LOCALIZED INITIAL DATA FOR THE EINSTEIN EQUATIONS

YUCHEN MAO AND ZHONGKAI TAO

Abstract. We apply a new method with explicit solution operators to construct asymptotically flat initial data sets of the vacuum Einstein equation with new localization properties. Applications include an improvement of the decay rate in Carlotto–Schoen [CS16] to $\mathcal{O}(|x|^{2-d})$ and a construction of nontrivial asymptotically flat initial data supported in a degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha, x_d \geq 0\}$ for $\frac{3}{d+1} < \alpha < 1$.

1. Introduction

In this note we provide a simple way to construct asymptotically flat initial data of the Einstein equation with new localization properties. The vacuum Einstein equation reads

$$Ric_{\mathbf{g}} = 0$$

where $\mathbf{g}$ is a Lorentzian metric and $Ric_{\mathbf{g}}$ is the Ricci curvature. When we restrict to a spacelike hypersurface, we get the Einstein constraint equation

$$\begin{cases} R_g + (\mathrm{tr}_g k)^2 - |k|_g^2 = 0 \\ \mathrm{div}_g(k - (\mathrm{tr}_g k)g) = 0 \end{cases} \tag{1.1}$$

It is a system of nonlinear underdetermined PDEs for initial data (g, k) on a spacelike hypersurface. When $k = 0$, it specializes to a problem in Riemannian geometry, namely vanishing of scalar curvature. In particular, we are interested in the following question.

Question 1. *What localization of asymptotically flat solutions to the Einstein constraint equation* (1.1) *is possible?*

This question has surprisingly nontrivial answers. The famous positive mass theorem [SY79; SY81; Wit81] says localization to a compact set is impossible. On the positive direction, Carlotto–Schoen [CS16] gives a gluing construction which gives a localized solution inside a cone. Aretakis–Czimek–Rodnianski [ACR23a; ACR21; ACR23b] gives an alternative gluing construction based on the characteristic gluing.

An interesting problem that was left open in [CS16] was whether the localized solution $g_{ij}(x)$ decays to the flat metric δ_{ij} as $x \to \infty$ with the rate $\mathcal{O}(|x|^{2-d})$, which is ideal in view of the positive mass theorem (see also Proposition 1 below); see Carlotto [Car21, Open Problem 3.18]. The construction in [CS16] achieves $\mathcal{O}(|x|^{2-d+\varepsilon})$ for arbitrary $\varepsilon > 0$. The characteristic gluing approach of Aretakis–Czimek–Rodnianski [ACR21] gives an affirmative answer to [Car21, Open Problem 3.18], but only for finite regularity. Here we give an alternative spacelike proof, which can also achieve C^∞ regularity. Moreover, we can construct nontrivial solutions localized in an even smaller domain, namely in a degenerate sector $\{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha,\ x_d \geq 0\}$ for $\frac{3}{d+1} < \alpha < 1$, with sharp decay rate consistent with the positive mass theorem. We state our main results below.

Main results. As the first application of our method, we give a simple spacelike construction of localized solutions in a cone with optimal decay, answering [Car21, Open Problem 3.18].

In order to state our results, we define the cones in $\mathbb{R}^d$ as follows. For $y \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$, let

$$\Omega_{y,\omega,\theta} := \{x \in \mathbb{R}^d : \angle(x - y, \omega) \leq \theta\}$$

be the cone in $\mathbb{R}^d$ with center at y, center vector ω and angle θ. When $y = 0$, we introduce the abbreviated notation $\Omega_{\omega,\theta} := \Omega_{0,\omega,\theta}$. We also recall the linearized equation of (1.1) at the trivial metric $g_{ij} = \delta_{ij}$ and second fundamental form $k_{ij} = 0$ (under a change of variables, see §2.1) is

$$\partial_i \partial_j h^{ij} = 0, \quad \partial_i \pi^{ij} = 0 \tag{1.2}$$

where $h^{ij} = h^{ji}$ and $\pi^{ij} = \pi^{ji}$ are two symmetric tensors.

Theorem 1. *Let $d \geq 3$. For $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$, there exists a nontrivial asymptotically flat solution $(g, k) \in C^\infty(\mathbb{R}^d)$ of equation (1.1) on $\mathbb{R}^d$ supported in the cone $\Omega_{\omega,\theta}$, i.e. $\operatorname{supp}(g_{ij} - \delta_{ij}, k_{ij}) \subset \Omega_{\omega,\theta}$ and the decay rate of (g, k) is give by*

$$\partial^\ell (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{2-d-\ell}), \quad \partial^\ell k_{ij}(x) = \mathcal{O}(\langle x \rangle^{1-d-\ell}), \quad \ell \in \mathbb{N}. \tag{1.3}$$

Here $\langle x \rangle = (1 + |x|^2)^{1/2}$. The nontriviality of the data can be interpreted as follows. For any solution $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized constraint equation (1.2) supported in $\Omega_{\omega,\theta}$ with the same decay rate

$$\partial^\ell h_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{2-d-\ell}), \quad \partial^\ell \pi_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-d-\ell}), \quad \ell \in \mathbb{N} \tag{1.4}$$

such that (h_0, π_0) is small in certain b-Sobolev spaces (see §3), we can find such a solution to the nonlinear equation (1.1) as a perturbation of (h_0, π_0), and will obey bounds of the form

$$\|(h - h_0, \pi - \pi_0)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim \|(h_0, \pi_0)\|^2_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}$$

on certain b-Sobolev spaces, where (h, π) is a change of variables of (g, k) by (2.1)(2.2). So a small nontrivial solution of the linearized equation (1.2) will lead to a nontrivial solution of (1.1). For more details, see the proof in §4.2.

We can also prove a similar gluing result as in [CS16] following the same strategy.

Theorem 2. *Let $d \geq 3$, $0 < \theta_0 < \theta < \pi/2$ and $\omega \in \mathbb{S}^{d-1}$. Suppose $(g_0, k_0) \in C^\infty(\mathbb{R}^d)$ solves (1.1) inside $\Omega_{\omega,\theta}$ and satisfies (1.3). Then there exists $y_0 \in \Omega_{\omega,\theta_0}$, $|y_0| \gg 1$ and an asymptotically flat solution $(g, k) \in C^\infty(\mathbb{R}^d)$ of equation (1.1) on $\mathbb{R}^d$ such that*

$$(g,k) = \begin{cases} (g_0, k_0), & \Omega_{y_0,\omega,\theta_0} \setminus B_1(y_0), \\ (\delta, 0), & \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0)), \end{cases}$$

and (g, k) also has decay rate in (1.3) on $\mathbb{R}^d$.

Another natural conjecture that Carlotto made in [Car21, Open Problem 3.14] is that whether we can construct solutions localized in a smaller region as long as we do not violate the constraint of the positive mass theorem. We give a partial answer for the case of a degenerate sector.

Theorem 3. *Let $d \geq 3$, $\frac{3}{d+1} < \alpha < 1$. Consider the degenerate sector*

$$\Omega_\alpha = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha, x_d \geq 1\}.$$

Then there exists a nontrivial asymptotically flat solution $(g, k) \in C^\infty(\mathbb{R}^d)$ of (1.1) supported in Ω_α, i.e.

$$\mathrm{supp}(g_{ij} - \delta_{ij}, k_{ij}) \subset \Omega_\alpha,$$

and the decay rate of the solution is given by

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{1.5}$$

and

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} k_{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{1.6}$$

Again the solutions to the nonlinear equation (1.1) are obtained as perturbations of solutions $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized constraint equation (1.2) supported in Ω_α with the decay rate

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} h_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{1.7}$$

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} \pi_0^{ij}(x) = \mathcal{O}(\langle x \rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}), \quad \beta = (\beta', \beta_d) \in \mathbb{N}^d. \tag{1.8}$$

When $\alpha = 1$, this reduces to Theorem 1. The constraint $\alpha > \frac{3}{d+1}$ comes from the proof and we do not expect it to be optimal. The decay rate (1.5) is optimal for localized solutions in degenerate sectors with this type of anisotropic decay, in the following sense:

Proposition 1. *Suppose $g \in C^2(\mathbb{R}^d)$ is a metric such that $g_{ij}(x) = \delta_{ij}$ for $x \notin \Omega = \{(x', x_d) \in \mathbb{R}^d : |x'| \leq x_d^\alpha,\ x_d \geq 0\}$ with $0 < \alpha \leq 1$. Suppose that for some $\varepsilon > 0$ we have*

$$\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} (g_{ij}(x) - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{1-\varepsilon-\alpha(d-1)-|\beta'|\alpha-\beta_d}), \quad |\beta'| + \beta_d \leq 2.$$

Then the ADM energy vanishes:

$$E := \lim_{R\to\infty} \frac{1}{2}\int_{|x|=R} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{x_j}{|x|} dS = 0.$$

Proof. We claim $\int_{|x|=R} \partial_i g(x) \frac{x_j}{|x|} dS = \mathcal{O}(R^{-\varepsilon})$. This is because

- For $i = j = d$,
$$\left| \int_{|x|=R} \partial_d g(x) \frac{x_d}{|x|} dS \right| \lesssim R^{-\alpha(d-1)-\varepsilon} R^{\alpha(d-1)} = R^{-\varepsilon}.$$
- For $i \neq d$ and $j \neq d$,
$$\left| \int_{|x|=R} \partial_i g(x) \frac{x_j}{|x|} dS \right| \lesssim R^{1-\alpha d-\varepsilon} R^{\alpha-1} R^{\alpha(d-1)} = R^{-\varepsilon}.$$
- Since
$$\int_{|x|=R} \partial_j g(x) \frac{x_d}{|x|} dS = \int_{|x|\le R} \partial_d \partial_j g(x) dx = \int_{|x|=R} \partial_d g(x) \frac{x_j}{|x|} dS,$$
we have for $j \neq d$,

$$\left| \int_{|x|=R} \partial_j g(x) \frac{x_d}{|x|} dS \right| = \left| \int_{|x|=R} \partial_d g(x) \frac{x_j}{|x|} dS \right| \lesssim R^{-\alpha(d-1)-\varepsilon} R^{\alpha-1} R^{\alpha(d-1)} = R^{\alpha-1-\varepsilon}. \quad \square$$

If the positive mass theorem holds and $R_g = 0$, then $g_{ij}(x)$ must be isometric to the flat Euclidean space. In this case, Theorem 3 has the optimal decay. We note that the positive mass theorem (see Carlotto [Car21, Appendix B]) is usually stated without considering the anisotropic behavior. For example, [Car21, Theorem B.9] works for $\alpha > \frac{d+4}{2(d+1)}$, which does not include the full range $\alpha > \frac{3}{d+1}$. We expect that the positive mass theorem would actually hold for a larger region of α if one considers solutions supported in a degenerate sector with anisotropic asymptotic behavior like (1.5)(1.6).

We remark that our proof also gives a finite regularity version of Theorem 1-3, which can deal with conic gluings for $(g_{ij} - \delta_{ij}, k_{ij}) \in H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ (those spaces will be introduced in §3) when $s > d/2$, which is optimal modulo the end point. We choose to state the smooth version here since it is easier to understand.

We restricted to the $k = 0$ case for Theorem 3 in an earlier version of this paper but Philip Isett pointed to us the method of Reshetnyak [Res70] can be used to construct nice solution operators for the symmetric divergence equation. We will give a detailed presentation of the method of solution operators in a joint paper with Sung-Jin Oh and Philip Isett [Ise+24].

The gluing technique in studying (1.1) was first studied by Corvino [Cor00] and Corvino–Schoen [CS06], and was generalized by Chruściel–Delay [CD03]. They studied the gluing problem in compact regions, in which case the linearized equation (1.2) has cokernel. For an account of gluing with cokernels using similar solution operators, see

[MOT23]. In the pioneering paper of Carlotto–Schoen [CS16], which was discussed earlier, the gluing technique was extended to noncompact regions and surprising first examples of localized solutions to (1.1) were exhibited. Aretakis–Czimek–Rodnianski [ACR23a; ACR21; ACR23b] (see also [Are15; Are17]) introduced and studied the characteristic gluing problem, and related it to the spacelike gluing problems to show the optimal decay. We refer to Chruściel [Chr19] and Carlotto [Car21] for reviews on the conic gluing method and some open problems. See also [Hin22; Hin23] for a microlocal approach.

Main ideas. Our construction is different from that of [CS16]. Recall the linearized constraint equation (1.2) is a divergence type equation $Pu = 0$. In [CS16], Carlotto–Schoen use PP^* to reduce to an elliptic system, which makes it hard to control the support of solutions. The key idea of our construction is to find a right inverse S of P such that S does not increase the support of a function. Such solution operators of divergence type operators with good support properties is motivated by Oh–Tataru [OT19, Section 4], in which they constructed nice solution operators for the divergence equation $\partial_j v^j = 0$. We generalize their construction here to (1.2). Then we use Picard iteration in appropriate Sobolev spaces to get the solution of the nonlinear equation (1.1). In the case of the degenerate sector (Theorem 3), we develop a new fundamental solution and introduce anisotropic Sobolev spaces adapted to the degenerate sector. The sharp decay rate is obtained by representing the solution as the solution operator applied to the nonlinearity.

Organization of the paper. In §2 we give the construction of the solution operator adapted to a cone. In §3 we recall several estimates for the b-Sobolev spaces. In §4 we prove Theorem 1 and 2 using our solution operator. In §5 we adapt our method to the degenerate sector to give a proof of Theorem 3.

Acknowledgement. We would like to thank Sung-Jin Oh for suggesting the idea of this simple solution operator and for his numerous help throughout this program. We would like to thank Philip Isett for pointing to us the paper [Res70]. We would also like to thank Peter Hintz and Ethan Sussman for helpful discussions about the b-Sobolev spaces. We would like to thank the anonymous referees for helpful comments to improve the paper. Y. Mao was partially supported by a National Science Foundation CAREER Grant under NSF-DMS-1945615. Z. Tao was partially supported by the NSF grant DMS-1952939 and by the Simons Targeted Grant Award No. 896630.

