LOCALIZED INITIAL DATA FOR EINSTEIN EQUATIONS

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Abstract. We apply a new method with explicit solution operators to construct asymptotically flat initial data sets of the vacuum Einstein equation with new localization properties. Applications include an improvement of the decay rate in Carlotto–Schoen [CS16] to $\mathcal{O}(|x|^{-(d-2)})$. Also we construct nontrivial asymptotically flat initial data supported in a degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \le x_d^\alpha\}$ for $\frac{3}{d+1} < \alpha < 1$.

1. Introduction

In this note we provide a simple way to construct asymptotically flat initial data of the Einstein equation with new localization properties. The vacuum Einstein equation reads

$$Ric_g = 0$$

where g is a Lorentzian metric and Ric_g is the Ricci curvature. When we restrict to a spacelike hypersurface, we get the constraint equation

$$\begin{cases} R_g + (\mathrm{tr}_g k)^2 - |k|_g^2 = 0 \\ \mathrm{div}_g(k - (\mathrm{tr}_g k) g) = 0 \end{cases} \tag{1.1}$$

It is a system of nonlinear underdetermined PDEs for initial data (g, k) on a spacelike hypersurface. When $k = 0$, it specializes to a problem in Riemannian geometry, namely vanishing of scalar curvature. In particular, we are interested in the following question.

Question 1. *What localization of the solutions to the Einstein constraint equation* (1.1) *is possible?*

This question has surprisingly nontrivial answers. The famous positive mass theorem [SY79][SY81][Wit81] says localization to a compact set is impossible. On the positive direction, Carlotto–Schoen [CS16] gives a gluing construction which gives a localized solution inside a cone. Aretakis–Czimek–Rodnianski [ACR21a][ACR21b][ACR21c] gives an alternative proof of the gluing construction.

The construction in [CS16] loses the decay rate a little, so they cannot get the ideal $\mathcal{O}(|x|^{2-d})$ decay. Carlotto [Car21, Open Problem 3.18] conjectured that we can get this optimal decay. Aretakis–Czimek–Rodnianski [ACR21b] gives an affirmative answer. Here we give a simple proof using our method. Moreover, we can construct solutions with better localization properties, namely in a degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \le x_d^\alpha\}$ for $\frac{3}{d+1} < \alpha < 1$. When α is close to 1, this is still in the range of positive mass theorem (see Carlotto [Car21, Appendix B]). We state our main results below.

Main results. As the first application of our method, we give a simple proof of [Car21, Open Problem 3.18].

Theorem 1. *Let $d \ge 3, s > \frac{d}{2}, -\frac{d}{2} < \delta < \frac{d}{2} - 2$. For $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi$, let*

$$\Omega = \Omega_{\omega,\theta} := \{x \in \mathbb{R}^d : |\arg x - \arg \omega| \le \theta\}$$

be the cone in $\mathbb{R}^d$ with center vector ω and angle θ. Then there exists a nontrivial asymptotically flat solution (g, k) of equation (1.1) on $\mathbb{R}^d$ supported in the cone Ω, in the sense that

$$(g^{ij} - \delta^{ij}, k^{ij}) \in H^s_\delta(\mathbb{R}^d) \times H^{s-1}_{\delta+1}(\mathbb{R}^d), \quad \operatorname{supp}(g^{ij} - \delta^{ij}, k^{ij}) \subset \Omega.$$

Moreover, the set of such solutions form a smooth submanifold in a neighbourhood of $0 \in H^s_\delta(\Omega) \times H^{s-1}_{\delta+1}(\Omega)$.

Moreover, we can make $(g, k) \in C^\infty(\mathbb{R}^d)$ and the decay rate of the solution can be made

$$\partial^l(g^{ij}(x) - \delta^{ij}) = \mathcal{O}(\langle x\rangle^{2-d-l}), \quad \partial^l k^{ij}(x) = \mathcal{O}(\langle x\rangle^{1-d-l}), \quad l \le s - d - 2. \tag{1.2}$$

The weighted Sobolev space H^s_δ is defined and studied in Section 3. When $s = 0$, $\|u\|_{H^0_\delta} \approx \|\langle x\rangle^\delta u\|_{L^2}$.

We can also prove a similar gluing result as in [CS16] following the same strategy.

Theorem 2. *Let $d \ge 3, s > \frac{d}{2}, -\frac{d}{2} < \delta < \frac{d}{2} - 2$. For $y \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi$, let*

$$\Omega = \Omega_{y,\omega,\theta} := \{x \in \mathbb{R}^d : |\arg(x - y) - \arg \omega| \le \theta\}$$

be the cone in $\mathbb{R}^d$ with center at y, center vector ω and angle θ. Let $0 < \theta_0 < \theta$. Suppose there is a solution $(g_0^{ij} - \delta^{ij}, k_0^{ij}) \in H^s_\delta(\Omega) \times H^{s-1}_{\delta+1}(\Omega)$ solving the equation (1.1), then for $|y| \gg 1$, there exists an asymptotically flat solution (g, k) of equation (1.1) on $\mathbb{R}^d$ such that

$$(g^{ij} - \delta^{ij}, k^{ij}) \in H^s_\delta(\mathbb{R}^d) \times H^{s-1}_{\delta+1}(\mathbb{R}^d), \quad (g, k) = \begin{cases} (g_0, k_0), & \Omega_{y,\omega,\theta_0} \setminus B_1(y), \\ (\delta, 0), & \mathbb{R}^n \setminus (\Omega_{y,\omega,\theta} \cup B_1(y)). \end{cases}$$

Suppose (g_0, k_0) has decay rate in (1.2), then (g, k) also has decay rate in (1.2). If $(g_0, k_0) \in C^\infty$, then for $|y| \gg 1$, we also have $(g, k) \in C^\infty$.

Another natural conjecture that Carlotto made in [Car21, Open Problem 3.14] is that whether we can construct solutions localized in a smaller region as long as we do not violate the constraint of the positive mass theorem. We give an elegant answer for the case of a degenerate sector and the geometric case $k = 0$.

Theorem 3. *Let $d \geq 3$, $\frac{3}{d+1} < \alpha < 1$, consider the degenerate sector*

$$\Omega = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^{\alpha}\}.$$

If $s > \frac{d}{2} + 2$, $\frac{3-(d+3)\alpha}{2} < \delta < \frac{\alpha(d-1)-3}{2}$, then there exists a smooth nontrivial asymptotically flat solution $(g, 0)$ of (1.1) supported in Ω, in the sense that

$$g^{ij} - \delta^{ij} \in H^s_{\delta,\alpha}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d), \quad \mathrm{supp}(g^{ij} - \delta^{ij}) \subset \Omega.$$

The decay rate of the solution is given by

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d}(g^{ij} - \delta^{ij}) = \mathcal{O}(\langle x\rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad |\beta| \leq s - d - 2. \tag{1.3}$$

The anisotropic Sobolev space $H^s_{\delta,\alpha}$ is defined in section 5. Note there is a natural constraint of the range of α we can get. When $\alpha = 0$, the localization is impossible due to positive mass theorem. When $\alpha = 1$, this is the result of [CS16]. The reason we put $k = 0$ is that we did not find a nice solution operator for the symmetric divergence equation (1.4) below for k. It is expected that a nice solution operator will lead to a well-localized solution for nontrivial k.

The gluing technique in studying (1.1) appeared much earlier in Corvino [Cor00] and Corvino-Schoen [CS06], and was generalized by Chruściel–Delay [CD03]. It has been developed to give a version of our Theorem 1 in Carlotto-Schoen [CS16]. Aretakis–Czimek–Rodnianski [ACR21a][ACR21b][ACR21c] introduced and studied the characteristic gluing problem. Chruściel [Chr19] and Carlotto [Car21] give reviews of current situation and some open problems.

Main ideas. Our construction is different from and shorter than that that of [CS16]. The key to our construction is a solution operator of the linearized equation with good support properties following Oh-Tataru [OT19, Section 4]. The linearized equation at the trivial metric δ_{ij} (under a change of variables) is

$$\begin{cases} \partial_i \partial_j h^{ij} = 0 \\ \partial_i \pi^{ij} = 0 \end{cases} \tag{1.4}$$

The basic idea behind the proofs of Theorems 1 and 2 is to construct a fundamental solution of (1.4) generalizing the fundamental solution for $\partial_j v^j = 0$ in [OT19]. Then we use Picard iteration in appropriate weighted Sobolev spaces to get the solution of the nonlinear equation (1.1). In the case of the degenerate sector (Theorem 3), we develop a new fundamental solution and introduce anisotropic Sobolev spaces adapted to the degenerate sector. The sharp decay rate is obtained by representing the solution as the solution operator applied to nonlinearity.

Organization of the paper. In section 2 we give the construction of the solution operator adapted to a cone. In section 3 we introduce the fractional weighted Sobolev spaces and provide basic properties of it (some technical proofs are put in the appendix). In particular we prove the boundedness of the solution operator in this weighted Sobolev space. In section 4 we prove Theorem 1 and 2 using our solution operator. In section 5 we adapt our method to the degenerate sector to give a proof of Theorem 3.

Acknowledgement. We would like to thank Sung-Jin Oh for suggesting the idea of this simple solution operator and for his numerous help throughout this program.

