THE FRACTAL UNCERTAINTY PRINCIPLE VIA DOLGOPYAT'S METHOD IN HIGHER DIMENSIONS

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Abstract. We prove a fractal uncertainty principle with exponent $\frac{d}{2} - \delta + \varepsilon$, $\varepsilon > 0$, for Ahlfors–David regular subsets of $\mathbf{R}^d$ with dimension δ which satisfy a suitable "nonorthogonality condition." This generalizes the application of Dolgopyat's method by Dyatlov–Jin [DJ18] to prove the same result in the special case $d = 1$. As a corollary, we get a quantitative spectral gap for the Laplacian on convex cocompact hyperbolic manifolds of arbitrary dimension with Zariski dense fundamental groups.

1. Introduction

The *fractal uncertainty principle* is the informal assertion that a function cannot be microlocalized to a neighborhood of a sufficiently self-similar set in phase space. Such assertions have applications in hyperbolic geometry and quantum chaos, where one can apply microlocal methods to show that fractal uncertainty principles imply the existence of essential spectral gaps [DZ16]. In particular, one can obtain $L^2 \to L^2$ bounds on the scattering resolvents of the Laplacian on convex cocompact hyperbolic manifolds, as well as improvements on the size of the maximal region in which certain zeta functions admit analytic continuation [BD18].

To make the fractal uncertainty principle more precise, we introduce the *semiclassical Fourier transform*

$$\mathscr{F}_h f(\xi) := (2\pi h)^{-d/2} \int_{\mathbf{R}^d} e^{-ix\cdot\xi/h} f(x)\, \mathrm{d}x$$

where $h > 0$ is a small parameter. If we have sets X, Y, and we write X_h, Y_h for the sumsets $X_h := X + B_h$, $Y_h := Y + B_h$, $B_h := B(0, h)$, then the fractal uncertainty principle for X, Y asserts bounds of the form

$$\|1_{X_h} \mathscr{F}_h 1_{Y_h}\|_{L^2 \to L^2} \lesssim h^\beta \tag{1.1}$$

in the limit $h \to 0$. We will be interested in the case that X, Y are Ahlfors–David regular sets:

Definition 1.1. A compactly supported finite Borel measure μ on $\mathbf{R}^d$ is *Ahlfors–David regular* of dimension $\delta \in [0, d]$, on scales $[\alpha, \beta]$, with regularity constant $C_R \geq 1$, if for every closed square box I with side length $r \in [\alpha, \beta]$, or closed ball I with radius $r \in [\alpha, \beta]$,

$$\mu(I) \leq C_R r^\delta,$$

and if in addition I is centered on a point in $X := \operatorname{supp} \mu$,

$$C_R^{-1} r^\delta \leq \mu(I).$$

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In short we say that (X, μ) is *δ-regular*.

Applying Plancherel's theorem and Hölder's inequality, one can easily check that if X is δ-regular and Y is δ'-regular on scales $[h, 1]$, then

$$\|1_{X_h} \mathscr{F}_h 1_{Y_h}\|_{L^2 \to L^2} \lesssim h^{\max\left(0, \frac{d-\delta-\delta'}{2}\right)}; \tag{1.2}$$

this estimate is a straightforward modification of [Dya19, (2.7)]. In fact, (1.2) is sharp if δ or δ' are either 0 or d, or if X, Y are orthogonal line segments in $\mathbf{R}^2$.

Thus we say that X, Y satisfy the *fractal uncertainty principle* if (1.1) holds for some $\beta > \max(0, \frac{d-\delta-\delta'}{2})$. There are several cases in which the fractal uncertainty principle is known:

(1) If $d = 1$ and $0 < \delta, \delta' < 1$, then the fractal uncertainty principle holds [DZ16; BD18; DJ18].
(2) If $d < \delta + \delta' < 2d$ and Y can be decomposed as a product of Ahlfors–David fractals in $\mathbf{R}$, then the fractal uncertainty principle holds [HS20].
(3) If d is odd and δ, δ' are very close to $d/2$, then the fractal uncertainty principle holds [CT21].
(4) If X, Y are arithmetic Cantor sets[1], then the fractal uncertainty principle holds for $d = 1$ [DJ17] and $d = 2$, $\delta + \delta' \geq 1$ under the condition that X does not contain any line [Coh22].

1.1. **The main theorem.**

In this paper we establish the fractal uncertainty principle for $0 < \delta + \delta' \leq d$ under the following additional hypothesis which rules out the possibility that X, Y are orthogonal line segments. For $\Phi(x, y) := -x \cdot y$ it is a quantitative form of the statement that "X and Y do not lie in submanifolds which have orthogonal tangent spaces."

Definition 1.2. Let $X, Y \subseteq \mathbf{R}^d$ and let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$. We say that (X, Y) is *Φ-nonorthogonal* with constant $0 < c_N \leq 1$ from scales (α_0^X, α_0^Y) to (α_1^X, α_1^Y) if for any $x_0 \in X$, $y_0 \in Y$, and $r_X \in (\alpha_0^X, \alpha_1^X)$ and $r_Y \in (\alpha_0^Y, \alpha_1^Y)$, there exists $x_1, x_2 \in X \cap B(x_0, r_X), y_1, y_2 \in Y \cap B(y_0, r_Y)$ such that

$$|\Phi(x_1, y_1) - \Phi(x_2, y_1) - \Phi(x_1, y_2) + \Phi(x_2, y_2)| \geq c_N r_X r_Y. \tag{1.3}$$

The motivation for this definition is as follows: we want nonorthogonality to be visible on virtually all scales; after all, orthogonality of fractals is a local property, so we want non-orthogonal examples on most balls centered on a point in X and Y. The Ahlfors–David regularity condition guarantees that each such ball contributes roughly the same amount of fractal mass, and is hence the reason why we upgrade "most" to "all". At the same time, we don't want nonorthogonal points to lie too close to each other. This is why we take the right hand side to be $r_X r_Y$ instead of $|x_1 - x_0| \cdot |y_1 - y_0|$. One can verify that this definition of nonorthogonality implies the nonorthogonality hypothesis of [Dya19, Proposition 6.5].

The nonorthogonality condition (1.3) is based on the *local nonintegrability condition* (LNI) of [Nau05; Sto11], which itself can be traced back to the *uniform nonintegrability condition* of [Che98; Dol98]. In such papers one is concerned with the nonintegrability of the stable and unstable foliations of an Axiom A (or perhaps even Anosov) flow. Roughly speaking, given fractals X, Y one may define two laminations (in the sense of Thurston [Thu79, Chapter 8]) in

[1] We define these fundamental examples in §1.2.1, but for now the reader may view them as Cantor sets where the removed boxes have rational vertices.

$\mathbf{R}^d_x \times \mathbf{R}^d_\xi$, the *vertical lamination* $\{x \in X\}$ and *horizontal lamination* $\{\xi = \partial_x \Phi(x,y) : y \in Y\}$, and then (1.3) essentially asserts that the vertical and horizontal laminations satisfy LNI.

Definition 1.3. A measure μ is *doubling* on scales $[h,1]$ if there exists $C_D > 0$ such that for every $r \in [h, \frac{1}{2}]$ and every cube I of side length r centered at $x \in \operatorname{supp}\mu$, $\mu(I \cdot 2) \leq C_D \mu(I)$.

Clearly every regular measure is doubling; we highlight that our main theorem only needs to assume doubling rather than regular. It is essential that we only consider cubes centered at $x \in \operatorname{supp}\mu$ in the definition. One can compare this doubling property with the Federer property in [Dol98, §7], in which case the Gibbs measure is supported everywhere.

What follows is our main theorem:

Theorem 1.4. *Let μ_X, μ_Y be doubling measures on scales $[h,1]$ with compact supports $X \subset I_0, Y \subset J_0$ where $I_0, J_0 \subset \mathbf{R}^d$ are rectangular boxes with unit length. Let $\mathcal{B}_h$ be the semiclassical Fourier integral operator*

$$\mathcal{B}_h f(x) = \int_Y \exp\left(\frac{i\Phi(x,y)}{h}\right) p(x,y) f(y) \, \mathrm{d}\mu_Y(y) \tag{1.4}$$

where the phase $\Phi \in C^2(I_0 \times J_0)$, X, Y are Φ-nonorthogonal from scales h to 1, and the symbol $p \in C^1(I_0 \times J_0)$. Then there exists $\varepsilon_0 > 0$ such that

$$\|\mathcal{B}_h\|_{L^2(\mu_Y) \to L^2(\mu_X)} \lesssim h^{\varepsilon_0}.$$

If one additionally assumes $d = 1$, and that μ_X, μ_Y are regular with dimension $\in (0,1)$, then Theorem 1.4 was proven by Dyatlov–Jin [DJ18], extending the method of Dolgopyat [Dol98] which had already been applied to construct spectral gaps. Using the construction of dyadic cubes in [Chr90], it might be possible that Theorem 1.4 can be generalized to doubling metric spaces. Since there is no immediate application for metric spaces, we have not attempted to write down the more general version.

Following the methods of [DJ18], Theorem 1.4 implies the following fractal uncertainty principle:

Corollary 1.5. *Let X and Y be Ahlfors–David regular sets in $\mathbf{R}^d$, which are nonorthogonal with respect to the dot product on $\mathbf{R}^d \times \mathbf{R}^d$. Assume that X is δ-regular, Y is δ'-regular, $0 < \delta, \delta' < d$. Then there exists $\varepsilon_0 > 0$ such that*

$$\|1_{X_h} \mathscr{F}_h 1_{Y_h}\|_{L^2 \to L^2} \lesssim h^{\frac{d-\delta-\delta'}{2} + \varepsilon_0}.$$

1.1.1. *Lower bounds on the uncertainty exponent.* If we let

$$L := \frac{10^{13} d^3}{c_N^3} \max(1, \|\partial^2_{xy}\Phi\|^3_{C^0}) \tag{1.5}$$

then we can take in Theorem 1.4

$$\frac{1}{\varepsilon_0} \leq 6 \cdot 10^9 c_N^{-2} d^2 (C_D(X) C_D(Y))^{2\lceil \log_2(20L^{5/3}) \rceil} L^{2/3} \log L. \tag{1.6}$$

In the model case that $X = Y$ is regular, $d = 1$, and $\Phi(x,y) = -xy$, we can always take $c_N = C_R^{-\frac{4}{\delta}}$ and $C_D = 2^\delta C_R^2$, which gives a subexponential bound of the form $1/\varepsilon \lesssim e^{C(\delta)\log_2 C_R}$. This is because of the rather poor dependence of ε_0 on the doubling constant; if one modified our proof to use the Ahlfors-David regularity directly, they would obtain a bound of the form $1/\varepsilon \lesssim C_R^{O(1+1/\delta)}$, which is an improvement over the bound $1/\varepsilon_0 \lesssim C_R^{\frac{160}{\delta(1-\delta)}}$ of [DJ18].

In any case, it does not seem that one can use Dolgopyat's method to obtain sharp fractal uncertainty principles, which therefore remains an interesting and challenging open problem. To drive this point home, we recall that in the case $d = 1$, $\delta = 1/2$, an unpublished manuscript of Murphy claims $1/\varepsilon_0 \lesssim \log C_R \log\log C_R$ [CT21, §1].

1.1.2. *Applications to spectral gaps.* Suppose $M = \Gamma\backslash\mathbf{H}^{d+1}$ is a (noncompact) convex cocompact hyperbolic manifold and $\Lambda(\Gamma)$ is the limit set (see §5.2 for the definition). The Patterson–Sullivan measure μ on $\Lambda(\Gamma)$ is Ahlfors–David regular of dimension $\delta_\Gamma \in [0, d)$ [Sul79, Theorem 7]. Under the condition that Γ is Zariski dense[2] in the algebraic group $SO(d+1,1)_0$, $(\Lambda(\Gamma), \mu)$ satisfies the nonorthogonality condition (1.3) for very general $\Phi(x, y)$ (see Corollary 5.4). So we have the fractal uncertainty principle for $\Lambda(\Gamma)$ with very general phase functions.

Dyatlov–Zahl [DZ16] showed that fractal uncertainty principles can be used to prove essential spectral gaps. Let Δ be the Laplace–Beltrami operator on M, then the resolvent

$$R(\lambda) := \left(-\Delta - \frac{d^2}{4} - \lambda^2\right)^{-1} : L^2_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M)$$

is well-defined for $\mathrm{Im}(\lambda) \gg 1$ and has a meromorphic continuation to $\lambda \in \mathbf{C}$; see [MM87; Gui05] for (even) asymptotically hyperbolic manifolds and [GZ95] for manifolds with constant negative curvature near infinity. Vasy [Vas13a; Vas13b] had a new construction of the meromorphic continuation, which is the one used in [DZ16].