2. Construction of the solution operator for the linearized equation

The crux of our argument is an explicit solution operator to (1.2). In this section we will show how to construct a solution operator $S : C_c^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ $(d \geq 3)$ for the linearized equation (1.2) such that for a cone Ω,

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Sf \subset \Omega.$$

Unlike in Corvino [Cor00], the linearized equation (1.2) does not have cokernel (on appropriate weighted Sobolev spaces) since we consider (1.2) on a noncompact region Ω. The integration kernel of S will have an appropriate decay property.

2.1. **Linearized problem.** We begin by reformulating (1.1) in a sufficiently flat region. We introduce new variables (h, π), defined as follows:

$$(h_{ij}, \pi_{ij}) = (g_{ij} - \delta_{ij} - \delta_{ij} \operatorname{tr}_\delta(g - \delta), k_{ij} - \delta_{ij} \operatorname{tr}_\delta k) \tag{2.1}$$

Observe that the transformation is obviously invertible with the formulae

$$(g_{ij}, k_{ij}) = (\delta_{ij} + h_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta h, \pi_{ij} - \frac{1}{d-1}\delta_{ij} \operatorname{tr}_\delta \pi). \tag{2.2}$$

With respect to the new variables, the left-hand sides of (1.1) may be written as

$$R[g] = \partial_i \partial_j h^{ij} - M_h^{(2)}(h, \partial^2 h) - M_h^{(1)}(\partial h, \partial h), \tag{2.3}$$

$$(\operatorname{tr}_g k)^2 - |k|_g^2 = -M_h^{(0)}(\pi, \pi), \tag{2.4}$$

$$g^{jj'}(g^{ii'}\nabla_{g;i} k_{i'j'} - \partial_{j'} \operatorname{tr}_g k) = \partial_i \pi^{ij} - N_h^{(1)j}(h, \partial \pi) - N_h^{(0)j}(\partial h, \pi), \tag{2.5}$$

where each of $M_h^{(n)}(u, v)$ or $N_h^{(n)j}(u, v)$ is a linear combination of contraction of u and v with a smooth tensor field (of the appropriate order) on $\mathbb{R}^d$ that depends only on h. Here the index (n) is just a label and does not have any particular meaning, and $\partial u = (\partial_{x_1} u, \cdots, \partial_{x_d} u)$.

In conclusion, (1.1) takes the form

$$\partial_i \partial_j h^{ij} = M_h^{(2)}(h, \partial^2 h) + M_h^{(1)}(\partial h, \partial h) + M_h^{(0)}(\pi, \pi), \tag{2.6}$$

$$\partial_i \pi^{ij} = N_h^{(1)j}(h, \partial \pi) + N_h^{(0)j}(\partial h, \pi). \tag{2.7}$$

The right hand side is viewed as the nonlinearity and its precise form will not matter in this paper. In the following, we study how to solve the linearized equations given by the left hand side of (2.6)(2.7), which we call double divergence equation and symmetric divergence equation, respectively.

2.2. **Solution operator for the divergence equations.** The construction of the following solution operators is the key for our proof of Theorem 1 and 2. In the following δ_0 means Dirac delta function at 0.

Theorem 4. *Let $v \in \mathbb{S}^{d-1}$ and $\chi \in C^\infty(\mathbb{S}^{d-1})$ such that $\int_{\mathbb{S}^{d-1}} \chi = 1$. There exists $K_\chi, L_{\chi,v} \in \mathscr{D}'(\mathbb{R}^d)$ such that*

$$\begin{cases} \partial_i \partial_j K_\chi^{ij} = \delta_0 \\ \partial_i L_{\chi,v}^{ij} = \delta_0 v^j. \end{cases}$$

Moreover, they satisfy the following properties

- *K_χ and $L_{\chi,v}$ are symmetric 2-tensors;*
- *The supports of $K_\chi, L_{\chi,v}$ lie inside the convex hull of the cone $\overline{\{x \in \mathbb{R}^d : \frac{x}{|x|} \in \operatorname{supp} \chi\}}$;*
- *K_χ is homogeneous of degree $2-d$, $L_{\chi,v}$ is homogeneous of degree $1-d$.*
- *$K_\chi, L_{\chi,v}$ are smooth in $\mathbb{R}^d \setminus \{0\}$.*

Proof. **Step 1:** We first consider the case of the divergence equation

$$\partial_i h^i = \delta_0.$$

Let $\omega \in \mathbb{S}^{d-1}$, H be the Heaviside function, then

$$T_\omega(x) = \omega H(x \cdot \omega) \delta(\omega^\perp)$$

is a fundamental solution for the divergence equation, where $\delta(\omega^\perp)$ is the delta function on the line $\omega\mathbb{R}$. In other words, for a test function $f \in C_c^\infty(\mathbb{R}^d)$, we have

$$\langle T_\omega, f \rangle = \int_0^\infty \omega f(t\omega) dt$$

From this we can construct a smoother fundamental solution by averaging in ω. Indeed, let $\chi \in C^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} \chi(\omega) d\omega = 1$. Then

$$\begin{aligned} \left\langle \int_{\mathbb{S}^{d-1}} \chi(\omega) T_\omega d\omega, f \right\rangle &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \omega f(t\omega) \chi(\omega) d\omega dt \\ &= \int_{\mathbb{R}^d} f(x) \chi\left(\frac{x}{|x|}\right) \frac{x}{|x|^d} dx. \end{aligned}$$

Thus we have a fundamental solution

$$\int_{\mathbb{S}^{d-1}} \chi(\omega) T_\omega(x) d\omega = \chi\left(\frac{x}{|x|}\right) \frac{x}{|x|^d} \tag{2.8}$$

which is homogeneous of degree $1-d$ and smooth outside the origin.

Step 2: We can now apply the same idea to construct solution operators of the linearized constraint equations

$$\partial_i \partial_j h^{ij} = \delta_0, \tag{2.9}$$

$$\partial_i \pi^{ij} = \delta_0 v^j. \tag{2.10}$$

For the first equation (2.9) we may just apply the previous solution operator twice. The fundamental solution has the integration kernel

$$K_\chi^{ij}(x) = \left(\chi\left(\frac{x}{|x|}\right)\frac{x_i}{|x|^d}\right) * \left(\chi\left(\frac{x}{|x|}\right)\frac{x_j}{|x|^d}\right).$$

For the second equation (2.10), we need to first find singular fundamental solutions as before. For $v, w \in \mathbb{S}^{d-1}$, let

$$\pi^{jk} = \partial_\ell \phi(\omega^j \omega^\ell v^k + \omega^\ell \omega^k v^j - v^\ell \omega^k \omega^j), \tag{2.11}$$

the equation (2.10) becomes

$$\partial_j \pi^{jk} = \partial_j \partial_\ell \phi \omega^j \omega^\ell v^k = \delta_0 v^k.$$

Define $\phi \in \mathscr{D}'(\mathbb{R}^d)$ so that for a test function $f \in C_c^\infty(\mathbb{R}^d)$,

$$\langle \phi, f\rangle = \int_0^\infty t f(t\omega) dt.$$

Then $\partial_j \partial_\ell \phi \omega^j \omega^\ell = \delta_0$ and (2.11) gives a fundamental solution $L_{\omega,v}^{jk}$ such that $\partial_j L_{\omega,v}^{jk} = \delta v^k$ and $L_{v,w}^{jk}$ is a symmetric tensor. Averaging along ω as before, we get

$$\begin{aligned}
\left\langle \int_{\mathbb{S}^{d-1}} \chi(\omega) L_{\omega,v}^{jk} d\omega, f \right\rangle &= -\int_{\mathbb{S}^{d-1}} \chi(\omega) \int_0^\infty t(\partial_\ell f)(t\omega)(\omega^j\omega^\ell v^k + \omega^\ell \omega^k v^j - v^\ell \omega^k \omega^j) dt d\omega \\
&= -\int_{\mathbb{R}^d} \chi\left(\frac{x}{|x|}\right)(\partial_\ell f)(x) \frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d} dx \\
&= \left\langle \partial_\ell \left(\chi\left(\frac{x}{|x|}\right) \frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d}\right), f\right\rangle.
\end{aligned}$$

So the fundamental solution reads

$$L_{\chi,v}^{jk}(x) = \partial_\ell \left(\chi\left(\frac{x}{|x|}\right) \frac{x^j x^\ell v^k + x^\ell x^k v^j - v^\ell x^k x^j}{|x|^d}\right).$$

All the properties of the solution operators follow directly from the construction. □

3. Estimates on the b-Sobolev spaces

In order to capture the decay property of functions and get optimal regularity, we recall basic estimates for b-Sobolev spaces in this section, and prove our solution operators are bounded on b-Sobolev spaces.

3.1. The b-Sobolev space.

Definition 2. *For $s \in \mathbb{N}_0$, the* b*-Sobolev space H^s_{b} is defined by the norm*

$$\|u\|^2_{H^s_{\mathrm{b}}} := \sum_{k\le s} \|\langle x\rangle^k \partial^k u\|^2_{L^2(\mathbb{R}^d)}.$$

We extend the definition to $s \in \mathbb{R}$ by duality and (complex) interpolation. We further define for $\delta \in \mathbb{R}$, $H^{s,\delta}_{\mathrm{b}} := \langle x\rangle^{-\delta} H^s_{\mathrm{b}}$. Moreover, for $\Omega \subset \mathbb{R}^d$, we denote $H^{s,\delta}_{\mathrm{b}}(\Omega) := \{u \in H^{s,\delta}_{\mathrm{b}} : \operatorname{supp} u \subset \overline{\Omega}\}$ the distributions supported in $\overline{\Omega}$.

The b-Sobolev space captures the property that the decay rate of a function improves by $\langle x\rangle^{-1}$ after taking a derivative. When s is an integer, the b-Sobolev space is nothing but the weighted Sobolev space, which already appears in the classical work of Fischer and Marsden [FM75]. We use noninteger $s \in \mathbb{R}$ because we want to solve the problem with optimal regularity ($H^{s,\delta}_{\mathrm{b}} \times H^{s-1,\delta+1}_{\mathrm{b}}$ for $s > d/2$ in our case). Later we will use those spaces for vector valued functions, but we shall use the same notation $H^{s,\delta}_{\mathrm{b}}$.

We recall some properties of the b-Sobolev space below. The most important one is the Littlewood–Paley decomposition (3.1), which can serve as an alternative definition of the b-Sobolev space and is what we actually use in this paper. We write $A \lesssim B$ if there is some constant $C > 0$ such that $A \le CB$. The constant C might depend on various things like dimension d, Sobolev exponents s, δ and so on. We will usually suppress the dependence since it is not important in this paper. We use $A \approx B$ to mean $A \lesssim B$ and $B \lesssim A$.

Proposition 3. *We have the following properties.*

(a) *Littlewood–Paley decomposition: let $\mathcal{B}_r = B_r(0)$ be the ball with radius $r > 0$ and $\mathcal{D}_j = \{2^{j-2} < |x| < 2^{j+2}\} \subset \mathbb{R}^d$ be dyadic annuli for $j \ge 0$, and*

$$\phi_j : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto 2^j x, \quad j \ge 0.$$

Let $\sum_{j=0}^{\infty} \chi_j = 1$ be a dyadic partition of unity such that

$$\chi_0 \in C^\infty_c(\mathcal{B}_1; [0,1]), \quad \chi \in C^\infty_c(\mathcal{D}_0; [0,1]), \quad \chi_j = \phi_{j*}(\chi) := \chi \circ \phi_j^{-1},\ j \ge 1.$$

Then

$$\|u\|^2_{H^{s,\delta}_{\mathrm{b}}} \approx \sum_{j=0}^{\infty} 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j u)\|^2_{H^s}. \tag{3.1}$$

(b) *Sobolev embedding: for $s > d/2$,*

$$\|\langle x\rangle^{d/2+\delta} u\|_{L^\infty} \lesssim \|u\|_{H^{s,\delta}_{\mathrm{b}}}. \tag{3.2}$$

(c) *Bilinear estimate: the multiplication is bounded on the following spaces*

$$(u,v) \mapsto uv : H_{\mathrm{b}}^{s_1,\delta_1} \times H_{\mathrm{b}}^{s_2,\delta_2} \to H_{\mathrm{b}}^{s,\delta} \tag{3.3}$$

for $\delta_1 + \delta_2 = \delta - d/2$, $s_1 + s_2 > 0$,

$$s = \begin{cases} \min(s_1, s_2), & \max(s_1, s_2) > \frac{d}{2}, \\ s_1 + s_2 - \frac{d}{2}, & \max(s_1, s_2) < \frac{d}{2}. \end{cases}$$

Proof. We refer to [Hin22, §2.4] for a discussion on basic properties of b-Sobolev spaces.

(a) The Littlewood–Paley decomposition (3.1) is [Hin22, Lemma 2.3]. By interpolation and duality, it suffices to check for $s \in \mathbb{N}_0$, and it is direct to check (3.1) inductively for s. Note our convention on δ is shifted by $d/2$ compared to [Hin22, §2.4] due to a different choice of density near the boundary.

(b) The Sobolev embedding is essentially [Hin22, Corollary 2.4]. It is proved as a corollary of (3.1):

$$\|\langle x\rangle^{d/2+\delta} u\|_{L^\infty} \lesssim \sup_j 2^{j(d/2+\delta)} \|\chi_j u\|_{L^\infty} \lesssim \sup_j 2^{j(d/2+\delta)} \|\phi_j^*(\chi_j u)\|_{H^s} \lesssim \|u\|_{H_{\mathrm{b}}^{s,\delta}}.$$

(c) The bilinear estimate (3.3) is again a corollary of (3.1). We introduce a new cutoff $\tilde{\chi}_0 \in C_c^\infty(\mathcal{B}_1; [0,1])$ such that $\tilde{\chi}_0(x) = 1$ on $\operatorname{supp} \chi_0$. Similarly, let $\tilde{\chi}_j = \phi_{j*}\tilde{\chi}$ for $j \geq 1$ where $\tilde{\chi} \in C_c^\infty(\mathcal{D}_0; [0,1])$ and $\tilde{\chi}(x) = 1$ for $x \in \operatorname{supp} \chi$.

$$\begin{aligned} \|uv\|_{H_{\mathrm{b}}^{s,\delta}}^2 &\lesssim \sum_j 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j uv)\|_{H^s}^2 \\ &\lesssim \sum_j 2^{2j(\delta+d/2)} \|\phi_j^*(\chi_j u)\|_{H^{s_1}}^2 \|\phi_j^*(\tilde{\chi}_j v)\|_{H^{s_2}}^2 \\ &\lesssim \left(\sum_j 2^{2j(\delta_1+d/2)} \|\phi_j^*(\chi_j u)\|_{H^{s_1}}^2 \right) \left(\sum_j 2^{2j(\delta_2+d/2)} \|\phi_j^*(\tilde{\chi}_j u)\|_{H^{s_2}}^2 \right) \\ &\lesssim \|u\|_{H_{\mathrm{b}}^{s_1,\delta_1}}^2 \|v\|_{H_{\mathrm{b}}^{s_2,\delta_2}}^2. \end{aligned}$$

□

We now show our solution operator is bounded on $H_{\mathrm{b}}^{s,\delta}$. One important property of our solution operator is that it is outgoing in the following sense. We will use the same notation K for an operator and its Schwartz kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$.