2. Construction of the solution operator for the linearized equation

The crux of our argument is an explicit solution operator. In this section we will show how to construct a solution operator $S : C_0^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ $(d \geq 3)$ such that

$$\operatorname{supp} f \subset \text{a cone} \Longrightarrow \operatorname{supp} Sf \subset \text{a cone}.$$

Unlike in Corvino [Cor00], S does not have cokernel (on appropriate weighted Sobolev spaces) since the support is noncompact. The integration kernel of S will have an appropriate decay property.

2.1. **Linearized problem.**

We begin by reformulating (1.1) in a sufficiently flat region. We introduce new variables (h, π), defined as follows:

$$(h_{ij}, \pi_{ij}) = (g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta(g - \delta), k_{ij} - \delta_{ij} \operatorname{tr}_\delta k) \tag{2.1}$$

Observe that the transformation is obviously invertible with the formulae

$$(g_{ij}, k_{ij}) = (\delta_{ij} + h_{ij} - \frac{1}{2}\delta_{ij} \operatorname{tr}_\delta h, \pi_{ij} - \frac{1}{2}\delta_{ij} \operatorname{tr}_\delta \pi). \tag{2.2}$$

With respect to the new variables, the left-hand sides of (1.1) may be written as

$$R[g] = \partial_i \partial_j h^{ij} - M_h^{(2)}(h, \partial^2 h) - M_h^{(1)}(\partial h, \partial h), \tag{2.3}$$

$$(\operatorname{tr}_g k)^2 - |k|_g^2 = -M_h^{(0)}(\pi, \pi), \tag{2.4}$$

$$g^{jj'}(g^{ii'} \nabla_{g;i} k_{i'j'} - \partial_{j'} \operatorname{tr}_g k) = \partial_i \pi^{ij} - N_h^{(1)j}(h, \partial \pi) + N_h^{(0)j}(\partial h, \pi), \tag{2.5}$$

where each of $M_h^{(n)}(u, v)$ or $N_h^{(n)j}(u, v)$ is a linear combination of contraction of u and v with a smooth tensor field (of the appropriate order) on $\mathbb{R}^d$ that depends only on h.

In conclusion, (1.1) takes the form

$$\partial_i\partial_j h^{ij} = M_h^{(2)}(h, \partial^2 h) + M_h^{(1)}(\partial h, \partial h) + M_h^{(0)}(\pi, \pi), \tag{2.6}$$

$$\partial_i \pi^{ij} = N_h^{(1)j}(h, \partial\pi) + N_h^{(0)j}(\partial h, \pi). \tag{2.7}$$

2.2. **Solution operator for the divergence equation.** The construction of the following solution operators is the main point for Theorem 1 and 2.

Theorem 4. *For any $w \in \mathbb{S}^{d-1}$ and $\chi \in C^\infty(\mathbb{S}^{d-1})$, there exists $K_\chi(x), L_{\chi,w} \in \mathcal{D}'(\mathbb{R}^d)$ such that*

$$\begin{cases} \partial_i\partial_j K_\chi^{ij} = \delta \\ \partial_i L_{\chi,w}^{ij} = \delta w^k. \end{cases}$$

Moreover, they satisfy the following properties

- *K_χ and $L_{\chi,\omega}$ are symmetric 2-tensors;*
- *The support of $K_\chi, L_{\chi,\omega}$ lies inside the cone $\overline{\{x \in \mathbb{R}^d : \frac{x}{|x|} \in \operatorname{supp}\chi\}}$;*
- *K_χ is homogeneous of degree $2-d$, $L_{\chi,\omega}$ is homogeneous of degree $1-d$.*
- *$K_\chi, L_{\chi,\omega}$ are smooth in $\mathbb{R}^d \setminus \{0\}$.*

Proof. **Step 1:** We first consider a the case of the divergence equation

$$\partial_i h^i = f.$$

Let $\omega \in \mathbb{S}^{d-1}$, H be the Heaviside function, then

$$T_\omega = \omega H(x\cdot\omega)\delta(\omega^\perp)$$

is a fundamental solution for the divergence equation. From this we can construct a smoother fundamental solution by averaging in ω. Indeed, Let $\chi \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \chi(\omega)d\omega = 1$, then

$$\langle T_\omega, f\rangle = \int_0^\infty \omega f(t\omega)dt$$

and

$$\begin{aligned}\left\langle \int_{\mathbb{S}^{d-1}} \chi(\omega)T_\omega d\omega, f\right\rangle &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \omega f(t\omega)\chi(\omega)d\omega dt \\ &= \int_{\mathbb{R}^d} f(x)\chi\left(\frac{x}{|x|}\right)\frac{x}{|x|^d}dx.\end{aligned}$$

Thus we have a fundmental solution

$$\tilde{K}_\chi(x) = \int_{\mathbb{S}^{d-1}} \chi(\omega)T_\omega d\omega = \chi\left(\frac{x}{|x|}\right)\frac{x}{|x|^d} \tag{2.8}$$

which is homogeneous of degree $1-d$ and smooth outside the origin.

Step 2: We can now apply the same idea to construct solution operators of the linearized constraint equations.

$$\partial_i\partial_j h^{ij} = f \tag{2.9}$$
$$\partial_i \pi^{ij} = g^j. \tag{2.10}$$

For the first equation (2.9) we may just apply the previous solution operator twice. The fundmental solution has the integration kernel

$$K_\chi^{ij} = \left(\chi\left(\frac{x}{|x|}\right)\frac{x_i}{|x|^d}\right) * \left(\chi\left(\frac{x}{|x|}\right)\frac{x_j}{|x|^d}\right).$$

For the second equation (2.10), we will first find singular fundamental solutions as before. For $v, w \in \mathbb{S}^{d-1}$, let

$$\pi^{jk} = \partial_l\phi(v^j v^l w^k + v^l v^k w^j - w^l v^k v^j),$$

the equation (2.10) becomes

$$\partial_j \pi^{jk} = \partial_j\partial_l\phi v^j v^l w^k = g^k.$$

Then

$$\langle \phi, f\rangle = \int_0^\infty t f(tv)dt$$

gives a fundamental solution $L_{v,w}$ such that $\partial_j\pi^{jk} = \delta w^k$ and π^{jk} is a symmetric tensor. Averaging along v as before, we get

$$\begin{aligned}\langle \int_{\mathbb{S}^{d-1}} \chi(v) L_{v,w} dv, f\rangle &= -\int_{\mathbb{S}^{d-1}}\chi(v)\int_0^\infty t(\partial_l f)(tv)(v^j v^l w^k + v^l v^k w^j - w^l v^k v^j)dtdv\\ &= -\int_{\mathbb{R}^d}\chi\left(\frac{x}{|x|}\right)(\partial_l f)(x)\frac{x^j x^l w^k + w^l w^k w^j - w^l x^k x^j}{|x|^d}dx\\ &= \left\langle \partial_l\left(\chi\left(\frac{x}{|x|}\right)\frac{x^j x^l w^k + w^l w^k w^j - w^l x^k x^j}{|x|^d}\right), f\right\rangle.\end{aligned}$$

So the fundamental solution reads

$$L_{\chi,w} = \partial_l\left(\chi\left(\frac{x}{|x|}\right)\frac{x^j x^l w^k + w^l w^k w^j - w^l x^k x^j}{|x|^d}\right).$$

All the properties of the solution operators follow directly from the construction.

□

3. Estimates on the H^s_δ space

In order to get the optimal regularity, we will use fractional weighted Sobolev spaces. In this section we introduce those weighted Sobolev spaces and prove certain properties of operators acting on those spaces.

3.1. **Definition of the weighted Sobolev space.** Let

$$\langle x\rangle_{2^{-k}} = \begin{cases} (|x|^2 + 2^{-2k})^{\frac{1}{2}}, & k < 0 \\ \langle x\rangle, & k \geq 0. \end{cases}$$

Definition 1. *Let $s, \delta \in \mathbb{R}$, the weighted fractional Sobolev space $H^s_\delta = H^s_\delta(\mathbb{R}^d)$ is defined by the following norm.*

$$\|u\|^2_{H^s_\delta} := \sum_{k\in\mathbb{Z}} 2^{2sk} \|\langle x\rangle^{s+\delta}_{2^{-k}} P_k u\|^2_{L^2}$$

where $P_k(u) = \psi(2^{-k}D)u$ for some $\psi \in C^\infty_0(\mathbb{R}^d)$ is the Littlewood–Paley projection at frequency 2^k. For $\Omega \subset \mathbb{R}^d$, we also denote $H^s_\delta(\Omega) = \{u \in H^s_\delta(\mathbb{R}^d) : \operatorname{supp} u \subset \overline{\Omega}\}$.

The motivation of the weight $\langle x\rangle_{2^{-k}}$ on different frequencies comes from the following lemma.