The standard Patterson–Sullivan gap [Pat76; Sul79] says

$$R(\lambda) \text{ has only finitely many poles in } \left\{\mathrm{Im}(\lambda) \geq -\max\left(0, \frac{d}{2} - \delta_\Gamma\right)\right\}. \tag{1.7}$$

Moreover, there is no pole in $\{\mathrm{Im}(\lambda) > \delta_\Gamma - d/2\}$ and there are conditions on δ_Γ such that $\lambda = i(\delta_\Gamma - d/2)$ is the first pole (see [Sul79; Pat88]). Using methods of [DZ16], we can improve the essential spectral gap when $\delta_\Gamma \leq d/2$.

Theorem 1.6. *Let M be a noncompact convex cocompact hyperbolic $d+1$-fold such that $\Gamma = \pi_1(M)$ is Zariski dense in $SO(d+1,1)_0$. Let $\delta_\Gamma \in (0, d)$ be the Hausdorff dimension of the Patterson–Sullivan measure associated to Γ. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon > 0$, $R(\lambda)$ has only finitely many poles λ with $\mathrm{Im}\,\lambda > \delta_\Gamma - \frac{d}{2} - \varepsilon_0 + \varepsilon$. Moreover, for any $\chi \in C_0^\infty(M)$, there exists $C_0 = C_0(\varepsilon) > 0$ and $C = C(\varepsilon, \chi) > 0$ such that*

$$\|\chi R(\lambda)\chi\|_{L^2\to L^2} \leq C|\lambda|^{-1-2\min(0,\mathrm{Im}\,\lambda)+\varepsilon}, \quad |\lambda| > C_0, \quad \mathrm{Im}\,\lambda \in \left[\delta_\Gamma - \frac{d}{2} - \varepsilon_0 + \varepsilon, 1\right]. \tag{1.8}$$

In [DJ18, Theorem 2], Dyatlov–Jin showed Theorem 1.6 with $d = 1$ by proving Theorem 1.4 for $d = 1$ and X and Y δ-regular and applying [DZ16, Theorem 3]; our result is the natural higher-dimensional generalization of this theorem.

The spectral gap in Theorem 1.6 was first proved by Naud [Nau05] in dimension 2 and generalized by Stoyanov [Sto11] to higher dimensions. The size of their gap is implicit but our method gives an explicit constant ε_0 as in (1.6) depending on the fractal dimension δ_Γ, the regularity constant and the nonorthogonality constant of the limit set $\Lambda(\Gamma)$. We give a method for computing nonorthogonality constants from the generators of a classical Schottky group $\Gamma \subset SL(2, \mathbf{C})$ in §5.3.

[2] We note carefully that all varieties in this paper are considered to be over $\mathbf{R}$, even when they have a natural structure as a $\mathbf{C}$-variety!

Another advantage of the method of [DZ16] is that we also get the resolvent estimate (1.8), which is hard to obtain using transfer operator techniques and in partiular is not included in [Nau05; Sto11].

Corollary 1.7. *Let M be convex cocompact with Γ Zariski dense. Let ζ_M be the* Selberg zeta function

$$\zeta_M(s) = \prod_{l\in\mathcal{L}_M}\prod_{k=0}^{\infty}(1-e^{-(s+k)l}), \quad s = \frac{d}{2} - i\lambda$$

where $\mathcal{L}_M$ consists of the lengths of all primitive closed geodesics on M (with multiplicity). Then $\zeta_M(s)$ has only finitely many singularities (i.e. zeroes or poles) in the half plane $\{\operatorname{Re} s > \delta_\Gamma - \epsilon_0 + \epsilon\}$ for any $\epsilon > 0$.

Proof. This follows from Theorem 1.6 and [BO99; PP01]. □

The spectral gap is closely related to asymptotics of closed geodesics and exponential decay of correlations, which are important and well-studied questions in dynamical systems. We list a few references.

- Chernov [Che98] gave the first dynamical proof showing sub-exponential decay of correlations for 3-dimensional contact Anosov flows. The groundbreaking work of Dolgopyat [Dol98] showed exponential decay of correlations for transitive Anosov flows with jointly nonintegrable C^1 stable/unstable foliations.
- Naud [Nau05] applied Dolgopyat's method to prove Theorem 1.6 in dimension 2.
- Stoyanov [Sto08; Sto11] showed exponential mixing for a general class of Axiom A flows satisfying his local non-integrability condition.
- Sarkar–Winter [SW21] used Dolgopyat's method to prove exponential mixing of the frame flow for convex cocompact hyperbolic manifolds. Chow–Sarkar [CS22] extended it to locally symmetric spaces.

All the above works require certain *nonintegrability conditions* which should be thought as the analogue of our nonorthogonality condition (1.3).

We would like to mention some other related works on the spectral gap for convex cocompact hyperbolic manifolds.

- Dyatlov–Zahl [DZ16], Dyatlov–Jin [DJ18] and Bourgain–Dyatlov [BD18] proved the fractal uncertainty principle for $d = 1$ and hence gave explicit essential spectral gaps.
- Bourgain–Dyatlov [BD17] used Fourier decay of the Patterson–Sullivan measure to get a spectral gap that only depends on δ_Γ when $d = 1, \delta_\Gamma \leq 1/2$. This is generalized to Kleinian Schottky groups when $d = 2$ by Li–Naud–Pan [LNP21] but in this case the spectral gap will depend on δ_Γ and another quantity related to our non-orthogonality constant c_N (see [LNP21, Lemma 4.4]).
- Oh–Winter [OW16] showed a uniform spectral gap for a large family of congruence arithmetic surfaces, which was then generalized to arbitrary dimensions by Sarkar [Sar22].

1.2. Idea of the proof.

1.2.1. *Model problem: Arithmetic Cantor sets.* We first describe the problem in the model case that X, Y are arithmetic Cantor sets. Let $M \geq 3$ be an integer and $A, B \subseteq \{1, \dots, M\}^d$

be sets with $\delta_A := \frac{\log|A|}{d\log(M)}, \delta_B := \frac{\log|B|}{d\log(M)} \leq \frac{d}{2}$. We let $N := M^k$ and define the *arithmetic Cantor sets*

$$C_{k,A} := \{a_0 + a_1 M + \cdots + a_k M^k : a_i \in A\}$$
$$C_{k,B} := \{b_0 + b_1 M + \cdots + b_k M^k : b_i \in B\}.$$

We introduce the *discrete Fourier transform*

$$\mathcal{F}_N f(j) := N^{-d/2} \sum_{\ell \in \{1,\ldots,N\}^d} \exp\left(2\pi i j \cdot \frac{\ell}{N}\right) f(\ell).$$

The fractal uncertainty principle states that there exists some $\varepsilon_0 > 0$ such that

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \to \ell^2} \lesssim N^{-\beta-\varepsilon_0} \tag{1.9}$$

where $\beta := \frac{d-\delta_A-\delta_B}{2}$ [DJ17, §3]. Analyzing the Hilbert-Schmidt norm, we have

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \to \ell^2} \leq \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{HS} \leq \sqrt{\frac{|A|^k |B|^k}{N^d}} = N^{-\beta}. \tag{1.10}$$

Thus, our goal is to obtain additional gain beyond β. To prove this, one can show as in [Dya19, Lemma 6.4] that if we let

$$r_k := \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}\|_{\ell^2 \to \ell^2},$$

then $r_{k_1+k_2} \leq r_{k_1} r_{k_2}$. This can be used to show that if we can get any gain at all at some scale k, then we get a gain on all further levels, so we suppose for the sake of contradiction that we cannot obtain any gain at any scale, or that the inequalities present in (1.10) are equalities. Then since the Hilbert-Schmidt norm measures the square root of the sum of squares of the eigenvalues and the operator norm measures the largest eigenvalue, it follows that the operator $N^{d/2} 1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}$ must be rank one. A simple computation then shows that the operator $N^{d/2} 1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,B}}$ is the matrix $(\exp(2\pi i \cdot j\ell/N))_{j \in C_{k,A}, \ell \in C_{k,B}}$ (and is zero in the unspecified entries). Computing the determinant of 2×2 minors, we see that

$$\left|\det \begin{pmatrix} \exp(2\pi i j \cdot \ell/N) & \exp(2\pi i j' \cdot \ell/N) \\ \exp(2\pi i j \cdot \ell'/N) & \exp(2\pi i j' \cdot \ell'/N) \end{pmatrix}\right| = \left|\exp\left(2\pi i \frac{\langle j - j', \ell - \ell'\rangle}{N}\right) - 1\right| = 0$$

for all $j, j' \in C_{k,A}$ and $\ell, \ell' \in C_{k,B}$. Thus, (1.9) holds as long as a *nonorthogonality* condition

$$\langle j - j', \ell - \ell' \rangle \neq 0$$

holds for some choice of $j, j' \in A, \ell, \ell' \in B$. This is also a necessary condition since the Fourier transform takes sets supported on a hyperplane H to a set supported on a dual hyperplane $H^\perp = \{x : x \cdot y = 0 \ \ \forall y \in H\}$.

1.2.2. *Nonorthogonality and Dolgopyat's method.* Our proof and the proof of [DJ18] lies in the continuous setting where the fractal is not necessarily self-similar. Thus, we must construct a tree of tiles that discretizes the doubling measure μ, and which is regular enough so that each tile has two children which are spaced far apart away. While very nice submultiplicativity does not hold as it does in the discrete case, we can still, via an induction on scales argument, propagate gain on one scale to gain on all scales. The key tool allowing us to obtain gain on all scales is nonorthogonality, which we formulated in (1.3); it asserts that we can find many points in the intersections of the vertical and horizontal laminations where the phase is "oscillating faster than the function $\mathcal{B}_h$ is being tested against" at every scale, and so we must obtain a gain at every scale. This technique, called *Dolgopyat's method*, has been used to obtain fractal uncertainty principles, spectral gaps, or exponential mixing in previous works, including [Dol98; Nau05; Sto08; Sto11; DJ18; TZ23].

The improvement on each child is measured in the spaces $C_\theta(I)$ that were introduced in [Nau05, Lemma 5.4]. Informally speaking, localizations of $\mathcal{B}_h$ to a tile I have roughly constant oscillation when normalized by $\theta \operatorname{diam}(I)$ for some appropriate choice of θ [DJ18, §2.2]. The $C_\theta(I)$ norms are meant to capture this fact and to measure cancellation on scale I, similar to how algebraic manipulations on M^k-dimensional vectors can be used to measure cancellation in the arithmetic Cantor case.

1.2.3. *Improvements over Dyatlov–Jin.* The method of Dyatlov–Jin [DJ18] does not immediately generalize to $d \geq 2$, for two reasons. First, in order to ensure that each interval has at least two children that are sufficiently far apart, Dyatlov–Jin allow intervals of varying length to appear in the tree by merging together consecutive intervals that intersect the fractal. However, in higher dimensions this leads to long, narrow, winding tiles appearing in the tree; these do not satisfy suitable doubling estimates, as exemplified by the following example.

Example 1.8. Let X be a Sierpiński carpet, and consider the merged discretization for X (see §3 or [DJ18, §2.1]). Since X is path-connected, every scale consists of a single tile, the only child of the single tile at the previous scale! It is impossible to prove that every tile has two children which enjoy phase cancellation.

However, our method must be able to handle the Sierpiński carpet, since it meets the hypotheses of Corollary 1.5 if it is embedded in $\mathbf{R}^4$. Indeed, $2\delta_X \approx 3.8 < 4$. Moreover, X is nonorthogonal to itself at one scale (see the figure), so it is at every scale by self-similarity.

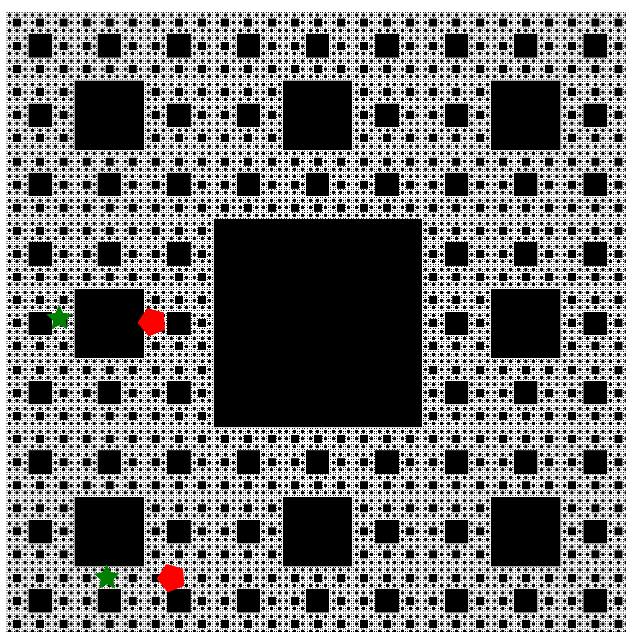

FIGURE 1. Nonorthogonality of the Sierpiński carpet X (the white region) to itself at scale $\frac{1}{3}$ (where $\operatorname{diam} X = \sqrt{2}$). Given any two green points $x_1, y_1 \in X$, we can find two red points $x_2, y_2 \in X$ such that $|x_1 - x_2|$ and $|y_1 - y_2|$ are both ≈ 0.15, and $|\sin \angle(x_2 - x_1, y_2 - y_1)| \ll 1$, so (X, X) is nonorthogonal with constant $(3 \cdot 0.14)^2 \approx 0.42$. Adapted from [Rös08].