Definition 4. *Let* $\Omega \subset \mathbb{R}^d$ *be a subset of* $\mathbb{R}^d$. $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ *be the Schwartz kernel of an operator which we also denote by* K. *Suppose* K *has the property*

$$u \in C_c^\infty(\mathbb{R}^d), \ \operatorname{supp} u \subset \overline{\Omega} \Longrightarrow \operatorname{supp} Ku \subset \overline{\Omega}.$$

We say K *is outgoing (incoming, respectively) on* Ω *if there exists a constant* $C > 0$ *such that* $K(x,y) = 0$ *for* $x, y \in \Omega$ *and* $|y| > C|x| + C$ *(for* $|x| > C|y| + C$*, respectively). We also say* K *is diagonal if* K *is both incoming and outgoing.*

Example 1. *Let $\Omega = \Omega_{\omega,\theta}$ for some $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta < \pi/2$. Let $\chi \in C_c^\infty(\Omega \cap \mathbb{S}^{d-1})$. The solution operator S constructed in Theorem 4 with integration kernel $(K_\chi^{ij}(x-y), L_{\chi,e_k}^{ij}(x-y))$ is outgoing on Ω, where e_k is the unit vector in the x_k direction. Indeed we have the following stronger property: $S(x,y) = 0$ for $|x| < |y|$, i.e. the value of $Sf(x)$ only depends on $f(y)$ for $|y| \leq |x|$.*

Lemma 5. *Suppose K is a diagonal operator on Ω such that for some $s_1, s_2, k \in \mathbb{R}$, K is bounded on $H^{s_1}_{\mathrm{comp}} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi(x) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, the operator with Schwartz kernel*

$$K(2^j x, 2^j y)\chi(x)\chi(y) \tag{3.4}$$

is bounded by $C2^{j(k-d)}$ on $H^{s_1} \to H^{s_2}$ where C is independent of $j \in \mathbb{N}$. Then

$$K : H_{\mathrm{b}}^{s_1,\delta}(\Omega) \to H_{\mathrm{b}}^{s_2,\delta-k}(\Omega)$$

is bounded for any $\delta \in \mathbb{R}$. In particular, if

$$|\partial_{x,y}^\ell K(x,y)| \lesssim \langle x \rangle^{k-d-\ell},$$

then $K : H_{\mathrm{b}}^{-N,\delta} \to H_{\mathrm{b}}^{N,\delta-k}$ is bounded for any $N \in \mathbb{N}$, $\delta \in \mathbb{R}$.

Proof. Recall for $u \in C_c^\infty(\Omega)$,

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \approx \sum_j 2^{2j(\delta+d/2-k)} \|\phi_j^*(\chi_j K u)\|^2_{H^{s_2}}$$

Let

- $\tilde{\chi}_0 \in C_c^\infty(\mathbb{R}^d; [0,1])$;
- $\tilde{\chi} \in C_c^\infty(\{2^{-C} < |x| < 2^C\}; [0,1])$ and $\tilde{\chi}_j(x) = \phi_{j*}\tilde{\chi}(x) = \tilde{\chi}(2^{-j}x)$ for $j \geq 1$

be another family of cutoffs such that for any $(x,y) \in \operatorname{supp} K(x,y)$, $x \in \operatorname{supp} \chi_j$ implies $\tilde{\chi}_j(y) = 1$ (this is possible because K is diagonal). Therefore by (3.4)

$$\|\phi_j^*(\chi_j K u)\|_{H^{s_2}} = \|\phi_j^*(\chi_j K(\tilde{\chi}_j u))\|_{H^{s_2}} \lesssim 2^{jk} \|\phi_j^*(\tilde{\chi}_j u)\|_{H^{s_1}}$$

and

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \lesssim \sum_j 2^{2j(\delta+d/2-k)} 2^{2jk} \|\phi_j^*(\tilde{\chi}_j u)\|^2_{H^{s_1}} \approx \|u\|^2_{H_{\mathrm{b}}^{s_1,\delta}}.$$ ∎

Lemma 6. *Suppose K is an outgoing (incoming, respectively) operator on Ω such that for some $s_1, s_2, k \in \mathbb{R}$, K is bounded on $H^{s_1}_{\mathrm{comp}} \to H^{s_2}_{\mathrm{loc}}$, and for any $\chi(x) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, the operator with Schwartz kernel*

$$K(2^j x, 2^{j'} y)\chi(x) \quad (K(2^j x, 2^{j'} y)\chi(y), \text{respectively})$$

is bounded by $C2^{j(k-d)}$ ($C2^{j'(k-d)}$, respectively) on $H^{s_1} \to H^{s_2}$ where C is independent of $j, j' \in \mathbb{N}$. Then

$$K : H_{\mathrm{b}}^{s_1,\delta}(\Omega) \to H_{\mathrm{b}}^{s_2,\delta-k}(\Omega)$$

is bounded for $\delta < d/2$ ($\delta > -d/2 + k$, respectively).

Proof. We will only show the outgoing case since the incoming case is similar. For K outgoing, there exists a constant $C > 0$ such that $(x, y) \in \operatorname{supp} K(x, y)$, $x \in \operatorname{supp} \chi_j$, $y \in \operatorname{supp} \chi_{j'}$ implies that $j' \leq j + C$. Thus for $\beta \in \mathbb{R}$ be to determined later and $u \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} &\approx \sum_j 2^{2j(\delta+d/2-k)} \|\phi_j^*(\chi_j K u)\|^2_{H^{s_2}} \\
&\approx \sum_j 2^{2j(\delta+d/2-k)} \left\| \sum_{j'\leq j+C} \phi_j^*(\chi_j K \chi_{j'} u) \right\|^2_{H^{s_2}} \\
&\lesssim \sum_j 2^{2j(\delta+d/2-k)} \left(\sum_{j'\leq j+C} 2^{j(k-d)} 2^{j'd} \|\phi_{j'}^*(\chi_{j'} u)\|_{H^{s_1}} \right)^2 \\
&\lesssim \sum_j 2^{2j(\delta+d/2-k)} \left(\sum_{j'\leq j+C} 2^{j(k-d)} 2^{j'd} 2^{\beta(j-j')} \right) \left(\sum_{j'\leq j+C} 2^{j(k-d)} 2^{j'd} 2^{\beta(j'-j)} \|\phi_{j'}^*(\chi_{j'} u)\|^2_{H^{s_1}} \right).
\end{aligned}$$

Since $\delta < d/2$, we may choose $\beta \in \mathbb{R}$ such that $2\delta < \beta < d$, so (in the first term on sums over j', while in the second term on sums over j)

$$\sum_{j'\leq j+C} 2^{j(k-d)} 2^{j'd} 2^{\beta(j-j')} \lesssim 2^{jk}, \quad \sum_{j'\leq j+C} 2^{2j(\delta+d/2-k)} 2^{jk} 2^{j(k-d)} 2^{j'd} 2^{\beta(j'-j)} \lesssim 2^{2j'(\delta+d/2)}.$$

So we conclude that

$$\|Ku\|^2_{H_{\mathrm{b}}^{s_2,\delta-k}} \lesssim \sum_{j'} 2^{2j'(\delta+d/2)} \|\phi_{j'}^*(\chi_{j'} u)\|^2_{H^{s_1}} \approx \|u\|^2_{H_{\mathrm{b}}^{s_1,\delta}}. \qquad \square$$

Remark 1. *The threshold $\delta < d/2$ is natural. Since for an outgoing operator K with Schwartz kernel $\langle x - y\rangle^{k-d}$ and compactly supported input $f \in C_c^\infty$, the best decay one can hope is*

$$|Kf(x)| \lesssim \langle x\rangle^{k-d} \notin H_{\mathrm{b}}^{s,\delta-k}, \quad \delta \geq d/2.$$

Proposition 7. *Let $K \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap \mathscr{D}'(\mathbb{R}^d)$ be a homogeneous distribution of degree $k - d$, $0 < k < d$. For $-d/2 < \delta < d/2 - k$, the map*

$$u \mapsto K * u(x) := \int_{\mathbb{R}^d} K(x-y)u(y)dy \tag{3.5}$$

is bounded on $H_{\mathrm{b}}^{s-k,\delta+k} \to H_{\mathrm{b}}^{s,\delta}$. Indeed, the operator $u \mapsto K * u$ can be decomposed into three parts $K_{\mathrm{diag}} + K_{\mathrm{in}} + K_{\mathrm{out}}$ such that

- *(a) $K_{\mathrm{diag}}(x, y)$ is supported in $\{2^{-1}|y| \leq |x| \leq 2|y|\} \cup \mathcal{B}_1 \times \mathcal{B}_1$ and $K_{\mathrm{diag}} : H_{\mathrm{b}}^{s-k,\delta+k} \to H_{\mathrm{b}}^{s,\delta}$ is bounded for $s \in \mathbb{R}$ and $\delta \in \mathbb{R}$;*
- *(b) $K_{\mathrm{out}}(x, y)$ is supported in $\{|x| \geq (1 + c)|y|\} \setminus \mathcal{B}_c \times \mathcal{B}_c$ for some $c > 0$, and $K_{\mathrm{out}} : H_{\mathrm{b}}^{-N,\delta+k} \to H_{\mathrm{b}}^{N,\delta}$ is bounded for any $N \in \mathbb{N}$ and $\delta < d/2 - k$;*

(c) *$K_{\mathrm{in}}(x,y)$ is supported in $\{|x| \le (1-c)|y|\} \setminus \mathcal{B}_c \times \mathcal{B}_c$ for some $c > 0$, and $K_{\mathrm{in}} : H_{\mathrm{b}}^{-N,\delta+k} \to H_{\mathrm{b}}^{N,\delta}$ is bounded for any $N \in \mathbb{N}$ and $\delta > -d/2$.*

Moreover, if we suppose $K(x-y)$ is outgoing (incoming, respectively), then for $\delta < d/2 - k$ (for $\delta > -d/2$, respectively), the map (3.5) is bounded on $H_{\mathrm{b}}^{s-k,\delta+k}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega)$.

Proof. We decompose $K(x-y) = K_{\mathrm{diag}}(x,y) + K_{\mathrm{in}}(x,y) + K_{\mathrm{out}}(x,y)$, where

$$K_{\mathrm{diag}}(x,y) = K(x-y)\varphi(x,y)$$

where $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a cutoff function such that $\varphi(x,y) = 1$ near $\{|x| = |y|\}$ and

$$\varphi(x,y) = \psi\left(\frac{|x|-|y|}{|y|}\right), \quad |x|+|y| > 1 \tag{3.6}$$

for some other cutoff $\psi \in C_c^\infty(-1/2, 1/2)$ such that $\psi = 1$ near 0; $K_{\mathrm{in}} = (K - K_{\mathrm{diag}})\mathbb{1}_{|x|<|y|}$ is incoming part of K, and $K_{\mathrm{out}} = (K - K_{\mathrm{diag}})\mathbb{1}_{|x|>|y|}$ is the outgoing part of K. One can check K_{in} is incoming, K_{out} is outgoing and K_{diag} is diagonal in the sense of Definition 4.

(a) For the diagonal part K_{diag}, we notice for $2^{-j} < |x|+|y| < C$,

$$K_{\mathrm{diag}}(2^j x, 2^j y) = K(2^j(x-y))\varphi(2^j x, 2^j y) = 2^{j(k-d)}K(x-y)\varphi(x,y)$$

is a singular integral operator bounded by $C2^{j(k-d)}$ on $H^{s-k} \to H^s$. So the result follows from Lemma 5.

(b) For the outgoing part K_{out}, we notice that for $|x| > 2^{-j}$ and $|x|+|y| < C$,

$$\begin{aligned} K_{\mathrm{out}}(2^j x, 2^{j'} y) &= K(2^j x - 2^{j'} y)(1 - \varphi(2^j x, 2^{j'} y))\mathbb{1}_{2^j|x|>2^{j'}|y|} \\ &= 2^{j(k-d)}K(x - 2^{j'-j}y)(1-\varphi(x, 2^{j'-j}y))\mathbb{1}_{|x|>2^{j'-j}|y|} \end{aligned}$$

satisfies

$$|\partial_{x,y}^\ell K_{\mathrm{out}}(2^j x, 2^{j'} y)| \lesssim 2^{j(k-d)}$$

and is thus bounded on $H^{-N} \to H^N$ for any $N \in \mathbb{N}$.

(c) Similarly for the incoming parting K_{in}, for $|y| > 2^{-j'}$ and $|x|+|y| < C$,

$$\begin{aligned} K_{\mathrm{in}}(2^j x, 2^{j'} y) &= K(2^j x - 2^{j'} y)(1 - \varphi(2^j x, 2^{j'} y))\mathbb{1}_{2^j|x|<2^{j'}|y|} \\ &= 2^{j'(k-d)}K(2^{j-j'}x - y)(1-\varphi(2^{j-j'}x, y))\mathbb{1}_{|y|>2^{j-j'}|x|} \end{aligned}$$

satisfies

$$|\partial_{x,y}^\ell K_{\mathrm{in}}(2^j x, 2^{j'} y)| \lesssim 2^{j'(k-d)}$$

and is bounded on $H^{-N} \to H^N$ for any $N \in \mathbb{N}$.