Lemma 2. *Let $\delta > -d$, there exists a function $\psi \in C^\infty_0(\mathbb{R}^d)$, such that*

$$\psi(2^{-k}D)\langle x\rangle^\delta \approx \langle x\rangle^\delta_{2^{-k}}.$$

Proof. Take $\phi \in C^\infty_0(\mathbb{R}^d)$ such that $\check{\phi}(x) \neq 0$ for $|x| < 1$, and $\psi = \phi(x) * \phi(-x)$, then $\psi \in C^\infty_0(\mathbb{R}^d)$ and $\check{\psi} = (2\pi)^d|\check{\phi}|^2$. For $k < 0$,

$$\begin{aligned} \psi(2^{-k}D)\langle x\rangle^\delta &= 2^{kd}\int \check{\psi}(2^k(x-y))\langle y\rangle^\delta dy \\ &\approx \begin{cases} 2^{-k\delta}, & |x| \leq 2^{-k}, \\ |x|^\delta, & |x| > 2^{-k}, \end{cases} \\ &\approx \langle x\rangle^\delta_{2^{-k}}. \end{aligned}$$

The proof for the $k \geq 0$ case is similar. □

The following Lemma on boundedness of multipliers is useful later.

Lemma 3. *Let $m \in C^\infty_0(\mathbb{R}^d)$, then for $0 < h \leq 1$,*

$$\|\langle hD\rangle^s(m(h^{-1}x)u)\|_{L^2} \lesssim \|\langle hD\rangle^s u\|_{L^2}$$

Proof. By commuting terms we may assume $s > 0$ without loss of generality. Then by fractional Leibniz rule (see [Gra09, Theorem 7.6.1]),

$$\begin{aligned} \|\langle hD\rangle^s(m(h^{-1}x)u)\|_{L^2} &\lesssim \|m(h^{-1}x)u\|_{L^2} + \||hD|^s(m(h^{-1}x)u)\|_{L^2} \\ &\lesssim \|u\|_{L^2} + h^s\||D|^s(m(h^{-1}x))\|_{L^\infty}\|u\|_{L^2} + h^s\|m(h^{-1}x)\|_{L^\infty}\||D|^s u\|_{L^2} \\ &\lesssim \|u\|_{L^2} + \||hD|^s u\|_{L^2} \\ &\lesssim \|\langle hD\rangle^s u\|_{L^2}. \end{aligned}$$

□

The following lemmas help understand our H^s_δ spaces. But the proofs are technical so we put them in the appendix.

Lemma 4. *Let $|\delta| < \frac{d}{2}$, then*

$$\|\langle x\rangle^\delta u\|_{L^2} \approx \|u\|_{H^0_\delta}.$$

Lemma 5. *Let $\psi(x), \phi(x)$ be nonnegative cutoff functions in an annulus, and $\psi_j(x) = \psi(x/2^j)$, $\phi_{j'}(x) = \phi(x/2^{j'})$, $j, j' = 1, 2, \cdots$. Moreover let ψ_0, ϕ_0 be cutoff functions in the unit ball such that*

$$1 \approx \sum_{j=0}^{\infty} \psi_j(x) \approx \sum_{j'=0}^{\infty} \phi_{j'}(x).$$

Let $P_k, \tilde{P}_{k'}$, $k, k' \in \mathbb{Z}$ be the Littlewood–Paley projection at frequency 2^k, then for $s \in \mathbb{R}$ and $|\delta| < \frac{d}{2}$, we have

$$\|u\|^2_{H^s_\delta} \approx \sum_{j+k>0} 2^{2ks}\langle 2^j\rangle^{2(\delta+s)}\|\psi_j P_k u\|^2_{L^2} + \sum_{k\le 0} 2^{2ks}\langle 2^{-k}\rangle^{2(\delta+s)}\|\psi_{\le -k}P_k u\|^2_{L^2} \tag{3.1}$$

$$\approx \sum_{j'+k'>0} 2^{2k's}\langle 2^{j'}\rangle^{2(\delta+s)}\|\tilde{P}_{k'}\phi_{j'}u\|^2_{L^2} + \sum_{j'\ge 0} 2^{-2j's}\langle 2^{j'}\rangle^{2(\delta+s)}\|\tilde{P}_{\le -j'}\phi_{j'}u\|^2_{L^2}. \tag{3.2}$$

Proof. See Appendix. □

In particular, Lemma 5 gives a phase space characterization of our weighted Sobolev space. It implies the following Sobolev embedding result.

Lemma 6. *Let $s > d/2$ and $|\delta| < d/2$, then*

$$\|\langle x\rangle^{\frac{d}{2}+\delta}u\|_{L^\infty} \lesssim \|u\|_{H^s_\delta}.$$

Proof. This follows from our phase space decomposition.

$$\begin{aligned}
\|\phi_j u\|_{L^\infty} &\lesssim \sum_{j+k>0} \|P_k\phi_j u\|_{L^\infty} + \|P_{\le -j}\phi_j u\|_{L^\infty}\\
&\lesssim \sum_{j+k>0} 2^{\frac{d}{2}k}\|P_k\phi_j u\|_{L^2} + 2^{-\frac{d}{2}j}\|P_{\le -j}\phi_j u\|_{L^2}\\
&\lesssim \sum_{j+k>0} 2^{(\frac{d}{2}-s)k}2^{sk}\|P_k\phi_j u\|_{L^2} + 2^{-\frac{d}{2}j}\|P_{\le -j}\phi_j u\|_{L^2}\\
&\lesssim 2^{-j(\delta+\frac{d}{2})}\|u\|_{H^s_\delta}.
\end{aligned}$$

□

3.2. **Smoothness of curvature.** The following bilinear estimate is very useful. Since the proof is long and technical we put it in the appendix.

Proposition 7. *Let $s > d/2$, $t_1, t_2 \geq 0$, $-d/2 < \delta \leq \delta + t_1 + t_2 < d/2$. Let $\varepsilon \geq 0$ such that $\delta + \frac{d}{2} - 2\varepsilon > 0$, then $(h, g) \mapsto hg$ is continuous $H^{s-t_1}_{\delta+t_1-\varepsilon} \times H^{s-t_2}_{\delta+t_2-\varepsilon} \to H^{s-t_1-t_2}_{\delta+t_1+t_2}$.*

Proof. See Appendix. □

Only the case $\varepsilon = 0$ will matter for us, but we include the general case for completeness. As a corollary, we know the maps we consider will be smooth.

Corollary 8. *Let $s > d/2$, $|\delta| < d/2$. In the neighbourhood of δ_{ij}, the inverse matrix map*

$$(g_{ij}) \mapsto (g^{ij}) : H^s_\delta \to H^s_\delta$$

is a smooth isomorphism.

Proof. We only need to prove the map

$$T : h \mapsto \frac{1}{1-h} = 1 + h + h^2 + h^3 + \cdots$$

is smooth for $\|h\|_{H^s_\delta} \ll 1$. The boundedness is a corollary of the bilinear estimate. To prove the boundedness of the derivatives, just observe

$$DT_g(h) = (1 + 2g + 3g^2 + \cdots)h, D^2T_g(h_1, h_2) = (2 + 3g + 4g^2 + \cdots)h_1h_2, \cdots$$

are all continuous multilinear maps in H^s_δ. □

Proposition 9. *The functional*

$$(h, \pi) \mapsto (M^{(2)}_h(h, \partial^2 h), M^{(1)}_h(\partial h, \partial h), M^{(0)}_h(\pi, \pi), N^{(1)j}(h, \partial\pi), N^{(0)j}(\partial h, \pi))$$

is smooth $H^s_\delta \times H^{s-1}_{\delta+1} \to H^{s-2}_{\delta+2}$.

Proof. The map is the composition of the inverse matrix map and polynomial maps, which are all smooth by Proposition 7 and Corollary 8. □

3.3. **Boundedness of the solution operator.** It is not hard to prove the solution operator is bounded in our weighted Sobolev spaces, using the homogeneity.

Proposition 10. *Let $s \in \mathbb{R}, -\frac{d}{2} < \delta < \frac{d}{2} - 2$, then the solution operators K_χ and $L_{\chi,w}$ satisfy the following bounds.*

$$\|K_\chi * u\|_{H^s_\delta} \lesssim \|u\|_{H^{s-2}_{\delta+2}}, \quad \|L_{\chi,w} * u\|_{H^{s-1}_{\delta+1}} \lesssim \|u\|_{H^{s-2}_{\delta+2}}.$$

Proof. For $K = K_\chi$, Lemma 3 implies

$$\sum_k 2^{2ks} \|\langle x\rangle_{2^{-k}}^{\delta+s}(K*u)_k\|_{L^2}^2 \lesssim \sum_k 2^{2ks}(2^{-2k}\|\langle x\rangle_{2^{-k}}^{\delta+s}u_k\|_{L^2})^2$$
$$\lesssim \sum_k 2^{2k(s-2)}\|\langle x\rangle_{2^{-k}}^{\delta+s}u_k\|_{L^2}^2.$$

The proof for L is similar. □

4. Solving the nonlinear equation

In this section we use our solution operators from Theorem 4 to prove Theorem 1 and 2.