Secondly, as remarked above, one cannot obtain cancellation for arbitrary children I_1, I_2, but only those which are "not orthogonal to each other". Otherwise, even if we construct I_1, I_2 to be the appropriate distance each other to impose cancellation, it will not follow that the phases actually cancel each other.

Example 1.9. Let $X := [-5, 5] \times \{0\}$ and $Y := \{0\} \times [-5, 5]$. The Gaussian

$$f(x, y) := e^{-\frac{x^2}{2} - \frac{y^2}{2h^2}}$$

is localized to X_{5h} and its Fourier transform is localized to Y_{5h}. So the fractal uncertainty principle is simply false for (X, Y), even though $\delta_X + \delta_Y = 2 \leq 2$, and we must use the nonorthogonality hypothesis somehow.

It is not enough to impose that X, Y are "fractalline" in some sense (certainly line segments are not fractals in the informal sense of the word), since we could replace X, Y with standard Cantor sets in $[-5, 5]$ and f with a linear combination of finitely many wavelets, each of which is concentrated on a 3-adic interval $I \times \{0\}$ in $[-5, 5] \times \{0\}$ and modulated so that its Fourier transform is concentrated in $\{0\} \times I$, and we would run into the same issue.

To overcome these difficulties, we improve on Dyatlov–Jin as follows:

(1) We carefully construct the tree, so that tiles in the tree are very close to cubes, and therefore satisfy good doubling estimates, but also so that each tile contains two children a suitable distance from each other. Crucially, our tree does not depend on the doubling constant of the measure. This is largely the reason why we are able to work with the weaker assumption of doubling rather than Ahlfors-David regular.
(2) We prove that if X, Y are nonorthogonal, then tangent vectors to X, Y satisfy a *reverse Cauchy–Schwarz inequality* which ensures that the phases cannot decouple.

These goals are accomplished by Proposition 3.3, which asserts that we can construct the so-called *perturbed standard discretization* of μ, and Proposition 3.10, which asserts that many quadruples of tiles in the perturbed standard discretization satisfy the desired spacing and reverse Cauchy–Schwarz inequality.

We found it convenient to use the language of probability theory to state Proposition 3.10, as we then could interpret the various quantities appearing in the induction on scale (Proposition 4.3) as expected values or variances of certain averages of $\mathcal{B}_h f$ taken over random tiles. The necessary estimates needed to obtain a contradiction then follow from the *second moment method* – namely, the observation that, if Proposition 4.3 is false, then the variance of such random variables is impossibly small given the large size of their tails. A similar approach was taken by [DJ18], which used the strict convexity of balls in Hilbert spaces [DJ18, Lemma 2.7] to accomplish the same goals, but we believe that the second moment method gives a cleaner proof.

1.3. **Outline of the paper.** In §2 we recall preliminaries on probability theory and differential geometry.

In §3 we construct our discretization and show that it has good statistical properties, as made precise by Proposition 3.10.

In §4 we carry out our inductive argument. The main proposition is the iterative step, Proposition 4.3; we then use this to prove Theorem 1.4.

We then turn to the applications in §5 where we reduce Corollary 1.5 and Theorem 1.6 to Theorem 1.4 by standard techniques.

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2. Preliminaries

2.1. **Probability theory.** We shall have probability spaces A, B, and will denote by a, a', a'' and b, b', b'' outcomes in those spaces (or equivalently random variables with values in A, B). The expected value of a random variable X is denoted $\mathbf{E}\, X$, while $\mathbf{E}(X|E)$ refers to the conditional expectation of X assuming an event E. The probability of the event E is denoted $\Pr(E)$, and the variance of a random variable is

$$\operatorname{Var} X := \mathbf{E}(X^2) - (\mathbf{E}\, X)^2.$$

We shall use the following elementary consequence of the Cauchy–Schwarz inequality.

Lemma 2.1. *Let A, B be probability spaces, let $(a, b) \mapsto f_{ab}$ be a random variable and draw $a \in A$ and $b, b' \in B$ independently. Then*

$$\mathbf{E}\, |f_{ab} - f_{ab'}|^2 \le 2 \operatorname{Var} f_{ab}.$$

Proof. If a is held fixed, then f_{ab} and $f_{ab'}$ are iid, so

$$\mathbf{E}(f_{ab} f_{ab'}) = \mathop{\mathbf{E}}_{a} \left(\mathop{\mathbf{E}}_{b,b'} (f_{ab} f_{ab'}) \right) = \mathop{\mathbf{E}}_{a} \left(\mathop{\mathbf{E}}_{b} f_{ab} \right)^2.$$

By the Cauchy–Schwarz inequality, it follows that

$$\mathbf{E}(f_{ab} f_{ab'}) \ge (\mathbf{E}\, f_{ab})^2$$

and so

$$\mathbf{E}\, |f_{ab} - f_{ab'}|^2 = 2\, \mathbf{E}\, |f_{ab}|^2 - 2\, \mathbf{E}(f_{ab} f_{ab'}) \le 2\, \mathbf{E}\, |f_{ab}|^2 - 2(\mathbf{E}\, f_{ab})^2 \le 2 \operatorname{Var} f_{ab}. \qquad \square$$

Lemma 2.2 (Cantelli's inequality)**.** *Let X be a random variable with $\operatorname{Var} X < \infty$. Then*

$$\Pr(X \ge \mathbf{E}\, X + \lambda) \le \frac{\operatorname{Var} X}{\lambda^2 + \operatorname{Var} X}.$$

Proof. See [Lug09, Theorem 1]. □

2.2. **A geometric mean value theorem.** We shall need an analogue of the mean value theorem for phase functions [DJ18, Lemma 2.5]. To formulate it, we shall recall some differential geometry.

If R is a nondegenerate rectangle in $\mathbf{R}^d_x \times \mathbf{R}^d_y$, and v, w are unit tangent to the edges of R, then we write $\gamma_R := v \otimes w$ for the *unit bitangent* to R[3] and $\mathrm{d}A_R$ for the area element on

[3] Strictly speaking, the unit bitangent should be defined using the exterior algebra, but since R is assumed nondegenerate this adds more complication for no gain.

R. We will consider the case that $v \in \mathbf{R}^d_x$ and $w \in \mathbf{R}^d_y$. In that case, γ_R and the off-diagonal Hessian $\partial^2_{xy}\Phi$ both lie in $\mathbf{R}^d_x \otimes \mathbf{R}^d_y$, so we can consider their contraction

$$\langle \partial^2_{xy}\Phi, \gamma_R\rangle = \partial_v \partial_w \Phi.$$

Lemma 2.3. *Let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$. Let $x_0, x_1, y_0, y_1 \in \mathbf{R}^d$, and let R be the rectangle with vertices (x_i, y_j), $i, j \in \{0, 1\}$. Then*

$$\int_R \langle \partial^2_{xy}\Phi, \gamma_R\rangle \, \mathrm{d}A_R = \Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1). \tag{2.1}$$

Proof. Both sides of (2.1) are preserved by orientation-preserving isometries which preserve the product structure on $\mathbf{R}^d \times \mathbf{R}^d$. In particular, we may take $x_0, y_0 = 0$, $x_1 = (\xi^*, 0, \dots, 0)$, and $y_1 = (\eta^*, 0, \dots, 0)$ for some $\xi^*, \eta^* \in \mathbf{R}$. We then set

$$\varphi(\xi, \eta) := \Phi((\xi, 0, \dots, 0), (\eta, 0, \dots, 0)).$$

Then by Fubini's theorem,

$$\begin{aligned}\int_R \langle \partial^2_{xy}\Phi, \gamma_R\rangle \, \mathrm{d}A_R &= \int_0^{\xi^*} \int_0^{\eta^*} \partial_\xi \partial_\eta \varphi(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \\ &= \int_0^{\xi^*} \partial_\xi \varphi(\xi, \eta^*) - \partial_\xi \varphi(\xi, 0) \, \mathrm{d}\xi \\ &= \varphi(\xi^*, \eta^*) - \varphi(\xi^*, 0) - (\varphi(0, \eta^*) - \varphi(0, 0)) \\ &= \Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1).\end{aligned}$$ □

We now estimate the difference between (2.1) evaluated over two different rectangles. This estimate will be useful when applying the nonorthogonality hypothesis.

Lemma 2.4. *Let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$ and let $R = I \times J, \tilde{R} = \tilde{I} \times \tilde{J}$ be rectangles in $\mathbf{R}^d \times \mathbf{R}^d$ such that $\gamma_R, \gamma_{\tilde{R}} \in \mathbf{R}^d_x \otimes \mathbf{R}^d_y$ as above. Assume that for some $\varepsilon_x, \varepsilon_y, c_x, c_y > 0$:*

1. *The vertices of I correspond to vertices of $\tilde{I}$ of distance $\leq \varepsilon_x$, and similarly for $J, \tilde{J}, \varepsilon_y$.*
2. $\max(|I|, |\tilde{I}|) \leq c_x$ *and* $\max(|J|, |\tilde{J}|) \leq c_y$.

Then

$$\left| \int_R \langle \partial^2_{xy}\Phi, \gamma_R\rangle \, \mathrm{d}A_R - \int_{\tilde{R}} \langle \partial^2_{xy}\Phi, \gamma_{\tilde{R}}\rangle \, \mathrm{d}A_{\tilde{R}} \right| \leq \|\partial^2_{xy}\Phi\|_{C^0} (2c_x\varepsilon_y + 2c_y\varepsilon_x + 16\varepsilon_x\varepsilon_y). \tag{2.2}$$

Proof. The left-hand side of (2.2) is

$$\left| \int_{R - \tilde{R}} \langle \partial^2_{xy}\Phi, \gamma_R\rangle \, \mathrm{d}A_R + \int_{\tilde{R}} \langle \partial^2_{xy}\Phi, \gamma_R - \gamma_{\tilde{R}}\rangle \, \mathrm{d}A_{\tilde{R}} \right| \leq \|\partial^2_{xy}\Phi\|_{C^0} (|R \Delta \tilde{R}| + |\tilde{R}||\gamma_R - \gamma_{\tilde{R}}|).$$

We then estimate

$$|R \Delta \tilde{R}| = |R \setminus \tilde{R}| + |\tilde{R} \setminus R|$$

and

$$|R \setminus \tilde{R}| \leq |I \times (J \setminus \tilde{J})| + |J \times (I \setminus \tilde{I})| \leq |I||J \setminus \tilde{J}| + |J||I \setminus \tilde{I}|.$$

An analogous estimate holds for $|\tilde{R} \setminus R|$ and implies

$$|R \Delta \tilde{R}| \leq \max(|I|, |\tilde{I}|)|J \Delta \tilde{J}| + \max(|J|, |\tilde{J}|)|I \Delta \tilde{I}| \leq 2c_x\varepsilon_y + 2c_y\varepsilon_x.$$

Now if $v, \tilde{v}$ are unit tangent to $I = [x_0, x_1]$ and $\tilde{I} = [\tilde{x}_0, \tilde{x}_1]$,

$$|v - \tilde{v}| = \left| \frac{x_1 - x_0}{|I|} - \frac{\tilde{x}_1 - \tilde{x}_0}{|\tilde{I}|} \right| \leq \frac{|x_1 - \tilde{x}_1| + |x_0 - \tilde{x}_0|}{|\tilde{I}|} + |I| \left| \frac{1}{|\tilde{I}|} - \frac{1}{|I|} \right|.$$

Estimating $|x_i - \tilde{x}_i| \leq \varepsilon_x$ and

$$|I| \left| \frac{1}{|\tilde{I}|} - \frac{1}{|I|} \right| = \frac{|I| - |\tilde{I}|}{|\tilde{I}|} \leq \frac{2\varepsilon_x}{|\tilde{I}|},$$

we conclude

$$|v - \tilde{v}| \leq \frac{4\varepsilon_x}{|\tilde{I}|}.$$

A similar estimate holds for J and $\tilde{J}$ and implies

$$|\gamma_R - \gamma_{\tilde{R}}| \leq \frac{16\varepsilon_x\varepsilon_y}{|\tilde{R}|}.$$

Adding all these terms up we obtain (2.2). □

3. Discretization of sets and measures

3.1. A new discretization. As in previous works on the fractal uncertainty principle, such as [BD18; DJ18], we will discretize fractals as trees.