Finally, if $K(x-y)$ is outgoing (incoming, respectively), the same proof works without dealing with the incoming (outgoing, respectively) part. $\square$

3.2. **Smoothness of curvature.** As a corollary of the bilinear estimate in Proposition 3(c), we can now conclude the maps we consider are smooth on b-Sobolev spaces.

Lemma 8. *Let $s > d/2$, $\delta > -d/2$. In a small neighbourhood of δ_{ij}, the inverse matrix map*

$$(g_{ij}) \mapsto (g^{ij}) : (\delta_{ij}) + H_{\mathrm{b}}^{s,\delta} \to (\delta_{ij}) + H_{\mathrm{b}}^{s,\delta}$$

is a smooth map.

Proof. By Cramer's rule, the inverse matrix map can be obtained as compositions of multiplications and the following map (which appeals in $\det(g_{ij})^{-1}$ if we write $\det(g_{ij}) = 1 - h$):

$$T : h \mapsto \frac{1}{1-h} - 1 = h + h^2 + h^3 + \cdots$$

Since multiplication is bounded by Proposition 3(c), it suffices to prove $T : H_{\mathrm{b}}^{s,\delta} \to H_{\mathrm{b}}^{s,\delta}$ is smooth for $\|h\|_{H_{\mathrm{b}}^{s,\delta}} \ll 1$. The boundedness is a corollary of the bilinear estimate (3.3) again. To prove the boundedness of the derivatives, just observe

$$DT_{h_0}(h) = (1 + 2h_0 + 3h_0^2 + \cdots)h, \quad D^2T_{h_0}(h_1, h_2) = (2 + 6h_0 + 12h_0^2 + \cdots)h_1h_2, \cdots$$

are all continuous multilinear maps in $H_{\mathrm{b}}^{s,\delta}$. □

Proposition 9. *For $s > d/2$, $\delta > -d/2$, the operator*

$$(h, \pi) \mapsto (M_h^{(2)}(h, \partial^2 h), M_h^{(1)}(\partial h, \partial h), M_h^{(0)}(\pi, \pi), N_h^{(1)j}(h, \partial\pi), N_h^{(0)j}(\partial h, \pi))$$

is smooth $H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1} \to H_{\mathrm{b}}^{s-2,\delta+2}$ in a small neighbourhood of $(0,0)$.

Proof. Recall from (2.3)-(2.5) that $M_h^{(\ell)}$ and $N_h^{(\ell)j}$ are bilinear expressions which depend smoothly on h. Moreover, the dependence on h only involves multiplication and the inverse matrix map, which are both smooth by Proposition 3(c) and Lemma 8. To get the correct exponents, one also uses that $\partial : H_{\mathrm{b}}^{s,\delta} \to H_{\mathrm{b}}^{s-1,\delta+1}$ is bounded. □

4. Solving the nonlinear equation

In this section we use our solution operators from Theorem 4 to prove Theorem 1 and 2. The basic idea is to solve the linearized equation (1.2) first and then use the Banach fixed point theorem (or Picard iteration) to upgrade to a solution of the nonlinear system (1.1).

In §4.1, we give a simple construction of solutions of the linearized equation (1.2), which shows the linearized operator has a big kernel. In particular, one can choose a solution so that the linearized Riemann curvature does not vanish, which will ensure that the final solution to (1.1) we get is not isometric to Euclidean space. In §4.2, we solve the nonlinear system (1.1) and prove it is actually smooth and has the decay

(1.3). In §4.3 we prove Theorem 2 following a similar strategy by also using the Bogovskii-type operator introduced in [MOT23].

4.1. **Construction of solutions to the linearized equation.** Our construction of solutions of the constraint equation (1.1) relies on solutions of the linearized equation (1.2). There are many compactly supported solutions to the homogeneous linearized equations (1.2). A basic observation is that to solve the double divergence equation we only need to solve the symmetric divergence equation

$$\partial_j \pi^{jk} = 0 \tag{4.1}$$

since this would give $\partial_j \partial_k \pi^{jk} = 0$. For (4.1), we may take an arbitrary $\phi \in C_c^\infty(\mathbb{R}^d)$ and

$$\pi^{11} = \partial_2\partial_2\phi,\ \pi^{12} = \pi^{21} = -\partial_1\partial_2\phi,\ \pi^{22} = \partial_1\partial_1\phi,\ \pi^{jk} = 0 \text{ for } \{j,k\} \not\subset \{1,2\}.$$

Then we get a solution to $\partial_j \pi^{jk} = 0$.

4.2. **Proof of Theorem 1.** In this section we prove Theorem 1 by Banach fixed point theorem (or Picard iteration).

Proof of Theorem 1. The proof goes by three steps.

Step 1: Existence. Let

$$\begin{aligned} &P(h,\pi) = (\partial_i\partial_j h^{ij}, \partial_i \pi^{ij}) \\ &\Phi(h,\pi) = (M_h^{(2)}(h, \partial^2 h) + M_h^{(1)}(\partial h, \partial h) + M_h^{(0)}(\pi,\pi), N_h^{(1)j}(h, \partial\pi) + N_h^{(0)j}(\partial h, \pi)). \end{aligned}$$

Then the equations (1.1) become

$$P(h,\pi) = \Phi(h,\pi). \tag{4.2}$$

Let $\Omega = \Omega_{\omega,\theta}$ be a cone centered at 0 in $\mathbb{R}^d$ and $(h_0,\pi_0) \in C^\infty(\mathbb{R}^d)$ be a solution of the linearized equation $P(h_0,\pi_0) = 0$ supported in Ω with decay rate (1.4) (such solutions exist even with compact support, as discussed in §4.1). Let $s > d/2$, $-d/2 < \delta < d/2 - 2$ and S be the solution operator given by Theorem 4 with $\operatorname{supp}\chi \subset \Omega \cap \mathbb{S}^{d-1}$ as in Example 1, such that

$$f \in \mathscr{D}'(\mathbb{R}^d),\ \operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Sf \subset \Omega,\ PSf = f$$

due to the convexity of the cone Ω. We consider the following fixed point problem

$$(h_1,\pi_1) = S\Phi(h_0 + h_1, \pi_0 + \pi_1). \tag{4.3}$$

By Proposition 9, $\Phi : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s-2,\delta+2}(\Omega)$ is a smooth map with the differential at zero $D\Phi_0 = 0$. By choosing $\|(h,\pi)\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}} \le \varepsilon := \|(h_0,\pi_0)\|_{H_{\mathrm{b}}^{s,\delta}\times H_{\mathrm{b}}^{s-1,\delta+1}}$

for some sufficiently small $\varepsilon > 0$ we get

$$\|\Phi(h_0 + h, \pi_0 + \pi)\|_{H_{\mathrm{b}}^{s-2,\delta+2}} \lesssim \varepsilon^2,$$
$$\|\Phi(h_0 + h, \pi_0 + \pi) - \Phi(h_0 + \tilde{h}, \pi_0 + \tilde{\pi})\|_{H_{\mathrm{b}}^{s-2,\delta+2}} \lesssim \varepsilon \|(h, \pi) - (\tilde{h}, \tilde{\pi})\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}.$$

By Proposition 7, $S : H_{\mathrm{b}}^{s-2,\delta+2}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$ is bounded. For ε sufficiently small, the map

$$F(h, \pi) = S\Phi(h_0 + h, \pi_0 + \pi) : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \quad (4.4)$$

is a contraction on $\{\|(h, \pi)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \leq \varepsilon\}$. By the Banach fixed point theorem, there exists a unique fixed point $(h_1, \pi_1) \in H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$ such that

$$(h_1, \pi_1) = S\Phi(h_0 + h_1, \pi_0 + \pi_1), \quad \|(h_1, \pi_1)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}} \lesssim \|(h_0, \pi_0)\|^2_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}.$$

This implies that $(h_0 + h_1, \pi_0 + \pi_1)$ solves (4.2):

$$P(h_0 + h_1, \pi_0 + \pi_1) = P(h_1, \pi_1) = \Phi(h_0 + h_1, \pi_0 + \pi_1).$$

Remark 2. *An alternative way is to notice that*

$$P - \Phi : H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega) \to H_{\mathrm{b}}^{s-2,\delta+2}(\Omega)$$

is a smooth map with surjective differential at 0 *(with a right inverse given by our solution operator). The inverse function theorem then shows that locally* $(P-\Phi)^{-1}(0) \cap \mathrm{nbd}(0)$ *is diffeomorphic to* $\ker P \cap \mathrm{nbd}(0)$*. In other words, the set of solutions of* (4.2) *in* $H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$ *near the trivial solution forms a Hilbert submanifold of* $H_{\mathrm{b}}^{s,\delta}(\Omega) \times H_{\mathrm{b}}^{s-1,\delta+1}(\Omega)$*, parameterized by* $\ker P \cap \mathrm{nbd}(0)$*.*

Step 2: Smoothness. Now we show the solution we obtain is actually smooth, even though s is fixed. The regularity comes from applying the solution operator to the nonlinearity.

Let $s = d + 2$ and $-d/2 < \delta < d/2 - 2$. We claim for ε sufficiently small,

$$(h_1, \pi_1) \in H_{\mathrm{b}}^{\infty,\delta} \times H_{\mathrm{b}}^{\infty,\delta+1} \quad (4.5)$$

where $H_{\mathrm{b}}^{\infty,\delta} := \cap_s H_{\mathrm{b}}^{s,\delta}$.

We prove it inductively: suppose $(h_1, \pi_1) \in H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ for integer $s \geq d+2$, we will show $(h_1, \pi_1) \in H_{\mathrm{b}}^{s+1,\delta} \times H_{\mathrm{b}}^{s,\delta+1}$.

We decompose $S = S_{\mathrm{diag}} + S_{\mathrm{out}}$ as in the proof of Proposition 7 such that $S_{\mathrm{diag}}(x, y)$ is supported near the diagonal and S_{out} is supported in $\{|x| \geq (1 + c)|y|\} \setminus \mathcal{B}_c \times \mathcal{B}_c$ for some $c > 0$. Note there $S_{\mathrm{in}} = 0$ due to the outgoing property of S. Recall (h_1, π_1) solves the equation (4.3), which can be written as

$$(h_1, \pi_1) = S_{\mathrm{diag}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)) + S_{\mathrm{out}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)). \quad (4.6)$$

By bilinear estimate (3.3), $\Phi(h_0+h_1,\pi_0+\pi_1)\in H_{\mathrm{b}}^{s-2,\delta+2}$. By Proposition 7(b), $S_{\mathrm{out}}(\Phi(h_0+h_1,\pi_0+\pi_1))\in H_{\mathrm{b}}^{\infty,\delta}\times H_{\mathrm{b}}^{\infty,\delta+1}$. Now we differentiate (4.6),

$$\partial(h_1,\pi_1)=[\partial,S_{\mathrm{diag}}](\Phi(h_0+h_1,\pi_0+\pi_1))+S_{\mathrm{diag}}(\partial\Phi(h_0+h_1,\pi_0+\pi_1))+\partial S_{\mathrm{out}}(\Phi(h_0+h_1,\pi_0+\pi_1)).$$

Recall

$$S_{\mathrm{diag}}(x,y)=(S^{(1)}_{\mathrm{diag}}(x,y),S^{(2)}_{\mathrm{diag}}(x,y))=(K^{(1)}(x-y),K^{(2)}(x-y))\varphi(x,y)$$

where φ is defined as in (3.6) and $K^{(j)}$ is homogeneous of degree k_j-d with $k_1=2$, $k_2=1$. The Schwartz kernel of $[\partial,S^{(j)}_{\mathrm{diag}}]$ is given by

$$K^{(j)}(x-y)(\partial_x-\partial_y)\varphi(x,y)$$

which vanishes near $\{|x|=|y|\}$. Moreover, we have the following bound

$$\partial^\ell_{x,y}(K^{(j)}(x-y)(\partial_x-\partial_y)\varphi(x,y))\lesssim\langle x\rangle^{k_j-d-\ell-1},\quad \ell\in\mathbb{N},\ j=1,2.$$

By Lemma 5, this implies boundedness of (we actually have better regularity improvement but we write a weaker estimate here which works for other cases we consider later in Theorem 2)

$$[\partial,S_{\mathrm{diag}}]:H_{\mathrm{b}}^{s-2,\delta+2}\to H_{\mathrm{b}}^{s,\delta+1}\times H_{\mathrm{b}}^{s-1,\delta+2},\quad s\in\mathbb{R},\ \delta\in\mathbb{R}.$$

So $[\partial,S_{\mathrm{diag}}](\Phi(h_0+h_1,\pi_0+\pi_1))\in H_{\mathrm{b}}^{s,\delta+1}\times H_{\mathrm{b}}^{s-1,\delta+2}$. We are left with $S_{\mathrm{diag}}(\partial\Phi(h_0+h_1,\pi_0+\pi_1))$. Note we can differentiate (4.3) s times and get

$$\partial^s(h_1,\pi_1)=S_{\mathrm{diag}}(\partial^s\Phi(h_0+h_1,\pi_0+\pi_1))+H_{\mathrm{b}}^{1,\delta+s}\times H_{\mathrm{b}}^{0,\delta+s+1}.$$

Now we note

$$\partial^s\Phi(h_0+h_1,\pi_0+\pi_1)=(M_{h_0,h_1}(h_0+h_1,\partial^{s+2}h_1),N_{h_0,h_1}(h_0+h_1,\partial^{s+1}\pi_1))+H_{\mathrm{b}}^{-1,\delta+s+2}$$

where M,N are bilinear forms depending on h_0,h_1. So

$$\|\partial^s\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H_{\mathrm{b}}^{-1,\delta+s+2}}\lesssim\|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}(\|h_1\|_{H_{\mathrm{b}}^{s+1,\delta}}+\|\pi_1\|_{H_{\mathrm{b}}^{s,\delta+1}})+C_s.$$

Since $S_{\mathrm{diag}}:H_{\mathrm{b}}^{-1,\delta+s+2}\to H_{\mathrm{b}}^{1,\delta+s}\times H_{\mathrm{b}}^{0,\delta+s+1}$ is bounded by Proposition 7(a), this implies that

$$\|(h_1,\pi_1)\|_{H_{\mathrm{b}}^{s+1,\delta}\times H_{\mathrm{b}}^{s,\delta+1}}\lesssim\|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}(\|h_1\|_{H_{\mathrm{b}}^{s+1,\delta}}+\|\pi_1\|_{H_{\mathrm{b}}^{s,\delta+1}})+C_s.$$

Since $\|h_0+h_1\|_{H_{\mathrm{b}}^{d/2+1,\delta}}$ is small, this shows

$$(h_1,\pi_1)\in H_{\mathrm{b}}^{s+1,\delta}\times H_{\mathrm{b}}^{s,\delta-1}.$$

This finishes the induction and proves (4.5).