4.1. **Construction of solutions to the linearized equation.** In this section we construct solutions of the linearized equations (1.4) supported in a cone. A basic observation is to solve the double divergence equation we only need to solve the symmetric divergence equation

$$\partial_j \pi^{jk} = 0 \tag{4.1}$$

since this would give

$$\partial_j\partial_k \pi^{jk} = 0.$$

For (4.1), we take

$$\pi^{11} = \partial_2\partial_2 f, \pi^{12} = -\partial_1\partial_2 f, \pi^{22} = \partial_1\partial_1 f, \pi^{jk} = 0 \text{ for } \{j,k\} \not\subset \{1,2\}.$$

Then

$$\partial_j \pi^{jk} = 0.$$

Remark 1. *We can also construct the solution using our solution operators or functional analytic tools.*

4.2. **Fixed point theorem.** Let

$$P(h,\pi) = (\partial_i\partial_j h^{ij}, \partial_i \pi^{ij})$$
$$\Phi(h,\pi) = (M_h^{(2)}(h,\partial^2 h) + M_h^{(1)}(\partial h,\partial h) + M_h^{(0)}(\pi,\pi), N^{(1)j}(h,\partial\pi) + N^{(0)j}(\partial h,\pi)).$$

Then the equations (1.1) become

$$P(h,\pi) = \Phi(h,\pi).$$

Let Ω be a cone in $\mathbb{R}^d$ and $(h_0,\pi_0) \in C_0^\infty(\Omega)$ be a solution of the linearized equation $P(h_0,\pi_0) = 0$. Let $S : H_{\delta+2}^{s-2}(\Omega) \to H_\delta^s(\Omega) \times H_{\delta+1}^{s-1}(\Omega)$ be the solution operator, we consider the following fixed point problem

$$(h,\pi) = S\Phi(h_0+h, \pi_0+\pi).$$

Since $\Phi : H^s_\delta(\Omega) \times H^{s-1}_{\delta+1}(\Omega) \to H^{s-2}_{\delta+2}(\Omega)$ is a smooth map with $d\Phi_0 = 0$, by choosing $\|(h,\pi)\|_{H^s_\delta \times H^{s-1}_{\delta+1}} \le \varepsilon/2$, $\|(h_0,\pi_0)\|_{H^s_\delta \times H^{s-1}_{\delta+1}} \le \varepsilon$ for some sufficiently small $\varepsilon > 0$ we get

$$\|\Phi(h_0 + h, \pi_0 + \pi)\|_{H^{s-2}_{\delta+2}} \lesssim \varepsilon^2,$$
$$\|\Phi(h_0 + h, \pi_0 + \pi) - \Phi(h_0 + \tilde{h}, \pi_0 + \tilde{\pi})\|_{H^{s-2}_{\delta+2}} \lesssim \varepsilon \|(h,\pi) - (\tilde{h}, \tilde{\pi})\|_{H^s_\delta \times H^{s-1}_{\delta+1}}.$$

Since S is again bounded, by Banach fixed point theorem, there exists a unique fixed point (h_1, π_1) such that $\|(h_1,\pi_1)\|_{H^s_\delta \times H^{s-1}_{\delta+1}} \le \varepsilon/2$ and

$$(h_1, \pi_1) = S\Phi(h_0 + h_1, \pi_0 + \pi_1).$$

This implies

$$P(h_0 + h_1, \pi_0 + \pi_1) = P(h_1, \pi_1) = \Phi(h_0 + h_1, \pi_0 + \pi_1).$$

An alternative way is to notice that

$$P - \Phi : H^s_\delta(\Omega) \times H^{s-1}_{\delta+1}(\Omega) \to H^{s-2}_{\delta+2}(\Omega)$$

is a smooth map with surjective differential at 0 (with a right inverse given by our solution operator). Thus locally $(P - \Phi)^{-1}(0) \cap \mathrm{nbd}(0)$ is diffeomorphic to $\ker P \cap \mathrm{nbd}(0)$.

Next we show the solution we get can be actually smooth, even though s is fixed. The regularity comes from applying the solution operator to the nonlinearity to upgrade the regularity.

Proposition 11. *Let $s > \frac{d}{2}, d \ge 3$. In the above construction, if we choose $(h_0, \pi_0) \in C^\infty_0$ small in H^s_δ, then $(h,\pi) \in C^\infty$.*

Proof. We need to come back to the equation. Without loss of generality we may assume $\frac{d}{2} < s < \frac{d+1}{2}$. Observe that

$$\partial(h,\pi) = S(M^{(3)}_{h_0,h}(h + h_0, \partial^3 h), N^{(2)j}_{h_0,h}(h + h_0, \partial^2 \pi)) + \text{controlled terms in } H^{2s-1-d/2}_{\mathrm{loc}} \times H^{2s-2-d/2}_{\mathrm{loc}}.$$

Since $\|h + h_0\|_{H^s_\delta}$ is small, we can upgrade the regularity to

$$(h,\pi) \in H^{2s-\frac{d}{2}}_{\mathrm{loc}} \times H^{2s-1-\frac{d}{2}}_{\mathrm{loc}}.$$

Keep doing this we will get $(h,\pi) \in H^{d/2+2}_{\mathrm{loc}}$. Then the equation gives us

$$\partial(h,\pi) = S(M^{(3)}_{h_0,h}(h + h_0, \partial^3 h), N^{(2)j}_{h_0,h}(h + h_0, \partial^2 \pi)) + \text{controlled terms in } H^{\frac{d}{2}+2}_{\mathrm{loc}}.$$

Since $\|h + h_0\|_{H^s_{\mathrm{loc}}}$ is small, we get $(h,\pi) \in H^{\frac{d}{2}+3}_{\mathrm{loc}}$. Iterating this we have $(h,\pi) \in C^\infty$. □

4.3. **Tail estimate of the solution.** Now we prove the last part of Theorem 1, namely the decay rate estimate (1.2).

Proposition 12. *Suppose $d \geq 3, s > \frac{d}{2}, -\frac{d}{2} < \delta < \frac{d}{2} - 2$. Let $(g^{ij} - \delta^{ij}, k^{ij}) \in H^s_\delta(\Omega) \times H^{s-1}_{\delta+1}(\Omega)$ be a solution of equation (1.1) obtained in the previous section, then for $l < s - d - 2$, we have*

$$|\partial_l (g^{ij}(x) - \delta^{ij}(x))| \lesssim \langle x \rangle^{2-d-l}, \quad |\partial_l k^{ij}(x)| \lesssim \langle x \rangle^{1-d-l}. \tag{4.2}$$

Proof. We first consider the case $l = 0$.

By Lemma 6, we have

$$|h^{ij}(x)| \lesssim \langle x \rangle^{-(\frac{d}{2}+\delta)}, |\partial^2 h^{ij}(x)| \lesssim \langle x \rangle^{-(\frac{d}{2}+\delta+2)}.$$

Thus

$$|M_h^{(2)}(h, \partial^2 h)| \lesssim \langle x \rangle^{-(d+2+2\delta)}.$$

Similarly we can prove

$$\Phi(h, \pi) = \mathcal{O}(\langle x \rangle^{-(d+2+2\delta)}).$$

Now

$$h = S\Phi_1(h, \pi)$$

where $\Phi_1(h, \pi) = \mathcal{O}(\langle x \rangle^{-(d+2+2\delta)})$. Moreover, S is a convolution operator with kernel $K = K_\chi$ homogeneous of degree -1, thus

$$\begin{aligned} |h(x)| &= \left| \int K(x-y)\Phi_1(h, \pi)(y) dy \right| \\ &\lesssim \int |x-y|^{2-d} \langle y \rangle^{-(d+2+2\delta)} dy \\ &= \int_{|x-y|<|x|/2} |x-y|^{2-d} \langle y \rangle^{-(d+2+2\delta)} dy + \int_{|x-y|\geq|x|/2} |x-y|^{2-d} \langle y \rangle^{-(d+2+2\delta)} dy. \end{aligned}$$

If $2+2\delta > 0$, then $|h(x)| \lesssim \langle x \rangle^{2-d}$. Otherwise we can only conclude $|h(x)| \lesssim \langle x \rangle^{-(d+2\delta)}$. But we can iterate this process and still conclude

$$|h(x)| \lesssim \langle x \rangle^{2-d}.$$

Similarly we have

$$|\pi(x)| \lesssim \langle x \rangle^{1-d}$$

and the proposition follows. The case $l = 1, 2$ can be obtained similarly.

Now for general $3 \leq l \leq s - d - 2$, let us suppose (4.2) is true for up to $l - 1$, and prove for l. Recall

$$|\partial^j h(x)| \lesssim \langle x \rangle^{-(\frac{d}{2}+\delta+j)},$$

the induction hypothesis gives

$$|\partial^l \Phi_1(h,\pi)| \lesssim \langle x\rangle^{-(3d/2+\delta+l)}.$$

Recall $\partial^l h = S\partial^l \Phi_1(h,\pi)$, so

$$\begin{aligned}|\partial^l h(x)| &\lesssim \int_{|x-y|<\frac{|x|}{2}} |x-y|^{2-d}\langle y\rangle^{-(\frac{3d}{2}+\delta+l)}dy + \int_{|x-y|>\frac{|x|}{2}} |x-y|^{2-d-l}\langle y\rangle^{-(3d/2+\delta)}dy \\ &\lesssim \langle x\rangle^{2-d-l}.\end{aligned}$$

Similarly we have

$$|\partial^l \pi(x)| \lesssim \langle x\rangle^{1-d-l}.$$

□

4.4. **Gluing construction of the solution.** In this section we provide the proof of the gluing result in Theorem 2.