Definition 3.1. Let $X \subseteq \mathbf{R}^d$ be a set. A *discretization* of X is a family $V(X) = (V_n(X))_{n \in \mathbf{Z}}$ of sets, where $V_n(X)$ is a set of nonempty subsets of $\mathbf{R}^d$ such that

- $X = \bigcup \{I \cap X : I \in V_n(X)\}$ for each n and the union is disjoint;
- for any $I \in V_n(x)$, there exist $I_k \in V_{n+1}(X)$ such that $I = \bigcup\limits_k I_k$.

Given $I \in \cup_n V_n(X)$, the *height* of I is defined as $H(I) = \sup\{n : I \in V_n(X)\}$.

Definition 3.2. For a compact set $X \subset \mathbf{R}^d$ and base $L \geq 2$, its *standard L-adic discretization* $V^0 = (V_n^0)_{n \in \mathbf{Z}}$ is defined by: $I \in V_n^0(X)$ if and only if

$$I = I_n(q) := [q_1, L^{-n} + q_1) \times [q_2, L^{-n} + q_2) \times \cdots \times [q_d, L^{-n} + q_d)$$

for some $q \in L^{-n}\mathbf{Z}^d$ and $I \cap X \neq \varnothing$.

The standard discretization was used by Bourgain–Dyatlov [BD18] to prove the fractal uncertainty principle in the case $d = 1$, $\delta > 1/2$. The problem with the standard discretization is that a box in $V_n^0(X)$ may be too small for the fractal measure. Dyatlov–Jin [DJ18] addressed this issue in the case $d = 1$, $\delta \leq 1/2$, by considering a discretization that we call the *merged discretization*. Unfortunately, if $d \geq 2$ and $\delta \geq 1$, then the merged discretization does not satisfy the desirable estimates, as intimated by the fact that such estimates have a constant of the form $O(1)^{\frac{1}{\delta(1-\delta)}}$ for $\delta < 1$ in [DJ18].

We now construct a discretization which is more adapted to our setting.

Given a compact convex set I and a real number $\alpha > 0$, we denote by $I\alpha$ the dilation of I by α from its barycenter.

For $A, B \subset \mathbf{R}^d$, we use the ℓ^∞ Hausdorff distance

$$\mathrm{dist}_\infty(A, B) = \sup\{|a_i - b_i| : 1 \leq i \leq d, a = (a_i) \in A, b = (b_i) \in B\}.$$

Proposition 3.3. *For a compact set $X \in \mathbf{R}^d$, $N \in \mathbf{N}$, $L \geq 10^3$, there is a discretization $V(X)$ of X such that for $I \in V_n(X)$, $1 \leq n \leq N$,*

- *there exists $I^0 \in V_n^0(X)$ such that*
$$I^0(1 - L^{-2/3}) \subset I \subset I^0(1 + L^{-2/3}), \tag{3.1}$$
- *and there exists a point x_0 in $X \cap I$ such that*
$$\mathrm{dist}_\infty(x_0, \partial I) \geq \frac{1}{10} L^{-2/3-n}. \tag{3.2}$$

We call this discretization the *perturbed standard discretization*, and we call elements of the perturbed standard discretization *tiles* (to emphasize that they may not be cubes).

Remark 3.4. Christ [Chr90] constructed dyadic cubes with similar properties for metric spaces with a doubling measure μ as in Definition 1.3. It's possible that his construction can also be applied to prove Theorem 1.4. Our construction is less general but does not rely on the existence of a doubling measure. We include the detailed proof of Proposition 3.3 to keep track of the constants.

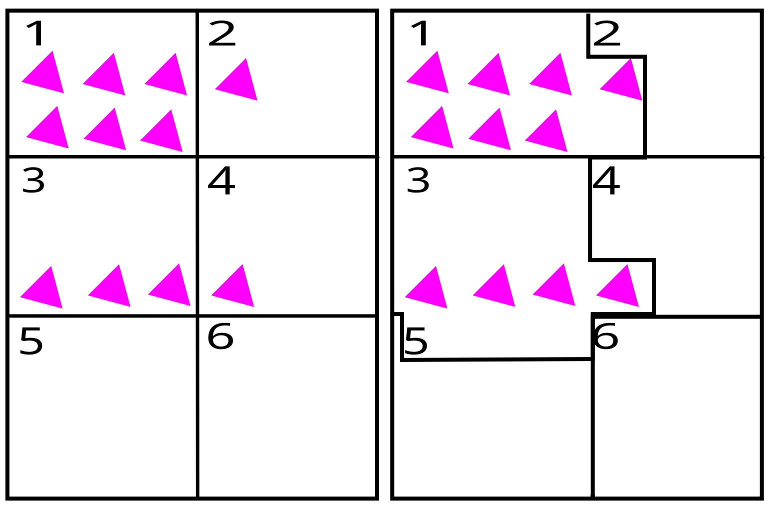

FIGURE 2. A standard (left) and perturbed standard (right) discretization. On the left, Cube 1 is type 2, Cubes 2 and 3 are type 1, Cube 4 is type 0, and Cubes 5 and 6 are type -1; on the right, Tiles 1 and 3 are good and all other tiles are type -1.

3.2. Constructing the new discretization.

3.2.1. *Preliminaries.* We establish some terminology and notation that we will use in the construction of the new discretization. Let I be a cube, such that $\bar{I} = [a_1, b_1] \times \cdots [a_d, b_d]$. For $1 \leq k \leq d$, define the k-boundary
$$\partial^k I := \bigcup_{j_1, \cdots, j_k} [a_1, b_1] \times \cdots \times \{a_{j_i}, b_{j_i}\} \times \cdots \times [a_d, b_d].$$
For a set $A \in \mathbf{R}^d$, $r > 0$, let the ℓ^∞ ball around A with radius r be
$$B_\infty(A, r) = \{x \in \mathbf{R}^d : \exists a \in A, |a - x|_{l^\infty} < r\}.$$

We stress that a B without a subscript refers to the ℓ^2 ball (and in particular, the balls in the definition of nonorthogonality are ℓ^2 balls!)

For a subset $P \subset \partial^k I$ of the k-boundary of a cube I, suppose without loss of generality that

$$P \subseteq \{a_1, b_1\} \times \cdots \{a_k, b_k\} \times [a_{k+1}, b_{k+1}] \times \cdots \times [a_d, b_d].$$

In that case, we define the *tubular neighbourhoods*

$$B^t_\infty(P, r) := \{x \in \mathbf{R}^d : \exists y = (y_i) \in P, |x_1 - y_1| < r, \cdots |x_k - y_k| < r, x_{k+1} = y_{k+1}, \cdots, x_d = y_d\}$$

and

$$B^t_\infty(P, r_1, r_2) := B^t_\infty(P, r_1) \cup B_\infty(P, r_2).$$

Let $V^0(X)$ be the standard discretization and $n \leq N$. We divide the cubes $I \in V^0_n(X)$ into the following types:

- I is of type d if there exists a point $x \in X \cap I$ such that $\operatorname{dist}_\infty(x, \partial I) > L^{-2/3-n}/2$;
- I is of type $d-1$ if there exists a point $x \in X \cap I$ with $\operatorname{dist}_\infty(x, \partial^2 I) > L^{-2/3-n}/2$ but it is not of type d;
- I is of type $d-2$ if there exists a point $x \in X \cap I$ with $\operatorname{dist}_\infty(x, \partial^3 I) > L^{-2/3-n}/2$, but not of type $\geq d-1$;
- $\cdots$;
- I is of type 0 if $X \cap I$ is nonempty and $\operatorname{dist}_\infty(X \cap I, \partial^d I) \leq L^{-2/3-n}/2$;
- I is of type -1 if $X \cap I$ is empty.

See Figure 2.

We want to modify the cubes $I \in V^0_n(X)$ into tiles T so that there exists $x_0 \in X \cap T$ satisfying

$$\operatorname{dist}_\infty(x_0, \partial T) \geq \frac{1}{5} L^{-2/3-n}. \tag{3.3}$$

We say that a tile T is *good* if (3.3) holds, and otherwise that it is *bad*.

For the remainder of the proof, we assume:

Invariant 3.5. *If a tile T constructed from a cube I is bad, then $T \subseteq I$.*

This invariant is true at the current stage of the proof; we necessarily have $T = I$, since we have not modified any tiles yet.

We want to do induction on the type of the tiles. In order to do so, we will need a notion of "type" for a bad tile. By Invariant 3.5, in order for type to be well-defined, it suffices to define the type of a tile T which was modified from a cube I such that $T \subseteq I$. In that case, we define the *type* of T to be k if I is of type k with respect to $X \cap T$; that is, if I has type k in $V_n(X \cap T)$.

3.2.2. *Induction on type.* We now induct backwards on the largest type k of a bad tile. We make the following inductive assumptions, which are vacuous for $k = d$:

Invariant 3.6. *Every bad tile has type $\leq k$.*

Invariant 3.7. *If a tile T was constructed from a cube I, then* $\operatorname{dist}_\infty(\partial T, \partial I) \leq L^{-n-2/3}/2$.

Lemma 3.8. *Assume that $0 \leq k \leq d-1$, and the above set of tiles satisfies Invariants 3.5, 3.6, and 3.7. Then we may modify each tile to obtain a new set of tiles satisfying Invariants 3.5, 3.6, and 3.7, but with k replaced by $k-1$.*

Proof. Let T be a bad tile of type k modified from some cube I, and let P be a connected component of $\partial^k I \setminus B_\infty(\partial^{k+1} I, L^{-2/3-n}/2)$ such that $B^t(P, L^{-n-2/3}/5) \cap X \cap T \neq \varnothing$. We modify the adjacent tiles to P:

(1) If there is a good tile $T' \neq T$ adjacent to P, then we enlarge T' to contain the tubular neighborhood $\mathcal{T} := B^t_\infty(P, L^{-2/3-n}/2) \cap (T' \cup T)$. Then:
  (a) T' is still good.
  (b) T no longer contains P.
  (c) Since $\mathcal{T}$ is contained in $T' \cup T$, no other tile is affected.
(2) Otherwise, by Invariant 3.6, every tile adjacent to P has type $\leq k$. In this case, we enlarge T by a tubular neighborhood $\mathcal{T} := B^t_\infty(P, L^{-2/3-n}/2, L^{-2/3-n}/4)$. Then:
  (a) $\mathcal{T}$ is disjoint from all other tubular neighborhoods of this form.
  (b) Prior to this step, every tile T' adjacent to P was bad, so by Invariant 3.5, T' was contained in the cube I' it was modified from. If $\mathcal{T}'$ is a tubular neighborhood transferred between tiles in a previous step, and $\mathcal{T} \cap \mathcal{T}'$ is nonempty, then there exists T' adjacent to P containing $\mathcal{T}'$, but then T' is not contained in I', contradiction. Therefore $\mathcal{T}$ is disjoint from all tubular neighborhoods transferred between tiles in a previous step.
  (c) T becomes good.
  (d) Every tile $T' \neq T$ adjacent to P no longer contains P.

We iterate the above procedure over all possible components P, stopping once there are no more components to consider. This happens after finitely many stages, because of the following facts:

(1) If a tubular neighborhood of a component P is absorbed by a tile T of type k, and its other neighboring tile is T', then T becomes good, and P can no longer witness that T' has type $\geq k$. Therefore we will not iterate over P again.
(2) At each stage, no new bad tiles are created, and no bad tiles are given more points and remain bad. Therefore Invariants 3.5 and 3.6 are preserved.
(3) Invariant 3.7 is preserved, because if T was constructed from I, then we only modify T in a neighborhood of distance $L^{-n-2/3}/2$ of ∂I.

After iterating over all possible components P, Invariant 3.6 is improved, so that every bad tile has type $\leq k-1$. Indeed, if T is still bad, and was type k, then every tubular neighborhood of a component P which could witness that T had type k was absorbed into a neighboring tile, so T must have type $\leq k-1$. □

After stage $k = 0$, every bad tile has type -1 by Invariant 3.6. However, if T is a tile of type -1, then by definition $X \cap T \cap I$ is empty. Then, by Invariant 3.5, $X \cap T$ is empty, and we may discard the tile T entirely.

Let $\tilde{V}_n(X)$ be the set of good tiles that were constructed from $V_n(X)$ by the above procedure. Then every tile in $\tilde{V}_n(X)$ satisfies (3.3), and

$$X = \bigsqcup_{T \in \tilde{V}_n(X)} T \cap X.$$

However, $\tilde{V}(X)$ may not have a tree structure, so it is not a discretization.