Step 3: Tail estimate. Now we prove the last part of Theorem 1, namely the decay rate estimate (1.3). Roughly speaking, if the nonlinearity is integrable, then the decay rate comes from applying the solution operator to the nonlinearity, and should be the same as the decay rate of the Schwartz kernel of the solution operator. For simplicity of the proof, we choose $\delta = d/2 - 2.1$. Since (h_0, π_0) satisfies (1.4), in order to show (1.3) it suffices to show the following

$$|\partial^\ell h_1(x)| \lesssim \langle x\rangle^{2-d-\ell}, \quad |\partial^\ell \pi_1(x)| \lesssim \langle x\rangle^{1-d-\ell}, \quad \ell \in \mathbb{N}. \tag{4.7}$$

First we note the bilinear estimate (3.3) actually implies

$$\Phi(h_0 + h_1, \pi_0 + \pi_1) \in H_{\mathrm{b}}^{\infty, 2\delta+2+d/2} \subset H_{\mathrm{b}}^{\infty, d/2+1/2}.$$

As in the previous step, we decompose $S = S_{\mathrm{diag}} + S_{\mathrm{out}}$. Consider (4.6):

$$(h_1, \pi_1) = S_{\mathrm{diag}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)) + S_{\mathrm{out}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)).$$

Since $S_{\mathrm{diag}} : H_{\mathrm{b}}^{s-2,\delta+2} \to H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}$ is bounded for any $s, \delta \in \mathbb{R}$, we have $S_{\mathrm{diag}}(\Phi(h_0 + h_1, \pi_0 + \pi_1)) \in H_{\mathrm{b}}^{\infty,d/2-3/2} \times H_{\mathrm{b}}^{\infty,d/2-1/2}$. By Sobolev embedding (3.2), this implies

$$|\partial^\ell S_{\mathrm{diag}}^{(j)}(\Phi(h_0 + h_1, \pi_0 + \pi_1))| \lesssim \langle x\rangle^{k_j-1/2-d-\ell}, \quad \ell \in \mathbb{N},\ j = 1, 2. \tag{4.8}$$

Let $S_{\mathrm{out}}^{(1)}$, $S_{\mathrm{out}}^{(2)}$ be the first and second component of S_{out}, and $k_1 = 2$, $k_2 = 1$. Then

$$|\partial^\ell S_{\mathrm{out}}^{(j)}(x, y)| \lesssim \langle x\rangle^{k_j-d-\ell}, \quad \ell \in \mathbb{N}.$$

This implies

$$|\partial^\ell S_{\mathrm{out}}^{(j)}(\Phi(h_0 + h_1, \pi_0 + \pi_1))| \lesssim \langle x\rangle^{k_j-d-\ell} \int_{\mathbb{R}^d} |\Phi(h_0 + h_1, \pi_0 + \pi_1)(y)| dy. \tag{4.9}$$

Since $\Phi(h_0 + h_1, \pi_0 + \pi_1) \in H_{\mathrm{b}}^{\infty,d/2+1/2}$, the integral on the right hand side is finite by Sobolev embedding (3.2). Putting (4.8) and (4.9) we conclude (4.7). □

4.3. **Gluing construction of the solution.** In this section we provide the proof of the gluing result in Theorem 2.

Let $\Omega_{y_0,\omega,\theta_0} \subset \Omega_{y_0,\omega,\theta}$ be the two cones centered at $y_0 \in \mathbb{R}^d$ in Theorem 2. Let $\chi \in C^\infty(\mathbb{R}^d; [0, 1])$ be a cutoff such that

$$\chi(x) = 1 \text{ near } \Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0), \quad \chi(x) = 0 \text{ in } \mathbb{R}^d \setminus (\Omega_{y_0,\omega,\theta} \cup B_1(y_0))$$

and $\chi(x) = \varphi((x - y_0)/|x - y_0|)$ for $x \in \Omega_{y_0,\omega,\theta} \setminus B_{10}(y_0)$ and $\varphi \in C^\infty(\mathbb{S}^{d-1})$. After cutting off we only need to solve the constraint equation

$$P(\chi h_0 + h, \chi\pi_0 + \pi) = \Phi(\chi h_0 + h, \chi\pi_0 + \pi) \tag{4.10}$$

with (h, π) supported inside the region $(\Omega_{y_0,\omega,\theta} \cup B_1(y_0)) \setminus (\Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0))$. By choosing $|y_0| \gg 1$, we may assume $\|(\chi h_0, \chi\pi_0)\|_{H_{\mathrm{b}}^{s,\delta} \times H_{\mathrm{b}}^{s-1,\delta+1}}$ is sufficiently small, and then we can apply the following solution operator to get a solution.

Proposition 10. *Let* $\Omega_{\text{int}} := \overline{(\Omega_{y_0,\omega,\theta} \cup B_1(y_0)) \setminus (\Omega_{y_0,\omega,\theta_0} \cup B_{1/2}(y_0))}$, *then there is a solution operator bounded on the following spaces*

$$S_{\text{int}} : H_{\text{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}})$$

for $s \in \mathbb{R}$ *and* $\delta < d/2 - 2$, *such that for any* $f \in C_c^\infty(\Omega_{\text{int}})$, $\operatorname{supp} S_{\text{int}} f \subset \Omega_{\text{int}}$ *and* $PS_{\text{int}} f = f$.

Proof. Let $f \in H_{\text{b}}^{s-2,\delta+2}(\Omega_{\text{int}})$ and we would like to solve $Pu = f$. Since Ω_{int} is not a conic region, we need to use the Bogovskii-type solution operator in [MOT23]. Let $\Omega_3 := \Omega_{\text{int}} \cap B_3(y_0)$ and $\Omega_2 := \Omega_{\text{int}} \cap B_2(y_0)$. By [MOT23, Lemma 2.2, 2.3], there exists an operator S_0 with the following properties for $f \in C_c^\infty(\Omega_3)$:

- For $f \in C_c^\infty(\Omega_3)$, $\operatorname{supp} S_0 f \subset \Omega_3$.
- For $f \in C_c^\infty(\Omega_3)$, $PS_0(f) = f$ inside Ω_2;
- $S_0 : H^{s-2}(\Omega_3) \to H^s(\Omega_3) \times H^{s-1}(\Omega_3)$ is bounded for any $s \in \mathbb{R}$.
- $[\partial, S_0] : H^{s-2}(\Omega_3) \to H^s(\Omega_3) \times H^{s-1}(\Omega_3)$ is bounded for any $s \in \mathbb{R}$.

We remark that [MOT23, Lemma 2.3] constructed such operators for a star-shaped region and [MOT23, Lemma 2.2] explains how to put them together on an annulus. However, a similar procedure (see also [MOT23, Lemma 5.8 Step 1]) can give such Bogovskii-type operators on finite union of star-shaped regions. We emphasize that we only require $PS_0(f) = f$ on Ω_2 instead of the whole Ω_3 since this cannot be achieved unless an integrability condition [MOT23, Lemma 2.2 (S2)(T2)] is satisfied.

Let $\chi_0 \in C_c^\infty(B_3(y_0))$ such that $\chi_0 = 1$ on Ω_2. Then $S_0(\chi_0 f)$ solves $Pu = f$ in Ω_2. Now the outside region is conic and we can use our conic solution operators from Theorem 4. Since $\Omega_{\text{int}} \setminus \Omega_2$ is not convex, we make a partition of unity in angular variables and use the conic solution operator on each piece.

Suppose $(\Omega_{y_0,\omega,\theta} \setminus \Omega_{y_0,\omega,\theta_0}) \cap \mathbb{S}_{y_0}^{d-1} = \cup U_i$ and each U_i is star-shaped with respect to an open subset $V_i \subset U_i$. Let $\chi_i \in C_c^\infty(V_i)$ be a cutoff function supported in V_i, then as in Theorem 4 we have solution operators S_{χ_i} with respect to U_i so that $\operatorname{supp} f \subset y_0 + \mathbb{R}_{>1}(U_i - y_0)$ implies $\operatorname{supp} S_{\chi_i} f \subset y_0 + \mathbb{R}_{>1}(U_i - y_0)$ and $PS_{\chi_j} f = f$. We can take $\mathbb{R}_{>1}$ because of the strong outgoing property of the solution operator. We take a partition of unity $\tilde{\chi}_j$ with respect to the covering $\{U_i\}$ and define

$$S_1 = \sum_j S_{\chi_j} \tilde{\chi}_j.$$

The final solution operator S_{int} is defined to be

$$S_{\text{int}} f = S_1(f - PS_0(\chi_0 f)) + S_0(\chi_0 f) \in H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}}).$$

One can check it is bounded on $H_{\text{b}}^{s-2,\delta+2}(\Omega_{\text{int}}) \to H_{\text{b}}^{s,\delta}(\Omega_{\text{int}}) \times H_{\text{b}}^{s-1,\delta+1}(\Omega_{\text{int}})$ and

$$PS_{\text{int}} f = PS_1(f - PS_0(\chi_0 f)) + PS_0(\chi_0 f) = f - PS_0(\chi_0 f) + PS_0(\chi_0 f) = f. \quad \square$$

Remark 3. *The norm of $H_{\rm b}^{s,\delta}(\Omega_{\rm int})$ should be defined with respect to the center y_0, to make uniform estimates in y_0. For example, when $s \in \mathbb{N}$, we can use*

$$\|u\|^2_{H_{\rm b}^{s,\delta}(\Omega_{\rm int})} = \sum_{|\alpha|\le s} \|\langle x - y_0\rangle^{|\alpha|}\partial^\alpha u(x)\|^2_{L^2}.$$

Proof of Theorem 2. Following the procedure of the previous section, we can get a gluing solution from the solution operator $S_{\rm int}$ as follows. Consider the fixed point problem

$$(h,\pi) = S_{\rm int}(\Phi(\chi h_0 + h, \chi\pi_0 + \pi) - P(\chi h_0, \chi\pi_0)), \quad (h,\pi) \in H_{\rm b}^{s,\delta}(\Omega_{\rm int}) \times H_{\rm b}^{s-1,\delta+1}(\Omega_{\rm int}) \tag{4.11}$$

for the map $F(h,\pi) = S_{\rm int}(\Phi(\chi h_0 + h, \chi\pi_0 + \pi) - P(\chi h_0, \chi\pi_0))$. We choose $|y_0| \gg 1$ so that $\varepsilon := \|\chi h_0, \chi\pi_0)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}}$ is sufficiently small. Since $\Phi(\chi h_0, \chi\pi_0) - P(\chi h_0, \chi\pi_0)$ is supported in $\Omega_{\rm int}$ and $S_{\rm int}$ is bounded, we have

$$\|S_{\rm int}(\Phi(\chi h_0, \chi\pi_0) - P(\chi h_0, \chi\pi_0))\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}} \le C_0\varepsilon$$

and

$$\|F(h,\pi)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}} \le C_0\varepsilon + C\varepsilon^2, \quad \text{for } \|(h,\pi)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}} \le 2C_0\varepsilon.$$

So F maps $\{\|(h,\pi)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}} \le 2C_0\varepsilon\}$ to itself when ε is sufficiently small. Moreover,

$$\|F(h,\pi) - F(\tilde h, \tilde\pi)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}} \lesssim \varepsilon\|(h,\pi) - (\tilde h,\tilde\pi)\|_{H_{\rm b}^{s,\delta}\times H_{\rm b}^{s-1,\delta+1}}.$$

So for ε sufficiently small, F is a contraction. By Banach fixed point theorem we conclude a unique solution $(h_1,\pi_1) \in H_{\rm b}^{s,\delta}(\Omega_{\rm int}) \times H_{\rm b}^{s-1,\delta+1}(\Omega_{\rm int})$ solving the fixed point problem (4.11). We remark the solution operator $S_{\rm int}$ also has a decomposition

$$S_{\rm int} = S_{\rm int,diag} + S_{\rm int,out}$$

where $S_{\rm int,diag}$ is supported near the diagonal and is bounded on $H_{\rm b}^{s-2,\delta+2}(\Omega_{\rm int}) \to H_{\rm b}^{s,\delta}(\Omega_{\rm int}) \times H_{\rm b}^{s-1,\delta+1}(\Omega_{\rm int})$ for any $s,\delta \in \mathbb{R}$, and $S_{\rm int,out} = (S^{(1)}_{\rm int,out}, S^{(2)}_{\rm int,out})$ is outgoing with the property

$$|\partial^\ell_{x,y} S^{(j)}_{\rm int,out}(x,y)| \lesssim \langle x\rangle^{k_j - d - \ell}, \quad k_1 = 2,\ k_2 = 1,\ \ell \in \mathbb{N}.$$

The proof for the smoothness and decay rate is thus similar to the previous argument. A slight difference is that ∂ does not commute with $S_{\rm int,diag}$ as before near $\{|x| = |y|\}$ for $x, y \in B_{10}(y_0)$, coming from the Bogovskii-type operator S_0. In this compact region, we recall that $S_0 \in \Psi^{-2} \times \Psi^{-1}$ is a pseudo-differential operator (see [MOT23, Lemma 2.3]) and we have $[\partial, S_0] : H^{s-2} \to H^s \times H^{s-1}$ (see [MOT23, Lemma 2.2]). $\square$

5. Solving the problem in a degenerate sector

In this section we construct solutions of the Einstein constraint equations in the degenerate sector

$$\Omega = \Omega_\alpha := \{(x', x_d) \in \mathbb{R}^d \,:\, |x'| \leq x_d^\alpha,\, x_d \geq 1\}.$$

5.1. **Anisotropic Sobolev space.** In order to analyze the regularity of the solution, we introduce the *anisotropic weighted Sobolev space* defined as follows.