Let $\Omega \subset \Omega'$ be two cones, after cutting off we only need to solve the constraint equation

$$\Psi(h,\pi) = f$$

inside the region $(\Omega' \cup B_1(y)) \setminus (\Omega \cup B_{1/2}(y))$. By choosing $|y| \gg 1$, we may assume $\|f\|_{H^{s-2}_{\delta+2}}$ is sufficiently small, and then we can apply the following solution operator to get a solution.

Proposition 13. *Let* $\Omega_{\text{int}} := (\Omega' \cup B_1(y)) \setminus (\Omega \cup B_{1/2}(y))$*, then there is a solution operator*

$$S_{\text{int}} : H^{s-2}_{\delta+2}(\Omega_{\text{int}}) \to H^s_\delta(\Omega_{\text{int}}) \times H^{s-1}_{\delta+1}(\Omega_{\text{int}}).$$

Proof. Let $f \in H^{s-2}_{\delta+2}(\Omega_{\text{int}})$. First we can move the support outside $B_2(y)$ using the explicit Bogovskii-type solution operator S_0 constructed in [MOT22]. Let $\chi_0 \in C_0^\infty(B_2(y))$, then

- $PS_0(\chi_0 f) = \chi_0 f$ inside $B_2(y)$;
- $\operatorname{supp} S_0(\chi_0 f) \subset B_3(y)$;
- $S_0(\chi_0 f) \in H^s_\delta(\Omega_{\text{int}}) \times H^{s-1}_{\delta+1}(\Omega_{\text{int}})$.

Then we make a partition of unity in angular variables and use of the previous solution operator on each piece. Suppose $\Omega_{\text{int}} \cap \mathbb{S}^{d-1}_y = \cup U_i$ and each U_i is star-shaped with respect to an open subset $V_i \subset U_i$. Now let $\chi_i \in C_0^\infty(V_i)$ be a cutoff function supported in V_i, then S_{χ_i} gives rise a solution operator with respect to U_i so that

$\operatorname{supp} u \subset y + \mathbb{R}_{>1}(U_i - y)$ implies $\operatorname{supp} S_{\chi_i} u \subset y + \mathbb{R}_{>1}(U_i - y)$. We take a partition of unity $\tilde{\chi}_j$ with respect to the covering $\{U_i\}$ and define

$$S_1 = \sum_j S_{\chi_j} \tilde{\chi}_j.$$

The final solution operator S_{int} is defined to be

$$S_{\text{int}} f = S_1(f - PS_0(\chi_0 f)) + S_0(\chi_0 f) \in H^s_\delta(\Omega_{\text{int}}) \times H^{s-1}_{\delta+1}(\Omega_{\text{int}}).$$

One can check it is bounded $H^{s-2}_{\delta+2}(\Omega_{\text{int}}) \to H^s_\delta(\Omega_{\text{int}}) \times H^{s-1}_{\delta+1}(\Omega_{\text{int}})$ and

$$PS_{\text{int}} f = PS_1(f - PS_0(\chi_0 f)) + PS_0(\chi_0 f) = f - PS_0(\chi_0 f) + PS_0(\chi_0 f) = f.$$

□

Following the procedure above, we can get a gluing solution from the solution operator S_{int}. The proof for the smoothness and decay rate is identical to previous argument.

5. Solving the problem in a degenerate sector

In this section we want to find solutions of the Einstein constraint equations of the form $(g, 0)$ in the degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha\}$ for some $\alpha < 1$. For technical convenience, we define

$$\Omega = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha : x_d \geq 1\}$$

and indeed construct the solution in Ω.

In order to analyze the regularity of the solution, we need an anisotropic weighted Sobolev space defined as follows. For simplicity, we will be only considering integer $s \in \mathbb{N}$ in this section.

Definition 14. *Let $\alpha \in (0, 1]$, $s \in \mathbb{N}$ be an integer.*

$$\|u\|^2_{H^s_{\delta,\alpha}} = \sum_{|\beta| \leq s} \|\langle x \rangle^{|\beta'|\alpha + \beta_d + \delta} \partial^{\beta'}_{x'} \partial^{\beta_d}_{x_d} u\|^2_{L^2}$$

We provide a few properties of this norm.

Proposition 15.
- *For $s > d/2$, $\|\langle x \rangle^{\frac{(d-1)\alpha+1}{2}+\delta} u\|_{L^\infty} \lesssim \|u\|_{H^s_{\delta,\alpha}}$. More generally, for $s > d/2 - d/p$, $\|\langle x \rangle^{((d-1)\alpha+1)(\frac{1}{2}-\frac{1}{p})+\delta} u\|_{L^p} \lesssim \|u\|_{H^s_{\delta,\alpha}}$.*
- *For $s > d/2$, $(g, h) \mapsto gh$ is is continuous $H^s_{\delta_1,\alpha} \times H^s_{\delta_2,\alpha} \to H^s_{\delta_1+\delta_2+\frac{(d-1)\alpha+1}{2},\alpha}$*

Proof. The first estimate follows from rescaling. The bilinear estiamte is also direct by choosing p_1, p_2 according to β for Hölder inequality and use the first Sobolev embedding estimate:

$$\begin{aligned}
&\|gh\|^2_{H^s_{\delta_1+\delta_2+\frac{(d-1)\alpha+1}{2},\alpha}} \\
&= \sum_{|\beta|\le s} \|\langle x\rangle^{|\beta'|\alpha+\beta_d+\delta_1+\delta_2+\frac{(d-1)\alpha+1}{2}} \partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}(gh)\|^2_{L^2} \\
&\lesssim \sum_{|\beta|+|\gamma|\le s} \|\langle x\rangle^{|\beta'|\alpha+\beta_d+\delta_1+\frac{(d-1)\alpha+1}{p_2}} \partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d} g\|^2_{L^{p_1}} \|\langle x\rangle^{|\gamma'|\alpha+\gamma_d+\delta_2+\frac{(d-1)\alpha+1}{p_1}} \partial_{x'}^{\gamma'}\partial_{x_d}^{\gamma_d} h\|^2_{L^{p_2}} \\
&\lesssim \|g\|^2_{H^s_{\delta_1,\alpha}}\|h\|^2_{H^s_{\delta_2,\alpha}}.
\end{aligned}$$

In the second last step we choose $1/p_1 + 1/p_2 = 1/2$ such that $|\beta| > d/2 - 1/p_1$ and $|\gamma| > d/2 - 1/p_2$. □

Next, we turn to the construction of a solution operator for the double divergence equation $\partial_i\partial_j h^{ij}$. We start with the divergence equation $\partial_j v^j = 0$. There are many fundamental solutions of the divergence equation supported on a half-curve.

Proposition 16. *For any smooth curve $\gamma : [0,\infty) \to \mathbb{R}^d$ starting at y, the distribution δ_γ defined as*

$$(\delta_\gamma, \varphi) = \int_0^\infty \varphi(\gamma(t))\gamma'(t)dt$$

is a fundamental solution for the divergence equation

$$\partial_j u^j = \delta_y.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, then

$$(\partial_j\delta^j_\gamma, \varphi) = -\int_0^\infty \partial_j\varphi(\gamma(t))\gamma'^j(t)dt = -\int_0^\infty \partial_t\varphi(\gamma(t))dt = \varphi(\gamma(0)) = \varphi(y).$$

□

As before, we average a family of fundamental solutions to get a more regular solution supported in Ω.

Lemma 17. *Suppose $\delta < \frac{\alpha(d-1)-1}{2}$. We have a solution operator $\tilde{S}_0 : H^{s-1}_{\delta+1,\alpha}(\Omega) \to H^s_{\delta,\alpha}(\Omega)$ for the divergence equation, i.e. $\partial_j\tilde{S}_0^j f = f$.*

Proof. We construct the solution operator in two steps.

Step 1: Let $\gamma^{(1)}_{y,\omega} = y + (\omega y_d^\alpha, y_d)t$, $\chi_1 \in C_0^\infty(\mathbb{R}^{d-1})$, and define

$$K_1 = \int_{\mathbb{R}^{d-1}} \chi_1(\omega)\delta_{\gamma^{(1)}_{y,\omega}} d\omega = \chi_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d}\right)\frac{x-y}{y_d^{\alpha(d-1)+1}|\frac{x_d-y_d}{y_d}|^d}.$$

Then $\operatorname{div} K_1 = \delta(x-y)$. Let χ_2 be a cutoff and

$$\tilde{K}_1 = \chi_2\left(\frac{x_d - y_d}{y_d}\right) K_1.$$

Then $\operatorname{div} \tilde{K}_1 f = f + \frac{1}{y_d}\chi_2'\left(\frac{x_d - y_d}{y_d}\right) K_1$. Let $\gamma^{(2)}_{y,\omega}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$ and

$$\begin{aligned} K_2 &= \int_{\mathbb{S}^{d-1}} \chi_1(\omega)\delta_{\gamma^{(2)}_{y,\omega}} d\omega \\ &= \chi_1\left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1}\right)(|1+x_d-y_d|-1)^{-\alpha(d-1)}\left(\alpha(x'-y')\frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1\right). \end{aligned}$$

Then $K(x,y) = \tilde{K}_1(x,y) - \int K_2(x,z)\frac{1}{y_d}\chi_2'\left(\frac{z_d - y_d}{y_d}\right) K_1(z,y)dz$ is a solution operator of the divergence equation. Let $\tilde{K}_2(x,y) = -\int K_2(x,z)\frac{1}{y_d}\chi_2'\left(\frac{z_d-y_d}{y_d}\right) K_1(z,y)dz$, it is straightforward to verify

- K is outgoing;
- $\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}\tilde{K}_2(x,y) \lesssim |x_d - y_d|^{-\alpha(d-1)-|\beta'|\alpha-\beta_d}$;

We need to estimate the solution operator in both regions.