3.2.3. *Obtaining a tree structure.* We now modify $\tilde{V}(X)$ to a discretization $V(X)$. We again proceed by induction. For $n > N$, let $V_n(X) = V_n^0(X)$. Now suppose that $n \leq N$ and we have constructed $(V_m(X))_{m \geq n+1}$, to be a discretization of X.

We partition $V_{n+1}(X)$ into sets $\mathcal{C}(T)$, for each $T \in \tilde{V}_n(X)$, so that $I := \bigcup \mathcal{C}(T)$ satisfies

$$\operatorname{dist}_\infty(\partial T, \partial I) \leq L^{-n-1} \leq \frac{1}{10} L^{-n-2/3}$$

(where the second inequality is because $L \geq 10^3$), and for $x \in T$ satisfying (3.3), $x \in I$. We let $V_n(X)$ be the set of all such I arising from tiles in $\tilde{V}_n(X)$. Then for every $x \in X$ there exists a unique $I' \in V_{n+1}(X)$ containing x by our inductive assumption, and a unique $I \in V_n(X)$ which is a superset of I', by the fact that $\{\mathcal{C}(T) : T \in \tilde{V}_n(X)\}$ is a partition of $V_{n+1}(X)$. It follows that $(V_m(X))_{m \geq n}$ is a discretization of X.

By construction, there exists $x_0 \in X \cap T$ satisfying (3.3), hence

$$\operatorname{dist}_\infty(x_0, \partial I) \geq \operatorname{dist}_\infty(x_0, \partial T) - \operatorname{dist}_\infty(\partial I, \partial T) \geq \left(\frac{1}{5} - \frac{1}{10}\right) L^{-n-2/3} = \frac{1}{10} L^{-n-2/3}$$

and hence x_0 satisfies (3.2). If we denote by I^0 the cube that we modified to create T, then by Invariant 3.7,

$$\operatorname{dist}_\infty(\partial I^0, \partial I) \leq \operatorname{dist}_\infty(\partial I^0, \partial T) + \operatorname{dist}_\infty(\partial T, \partial I) \leq \left(\frac{1}{2} + \frac{1}{10}\right) L^{-n-2/3},$$

which one can use to show (3.1). This completes the proof of Proposition 3.3.

3.3. **Regularity of the discretization.** We now show that if the compact set X is the support of a doubling measure, then its perturbed standard discretization $V(X)$ satisfies regularity conditions similar to those established in [DJ18, Lemma 2.1] for the merged discretization in the case $d = 1$.

We begin by showing that every pair of tiles $(I, J) \in V_n(X) \times V_m(Y)$ have children which contain points for which the analogue

$$|\Phi(x_0, y_0) - \Phi(x_0, y_1) - \Phi(x_1, y_0) + \Phi(x_1, y_1)| \gtrsim |x_0 - x_1| \cdot |y_0 - y_1|$$

of the reverse Cauchy–Schwarz inequality for the indefinite inner product $\partial^2_{xy}\Phi$ holds. This is the key new estimate needed in the higher-dimensional case:

Lemma 3.9. *Let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$, and let $X, Y \subseteq \mathbf{R}^d$ be Φ-nonorthogonal with constant c_N from scales (L^{-K_X}, L^{-K_Y}) to 1. Let $V(X), V(Y)$ be the perturbed standard discretizations of X, Y. Then for*

$$L \geq \max(180^3, 10^9 c_N^{-3} \|\partial^2_{xy}\Phi\|^3_{C^0}) d^{3/2}, \tag{3.4}$$

and every $n < K_X$, $m < K_Y$, $I \in V_n(X)$, $J \in V_m(Y)$, there exist children $I_a, I_{a'}$ of I and $J_b, J_{b'}$ of J such that for every $x_\alpha \in I_\alpha$, $y_\beta \in J_\beta$, and $\omega_{\alpha\beta} := \Phi(x_\alpha, y_\beta)$, we have the reverse Cauchy–Schwarz inequality

$$\frac{c_N}{1000} \leq L^{m+n+4/3} |\omega_{ab} - \omega_{a'b} - \omega_{ab'} + \omega_{a'b'}| \leq \frac{\|\partial^2_{xy}\Phi\|_{C^0}}{20}. \tag{3.5}$$

and the even spacing condition

$$L^{n+2/3} |x_a - x_{a'}|, L^{m+2/3} |y_b - y_{b'}| \leq \frac{1}{2}. \tag{3.6}$$

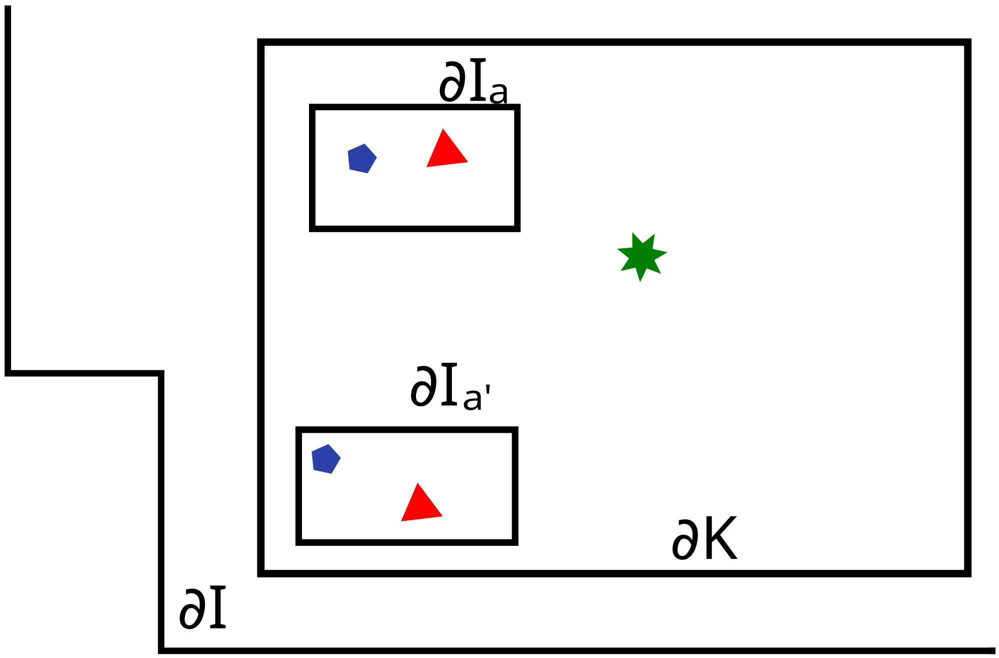

FIGURE 3. A typical situation in the proof of Lemma 3.9. The child tiles $I_a, I_{a'}$ are contained in the cube $K \subset I$, and are much smaller than I. The green 7-gon denotes x, the red triangles denote $\tilde{x}_a, \tilde{x}_{a'}$, and the blue pentagons denote $x_a, x_{a'}$.

Moreover, we may assume

$$\text{for any } x_a \in I_a, x_{a'} \in I_{a'}, \text{ the line segment } \overline{x_a x_{a'}} \text{ always lies in } I. \tag{3.7}$$

Proof. Choose $\underline{x} \in X \cap I$ and $\underline{y} \in Y \cap J$ such that

$$\min\left(L^{n+2/3}\mathrm{dist}_\infty(\underline{x}, \partial I), L^{m+2/3}\mathrm{dist}_\infty(\underline{y}, \partial J)\right) \geq \frac{1}{10}.$$

Let $r_X = \frac{1}{20}L^{-n-2/3}$, $r_Y = \frac{1}{20}L^{-m-2/3}$. One can show that if (3.4) holds, then $L \geq 20^3$ and

$$(1 + 2L^{-2/3})^4 \leq \frac{9}{8}. \tag{3.8}$$

Since $n \leq K_X - 1$ and $L \geq 20^3$,

$$r_X = \frac{1}{20}L^{-n-2/3} \geq \frac{1}{20}L^{-K_X}L^{1/3} \geq L^{-K_X},$$

and similarly $r_Y \geq L^{-K_Y}$. So by nonorthogonality, there exist $\tilde{x}_a, \tilde{x}_{a'} \in X \cap B(\underline{x}, r_X)$ and $\tilde{y}_b, \tilde{y}_{b'} \in Y \cap B(\underline{y}, r_Y)$ such that for $\tilde{\omega}_{\alpha\beta} := \Phi(\tilde{x}_\alpha, \tilde{y}_\beta)$,

$$|\tilde{\omega}_{ab} - \tilde{\omega}_{a'b} - \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b'}| \geq c_N r_X r_Y. \tag{3.9}$$

In the other direction, (2.1) and the triangle inequality gives

$$|\tilde{\omega}_{ab} - \tilde{\omega}_{a'b} - \tilde{\omega}_{ab'} + \tilde{\omega}_{a'b'}| \leq \|\partial^2_{xy}\Phi\|_{C^0} \cdot |\tilde{x}_a - \tilde{x}_{a'}| \cdot |\tilde{y}_b - \tilde{y}_{b'}|. \tag{3.10}$$

Let I_α the the children of I containing $\tilde{x}_\alpha$ and J_β be the children of J containing y_β.

Let $x_\alpha \in I_\alpha$ and $y_\beta \in J_\beta$. We first use (3.1), (3.8), and (3.4) to bound

$$\begin{aligned}|x_a - x_{a'}| &\le 2r_X + \operatorname{diam} I_a + \operatorname{diam} I_{a'}\\ &\le \frac{1}{10}L^{-n-2/3} + 2d^{1/2}L^{-n-1}(1+2L^{-2/3})^2\\ &\le \frac{1}{10}L^{-n-2/3} + 5d^{1/2}L^{-n-1}\\ &\le \frac{1}{2}L^{-n-2/3}.\end{aligned}$$

A similar estimate holds on $|y_b - y_{b'}|$, which proves the upper bound in (3.6).

To prove (3.5), let $c_x := 2r_X$, $c_y := 2r_Y$, $\varepsilon_x := \max(\operatorname{diam} I_a, \operatorname{diam} I_{a'})$, and $\varepsilon_y := \max(\operatorname{diam} J_b, \operatorname{diam} J_{b'})$. Then by (2.1), (2.2), (3.1), (3.8), and (3.4),

$$\begin{aligned}&|\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'} - \tilde\omega_{ab} + \tilde\omega_{ab'} + \tilde\omega_{a'b} - \tilde\omega_{a'b'}|\\ &\quad\le \|\partial^2_{xy}\Phi\|_{C^0}(2c_x\varepsilon_y + 2c_y\varepsilon_x + 16\varepsilon_x\varepsilon_y)\\ &\quad\le \|\partial^2_{xy}\Phi\|_{C^0}\left(\frac{4}{5}d^{1/2}L^{-n-m-5/3}(1+2L^{-2/3})^2 + 16dL^{-n-m-2}(1+2L^{-2/3})^4\right)\\ &\quad\le \|\partial^2_{xy}\Phi\|_{C^0}\left(\frac{9}{10}d^{1/2}L^{n-m-5/3} + 18dL^{-n-m-2}\right)\\ &\quad\le \|\partial^2_{xy}\Phi\|_{C^0}d^{1/2}L^{-n-m-5/3}.\end{aligned}$$

Combining this estimate with (3.9) and (3.4),

$$\begin{aligned}&|\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}|\\ &\quad\ge |\tilde\omega_{ab} - \tilde\omega_{a'b} - \tilde\omega_{ab'} + \tilde\omega_{a'b'}|\\ &\qquad - |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'} - \tilde\omega_{ab} + \tilde\omega_{ab'} + \tilde\omega_{a'b} - \tilde\omega_{a'b'}|\\ &\quad\ge \frac{c_N}{400}L^{-n-m-4/3} - \|\partial^2_{xy}\Phi\|_{C^0}d^{1/2}L^{-n-m-5/3}\\ &\quad\ge \frac{c_N}{1000}L^{-n-m-4/3}\end{aligned}$$

which is the desired lower bound in (3.5). For the upper bound, we similarly use (3.10) and (3.4):

$$\begin{aligned}&|\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}|\\ &\quad\le |\tilde\omega_{ab} - \tilde\omega_{a'b} - \tilde\omega_{ab'} + \tilde\omega_{a'b'}|\\ &\qquad + |\omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'} - \tilde\omega_{ab} + \tilde\omega_{ab'} + \tilde\omega_{a'b} - \tilde\omega_{a'b'}|\\ &\quad\le 4\|\partial^2_{xy}\Phi\|_{C^0}r_Xr_Y + \|\partial^2_{xy}\Phi\|_{C^0}d^{1/2}L^{-n-m-5/3}\\ &\quad\le \|\partial^2_{xy}\Phi\|_{C^0}\left(\frac{1}{100}L^{-n-m-4/3} + d^{1/2}L^{-n-m-5/3}\right)\\ &\quad\le \frac{\|\partial^2_{xy}\Phi\|_{C^0}}{20}L^{-n-m-4/3}.\end{aligned}$$