Definition 11. *Let $\alpha \in (0,1]$, $s \in \mathbb{R}$ and $\delta \in \mathbb{R}$, $H^{s,\delta}_\alpha(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm*

$$\|u\|^2_{H^{s,\delta}_\alpha} := \sum_{j=0}^\infty 2^{2j\left(\delta + \frac{(d-1)\alpha+1}{2}\right)} \|\phi_j^*(\chi_j u)\|^2_{H^s}$$

where

$$\phi_0 = \mathrm{id}, \quad \phi_j(x) = (2^{\alpha j} x', 2^j x_d), \quad j \geq 1$$

and $\sum\limits_{j=0}^\infty \chi_j = 1$ be a dyadic partition of unity such that

$$\chi_0 \in C_c^\infty(\mathcal{B}_1; [0,1]), \quad \chi \in C_c^\infty(\mathcal{D}_0; [0,1]), \quad \chi_j = \phi_{j*}(\chi) := \chi \circ \phi_j^{-1},\ j \geq 1.$$

It is easy to check that when $s \in \mathbb{N}$, we have an alternative description

$$\|u\|^2_{H^{s,\delta}_\alpha} \approx \sum_{|\beta'|+\beta_d \leq s} \|\langle x\rangle^{|\beta'|\alpha + \beta_d + \delta} \partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} u\|^2_{L^2}.$$

We provide a few properties of this norm.

Proposition 12. *Let $u, v \in C_c^\infty(\Omega)$, then*

(a) For $s > d/2$ and $\delta \in \mathbb{R}$,

$$\|\langle x\rangle^{\frac{(d-1)\alpha+1}{2}+\delta} u\|_{L^\infty} \lesssim \|u\|_{H^{s,\delta}_\alpha}. \tag{5.1}$$

(b) For $s_1 + s_2 > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$, we have

$$\|uv\|_{H^{s,\delta}_\alpha} \lesssim \|u\|_{H^{s_1,\delta_1}_\alpha} \|v\|_{H^{s_2,\delta_2}_\alpha}, \tag{5.2}$$

where $\delta = \delta_1 + \delta_2 + \frac{(d-1)\alpha+1}{2}$ and

$$s = \begin{cases} \min(s_1, s_2), & \max(s_1, s_2) > \frac{d}{2}, \\ s_1 + s_2 - \frac{d}{2}, & \max(s_1, s_2) < \frac{d}{2}. \end{cases}$$

Proof. The Sobolev embedding estimate (a) follows from the definition. The bilinear estimate (b) is proved similarly as Proposition 3(c):

$$\begin{aligned}\|uv\|^2_{H^{s,\delta}_\alpha} &\approx \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}\right)}\|\phi_j^*(\chi_j uv)\|^2_{H^s}\\ &\lesssim \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}\right)}\|\phi_j^*(\chi_j u)\|^2_{H^{s_1}}\|\phi_j^*(\tilde\chi_j v)\|^2_{H^{s_2}}\\ &\lesssim \left(\sum_j 2^{2j\left(\delta_1+\frac{(d-1)\alpha+1}{2}\right)}\|\phi_j^*(\chi_j u)\|^2_{H^{s_1}}\right)\left(\sum_j 2^{2j\left(\delta_2+\frac{(d-1)\alpha+1}{2}\right)}\|\phi_j^*(\tilde\chi_j u)\|^2_{H^{s_2}}\right)\\ &\lesssim \|u\|^2_{H^{s_1,\delta_1}_\alpha}\|v\|^2_{H^{s_2,\delta_2}_\alpha}.\end{aligned}$$

□

There are also analogues of Lemma 5 and Lemma 6:

Lemma 13. *Suppose K is a diagonal operator on Ω such that for some $s_1, s_2, k \in \mathbb{R}$, K is bounded on $H^{s_1}_{\text{comp}} \to H^{s_2}_{\text{loc}}$, and for any $\chi(x) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, the operator with the Schwartz kernel*

$$K(\phi_j(x), \phi_j(y))\chi(x)\chi(y) \tag{5.3}$$

is bounded by $C2^{j(k-\alpha(d-1)-1)}$ on $H^{s_1} \to H^{s_2}$ where C is independent of $j \in \mathbb{N}$. Then

$$K : H^{s_1,\delta}_\alpha(\Omega) \to H^{s_2,\delta-k}_\alpha(\Omega)$$

is bounded for any $\delta \in \mathbb{R}$.

Proof. Recall for $u \in C_c^\infty(\Omega)$,

$$\|Ku\|^2_{H^{s_2,\delta-k}_\alpha} \approx \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)}\|\phi_j^*(\chi_j Ku)\|^2_{H^{s_2}}$$

Let

- $\tilde\chi_0 \in C_c^\infty(\mathbb{R}^d; [0,1])$;
- $\tilde\chi \in C_c^\infty(\{2^{-C} < |x| < 2^C\}; [0,1])$ and $\tilde\chi_j(x) = \phi_{j*}\tilde\chi(x) = \tilde\chi(2^{-j}x)$ for $j \geq 1$

be another family of cutoffs such that for any $(x,y) \in \operatorname{supp} K(x,y)$, $x \in \operatorname{supp}\chi_j$ implies $\tilde\chi_j(y) = 1$ (this is possible because K is diagonal). Therefore by (5.3),

$$\|\phi_j^*(\chi_j Ku)\|_{H^{s_2}} = \|\phi_j^*(\chi_j K(\tilde\chi_j u))\|_{H^{s_2}} \lesssim 2^{jk}\|\phi_j^*(\tilde\chi_j u)\|_{H^{s_1}}$$

and

$$\|Ku\|^2_{H^{s_2,\delta-k}_\alpha} \lesssim \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)}2^{2jk}\|\phi_j^*(\tilde\chi_j u)\|^2_{H^{s_1}} \approx \|u\|^2_{H^{s_1,\delta}_\alpha}.$$

□

Lemma 14. *Suppose K is an outgoing (incoming, respectively) operator on Ω such that for some $s_1, s_2, k \in \mathbb{R}$, K is bounded on $H^{s_1}_{\text{comp}} \to H^{s_2}_{\text{loc}}$, and for any $\chi(x) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, the operator with Schwartz kernel*

$$K(\phi_j(x), \phi_{j'}(y))\chi(x) \quad (K(\phi_j(x), \phi_{j'}(y))\chi(y), \textit{respectively})$$

is bounded by $C2^{j(k-\alpha(d-1)-1)}$ *(*$C2^{j'(k-\alpha(d-1)-1)}$*, respectively) on* $H^{s_1} \to H^{s_2}$ *where* C *is independent of* $j, j' \in \mathbb{N}$*. Then*

$$K : H^{s_1,\delta}_\alpha(\Omega) \to H^{s_2,\delta-k}_\alpha(\Omega)$$

is bounded for $\delta < \frac{(d-1)\alpha+1}{2}$ *(*$\delta > -\frac{(d-1)\alpha+1}{2} + k$*, respectively).*

Proof. We will only show the outgoing case since the incoming case is similar. For K outgoing, there exists a constant $C > 0$ such that $(x, y) \in \operatorname{supp} K(x, y)$, $x \in \operatorname{supp} \chi_j$, $y \in \operatorname{supp} \chi_{j'}$ implies that $j' \le j + C$. Thus for $\beta \in \mathbb{R}$ be to determined later and $u \in C^\infty_c(\Omega)$,

$$\begin{aligned}
&\|Ku\|^2_{H^{s_2,\delta-k}_\alpha} \approx \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)} \|\phi_j^*(\chi_j K u)\|^2_{H^{s_2}} \\
&\approx \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)} \left\| \sum_{j' \le j+C} \phi_j^*(\chi_j K \chi_{j'} u) \right\|^2_{H^{s_2}} \\
&\lesssim \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)} \left(\sum_{j' \le j+C} 2^{j(k-(d-1)\alpha-1)} 2^{j'((d-1)\alpha+1)} \|\phi_{j'}^*(\chi_{j'} u)\|_{H^{s_1}} \right)^2 \\
&\lesssim \sum_j 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)} \left(\sum_{j' \le j+C} 2^{j(k-(d-1)\alpha-1)} 2^{j'((d-1)\alpha+1)} 2^{\beta(j-j')} \right) \\
&\left(\sum_{j' \le j+C} 2^{j(k-(d-1)\alpha-1)} 2^{j'((d-1)\alpha+1)} 2^{\beta(j'-j)} \|\phi_{j'}^*(\chi_{j'} u)\|^2_{H^{s_1}} \right).
\end{aligned}$$

Since $\delta < \frac{(d-1)\alpha+1}{2}$, we may choose $\beta \in \mathbb{R}$ such that $2\delta < \beta < (d-1)\alpha + 1$, so (in the first term on sums over j', while in the second term on sums over j)

$$\sum_{j' \le j+C} 2^{j(k-(d-1)\alpha-1)} 2^{j'((d-1)\alpha+1)} 2^{\beta(j-j')} \lesssim 2^{jk},$$

$$\sum_{j' \le j+C} 2^{2j\left(\delta+\frac{(d-1)\alpha+1}{2}-k\right)} 2^{jk} 2^{j(k-(d-1)\alpha-1)} 2^{j'((d-1)\alpha+1)} 2^{\beta(j'-j)} \lesssim 2^{2j'\left(\delta+\frac{(d-1)\alpha+1}{2}\right)}.$$

So we conclude that

$$\|Ku\|^2_{H^{s_2,\delta-k}_\alpha} \lesssim \sum_{j'} 2^{2j'\left(\delta+\frac{(d-1)\alpha+1}{2}\right)} \|\phi_{j'}^*(\chi_{j'} u)\|^2_{H^{s_1}} \approx \|u\|^2_{H^{s_1,\delta}_\alpha}.$$ □

5.2. **Solution operators on a degenerate sector.** Now we turn to the construction of solution operators for the linearized constraint equation (1.2). We start with the divergence equation $\partial_j v^j = f$ as before. We can find fundamental solutions of the divergence equation supported on a curve.

Lemma 15. *For any smooth curve $\gamma : [0,\infty) \to \mathbb{R}^d$ with $\gamma(0) = y$ and $\lim_{t\to\infty} \gamma(t) = \infty$, the distribution δ_γ defined as*

$$(\delta_\gamma, \varphi) = \int_0^\infty \varphi(\gamma(t))\gamma'(t)dt$$

is a fundamental solution for the divergence equation

$$\partial_j u^j = \delta_y.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, then

$$(\partial_j \delta_\gamma^j, \varphi) = -\int_0^\infty \partial_j\varphi(\gamma(t))\gamma'^j(t)dt = -\int_0^\infty \partial_t\varphi(\gamma(t))dt = \varphi(\gamma(0)) = \varphi(y). \qquad \square$$

As before, we average a family of fundamental solutions to get a more regular solution supported in Ω. Note the solution operators will no longer be convolution operators due to the geometry of the degenerate sector Ω. In other words, the shape of the curves we are taking average on will depend on the initial point y (previously they were just translations of straight lines).

Lemma 16. *We have a solution operator $\tilde{S}_0$ for the divergence equation, i.e. $\partial_j \tilde{S}_0^j f = f$ such that for $\delta < \frac{\alpha(d-1)-1}{2}$,*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} \tilde{S}_0 f \subset \Omega, \quad \textit{and } \tilde{S}_0 : H_\alpha^{s-1,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega) \textit{ is bounded.}$$

Moreover, $\tilde{S}_0 = \tilde{S}_{0,\mathrm{diag}} + \tilde{S}_{0,\mathrm{out}}$ such that

$$\tilde{S}_{0,\mathrm{diag}} : H_\alpha^{s-1,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{5.4}$$

$$\tilde{S}_{0,\mathrm{out}} : H_\alpha^{-N,\delta+1}(\Omega) \to H_\alpha^{N,\delta}(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d-1)-1}{2} \tag{5.5}$$

are both bounded. Moreover,

$$[\partial, \tilde{S}_{0,\mathrm{diag}}] : H_\alpha^{s-1,\delta}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{5.6}$$

is also bounded.

Proof. We construct the solution operator in two steps.

Let $\gamma_{y,\omega}^{(1)} = y + (\omega y_d^\alpha, y_d)t$, $\chi_1 \in C_c^\infty(\mathbb{R}^{d-1})$ with $\int_{\mathbb{R}^{d-1}} \chi_1 = 1$. We define

$$K_1(x,y) = \int_{\mathbb{R}^{d-1}} \chi_1(\omega)\delta_{\gamma_{y,\omega}^{(1)}} d\omega = \chi_1\left(\frac{(x'-y')/y_d^\alpha}{(x_d-y_d)/y_d}\right) \frac{(x-y)}{y_d^{\alpha(d-1)+1}\left|\frac{x_d-y_d}{y_d}\right|^d}. \tag{5.7}$$

Then $\operatorname{div} K_1 = \delta(x-y)$. Let $\chi_2 \in C_c^\infty(\mathbb{R})$ be a cutoff with $\chi_2 = 1$ near 0 and

$$\tilde{K}_1(x,y) = \chi_2\left(\frac{x_d-y_d}{y_d}\right) K_1(x,y).$$

Then $\operatorname{div} \tilde{K}_1 f = f + \frac{1}{y_d}\chi_2'\left(\frac{x_d - y_d}{y_d}\right) K_1$. Let $\gamma^{(2)}_{y,\omega}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$ and

$$\begin{aligned} K_2(x,y) &= \int_{\mathbb{R}^{d-1}} \chi_1(\omega)\delta_{\gamma^{(2)}_{y,\omega}} d\omega \\ &= \chi_1\left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1}\right)((1+x_d-y_d)^\alpha - 1)^{-(d-1)}\left(\alpha(x'-y')\frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha-1}, 1\right). \end{aligned}$$

Then $K(x,y) = \tilde{K}_1(x,y) - \int K_2(x,z)\frac{1}{y_d}\chi_2'\left(\frac{z_d-y_d}{y_d}\right) K_1(z,y)dz$ satisfies

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} Kf \subset \Omega, \quad \operatorname{div} Kf = f.$$

Write $K = \tilde{K}_1 + \tilde{K}_2$ where

$$\tilde{K}_2(x,y) = -\int K_2(x,z)\frac{1}{y_d}\chi_2'\left(\frac{z_d-y_d}{y_d}\right) K_1(z,y)dz.$$

It is straightforward to verify

- $\tilde{K}_1$ is diagonal, $\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d}\tilde{K}_1(x,y) \lesssim y_d^{(1-\alpha)(d-1)}|x_d - y_d|^{1-d-|\beta'|\alpha-\beta_d}$;
- $\tilde{K}_2$ is outgoing and smooth, with the property

$$\partial^{\beta'}_{x'}\partial^{\beta_d}_{x_d}\tilde{K}_2(x,y) \lesssim |x_d|^{-\alpha(d-1)-|\beta'|\alpha-\beta_d}. \tag{5.8}$$

We then define $\tilde{S}_{0,\text{diag}}$ to be the operator with Schwartz kernel $\tilde{K}_1$ and $\tilde{S}_{0,\text{out}}$ to be the operator with Schwartz kernel $\tilde{K}_2$. It suffices to show their boundedness.

