t_0 > 0$ smaller than the least length of the closed orbits. For any $A, B \in \Psi_h^{\mathrm{comp}}(X)$ recall

$$e^{-it_0P} - e^{-it_0h^{-1}(hP-iQ)} = h^{-1}\int_0^{t_0} e^{-ith^{-1}(hP-iQ)}Qe^{-i(t_0-t)P}dt,$$

we have

$$\begin{aligned}&\|Ae^{-it_0h^{-1}\widetilde{P}_h(z)}\widetilde{R}_h(z)B\|_{L^2\to L^2}\\ &\lesssim \|Ae^{-it_0P}\widetilde{R}_h(z)B\|_{L^2\to L^2} + h^{-1}\int_0^{t_0}\|Ae^{-ith^{-1}(hP-iQ)}Qe^{-i(t_0-t)P}\widetilde{R}_h(z)B\|_{L^2\to L^2}dt\\ &\lesssim \|Ae^{-it_0P}\widetilde{R}_h(z)B\|_{\mathcal{H}_h^s\to\mathcal{H}_h^s} + h^{-1}\int_0^{t_0}\|Qe^{-i(t_0-t)P}\widetilde{R}_h(z)B\|_{L^2\to L^2}dt\\ &= \mathcal{O}(h^{-1}) + h^{-1}\int_0^{t_0}\|Qe^{-i(t_0-t)P}\widetilde{R}_h(z)B\|_{\mathcal{H}_h^s\to\mathcal{H}_h^s}dt\\ &= \mathcal{O}(h^{-2}).\end{aligned}$$

Here we use the fact that on compact sets in the phase space L^2 norm is equivalent to any $\mathcal{H}^s$ norm. Now the claim follows from Lemma 3.2. $\square$

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