Finally we prove (3.7). We use (3.1), (3.8), (3.4), and the fact that $\operatorname{dist}_\infty(a,b) \le |a-b|$ to estimate

$$\operatorname{dist}_\infty(x_a, \underline{x}) \le \operatorname{dist}_\infty(x_a, \tilde x_a) + \operatorname{dist}_\infty(\tilde x_a, \underline{x}) \le 2L^{-n-1} + r_X \le \frac{1}{15}L^{-n-2/3}.$$

The same bound holds for $x_{a'}$ and it follows that $x_a, x_{a'}$ are contained in the convex set $K := B_\infty(\underline{x}, L^{-n-2/3}/15)$. In particular, $\ell := \overline{x_a x_{a'}}$ satisfies $\ell \subset K$. This implies $\ell \subset I$, since

$$\mathrm{dist}_\infty(\partial K, \partial I) \geq \mathrm{dist}_\infty(\underline{x}, \partial I) - \frac{L^{-n-2/3}}{15} \geq \frac{L^{-n-2/3}}{30}$$

so that $K \subseteq I$. □

We now give a probabilistic interpretation of the above lemmata. To establish notation, suppose that $I \in V_n(X)$ for some compact set X and some n. We write $\{I_a : a \in A\}$ for the set of children of I. This induces the structure of a probability space on A: namely,

$$\Pr(a) := \frac{\mu_X(I_a)}{\mu_X(I)}.$$

Proposition 3.10. *Let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$, and suppose that L satisfies (3.4). Let (X, μ_X) be doubling with constant $C_D(X)$ on scales $[L^{-K_X}, 1]$, let (Y, μ_Y) be doubling with constant $C_D(Y)$ on scales $[L^{-K_Y}, 1]$, let $V(X), V(Y)$ be their perturbed standard discretizations, and assume that (X, Y) is Φ-nonorthogonal with constant c_N from scales (L^{-K_X}, L^{-K_Y}) to 1, $n < K_X$, $m < K_Y$, $I \in V_n(X)$, and $J \in V_m(Y)$, and $\{I_a : a \in A\}$ and $\{J_b : b \in B\}$ the sets of children of I, J. Furthermore, choose for each $a \in A$ and $b \in B$, $x_a \in I_a$ and $y_b \in J_b$, and set $\omega_{ab} := \Phi(x_a, y_b)$.*

Draw independent random outcomes $a, a' \in A$ and $b, b' \in B$. Then with probability

$$\rho \geq C_D(X)^{-2\lceil \log_2(20L^{5/3})\rceil} C_D(Y)^{-2\lceil \log_2(20L^{5/3})\rceil}, \tag{3.11}$$

we have the reverse Cauchy–Schwarz inequality

$$\frac{c_N}{1000} L^{-1/3} \leq L^{n+m+1} |\omega_{ab} - \omega_{a'b} - \omega_{ab'} + \omega_{a'b'}| \leq \pi \tag{3.12}$$

and the even spacing condition

$$L^{n+2/3}|x_a - x_{a'}|, L^{m+2/3}|y_b - y_{b'}| \leq \frac{1}{2}. \tag{3.13}$$

Moreover, we may assume

$$\text{for any } x_a \in I_a, x_{a'} \in I_{a'}, \text{ the line segment } \overline{x_a x_{a'}} \text{ always lies in } I. \tag{3.14}$$

Proof. By Lemma 3.9, there exist a, b, a', b' satisfying (3.5) and (3.6). By definition of the perturbed standard discretization, there exists $x_* \in I_a \cap X$ with $I_* := \frac{1}{10} B_\infty(x_*, L^{-n-5/3}) \subset I_a$. Moreover, $I \subset B_\infty(x_0, 2L^{-n}) = I_*(20L^{5/3})$. Therefore,

$$\Pr(a) = \frac{\mu_X(I_a)}{\mu_X(I)} \geq \frac{\mu_X(I_*)}{\mu_X(I_*(20L^{5/3}))} \geq C_D(X)^{-\lceil \log_2(20L^{5/3})\rceil}.$$

We have analogous lower bound on $\Pr(b), \Pr(a'), \Pr(b')$. Then by independence,

$$\rho \geq \Pr(a)\Pr(a')\Pr(b)\Pr(b'),$$

which gives (3.11), and (3.5) and (3.6) clearly imply (3.13) and the lower bound on (3.12). The condition (3.14) comes from (3.7). For the upper bound we apply (3.5) and (3.4). □

4. The induction on scales

We now begin the proof of Theorem 1.4. Let $\Phi \in C^2(\mathbf{R}^d \times \mathbf{R}^d)$ and $p \in C^1(\mathbf{R}^d \times \mathbf{R}^d)$ be the phase and symbol of $\mathcal{B}_h$, and let $K := \lfloor -\log_L h \rfloor$.

Let (X, μ_X) and (Y, μ_Y) be doubling with constants $C_D(X), C_D(Y)$ on scales $\geq h$, let $V(X), V(Y)$ be their perturbed standard discretizations, and assume that (X, Y) is Φ-nonorthogonal with constant c_N from scales (h, h) to 1.

For $I \in V_n(X)$ and $J \in V_m(Y)$, where $n + m + 1 = K$, we set

$$F_J(x) = \frac{1}{\mu_Y(J)} \int_J \exp\left(i \frac{\Phi(x,y) - \Phi(x, y_J)}{h}\right) p(x,y) f(y) \, \mathrm{d}\mu_Y(y).$$

Here y_J is the center of J^0, the box in the standard discretization associated to J. Let $\{I_a : a \in A\}$ and $\{J_b : b \in B\}$ be sets of children with their usual probability measures. Let $x_a := \arg\max_{I_a} |F_J|$ and $y_b := y_{J_b}$.

4.1. Mean value space.

We need to generalize the space $C_\theta(I)$ where $d = 1$ (see [DJ18, §2.2] and also [Nau05, Lemma 5.4]), which is supposed to locally measure oscillation on I whilst also being "scale-invariant."[4] This will allow us to get some gain out of the cancellation obtained from nonorthogonality while performing induction on scales.

Definition 4.1. Given $I \in V_n(X)$ and $\theta \in (0,1)$, we define the $C_\theta(I)$ norm for functions $f \in C^1(I)$ by

$$\|f\|_{C_\theta(I)} := \max\left(\|f\|_{C^0(I)}, \theta \operatorname{diam}(I) \|\nabla f\|_{C^0(I)}\right).$$

Given $J \in V_m(Y)$, we set $\Psi_b : I \to \mathbf{R}$ as

$$\Psi_b(x) := \frac{\Phi(x, y_{J_b}) - \Phi(x, y_J)}{h}.$$

Lemma 4.2. *Let* $\theta \leq \frac{1}{4\max(1, \|\partial^2_{xy}\Phi\|_{C^0(I_{conv})})}$ *(where I_{conv} is the convex hull of I) and $L \geq 5$. Then for $f \in C_\theta(I)$,*

$$\|e^{i\Psi_b} f\|_{C_\theta(I_a)} \leq \|f\|_{C_\theta(I)}, \tag{4.1}$$

Proof. Observe that if ψ is a smooth function on $I_{a,conv}$, then any $f \in C_\theta(I_a)$ satisfies

$$|\nabla(e^{i\psi} f)| = |ie^{i\psi} f \nabla\psi + e^{i\psi} \nabla f| \leq |f \nabla\psi| + |\nabla f|.$$

Hence

$$\theta \operatorname{diam}(I_a) |\nabla(e^{i\Psi_b} f)(x)| \leq \theta \operatorname{diam}(I_a) \|\nabla\Psi_b\|_{C^0(I_a)} \|f\|_{C^0(I_a)} + \theta \operatorname{diam}(I_a) \|\nabla f\|_{C^0(I_a)}.$$

We estimate that

$$\begin{aligned} |\nabla\Psi_b(x)| &= \frac{1}{h} |\partial_x(\Phi(x, y_{J_b}) - \Phi(x, y_J))| \leq \frac{1}{h} |y_{J_b} - y_J| \|\partial^2_{xy}\Phi\|_{C^0(I_{conv})} \\ &\leq \frac{\operatorname{diam}(J)}{h} \|\partial^2_{xy}\Phi\|_{C^0(I_{conv})}. \end{aligned}$$

[4] We cannot use the space $C^1(I)$ with its norm $\|f\|_{C^1(I)} := \|f\|_{C^0(I)} + \|\nabla f\|_{C^0(I)}$, because the first and second terms in the norm will scale differently if we rescale I.

So by hypothesis on θ and L,

$$\begin{aligned}\theta \operatorname{diam}(I_a)\|\nabla\Psi_b\|_{C^0(I_a)}\|f\|_{C^0(I_a)} &\leq \frac{\operatorname{diam}(I_a)\operatorname{diam}(J)}{h}\|\partial^2_{xy}\Phi\|_{C^0(I_{conv})}\|f\|_{C^0(I)} \\ &\leq \theta(1+L^{-2/3})^2\|\partial^2_{xy}\Phi\|_{C^0(I_{conv})}\|f\|_{C_\theta(I)} \\ &\leq \frac{\|f\|_{C_\theta(I)}}{2}.\end{aligned}$$

In addition, by hypothesis on L,

$$\theta \operatorname{diam}(I_a)\|\nabla f\|_{C^0(I_a)} \leq \frac{2}{L}\theta \operatorname{diam}(I)\|\nabla f\|_{C^0(I)} \leq \frac{\|f\|_{C_\theta(I)}}{2}.$$

Summing up,

$$\theta \operatorname{diam}(I_a)\|\nabla(e^{i\Psi_b}f)\|_{C^0(I_a)} \leq \|f\|_{C_\theta(I)}.$$

We also trivially have

$$\|f\|_{C^0(I_a)} \leq \|f\|_{C^0(I)} \leq \|f\|_{C_\theta(I)},$$

which proves (4.1). $\square$

4.2. **Inductive step.** Our next task is to prove the following analogue of [DJ18, Lemma 3.2].

Proposition 4.3. *Let $I \in V_n(X)$, $J \in V_m(Y)$, where $n+m+1 = K$. Draw a random $b \in B$, and assume that (3.12) and (3.13) hold with probability ρ. Assume that*

$$L \geq \max\left(\frac{10^{12}d^3}{c_N^3\theta^{3/2}}, \frac{10^9\|\partial^2_{xy}\Phi\|^3_{C^0}d^{3/2}}{c_N^3}\right), \tag{4.2}$$

$$\varepsilon_1 \leq \frac{\rho c_N^2}{10^9 d^2 L^{2/3}}. \tag{4.3}$$

Then we have the improvement

$$\|F_J\|^2_{C_\theta(I)} \leq (1-\varepsilon_1)\,\mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)}. \tag{4.4}$$

4.2.1. *The contradiction assumption.* We set up the proof of Proposition 4.3 with the following lemma which is nearly identical to [DJ18, Lemma 3.3].

Lemma 4.4. *One has*

$$\|F_J\|^2_{C_\theta(I_a)} \leq \left(\mathbf{E}\,\|F_{J_b}\|_{C_\theta(I)}\right)^2 \leq \mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)}. \tag{4.5}$$

Proof. By (4.1),

$$\|e^{i\Psi_b}F_{J_b}\|_{C_\theta(I_a)} \leq \|F_{J_b}\|_{C_\theta(I)},$$

and from the definitions,

$$F_J = \mathbf{E}\,e^{i\Psi_b}F_{J_b}.$$

The assertions of (4.5) now follow from the Cauchy–Schwarz inequality. $\square$

We set $R := \mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)}$. Draw $a \in A$ independently of b. Taking expectations in (4.5), we obtain

$$\sigma^2 := \mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)} - \mathbf{E}\,\|F_J\|^2_{C_\theta(I_a)} \geq \operatorname{Var}\|F_{J_b}\|_{C_\theta(I)}. \tag{4.6}$$

In particular, (4.6) can be written

$$\sigma^2 = R - \|F_J\|^2_{C_\theta(I)}.$$

If we knew that $\sigma^2 \geq \varepsilon_1 R$, then the improvement (4.4) would follow. So, we assume towards contradiction that

$$\sigma^2 < \varepsilon_1 R. \tag{4.7}$$

Let $F_{ab} := F_{J_b}(x_a)$, $\omega_{ab} := \Psi_b(x_a)$, and $f_{ab} := e^{i\omega_{ab}} F_{ab}$, so that

$$F_J(x_a) = \mathop{\mathbf{E}}_b f_{ab}$$

and

$$\mathbf{E}\,|F_{ab}|^2 \leq \mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)} = R. \tag{4.8}$$

4.2.2. *Outline of the proof.* By our contradiction assumption (4.7) and variance bound (4.6), the $C_\theta(I)$ norms of the functions F_{J_b} are all almost independent of b. In particular, $|f_{ab}|$ is almost independent of b. However, the events (3.12) and (3.13) have positive probability, so we may condition on them without losing too much, and after conditioning, the phases of f_{ab} and $f_{a'b'}$ cannot be too correlated by (3.12) and (3.13). So we expect cancellation between f_{ab} and $f_{a'b'}$ whenever a, a', b, b' are drawn at random, by the square-root cancellation heuristic. This cancellation implies that the conditional expectation of $|F_{ab}|$ is both very small and comparable to R, a contradiction.