(a) First we show (5.4), i.e. $\tilde{K}_1 : H^{s-1,\delta+1}_\alpha \to H^{s,\delta}_\alpha$ is bounded for any $s, \delta \in \mathbb{R}$. Let $\tilde{K}_1 = (\tilde{K}_1', \tilde{K}_1^d)$, then

$$\tilde{K}_1(2^{j\alpha}x', 2^j x_d, 2^{j\alpha}y', 2^j y_d) = 2^{-(d-1)\alpha-1}(2^{j\alpha}\tilde{K}_1', 2^j\tilde{K}_1^d).$$

By Lemma 13, it suffices to show $\tilde{K}_1$ is bounded on $H^{s-1}_{\text{comp}} \to H^s_{\text{loc}}$. Indeed, $\tilde{K}_1(x,y)$ is a pseudodifferential operator of order -1. Let

$$b(y_d, z) = \chi_1\left(\frac{z'/y_d^\alpha}{z_d/y_d}\right)\frac{(z'/y_d^\alpha, z_d/y_d)}{y_d^{\alpha(d-1)+1}|\frac{z_d}{y_d}|^d}, \quad a(\xi) = \mathcal{F}(b(1,\cdot)).$$

Then a is homogeneous of degree -1 and

$$(y_d^{-\alpha}\tilde{K}_1'(x,y), y_d^{-1}\tilde{K}_1^d(x,y)) = \frac{1}{(2\pi)^d}\chi_2\left(\frac{x_d-y_d}{y_d}\right)\int e^{i(x-y)\cdot\xi}a(y_d^\alpha\xi', y_d\xi_d)d\xi.$$

Let χ be a cutoff near $\xi = 0$, then $(1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d)$ is a symbol in the sense that

$$|\partial_y^\beta\partial_\xi^\gamma((1-\chi(\xi))a(y_d^\alpha\xi', y_d\xi_d))| \lesssim_{\beta,\gamma} \langle\xi\rangle^{-1-\gamma}.$$

Thus the pseudodifferential operator maps from H^{s-1}_{comp} to H^s_{loc}. Moreover, the rest part $\int e^{i(x-y)\cdot\xi}\chi(\xi)a(y_d^\alpha\xi', y_d\xi_d)d\xi \in C^\infty$ has smooth Schwartz kernel, thus gives a smoothing operator.

Now the bound (5.6) is similar, since from the expression (5.7) one sees that the commutator only comes from differentiating in the y_d terms, which will give a similar expression with improvement of decay by y_d^{-1}.

(b) For the outgoing part $\tilde{K}_2$, the bound (5.5) follows from (5.8). Lemma 14 implies that $\tilde{K}_2 : H_\alpha^{-N,\delta+1} \to H_\alpha^{N,\delta}$ is bounded for any $N \in \mathbb{N}$ and $\delta < \frac{(d-1)\alpha+1}{2} - 1 = \frac{(d-1)\alpha-1}{2}$. □

From the solution operator for the divergence equation, we also get the solution operator for the double divergence equation.

Proposition 17. *Suppose $\delta < \frac{\alpha(d-1)-3}{2}$, then there is a solution operator $\tilde{S}$ for the double divergence equation, i.e. $\partial_i\partial_j\tilde{S}^{ij}f = f$ and $\tilde{S}^{ij} = \tilde{S}^{ji}$ such that*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} \tilde{S}f \subset \Omega, \quad \text{and } \tilde{S} : H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s,\delta}(\Omega) \text{ is bounded.}$$

Moreover, $\tilde{S} = \tilde{S}_{\text{diag}} + \tilde{S}_{\text{out}}$ such that

$$\tilde{S}_{\text{diag}} : H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{5.9}$$

$$\tilde{S}_{\text{out}} : H_\alpha^{-N,\delta+2}(\Omega) \to H_\alpha^{N,\delta}(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d-1)-3}{2}, \tag{5.10}$$

$$[\partial, \tilde{S}_{\text{diag}}] : H_\alpha^{s-2,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{5.11}$$

are all bounded. Moreover, the integration kernel $\tilde{S}_{\text{out}}(x,y)$ of $\tilde{S}_{\text{out}}$ satisfies

$$|\partial_{x'}^{\beta'}\partial_{x_d}^{\beta_d}\tilde{S}_{\text{out}}(x,y)| \lesssim \langle x\rangle^{1-\alpha(d-1)-|\beta'|\alpha-\beta_d}. \tag{5.12}$$

Proof. We just need to apply $\tilde{S}_0$ twice and symmetrize it:

$$\tilde{S}^{ij}f = \frac{1}{2}(\tilde{S}_0^i\tilde{S}_0^j f + \tilde{S}_0^j\tilde{S}_0^i f).$$

The boundedness on $H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s,\delta}(\Omega)$ follows from that $\tilde{S}_0 : H_\alpha^{s-1,\delta+1}(\Omega) \to H_\alpha^{s,\delta}(\Omega)$ and $\tilde{S}_0 : H_\alpha^{s-2,\delta+2}(\Omega) \to H_\alpha^{s-1,\delta+1}(\Omega)$ are bounded for $\delta < \frac{(d-1)\alpha-3}{2}$.

We define

$$\tilde{S}^{ij}_{\text{diag}}f = \frac{1}{2}(\tilde{S}^i_{0,\text{diag}}\tilde{S}^j_{0,\text{diag}}f + \tilde{S}^j_{0,\text{diag}}\tilde{S}^i_{0,\text{diag}}f), \quad \tilde{S}_{\text{out}} = \tilde{S} - \tilde{S}_{\text{diag}}.$$

The bounds (5.9)(5.10)(5.11) follows from (5.4)(5.5)(5.6). The bound of the tail (5.12) follows from (5.8). □

For the symmetric divergence equation, the construction is trickier. We use methods from [Res70] and refer to [Ise+24] for further discussions.

Proposition 18. *Suppose $\delta < \frac{\alpha(d+1)-3}{2}$, then there exists a solution operator $\tilde{L}$ for the symmetric divergence equation, i.e. $\partial_i \tilde{L}_k^{ij} f_j = f_k$ and $\tilde{L}_k^{ij} = \tilde{L}_k^{ji}$ such that*

$$\operatorname{supp} f \subset \Omega \Longrightarrow \operatorname{supp} \tilde{L} f \subset \Omega, \quad \text{and } \tilde{L} : H_\alpha^{s-1,\delta+2-\alpha}(\Omega) \to H_\alpha^{s,\delta}(\Omega) \text{ is bounded.}$$

Moreover, $\tilde{L} = \tilde{L}_{\text{diag}} + \tilde{L}_{\text{out}}$ such that

$$\tilde{L}_{\text{diag}} : H_\alpha^{s-1,\delta+2-\alpha}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R}, \tag{5.13}$$

$$\tilde{L}_{\text{out}} : H_\alpha^{-N,\delta+2-\alpha}(\Omega) \to H_\alpha^{N,\delta}(\Omega), \quad N \in \mathbb{N},\ \delta < \frac{\alpha(d+1)-3}{2}, \tag{5.14}$$

$$[\partial, \tilde{L}_{\text{diag}}] : H_\alpha^{s-1,\delta+1-\alpha}(\Omega) \to H_\alpha^{s,\delta}(\Omega), \quad s \in \mathbb{R},\ \delta \in \mathbb{R} \tag{5.15}$$

are all bounded. Moreover, the integration kernel $\tilde{L}_{\text{out}}(x,y)$ of $\tilde{L}_{\text{out}}$ satisfies

$$|\partial_{x'}^{\beta'} \partial_{x_d}^{\beta_d} \tilde{L}_{\text{out}}(x,y)| \lesssim \langle x \rangle^{1-\alpha d-|\beta'|\alpha-\beta_d}. \tag{5.16}$$

Proof. Fix a smooth curve $\gamma(t) : [0,\infty) \to \mathbb{R}^d$ such that $\gamma(0) = y$ and $\lim_{t\to\infty} \gamma(t) = \infty$. We want to find $L \in \mathscr{D}'(\mathbb{R}^d)$ such that

$$\varphi_k(y) = \langle \partial_i L_k^{ij}, \varphi_j \rangle = -\frac{1}{2} \langle L_k^{ij}, \partial_i \varphi_j + \partial_j \varphi_i \rangle, \quad \varphi_j \in C_c^\infty(\mathbb{R}^d).$$

In order to recover $\varphi_k(y)$ from $\zeta_{ij} = -\frac{1}{2}(\partial_i \varphi_j + \partial_j \varphi_i)$, we let $\eta_{ij} = \frac{1}{2}(\partial_i \varphi_j - \partial_j \varphi_i)$ so that

$$\partial_i \varphi_j = \eta_{ij} - \zeta_{ij}. \tag{5.17}$$

Since

$$\partial_j \zeta_{ik} = -\frac{1}{2} \partial_{ij}^2 \varphi_k - \frac{1}{2} \partial_{jk}^2 \varphi_i, \quad \partial_k \zeta_{ij} = -\frac{1}{2} \partial_{ik}^2 \varphi_j - \frac{1}{2} \partial_{jk}^2 \varphi_i,$$

we have

$$\partial_i \eta_{jk} = \frac{1}{2}(\partial_{ij}^2 \varphi_k - \partial_{ik}^2 \varphi_j) = \partial_k \zeta_{ij} - \partial_j \zeta_{ik}. \tag{5.18}$$

Now we can integrate (5.18) along γ and get

$$\eta_{jk}(\gamma(t)) = -\int_t^\infty (\gamma'(s)^i \partial_k \zeta_{ij}(\gamma(s)) - \gamma'(s)^i \partial_j \zeta_{ik}(\gamma(s))) ds.$$

Then we can integrate (5.17),

$$\begin{aligned}
\varphi_j(y) &= -\int_0^\infty (\gamma'(t)^i \eta_{ij}(\gamma(t)) - \gamma'(t)^i \zeta_{ij}(\gamma(t))) dt \\
&= \int_0^\infty (\gamma'(s)^l \partial_j \zeta_{li}(\gamma(s)) - \gamma'(s)^l \partial_i \zeta_{lj}(\gamma(s))) \left(\int_0^s \gamma'(t)^i dt \right) ds + \int_0^\infty \gamma'(t)^i \zeta_{ij}(\gamma(t)) dt \\
&= \int_0^\infty (\gamma'(t)^l \partial_j \zeta_{li}(\gamma(t)) - \gamma'(t)^l \partial_i \zeta_{lj}(\gamma(t)))(\gamma(t)^i - \gamma(0)^i) dt + \int_0^\infty \gamma'(t)^i \zeta_{ij}(\gamma(t)) dt.
\end{aligned}$$

So the fundamental solution L_k^{ij} supported on the curve γ is given by

$$\langle L_k^{ij}, \zeta_{ij}\rangle = \int_0^\infty (\gamma'(t)^j \partial_k \zeta_{ij}(\gamma(t)) - \gamma'(t)^j \partial_i \zeta_{jk}(\gamma(t)))(\gamma(t)^i - \gamma(0)^i)dt + \int_0^\infty \gamma'(t)^i \zeta_{ik}(\gamma(t))dt.$$

We then average along curves as in Lemma 16. For $\gamma_{y,\omega}^{(1)} = y + (\omega y_d^\alpha, y_d)t$, $\omega \in \mathbb{R}^{d-1}$, $\chi_1 \in C_c^\infty(\mathbb{R}^{d-1})$, we have

$$\begin{aligned} L_1 := \int_{\mathbb{R}^{d-1}} \chi_1(\omega) L_{y,\omega}^{(1)} d\omega = -\partial_k \left(\chi_1 \left(\frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d} \right) \frac{(x_j - y_j)(x_i - y_i)}{y_d^{\alpha(d-1)+1} |x_d - y_d|^d / y_d^d} \right) \\ + \frac{1}{2} \partial_l \left(\chi_1 \left(\frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d} \right) \frac{(x_l - y_l)((x_j - y_j)\delta_{ik} + (x_i - y_i)\delta_{jk})}{y_d^{\alpha(d-1)+1} |x_d - y_d|^d / y_d^d} \right) \\ + \frac{1}{2} \chi_1 \left(\frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d} \right) \frac{(x_i - y_i)\delta_{jk} + (x_j - y_j)\delta_{ik}}{y_d^{\alpha(d-1)+1} |x_d - y_d|^d / y_d^d}. \end{aligned} \tag{5.19}$$

For $\gamma_{y,\omega}^{(2)}(t) = (y' + \omega((1+t)^\alpha - 1), y_d + t)$, we have

$$\begin{aligned} L_2 :&= \int_{\mathbb{R}^{d-1}} \chi_1(\omega) L_{y,\omega}^{(2)} d\omega \\ &= -\frac{1}{2} \partial_k \left(\chi_1 \left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) ((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \right. \\ &\left. \left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j (x_i - y_i) + \left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i (x_j - y_j) \right) \\ &+ \frac{1}{2} \partial_l \left(\chi_1 \left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) (x_l - y_l)((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \right. \\ &\left. \left(\left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j \delta_{ik} + \left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i \delta_{jk} \right) \right) \\ &+ \frac{1}{2} \chi_1 \left(\frac{x'-y'}{(1+x_d-y_d)^\alpha - 1} \right) ((1+x_d-y_d)^\alpha - 1)^{-(d-1)} \\ &\left(\left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^i \delta_{jk} + \left(\alpha(x'-y') \frac{(1+x_d-y_d)^{\alpha-1}}{(1+x_d-y_d)^\alpha - 1}, 1 \right)^j \delta_{ik} \right). \end{aligned}$$

We then define the solution operator $\tilde{L}$ from L_1 and L_2 as in Lemma 16 and (5.13)-(5.16) follows from a similar proof. □

Remark 4. *Here the indices look different from before but this can be seen easily from the expressions. For example, for the first term in L_1:*

$$K_k^{ij}(x,y) = \partial_k \left(\chi_1 \left(\frac{(x'-y')/y_d^\alpha}{(x_d - y_d)/y_d} \right) \frac{(x_j - y_j)(x_i - y_i)}{y_d^{\alpha(d-1)+1} |x_d - y_d|^d / y_d^d} \right),$$

where we compute the rescaling as in Lemma 13 for $i = j = d$ and $1 \le k \le d-1$, we get

$$K_k^{dd}(\phi_j(x), \phi_j(y)) = 2^{-j((d-1)\alpha+1)} 2^{(2-\alpha)j} K_k^{dd}(x, y).$$

So the δ index is shifted by $2-\alpha$.