- In the nearby region, we would like to show
$$\left\|\int \tilde{K}_1(x,y)u(y)dy\right\|_{H^s_{\delta,\alpha}} \lesssim \|u\|_{H^{s-1}_{\delta+1,\alpha}}. \tag{5.1}$$
It suffices to prove it in a fixed annulus $R < |y_d| < 2R$. We claim by rescaling, we may assume $R = 1$ without loss of generality. Let $\tilde{K}_1(x,y) = (K_s(x,y), K_b(x,y))$, we observe that
$$\tilde{K}_1(R^\alpha x', Rx_d, R^\alpha y', Ry_d) = R^{-\alpha(d-1)-1}(R^\alpha K_s(x,y), RK_b(x,y)),$$
so given the estimate (5.1) for $R = 1$ we get

$$\begin{aligned} &\left\|\int \tilde{K}_1(x,y)u(y)dy\right\|^2_{H^s_{\delta,\alpha}(A_R)} \\ &\lesssim R^{\alpha(d-1)+1}R^{2\delta}\left\|R^{\alpha(d-1)+1}\int \tilde{K}_1(R^\alpha x', Rx_d, R^\alpha y', Ry_d)u(R^\alpha y', Ry_d)dy\right\|^2_{H^s(A_1)} \\ &\lesssim R^{\alpha(d-1)+1}R^{2+2\delta}\left\|\int \tilde{K}_1(x,y)u(R^\alpha y', Ry_d)dy\right\|^2_{H^s(A_1)} \\ &\lesssim R^{\alpha(d-1)+1}R^{2+2\delta}\|u(R^\alpha y', Ry_d)\|^2_{H^{s-1}(A_1)} \\ &\lesssim \|u\|^2_{H^{s-1}_{\delta+1,\alpha}(A_R)}. \end{aligned}$$

On the unit annulus, we notice $\tilde{K}_1(x,y)$ gives a pseudodifferential operator of order -1. If we let $b(y_d,z) = \chi_1\left(\frac{z'/y_d^\alpha}{z_d/y_d}\right)\frac{(z'/y_d^\alpha, z_d/y_d)}{y_d^{\alpha(d-1)+1}|\frac{z_d}{y_d}|^d}\chi_2(z_d/y_d)$ and $a(\xi) = \mathcal{F}^{-1}(b(1,\cdot))$, then a is homogeneous of degree -1 and

$$(y_d^{-\alpha}K_s(x,y), y_d^{-1}K_b(x,y)) = \frac{1}{(2\pi)^d}\int e^{i(x-y)\cdot\xi}a(y_d^\alpha\xi', y_d\xi_d)d\xi.$$

Let χ be a cutoff near $\xi = 0$, then $(1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d)$ is a symbol in the sense that

$$|\partial_y^\beta\partial_\xi^\gamma(1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d)| \lesssim_{\beta,\gamma} \langle\xi\rangle^{-1-\gamma|\xi|}.$$

Thus maps H^{s-1} to H^s. Moreover, $\int e^{i(x-y)\cdot\xi}\chi(\xi)a(y_d^\alpha\xi', y_d\xi_d)d\xi \in C_0^\infty$, thus gives a smoothing operator. So we conclude (5.1) from the discussion above.

- In the faraway region, we can estimate directly.

$$\begin{aligned}
\|\tilde{K}_2u\|^2_{H^s_{\delta,\alpha}(\Omega)} &= \sum_{|\beta|\le s}\|\langle x\rangle^{|\beta'|\alpha+\beta_d+\delta}\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}(\tilde{K}_2u)\|^2_{L^2}\\
&\lesssim \sum_{|\beta|\le s}\sum_{j'>j} 2^{2j'(|\beta'|\alpha+\beta_d+\delta-\alpha(d-1)-|\beta'|\alpha-\beta_d)}2^{((d-1)\alpha+1)(j+j')}\|\psi_j u\|^2_{L^2}\\
&\lesssim \sum_j 2^{j(2\delta-2\alpha(d-1)+2((d-1)\alpha+1))}\|\psi_j u\|^2_{L^2}\\
&\lesssim \|\langle x\rangle^{\delta+1}u\|^2_{L^2}.
\end{aligned}$$

□

From the solution operator for the divergence equation, we also get the solution operator for the double divergence equation.

Proposition 18. *Suppose $\delta < \frac{\alpha(d-1)-3}{2}$, then there is a solution operator $\tilde{S} : H^{s-2}_{\delta+2,\alpha}(\Omega) \to H^s_{\delta,\alpha}(\Omega)$ for the double divergence equation, i.e. $\partial_i\partial_j\tilde{S}^{ij}f = f$ and $\tilde{S}^{ij}$ is symmetric. Moreover, the integration kernel $K(x,y)$ of $\tilde{S}$ satisfies*

$$|\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}K(x,y)| \lesssim \langle x\rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}, \quad x_d > 2y_d.$$

Proof. We just need to apply $\tilde{S}_0$ twice and symmetrize it:

$$S^{ij}f = \frac{1}{2}(\tilde{S}_0^i\tilde{S}_0^jf + \tilde{S}_0^j\tilde{S}_0^if).$$

The bound of the tail follows from the construction in Lemma 17. □

As a corollary we get

Proof of Theorem 3. Let $d \geq 3, s > \frac{d}{2} + 2$ be an integer. Let $\frac{3}{d+1} < \alpha < 1$ and $\delta = \frac{\alpha(d-1)-3}{2} - > \frac{3-(d+3)\alpha}{2}$. We now choose small but nontrivial C_0^∞ solutions h_0 of the linearized equation $P(h_0) = 0$, and solve the fixed point problem

$$h = \tilde{S}\Phi(h_0 + h).$$

on the space $H^s_{\delta,\alpha}(\Omega)$. By fixed point theorem, we get a unique solution

$$g^{ij} - \delta^{ij} \in H^s_{\delta,\alpha}(\Omega).$$

We now claim h is actually smooth. To see this we come back to the equation.

$$\partial h = \tilde{S}\partial^2 M^{(3)}_{h_0,h}(h + h_0, \partial h) + \text{controlled terms in } H^s_{\text{loc}}.$$

Thus

$$\|\partial^{s+1} h\|_{L^2_{\text{loc}}} \leq C\|\langle x\rangle^{2(1-\alpha)}(h + h_0)\|_{L^\infty}\|\partial^{s+1} h\|_{L^2_{\text{loc}}} + C.$$

If we choose $\|h_0\|_{H^s_{\delta,\alpha}} \ll 1$ and thus $\|\langle x\rangle^{2(1-\alpha)}(h_0 + h)\|_{L^\infty} \ll 1$, we conclude

$$h \in H^{s+1}_{\text{loc}}.$$

By keeping doing this, we conclude h is actually smooth.

Now we compute the decay rate of the solution. By Sobolev embedding, we have

$$\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d} h^{ij}(x) = \mathcal{O}(\langle x\rangle^{1-(d-1)\alpha+\varepsilon-|\beta'|\alpha-\beta_d}), \quad |\beta| \leq s.$$

This implies

$$\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d}\Phi(h)(x) = \mathcal{O}(\langle x\rangle^{2-2d\alpha-|\beta'|\alpha-\beta_d+2\varepsilon}), \quad |\beta| \leq s - 2$$

and then

$$\begin{aligned}
&|\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d} h(x)| = |\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d}\tilde{S}\Phi(h)(x)| \\
&\lesssim \left|\int_{x_d<2y_d} \partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d} K(x,y)\Phi(h)(y)dy\right| + \left|\int_{x_d\geq 2y_d} \partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d} K(x,y)\Phi(h)(y)dy\right| \\
&\lesssim \sum_{\gamma+\kappa_d=\beta} \int_{x_d<2y_d} y_d^{(1-\alpha)(d-1)-\kappa_d}|x_d - y_d|^{1-d}|\partial^{\gamma'}_{y'}\partial^{\gamma_d}_{y_d}\Phi(h)(y)|dy \\
&+ \int_{x_d\geq 2y_d} \langle x\rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}|\Phi(h)(y)|dy \\
&= \mathcal{O}(\langle x\rangle^{1-(d-1)\alpha-|\beta'|\alpha-\beta_d}).
\end{aligned}$$

□

Appendix

In the appendix we provide proofs of some statements in Section 3. The proof of Lemma 4 and Lemma 5 requires a detailed study of the phase space decomposition adapted to uncertainty principle.