4.2.3. *Two unconditional moment estimates.* We now make two unconditional moment estimates; we shall later use Cantelli's inequality to show that weaker versions of the same moment estimates hold even when we condition on the unlikely events (3.12) and (3.13).

Lemma 4.5. *One has*

$$\operatorname{Var} f_{ab} \leq \mathbf{E}\,|F_{ab}|^2 - R + 2\sigma^2 \leq 2\sigma^2, \tag{4.9}$$

$$\mathbf{E}\,|F_{ab}| \geq (1 - 2\varepsilon_1)\sqrt{R}. \tag{4.10}$$

Proof. We follow [DJ18, Lemma 3.5]. By Lemma 4.2,

$$\theta\|\nabla(e^{i\Psi_b}F_{J_b})\|_{C^0} \operatorname{diam} I_a \leq \frac{\|F_{J_b}\|_{C_\theta(I)}}{2}.$$

From the definition of $C_\theta(I_a)$ and the triangle inequality,

$$\|F_J\|_{C_\theta(I_a)} \leq \max\left(\|F_J\|_{C^0(I_a)}, \frac{1}{2}\,\mathbf{E}\,\|F_{J_b}\|_{C_\theta(I)}\right).$$

By (4.5),

$$\|F_J\|^2_{C^0(I_a)} \leq \|F_J\|^2_{C_\theta(I_a)} \leq R$$

and by (4.6),

$$\frac{1}{2}\,\mathbf{E}\,\|F_{J_b}\|^2_{C_\theta(I)} \leq \frac{R}{2} \leq \frac{1}{2}\left(R + \|F_J\|_{C^0(I_a)}\right),$$

so

$$\|F_J\|^2_{C_\theta(I_a)} \leq \frac{1}{2}\left(R + \|F_J\|^2_{C^0(I_a)}\right).$$

After taking expectations and applying (4.6), we get

$$\mathbf{E}\,\|F_J\|^2_{C^0(I_a)} \geq 2\,\mathbf{E}\,\|F_J\|^2_{C_\theta(I_a)} - R = R - 2\sigma^2.$$

Therefore, since $|F_J(x_a)| = \|F_J\|_{C^0(I_a)}$,

$$|\,\mathbf{E}\,f_{ab}|^2 = \mathbf{E}\,|F_J(x_a)|^2 \geq R - 2\sigma^2.$$

Since

$$\mathbf{E}\,|F_{ab}| = \mathbf{E}\,|f_{ab}| \geq |\,\mathbf{E}\,f_{ab}|,$$

a Taylor expansion of the square root now gives (4.10). Moreover, by our contradiction assumption (4.7),

$$\begin{aligned}\mathbf{E}\,|F_{ab}|^2 = \mathbf{E}\,|f_{ab}|^2 &= |\,\mathbf{E}\,f_{ab}|^2 + \operatorname{Var} f_{ab} \\ &\geq R - 2\sigma^2 + \operatorname{Var} f_{ab}.\end{aligned}$$

Rearranging, we obtain

$$\operatorname{Var} f_{ab} \leq \mathbf{E}\,|F_{ab}|^2 - R + 2\sigma^2.$$

Then (4.9) follows from (4.8). □

4.2.4. *Drawing random nonorthogonal tiles.* By (4.5) and the Cauchy–Schwarz inequality,

$$\mathbf{E}\,\|F_{J_b}\|_{C_\theta(I)} \leq \sqrt{R}. \tag{4.11}$$

Let T be the event that $\|F_{J_b}\|_{C_\theta(I)} \leq 2\sqrt{R}$. By the moment bounds (4.11) and (4.6), and Cantelli's inequality,

$$\Pr(T) > 1 - \varepsilon_1. \tag{4.12}$$

We let T' be the respective event for b', where a', b' are drawn independently from a, b and uniformly at random. From (4.9), (4.12), and Lemma 2.1, we obtain

$$\mathbf{E}(|f_{ab} - f_{ab'}|^2 | T \cap T') \leq 5\sigma^2. \tag{4.13}$$

If T and (3.14) holds, then by Lemma 4.2,

$$|F_{ab} - F_{a'b}| \leq \frac{2\sqrt{R}}{\theta} L^{H(I)} |x_a - x_{a'}| \tag{4.14}$$

Let S be the intersection of T, T', and the events (3.12), (3.13) and (3.14). By (4.3), $\varepsilon_1 \leq \rho/10$, so by (4.12),

$$\frac{\Pr(S)}{\Pr(T)^2} \geq \frac{\rho - 2\Pr(T)}{\Pr(T)^2} \geq \frac{\rho}{2}. \tag{4.15}$$

If S holds, then by (4.14) and (3.13),

$$|F_{ab} - F_{a'b}| \leq \frac{\sqrt{R}}{L^{2/3}\theta}. \tag{4.16}$$

4.2.5. *Conditional second moment bounds.* We now use (4.15) and (4.16) to obtain lower and upper bounds on $\mathbf{E}(|F_{ab}|^2|S)$ which are not both tenable.

Lemma 4.6. *For $M := 8000000$,*

$$\mathbf{E}(|F_{ab}|^2|S) \leq Md^2 \left(\frac{R}{c_N^2 L^{2/3}\theta} + 2\frac{L^{2/3}\sigma^2}{c_N^2 \rho}\right). \tag{4.17}$$

Proof. Write $\tau := \omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}$, so if S holds then

$$|e^{i\tau} - 1|^2 \geq |\tau|^2 \geq 10^{-6} c_N^2 L^{-2/3}$$

by (3.12) and [DJ18, Lemma 2.6]. Following [DJ18, p. 19], we rewrite

$$\begin{aligned}|(e^{i\tau} - 1)F_{ab}| &= |e^{i(\omega_{ab} - \omega_{ab'})}F_{ab} - e^{i(\omega_{a'b} - \omega_{a'b'})}F_{ab}| \\ &= |e^{-i\omega_{ab'}}(f_{ab} - f_{ab'}) + F_{ab'} - F_{a'b'} - e^{-i\omega_{a'b'}}(f_{a'b} - f_{a'b'}) + F_{ab} - F_{ab'}|.\end{aligned}$$

So by the triangle inequality in L^2,

$$\mathbf{E}(|(e^{i\tau}-1)F_{ab}|^2|S) \leq 4\,\mathbf{E}(|F_{ab}-F_{a'b}|^2+|F_{a'b'}-F_{ab'}|^2+|f_{ab}-f_{ab'}|^2+|f_{a'b'}-f_{a'b}|^2|S).$$

So

$$\begin{aligned}\mathbf{E}(|F_{ab}|^2|S) &\leq 10^6\cdot\frac{d^2L^{2/3}}{c_N^2}\,\mathbf{E}(|(e^{i\tau}-1)F_{ab}|^2|S)\\ &\leq \frac{Md^2L^{2/3}}{2c_N^2}\,\mathbf{E}(|F_{ab}-F_{a'b}|^2+|F_{a'b'}-F_{ab'}|^2|S)\\ &\quad+\frac{Md^2L^{2/3}}{2c_N^2}\,\mathbf{E}(|f_{ab}-f_{ab'}|^2+|f_{a'b'}-f_{a'b}|^2|S).\end{aligned}$$

Applying (4.16),

$$|F_{ab}-F_{a'b}|^2+|F_{a'b'}-F_{ab'}|^2 \leq \frac{2R}{L^{4/3}\theta}.$$

Since S implies $T\cap T'$, and a, a' are independent,

$$\mathbf{E}(|f_{ab}-f_{ab'}|^2+|f_{a'b'}-f_{a'b}|^2|S) \leq 2\frac{\Pr(T)^2}{\Pr(S)}\,\mathbf{E}(|f_{ab}-f_{ab'}|^2|T\cap T').$$

By (4.15), $\Pr(T)^2/\Pr(S) \leq 2/\rho$. Summing all this up and applying (4.13), we conclude (4.17). ☐

Lemma 4.7. *One has*

$$\mathbf{E}(|F_{ab}|^2|S) \geq \frac{R}{6}. \tag{4.18}$$

Proof. By Cantelli's inequality and (4.10),

$$\Pr\left(|F_{ab}|^2 \leq \frac{R}{5}\right) \leq \Pr\left(|F_{ab}| \leq \mathbf{E}\,|F_{ab}| - \frac{\sqrt{R}}{2}\right) \leq \frac{\operatorname{Var}|F_{ab}|}{\operatorname{Var}|F_{ab}|+R/4}.$$

Since $|F_{ab}|=|f_{ab}|$, it follows from (4.9) and (4.7) that

$$\Pr\left(|F_{ab}|^2 \leq \frac{R}{5}\right) \leq \frac{\operatorname{Var} f_{ab}}{\operatorname{Var} f_{ab}+R/4} \leq \frac{2\sigma^2}{R/4} < 8\varepsilon_1.$$

But by (4.15),

$$\Pr\left(|F_{ab}|^2 \leq \frac{R}{5}\middle|S\right) = \frac{\Pr((|F_{ab}|^2\leq R/5)\cap S)}{\Pr(S)} \leq \frac{\Pr(|F_{ab}|^2 \leq R/5)}{\rho}.$$

The definition (4.3) of ε_1 then implies

$$\Pr\left(|F_{ab}|^2 \leq \frac{R}{5}\middle|S\right) \leq \frac{8\varepsilon_1}{\rho} < L^{-2/3}.$$

Therefore

$$\Pr\left(|F_{ab}|^2 \geq \frac{R}{5}\middle|S\right) \geq 1-L^{-2/3}$$

so by Markov's inequality and the assumption (4.2),

$$\mathbf{E}(|F_{ab}|^2|S) \geq \frac{R}{5}\Pr\left(|F_{ab}|^2 \geq \frac{R}{5}\middle|S\right) \geq \frac{R}{6}.$$ ☐

4.2.6. *Deriving a contradiction.* The two above conditional second moment bounds contradict (4.2, 4.3), and the the contradiction assumption (4.7). To be more precise, combining (4.17) with (4.18) and (4.7), we obtain

$$\frac{R}{6} \leq \mathbf{E}(|F_{ab}|^2|S) \leq Md^2\left(\frac{R}{c_N^2 L^{2/3}\theta} + \frac{2L^{2/3}\sigma^2}{c_N^2\rho}\right) < Md^2\left(\frac{R}{c_N^2 L^{2/3}\theta} + \frac{2L^{2/3}\varepsilon_1 R}{c_N^2\rho}\right).$$

Dividing both sides by RM and applying (4.2, 4.3), we obtain

$$2\cdot 10^{-8} < \frac{1}{48\cdot 10^6} = \frac{1}{6M} \leq \frac{d^2}{c_N^2 L^{2/3}\theta} + \frac{2d^2L^{2/3}\varepsilon_1}{c_N^2\rho} \leq \frac{1}{10^8} + \frac{2}{10^9} < 1.2\cdot 10^{-8}.$$

This is a contradiction that proves that $\sigma^2 \geq \varepsilon_1 R$, and so completes the proof of Proposition 4.3.

4.3. **Proof of main theorem.** To prove Theorem 1.4 we iterate Proposition 4.3. For each J, we define

$$\begin{aligned} E_J : V_{K-H(J)}(X) &\to \mathbf{R} \\ I &\mapsto \|F_J\|_{C_\theta(I)}. \end{aligned}$$

We endow $V_n(X)$ with the discrete measure induced by μ_X, namely $\mu_X(\{I\}) = \mu_X(I)$, and J with the restricted fractal measure μ_Y.