We can now give the proof of Theorem 3:

Proof of Theorem 3. Let $d \ge 3$, $s \ge d+2$, $\frac{3}{d+1} < \alpha < 1$ and $\frac{3-(d+3)\alpha}{2} < \delta < \frac{\alpha(d-1)-3}{2}$. We start with solutions $(h_0, \pi_0) \in C^\infty(\mathbb{R}^d)$ of the linearized equation $P(h_0, \pi_0) = 0$ supported in Ω with decay (1.7)(1.8) (again such solutions exist even with compact support, as discussed in §4.1). We want to solve the fixed point problem

$$(h, \pi) = (\tilde{S}, \tilde{L})\Phi(h_0 + h, \pi_0 + \pi). \tag{5.20}$$

on the space $H_\alpha^{s,\delta}(\Omega) \times H_\alpha^{s-1,\delta+\alpha}(\Omega)$. The condition $\delta > \frac{3-(d+3)\alpha}{2}$ ensures the multiplication is bounded on $H_\alpha^{s,\delta} \times H_\alpha^{s-2,\delta+2\alpha} \to H_\alpha^{s-2,\delta+2}$, $H_\alpha^{s-1,\delta+\alpha} \times H_\alpha^{s-1,\delta+\alpha} \to H_\alpha^{s-2,\delta+2}$ (see Proposition 12(b)) and the following bilinear estimate holds (note $\partial : H_\alpha^{s,\delta} \to H_\alpha^{s-1,\delta+\alpha}$ in general).

$$\|\Phi(h_0 + h, \pi_0 + \pi)\|_{H_\alpha^{s-2,\delta+2}} \lesssim_{\|h_0+h\|_{H_\alpha^{s,\delta}}} \|(h_0 + h, \pi_0 + \pi)\|^2_{H_\alpha^{s,\delta} \times H_\alpha^{s-1,\delta+\alpha}}.$$

The condition $\delta < \frac{\alpha(d-1)-3}{2}$ ensures the solution operators

$$\tilde{S} : H_\alpha^{s-2,\delta+2} \to H_\alpha^{s,\delta}, \quad \tilde{L} : H_\alpha^{s-2,\delta+2} \to H_\alpha^{s-1,\delta+\alpha}$$

are bounded (note this index is shifted from Proposition 18).

By Banach fixed point theorem, for $\|(h_0, \pi_0)\|_{H_\alpha^{s,\delta} \times H_\alpha^{s-1,\delta+\alpha}}$ sufficiently small, we get a unique solution to (5.20):

$$(h_1, \pi_1) \in H_\alpha^{s,\delta}(\Omega) \times H_\alpha^{s-1,\delta+\alpha}(\Omega), \quad \|(h_1, \pi_1)\|_{H_\alpha^{s,\delta} \times H_\alpha^{s-1,\delta+\alpha}} \lesssim \|(h_0, \pi_0)\|^2_{H_\alpha^{s,\delta} \times H_\alpha^{s-1,\delta+\alpha}}.$$

Applying P to (5.20), we conclude $(h_0 + h_1, \pi_0 + \pi_1)$ is a solution to the constraint equation (4.2).

Now we claim indeed

$$(h_1, \pi_1) \in H_\alpha^{\infty,\delta} \times H_\alpha^{\infty,\delta+\alpha} \tag{5.21}$$

where $H_\alpha^{\infty,\delta} := \cap_s H_\alpha^{s,\delta}$. In order to show the smoothness, we inductively prove the following statement: if $(h_1, \pi_1) \in H_\alpha^{s,\delta} \times H_\alpha^{s-1,\delta+\alpha}$ for some integer $s \ge d+2$, then $(h_1, \pi_1) \in H_\alpha^{s+1,\delta} \times H_\alpha^{s,\delta+\alpha}$.

We write (5.20) as

$$(h_1, \pi_1) = (\tilde{S}_{\text{diag}}, \tilde{L}_{\text{diag}})\Phi(h_0 + h_1, \pi_0 + \pi_1) + (\tilde{S}_{\text{out}}, \tilde{L}_{\text{out}})\Phi(h_0 + h_1, \pi_0 + \pi_1). \tag{5.22}$$

Differentiate it we get

$$\begin{aligned}\partial(h_1, \pi_1) &= (\tilde{S}_{\text{diag}}, \tilde{L}_{\text{diag}})\partial\Phi(h_0 + h_1, \pi_0 + \pi_1) \\ &\quad + [\partial, (\tilde{S}_{\text{diag}}, \tilde{L}_{\text{diag}})]\Phi(h_0 + h_1, \pi_0 + \pi_1) + \partial(\tilde{S}_{\text{out}}, \tilde{L}_{\text{out}})\Phi(h_0 + h_1, \pi_0 + \pi_1).\end{aligned}$$

By (5.10) and (5.14), we have

$$(\tilde{S}_{\mathrm{out}}, \tilde{L}_{\mathrm{out}})\Phi(h_0+h_1, \pi_0+\pi_1) \in H_\alpha^{\infty,\delta} \times H_\alpha^{\infty,\delta+\alpha}.$$

By (5.11) and (5.15), we have

$$[\partial, (\tilde{S}_{\mathrm{diag}}, \tilde{L}_{\mathrm{diag}})]\Phi(h_0+h_1, \pi_0+\pi_1) \in H_\alpha^{s,\delta+1} \times H_\alpha^{s-1,\delta+1+\alpha}.$$

So we conclude

$$\partial^\beta(h_1,\pi_1) = (\tilde{S}_{\mathrm{diag}}, \tilde{L}_{\mathrm{diag}})\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1) + H_\alpha^{1,\delta+|\beta'|\alpha+\beta_d} \times H_\alpha^{0,\delta+|\beta'|\alpha+\beta_d+\alpha}, \quad |\beta| = s.$$

We note as before, for $|\beta| = s$,

$$\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1) = (M_{h_0,h_1}(h_1+h_0, \partial^{\beta+2}h_1), N_{h_0,h_1}(h_1+h_0, \partial^{\beta+1}\pi_1)) + H_\alpha^{-1,\delta+|\beta'|\alpha+\beta_d+2}$$

where M, N are bilinear forms depending smoothly on h_0, h_1. The bilinear estimate (5.2) shows

$$\begin{aligned}
&\|(\tilde{S}_{\mathrm{diag}}, \tilde{L}_{\mathrm{diag}})\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H_\alpha^{1,\delta+|\beta'|\alpha+\beta_d} \times H_\alpha^{0,\delta+|\beta'|\alpha+\beta_d+\alpha}} \\
&\lesssim \|\partial^\beta\Phi(h_0+h_1,\pi_0+\pi_1)\|_{H_\alpha^{-1,\delta+|\beta'|\alpha+\beta_d+2}} \\
&\lesssim \|h_0+h_1\|_{H_\alpha^{d/2+1,\delta}}(\|h_1\|_{H_\alpha^{s+1,\delta}} + \|\pi_1\|_{H_\alpha^{s,\delta+\alpha}}) + C_s.
\end{aligned}$$

Thus we have

$$\|(h_1,\pi_1)\|_{H_\alpha^{s+1,\delta}\times H_\alpha^{s,\delta+\alpha}} \lesssim \|h_0+h_1\|_{H_\alpha^{d/2+1,\delta}}(\|h_1\|_{H_\alpha^{s+1,\delta}} + \|\pi_1\|_{H_\alpha^{s,\delta+\alpha}}) + C_s.$$

For $\|h_0+h_1\|_{H_\alpha^{d/2+1,\delta}}$ sufficiently small, we conclude $(h_1,\pi_1) \in H_\alpha^{s+1,\delta} \times H_\alpha^{s,\delta+\alpha}$. This finishes the induction and proves (5.21).

Finally we check the decay rate (1.5)(1.6). We estimate the two terms on the right hand side of (5.22) separately. For $\delta = \frac{(d-1)\alpha-3}{2}-$ and $(h,\pi) \in H_\alpha^{\infty,\delta} \times H_\alpha^{\infty,\delta+\alpha}$, the bilinear estimate (5.2) implies

$$\Phi(h_0+h_1,\pi_0+\pi_1) \in H_\alpha^{\infty,2\delta+2\alpha+\frac{(d-1)\alpha+1}{2}} \subset H_\alpha^{\infty,\frac{(d-1)\alpha+1}{2}+}$$

is integrable. Thus the tail estimates (5.12)(5.16) implies $(\tilde{S}_{\mathrm{out}}, \tilde{L}_{\mathrm{out}})\Phi(h_0+h_1,\pi_0+\pi_1)$ has the decay rate (1.5)(1.6). For the diagonal part $(\tilde{S}_{\mathrm{diag}}, \tilde{L}_{\mathrm{diag}})\Phi(h_0+h_1,\pi_0+\pi_1)$ we recall $\Phi(h_0+h,\pi_0+\pi) \in H_\alpha^{\infty,\frac{(d-1)\alpha+1}{2}+}$ and thus

$$(\tilde{S}_{\mathrm{diag}}, \tilde{L}_{\mathrm{diag}})\Phi(h_0+h_1,\pi_0+\pi_1) \in H_\alpha^{\infty,\frac{(d-1)\alpha-3}{2}+} \times H_\alpha^{\infty,\frac{(d-1)\alpha-1}{2}+}.$$

The Sobolev embedding (5.1) then implies the diagonal part has even stronger decay than (1.5)(1.6). ☐

References

[ACR21] S. Aretakis, S. Czimek, and I. Rodnianski, *The characteristic gluing problem for the Einstein equations and applications*, arXiv preprint arXiv:2107.02441 (2021).

[ACR23a] S. Aretakis, S. Czimek, and I. Rodnianski, *Characteristic gluing to the Kerr family and application to spacelike gluing*, Communications in Mathematical Physics (2023), no. 403, 275–327.

[ACR23b] S. Aretakis, S. Czimek, and I. Rodnianski, *The Characteristic Gluing Problem for the Einstein Vacuum Equations: Linear and Nonlinear Analysis*, in: *Annales Henri Poincaré*, Springer, 2023, 1–125.

[Are15] S. Aretakis, *On a foliation-covariant elliptic operator on null hypersurfaces*, International Mathematics Research Notices **2015** (2015), no. 15, 6433–6469.

[Are17] S. Aretakis, *The characteristic gluing problem and conservation laws for the wave equation on null hypersurfaces*, Annals of PDE **3** (2017), no. 1, 3.

[Car21] A. Carlotto, *The general relativistic constraint equations*, Living Reviews in Relativity **24** (2021), no. 1, 1–170.

[CD03] P. T. Chruściel and E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, tech. rep., Société mathématique de France, 2003.

[Chr19] P. T. Chrusćiel, *Anti-gravity à la Carlotto-Schoen [after Carlotto and Schoen]*. Astérisque (2019), no. 407, 1–25.

[Cor00] J. Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Communications in Mathematical Physics **214** (2000), no. 1, 137–189.

[CS06] J. Corvino and R. M. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, Journal of Differential Geometry **73** (2006), no. 2, 185–217.

[CS16] A. Carlotto and R. Schoen, *Localizing solutions of the Einstein constraint equations*, Inventiones mathematicae **205** (2016), no. 3, 559–615.

[FM75] A. E. Fischer and J. E. Marsden, *Deformations of the scalar curvature*, Duke Math. J. **42** (1975), no. 3, 519–547, ISSN: 0012-7094,1547-7398, URL: *http://projecteuclid.org/euclid.dmj/1077311199*.

[Hin22] P. Hintz, *Gluing small black holes into initial data sets*, arXiv preprint arXiv:2210.13960 (2022).

[Hin23] P. Hintz, *Underdetermined elliptic PDE on asymptotically Euclidean manifolds, and generalizations*, arXiv preprint arXiv:2302.12904 (2023).

[Ise+24] P. Isett, Y. Mao, S.-J. Oh, and Z. Tao, *Solution operators for divergence-type equations with prescribed support properties*, in preparation (2024).

[MOT23] Y. Mao, S.-J. Oh, and Z. Tao, *Initial data gluing in the asymptotically flat regime via solution operators with prescribed support properties*, arXiv preprint arXiv:2308.13031 (2023).

[OT19] S.-J. Oh and D. Tataru, *The hyperbolic Yang–Mills equation for connections in an arbitrary topological class*, Communications in Mathematical Physics **365** (2019), no. 2, 685–739.

[Res70] Y. G. Reshetnyak, *Estimates for certain differential operators with finite-dimensional kernel*, Siberian Mathematical Journal **11** (1970), no. 2, 315–326.

[SY79] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Communications in Mathematical Physics **65** (1979), no. 1, 45–76.

[SY81] R. Schoen and S.-T. Yau, *Proof of the positive mass theorem. II*, Communications in Mathematical Physics **79** (1981), no. 2, 231–260.

[Wit81] E. Witten, *A new proof of the positive energy theorem*, Communications in Mathematical Physics **80** (1981), no. 3, 381–402.

Email address: yuchen_mao@berkeley.edu

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA

Email address: ztao@math.berkeley.edu

Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720, USA