Proof of Lemma 4. Let

$$\alpha_{j,k} = \|\psi_j P_k u\|_{L^2}, \alpha'_{j',k'} = \|\tilde{P}_{k'}\psi_{j'} u\|_{L^2}$$

for $j+k>0$ and

$$\beta_k = \|\psi_{\le -k} P_k u\|_{L^2}, \beta'_{j'} = \|\tilde{P}_{\le -j'}\phi_{j'} u\|_{L^2}.$$

Then for $j+k>0$ we have

$$\|\psi_j P_k \phi_{j'}\|_{L^2\to L^2} \lesssim_N \begin{cases} 2^{-\frac{d}{2}|j-j'|}2^{-N(\max(j,j')+k)}, & |j-j'|>10, \\ 1, & |j-j'|\le 10. \end{cases}$$

Dually we have for $j'+k'>0$

$$\|P_k \psi_{j'} \tilde{P}_{k'}\|_{L^2\to L^2} \lesssim_N \begin{cases} 2^{-\frac{d}{2}|k-k'|}2^{-N(j'+\max(k,k'))}, & |k-k'|>10, \\ 1, & |k-k'|\le 10. \end{cases}$$

In the $j+k<0$ range we have

$$\|\psi_{\le -k} P_k \phi_{j'}\|_{L^2\to L^2} \lesssim_N \begin{cases} 2^{-N(j'+k)}, & j'+k>0, \\ 2^{-\frac{d}{2}|j'+k|}, & j'+k\le 0. \end{cases}$$

Dually we have

$$\|P_{\le -j} \psi_j \tilde{P}_{k'}\|_{L^2\to L^2} \lesssim_N \begin{cases} 2^{-N(j+k')}, & j+k'>0, \\ 2^{-\frac{d}{2}|j+k'|}, & j+k'\le 0. \end{cases}$$

So we have

$$\begin{aligned} \alpha_{j,k} &\le \Big\| \sum_{j'+k'>0} \psi_j P_k \tilde{\phi}_{j'} \bar{P}_{k'} \tilde{P}_{k'} \phi_{j'} u \Big\| + \Big\| \sum_{j'} \psi_j P_k \tilde{\phi}_{j'} \tilde{P}_{\le -j'} \phi_{j'} u \Big\| \\ &\lesssim \sum_{j'+k'>0} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k-k'|} 2^{-N(\max(j,j')+k)\mathbb{1}_{|j-j'|>10} - N(j'+\max(k,k'))\mathbb{1}_{|k-k'|>10}} \alpha'_{j',k'} \\ &+ \sum_{j'} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k+j'|} 2^{-N(\max(j,j')+k)\mathbb{1}_{|j-j'|>10} - N(j'+k)\mathbb{1}_{j'+k>0}} \beta'_{j'}, \end{aligned}$$

and

$$\begin{aligned}\beta_k &\leq \left\| \sum_{j'+k'>0} \psi_{\leq -k} P_k \tilde{\phi}_{j'} \bar{P}_{k'} \tilde{P}_{k'} \phi_{j'} u \right\| + \left\| \sum_{j'} \psi_{\leq -k} P_k \tilde{\phi}_{j'} \tilde{P}_{\leq -j'} \phi_{j'} u \right\| \\ &\lesssim \sum_{j'+k'>0} 2^{-(\frac{d}{2}-\varepsilon)|j'+k|-\varepsilon|k-k'|} 2^{-N(j'+k)\mathbb{1}_{j'+k>0} - N(j'+\max(k,k'))\mathbb{1}_{|k-k'|>10}} \alpha'_{j',k'} \\ &+ \sum_{j'} 2^{-\frac{d}{2}|j'+k|} 2^{-N(j'+k)\mathbb{1}_{j'+k>0}} \beta'_{j'}.\end{aligned}$$

Dually, $\alpha'_{j',k'}$ and $\beta'_{j'}$ can be controlled by envelopes of $\alpha_{j,k}$ and β_k similarly. This gives

$$\begin{aligned}\|\langle x\rangle^\delta u\|_{L^2}^2 &\approx \sum_{j'} \langle 2^{j'}\rangle^{2\delta} \|\phi_{j'} u\|_{L^2}^2 \\ &\approx \sum_{j'+k'>0} \langle 2^{j'}\rangle^{2\delta} \|\tilde{P}_{k'} \phi_{j'} u\|_{L^2}^2 + \sum_{j'} \langle 2^{j'}\rangle^{2\delta} \|\tilde{P}_{\leq -j'} \phi_{j'} u\|_{L^2}^2 \\ &\approx \sum_{j'+k'>0} \langle 2^{j'}\rangle^{2\delta} \alpha'^2_{j',k'} + \sum_{j'} \langle 2^{j'}\rangle^{2\delta} \beta'^2_{j'} \\ &\lesssim \sum_{j'+k'>0} \langle 2^{j'}\rangle^{2\delta} \left(\sum_{j+k>0} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k-k'|} \alpha_{j,k} + \sum_k 2^{-(\frac{d}{2}-\varepsilon)|j'+k|-\varepsilon|k-k'|} \beta_k \right)^2 \\ &+ \sum_{j'} \langle 2^{j'}\rangle^{2\delta} \left(\sum_{j+k>0} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|j'+k|} \alpha_{j,k} + \sum_k 2^{-\frac{d}{2}|j'+k|} \beta_k \right)^2 \\ &\lesssim \sum_{j+k>0} \langle 2^{j}\rangle^{2\delta} \alpha^2_{j,k} + \sum_k \langle 2^{-k}\rangle^{2\delta} \beta^2_k \\ &\approx \sum_k \|\langle x\rangle^\delta_{2^{-k}} P_k u\|_{L^2}^2.\end{aligned}$$

$$\begin{aligned}\sum_k \|\langle x\rangle^\delta_{2^{-k}} P_k u\|_{L^2}^2 &\approx \sum_{j+k>0} \langle 2^{j}\rangle^{2\delta} \alpha^2_{j,k} + \sum_k \langle 2^{-k}\rangle^{2\delta} \beta^2_k \\ &\lesssim \sum_{j+k>0} \langle 2^{j}\rangle^{2\delta} \left(\sum_{j'+k'>0} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k-k'|} \alpha'_{j',k'} + \sum_{j'} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k+j'|} \beta_{j'} \right)^2 \\ &+ \sum_k \langle 2^{-k}\rangle^{2\delta} \left(\sum_{j'+k'>0} 2^{-(\frac{d}{2}-\varepsilon)|j'+k|-\varepsilon|k-k'|} \alpha'_{j',k'} + \sum_{j'} 2^{-\frac{d}{2}|j'+k|} \beta'_{j'} \right)^2 \\ &\lesssim \sum_{j'+k'>0} \langle 2^{j'}\rangle^{2\delta} \alpha'^2_{j',k'} + \sum_{j'} \langle 2^{j'}\rangle^{2\delta} \beta'^2_{j'} \\ &\approx \|\langle x\rangle^\delta u\|_{L^2}^2.\end{aligned}$$

□

Proof of Lemma 5. The first line follows from the definition, the matter is to show that we can commute the space and frequency decomposition.

Let $\alpha_{j,k}, \alpha'_{j',k'}, \beta_k, \beta'_{j'}$ as in the proof of Lemma 4, then

$$\begin{aligned}
&\sum_{j'+k'>0} 2^{2k's}\langle 2^{j'}\rangle^{2(\delta+s)}\|\tilde{P}_{k'}\phi_{j'}u\|_{L^2}^2 + \sum_{j'\geq 0} 2^{-2j's}\langle 2^{j'}\rangle^{2(\delta+s)}\|\tilde{P}_{\leq -j'}\phi_{j'}u\|_{L^2}^2 \\
&\approx \sum_{j'+k'>0} 2^{2k's}\langle 2^{j'}\rangle^{2(\delta+s)}\alpha'^2_{j',k'} + \sum_{j'} 2^{2\delta j'}\beta'^2_{j'} \\
&\lesssim \sum_{j'+k'>0} 2^{2k's}\langle 2^{j'}\rangle^{2(\delta+s)}\Bigg(\sum_{j+k>0} 2^{-(\frac{d}{2}-\varepsilon)|j-j'|-\varepsilon|k-k'|}2^{-N(\max(j,j')+k)\mathbb{1}_{|j-j'|>10}-N(j'+\max(k,k'))\mathbb{1}_{|k-k'|>10}}\alpha_{j,k} \\
&+ \sum_k 2^{-(\frac{d}{2}-\varepsilon)|j'+k|-\varepsilon|k-k'|}2^{-N(\max(k,k')+j')\mathbb{1}_{|k-k'|>10}-N(j'+k)\mathbb{1}_{j'+k>0}}\beta_k\Bigg)^2 \\
&+ \sum_{j'} 2^{2\delta j'}\Bigg(\sum_{j+k>0} 2^{-(\frac{d}{2}-\varepsilon)|j'+k|-\varepsilon|j-j'|}2^{-N(j'+k)\mathbb{1}_{j'+k>0}-N(k+\max(j,j'))\mathbb{1}_{|j-j'|>10}}\alpha_{j,k} \\
&+ \sum_k 2^{-\frac{d}{2}|j'+k|}2^{-N(j'+k)\mathbb{1}_{j'+k>0}}\beta_k\Bigg)^2 \\
&\lesssim \sum_{j+k>0} 2^{2ks}\langle 2^j\rangle^{2(\delta+s)}\alpha_{j,k}^2 + \sum_k 2^{2ks}\langle 2^{-k}\rangle^{2(\delta+s)}\beta_k^2 \\
&\approx \sum_k 2^{2ks}\|\langle x\rangle_{2^{-k}}^{\delta+s}P_k u\|_{L^2}^2.
\end{aligned}$$

The other direction is similar. □

The bilinear estimate uses the paraproduct decomposition.

Proof of Proposition 7. Let $h \in H^{s-t_1}_{\delta+t_1-\varepsilon}$, $g \in H^{s-t_2}_{\delta+t_2-\varepsilon}$. We use paraproduct decomposition of hg and write

$$hg = h_{\text{low}}g_{\text{high}} + h_{\text{high}}g_{\text{low}} + h_{\text{high}}g_{\text{high}}.$$

Recall

$$\begin{aligned}
\|\langle x\rangle_{2^{-k}}^{\varepsilon} h_{<k-1}\|_{L^\infty} &\lesssim \sum_{k'<k-1} \|\langle x\rangle_{2^{-k}}^{\varepsilon} h_{k'}\|_{L^\infty} \\
&\lesssim \sum_{k'<k} 2^{\frac{d}{2}k'} \|\langle x\rangle_{2^{-k'}}^{\varepsilon} h_{k'}\|_{L^2} \\
&\lesssim \sum_{k'<k} 2^{\frac{d}{2}k'} \left(\sum_{j'+k'>0} 2^{-j'(\delta+s-2\varepsilon)} 2^{-k'(s-t_1)} + 2^{k'(\delta+t_1-2\varepsilon)} \mathbb{1}_{k'\leq 0} \right) \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \\
&\lesssim \sum_{k'<k} 2^{\frac{d}{2}k'} (2^{-k'(s-t_1)} \mathbb{1}_{k'>0} + 2^{k'(\delta+t_1-2\varepsilon)} \mathbb{1}_{k'\leq 0}) \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \\
&\lesssim \begin{cases} \max(2^{k(\frac{d}{2}-s+t_1)}, 1) \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}}, & k>0, \\ 2^{(\frac{d}{2}+t_1+\delta-2\varepsilon)k} \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}}, & k\leq 0. \end{cases}
\end{aligned}$$

For the low-high part, when $k>0$ we have

$$\begin{aligned}
2^{(s-t_1-t_2)k} \|\langle x\rangle_{2^{-k}}^{\delta+s} h_{<k-1} g_k\|_{L^2} &\lesssim 2^{(s-t_1-t_2)k} \|\langle x\rangle_{2^{-k}}^{\varepsilon} h_{<k-1}\|_{L^\infty} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2} \\
&\lesssim 2^{(s-t_1-t_2)k} \max(2^{(\frac{d}{2}-s+t_1)k}, 1) \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2} \\
&\lesssim 2^{(s-t_2)k} \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2}.
\end{aligned}$$

For $k<0$ we have

$$\begin{aligned}
2^{(s-t_1-t_2)k} \|\langle x\rangle_{2^{-k}}^{\delta+s} h_{<k-1} g_k\|_{L^2} &\lesssim 2^{(s-t_1-t_2)k} \|\langle x\rangle_{2^{-k}}^{\varepsilon} h_{<k-1}\|_{L^\infty} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2} \\
&\lesssim 2^{(s-t_1-t_2)k} 2^{(\frac{d}{2}+\delta+t_1-2\varepsilon)k} \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2} \\
&\lesssim 2^{(s-t_2)k} \|h\|_{H^{s-t_1}_{\delta+t_1-\varepsilon}} \|\langle x\rangle_{2^{-k}}^{\delta+s-\varepsilon} g_k\|_{L^2}.
\end{aligned}$$

Taking the ℓ^2 sum both sides yields the desired result. By symmetry the proof of the high-low part is similar. Now we consider the comparable frequencies.

- For $k<k'<0$ and $\chi_{|x|\leq 2^{-k}}$ the indicator function of $\{|x|\leq 2^{-k}\}$,

$$\begin{aligned}
2^{(s-t_1-t_2)k} \|\langle x\rangle_{2^{-k}}^{\delta+s} P_k(h_{k'} g_{k'})\|_{L^2} &\lesssim 2^{(s-t_1-t_2)k} 2^{kd/2} \|\langle x\rangle_{2^{-k}}^{\delta+s} h_{k'} g_{k'}\|_{L^1} \\
&\lesssim 2^{(s-t_1-t_2)k} 2^{kd/2} \|\langle x\rangle_{2^{-k}}^{\delta+s} (h_{k'} g_{k'}) \chi_{|x|\leq 2^{-k}}\|_{L^1} \\
&+ 2^{(s-t_1-t_2)k} 2^{kd/2} \|\langle x\rangle_{2^{-k}}^{\delta+s} (h_{k'} g_{k'})(1-\chi_{|x|\leq 2^{-k}})\|_{L^1} \\
&= I + II
\end{aligned}$$

We first estimate I:

$$
\begin{aligned}
I &\lesssim 2^{(s-t_1-t_2)k}2^{kd/2}2^{-k(\delta+s)}\|h_{k'}g_{k'}\chi_{|x|\le 2^{-k}}\|_{L^1}\\
&\lesssim 2^{-k(\delta+t_1+t_2-d/2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\|\langle x\rangle_{2^{-k'}}^{-2\delta-2s+2\varepsilon}\chi_{|x|\le 2^{-k}}\|_{L^\infty}\\
&\lesssim 2^{-k(\delta+t_1+t_2-d/2)}2^{2k'(\delta+s-\varepsilon)}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\\
&\lesssim 2^{(k'-k)(\delta+t_1+t_2-d/2)}2^{k'(\delta+\frac{d}{2}-2\varepsilon)}(2^{k'(s-t_1)}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}g_{k'}\|_{L^2}).
\end{aligned}
$$

This term is ok since $\delta+t_1+t_2-d/2<0$ and $\delta+\frac{d}{2}-2\varepsilon>0$. For the second term II:

$$
\begin{aligned}
II &\lesssim 2^{k(s-t_1-t_2+d/2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle_{2^{-k'}}^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\|\langle x\rangle^{-(\delta+s)+2\varepsilon}(1-\chi_{|x|\le 2^{-k}})\|_{L^\infty}\\
&\lesssim 2^{k(s-t_1-t_2+d/2+\delta+s-2\varepsilon)}2^{k'(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}g_{k'}\|_{L^2})\\
&\lesssim 2^{k(d/2+\delta-2\varepsilon)}2^{(k'-k)(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}g_{k'}\|_{L^2})\\
&\lesssim (2^{k'(s-t_1)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle_{2^{-k'}}^{\delta+s}g_{k'}\|_{L^2}).
\end{aligned}
$$

- For $k<0\le k'$, we have similar decomposition as above

$$2^{(s-t_1-t_2)k}\|\langle x\rangle_{2^{-k}}^{\delta+s}P_k(h_{k'}g_{k'})\|_{L^2}\lesssim I+II$$

and

$$
\begin{aligned}
I &\lesssim 2^{(s-t_1-t_2)k}2^{kd/2}2^{-k(\delta+s)}\|h_{k'}g_{k'}\chi_{|x|\le 2^{-k}}\|_{L^2}\\
&\lesssim 2^{-k(\delta+t_1+t_2-d/2)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\|\langle x\rangle^{-2\delta-2s+2\varepsilon}\chi_{|x|\le 2^{-k}}\|_{L^\infty}\\
&\lesssim 2^{-k(\delta+t_1+t_2-d/2)}2^{k'(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2})\\
&\lesssim 2^{(k'-k)(t_1+t_2+\delta-\frac{d}{2})}2^{k'(-\delta+\frac{d}{2}-2s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}),
\end{aligned}
$$

$$
\begin{aligned}
II &\lesssim 2^{k(s-t_1-t_2+d/2)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\|\langle x\rangle^{-(\delta+s-2\varepsilon)}(1-\chi_{|x|\le 2^{-k}})\|_{L^\infty}\\
&\lesssim 2^{k(s-t_1-t_2+d/2+\delta+s-2\varepsilon)}2^{k'(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2})\\
&\lesssim 2^{k(d/2+\delta-2\varepsilon)}2^{(k'-k)(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}).
\end{aligned}
$$

- For $0\le k\le k'$, we have

$$
\begin{aligned}
&2^{(s-t_1-t_2)k}\|\langle x\rangle^{\delta+s}P_k(h_{k'}g_{k'})\|_{L^2}\\
&\lesssim 2^{(s-t_1-t_2)k}2^{kd/2}\|\langle x\rangle^{\delta+s}h_{k'}g_{k'}\|_{L^1}\\
&\lesssim 2^{(s-t_1-t_2)k}2^{kd/2}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}\\
&\lesssim 2^{k(s-t_1-t_2+\frac{d}{2})}2^{k'(t_1+t_2-2s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2})\\
&\lesssim 2^{(k'-k)(t_1+t_2-2s)}2^{k(\frac{d}{2}-s)}(2^{k'(s-t_1)}\|\langle x\rangle^{\delta+s-\varepsilon}h_{k'}\|_{L^2})(2^{k'(s-t_2)}\|\langle x\rangle^{\delta+s-\varepsilon}g_{k'}\|_{L^2}).
\end{aligned}
$$

□

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