First suppose that $J \in V_K(Y)$. Then by the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} |\nabla F_J(x)| &= \frac{1}{\mu_Y(J)}\int_J i\partial_x\Psi_J(x,y)\exp(i(\Psi_J(x,y)))p(x,y)f(x,y) \\ &\quad + \exp(i(\Psi_J(x,y)))\partial_x p(x,y)f(y)\,\mathrm{d}\mu_Y(y) \\ &\leq \frac{1}{\sqrt{\mu_Y(J)}}\left(\frac{\operatorname{diam} J}{h}\|\partial_{xy}^2\Phi\|_{C^0}\|f\|_{L^2(J)}\|p\|_{C^0} + \|\partial_x p\|_{C^0}\|f\|_{L^2(J)}\right) \end{aligned}$$

since $|I| \leq 1$ and

$$\|F_J\|_{C^0} \leq \frac{\|p\|_{C^0}\|f\|_{L^2(J)}}{\sqrt{\mu_Y(J)}}.$$

Thus,

$$E_J(I) = \|F_J\|_{C_\theta(I)} \leq \frac{\|p\|_{C^1}\|f\|_{L^2(J)}}{\sqrt{\mu_Y(J)}}. \tag{4.19}$$

Taking L^2 norms of both sides of (4.19), we get

$$\|E_J\|_{L^2}^2 \leq \frac{\|p\|_{C^1}^2\mu_X(X)}{\mu_Y(J)}\|f\|_{L^2(J)}^2. \tag{4.20}$$

If we take L^2 norms of both sides of (4.4), we get

$$\|E_J\|_{L^2}^2 \leq (1-\varepsilon_1)\,\mathbf{E}\,\|E_{J_b}\|_{L^2}^2. \tag{4.21}$$

Inducting backwards on $H(J)$ with (4.20) as base case and (4.21) as inductive case, we conclude that if J is a tile in Y such that $H(J) = 0$,

$$\|E_J\|_{L^2}^2 \leq \frac{\|p\|_{C^1}^2\mu_X(X)}{\mu_Y(Y)}(1-\varepsilon_1)^K\|f\|_{L^2(J)}^2.$$

Summing both sides in J, we obtain

$$\|\mathcal{B}_h f\|_{L^2}^2 \le \|p\|_{C^1}^2 \mu_X(X)\mu_Y(Y)(1-\varepsilon_1)^K \|f\|_{L^2}^2.$$

We now can set

$$\varepsilon_0 := \frac{\varepsilon_1}{6\log L} \le \frac{\log(1-\varepsilon_1)^{-1}}{2\log L}$$

and plug in θ in (4.2) to obtain (1.5, (1.6). Then $(1-\varepsilon_1)^{K/2} \le h^{\varepsilon_0}$, so

$$\|\mathcal{B}_h\|_{L^2(\mu_Y)\to L^2(\mu_X)} \le \|p\|_{C^1}\sqrt{\mu_X(X)\mu_Y(Y)}h^{\varepsilon_0}$$

which completes the proof of Theorem 1.4.

5. Applications

5.1. **Classical fractal uncertainty principle.** We now prove Corollary 1.5, following [DJ18, Theorem 1, Remarks 1].

Lemma 5.1. *Let (X,μ) be δ-regular on scales $[h,1]$, $h>0$, where $\delta\in[0,d]$, and μ is the δ-dimensional Hausdorff measure. Let $X_h := X + B_h$ and*

$$\mu_h(A) := h^{\delta-d}|X\cap A|.$$

Then (X_h,μ_h) is δ-regular on scales $[2h,1]$ with constant

$$C_R(X_h) := 6^\delta |\mathbf{B}^d| C_R(X)^2.$$

Proof. Let $N = N_X(x,r,h)$ be the cardinality of a maximal h-separated set $X\cap B(x,r)$, for $x\in X$ and $r\ge 2h$. By [DZ16, Lemma 7.4], we have

$$C_R(X)^{-2}\frac{r^\delta}{h^\delta} \le N_X(x,r,h) \le C_R(X)^2\left(1+\frac{2r}{h}\right)^\delta.$$

If $\{x_1,\dots,x_N\}$ is such a maximal set, and $I_n := B(x_n,2h)$, then $X\cap B(x,r)\subseteq \bigcup_{n=1}^N I_n$, so

$$\mu_h(B(x,r)) \le h^{\delta-d}\sum_{n=1}^N |I_n| \le (2h)^\delta|\mathbf{B}^d|N \le 2^\delta|\mathbf{B}^d|C_R(X)^2(h+2r)^\delta \le C_R(X_h)r^\delta.$$

Conversely, if $J_n := B(x_n,h/2)$, then J_n, J_m are disjoint, and $\bigcup_{n=1}^N J_n \subseteq X\cap B(x,r)$, so

$$\mu_h(B(x,r)) \ge \sum_{n=1}^N B(x_n,h/2) \ge N\frac{h^\delta}{2^\delta} \ge C_R(X)^{-2}2^{-\delta}r^\delta \ge C_R(X_h)^{-1}r^\delta. \qquad \square$$

Lemma 5.2. *Let (X,Y) be Φ-nonorthogonal on scales $[h,1]$, $h>0$. Then (X_h,Y_h) is Φ-nonorthogonal on scales $[2h,1]$ with constant $c_N(X_h,Y_h) := c_N(X,Y)/4$.*

Proof. Let $x_0\in X_h$, $y_0\in Y_h$, and $r_X, r_Y\ge 2h$; then there exist $\tilde{x}_0\in X$ and $\tilde{y}_0\in Y$ with

$$\max(|x_0-\tilde{x}_0|,|y_0-\tilde{y}_0|)\le h.$$

Putting $\tilde{r}_X := r_X - h$ and $\tilde{r}_Y := r_Y - h$, we can find by Φ-nonorthogonality of (X,Y) points

$$x_1,x_2\in X\cap B(\tilde{x}_0,\tilde{r}_X)\subseteq X\cap B(x_0,r_X)$$

and

$$y_1,y_2\in Y\cap B(\tilde{y}_0,\tilde{r}_Y)\subseteq Y\cap B(y_0,r_Y)$$

such that

$$|\Phi(x_1, y_1) - \Phi(x_1, y_2) - \Phi(x_2, y_1) + \Phi(x_2, y_2)| \geq c_N(X)\tilde{r}_X\tilde{r}_Y \geq c_N(X_h)r_X r_Y. \qquad \square$$

Proof of Corollary 1.5. We introduce the Fourier integral operator

$$\mathcal{B}_h f(\xi) := \int_Y e^{ix\cdot\xi/h} f(x)\, \mathrm{d}\mu_{Y,h}(x).$$

By the above lemmata, $(X_h, \mu_{X,h})$ is δ-regular, $(Y_h, \mu_{Y,h})$ is δ'-regular, and (X_h, Y_h) is Φ-nonorthogonal. Thus by Theorem 1.4,[5] there exists $\varepsilon_0 > 0$ such that

$$\|1_{X_h}\mathscr{F}_h 1_{Y_h}\|_{L^2\to L^2} = \frac{h^{d/2-\delta}}{(2\pi)^{\delta/2}}\|\mathcal{B}_h\|_{L^2(\mu_{Y,h})\to L^2(\mu_{X,h})} \lesssim h^{d/2-\delta+\varepsilon_0}. \qquad \square$$

5.2. **Convex cocompact hyperbolic manifolds.** In this section we prove Theorem 1.6. First we recall some preliminaries for convex cocompact hyperbolic manifolds.

Let $\mathbf{H}^{d+1}$ be the $d+1$ dimensional hyperbolic space (with constant curvature -1). The orientation preserving isometry group is given by $G = SO(d+1,1)_0$. Let $K = SO(d+1)$ be a maximal compact subgroup, so that $\mathbf{H}^{d+1} = G/K$. We are interested in infinite volume hyperbolic manifolds given by $M = \Gamma\backslash G/K$ where $\Gamma \subset G$ is a convex cocompact Zariski dense torsion-free discrete subgroup.

Let $\mathfrak{o} = [\mathrm{id}]$ be the reference point in $\mathbf{H}^{d+1}$. The *limit set* is defined as $\Lambda(\Gamma) = \overline{\lim \Gamma\mathfrak{o}} \subset \partial_\infty(\mathbf{H}^{d+1}) \subset \overline{\mathbf{H}^{d+1}}$. Γ is called *convex cocompact* if the convex core $\mathrm{Core}(M) := \Gamma\backslash\mathrm{Hull}(\Lambda(\Gamma)) \subset M$ is compact. We say $\Gamma \subset G$ is *Zariski dense* if the closure of Γ is equal to G with respect to the Zariski topology of G viewed as an algebraic variety over $\mathbf{R}$. In the Poincaré upper half space model, the limit set $\Lambda(\Gamma) \subset \mathbf{R}^d \cup \{\infty\}$ is a compact set of dimension $\delta_\Gamma \in (0, d)$ (see [SW21, §2]), and we may assume that $\Lambda(\Gamma)$ is a compact subset of $\mathbf{R}^d$.

We recall the following non concentration property from Sarkar–Winter [SW21, Proposition 6.6].

Proposition 5.3. *Let $\Gamma \subset G$ be a convex cocompact subgroup such that Γ is Zariski dense in G. Then there exists $c_0 > 0$ so that for any $x \in \Lambda(\Gamma) \cap \mathbf{R}^d$, $\varepsilon \in (0,1)$ and $w \in \mathbf{R}^d$ with $|w| = 1$, there exists $y \in \Lambda(\Gamma) \cap B(x, \varepsilon)$ so that*

$$|\langle y - x, w\rangle| > c_0\varepsilon. \tag{5.1}$$

As a corollary we have

Corollary 5.4. *Let M be a convex cocompact hyperbolic $d+1$-fold such that Γ is Zariski dense in G. Then for any $\Phi \in C^3(\mathbf{R}^d \times \mathbf{R}^d; \mathbf{R})$ such that $\partial^2_{xy}\Phi(x, y)$ is nonvanishing, the pair $(\Lambda(\Gamma), \Lambda(\Gamma))$ is Φ-non-orthogonal with some constant $c_N > 0$ from scales 0 to 1.*

Proof. By the mean value theorem, for $x_1, x_2 \in B(x_0, r_X)$, $y_1, y_2 \in B(y_0, r_Y)$,

$$\begin{aligned}&|\Phi(x_0, y_0) - \Phi(x_1, y_0) - \Phi(x_0, y_1) + \Phi(x_1, y_1) - \langle\partial_{xy}\Phi(x_0, y_0)(x_1 - x_0), y_1 - y_0\rangle| \\ &\leq \|\Phi\|_{C^3} r_X r_Y(r_X + r_Y).\end{aligned}$$

[5] The fact that regularity and nonorthogonality only hold up to scale $2h$ cause us to incur a loss of a power of 2, but this is irrelevant.

Let $H = \ker(\partial^2_{xy}\Phi(x_0, y_0))$ and v be a unit normal vector to H (if $H = \{0\}$, the we choose v arbitrarily). By Proposition 5.3, there exists $x_1 \in \Lambda(\Gamma) \cap B(x_0, r_X)$ such that $|\langle x_1 - x_0, v\rangle| > c_0 r_X$. This would imply for some $c_1 \in (0, 1)$,

$$|\partial^2_{xy}\Phi(x_0, y_0)(x_1 - x_0)| > c_1 c_0 r_X.$$

By Proposition 5.3 again, there exists $y_1 \in \Lambda(\Gamma) \cap B(y_0, r_Y)$ such that

$$|\langle \partial^2_{xy}\Phi(x_0, y_0)(x_1 - x_0), y_1 - y_0\rangle| > c_1 c_0^2 r_X r_Y.$$

Thus we may choose $r_X, r_Y \leq c_1 c_0^2 \|\Phi\|^{-1}_{C^3}/10$ so that

$$|\Phi(x_0, y_0) - \Phi(x_1, y_0) - \Phi(x_0, y_1) + \Phi(x_1, y_1)| > \frac{c_1 c_0^2}{2} r_X r_Y,$$

i.e. nonorthogonality holds with $c_N = \frac{c_1^3 c_0^6}{200(1+\|\Phi\|_{C^3})^2} > 0$. □

Theorem 1.4 and Lemma 5.1 then implies $B_\chi(h) : L^2(S^d) \to L^2(S^d)$ defined by

$$B_\chi(h)u(x) = (2\pi h)^{-d/2} \int_{S^d} |x - y|^{2i/h} \chi(x, y) u(y) dy$$

where $\chi(x, y) \in C_0^\infty(S^d \times S^d \setminus \{(x, x) : x \in S^d\})$ satisfies the fractal uncertainty bound

$$\|\mathbb{1}_{\Lambda(\Gamma)(h)} B_\chi(h) \mathbb{1}_{\Lambda(\Gamma)(h)}\|_{L^2(S^d)\to L^2(S^d)} \leq C h^{\frac{d}{2} - \delta_\Gamma + \varepsilon_0}.$$

By a covering argument as in [BD18, Proposition 4.2], we have for $\rho \in (0, 1)$,

$$\|\mathbb{1}_{\Lambda(\Gamma)(h^\rho)} B_\chi(h) \mathbb{1}_{\Lambda(\Gamma)(h^\rho)}\|_{L^2(S^d)\to L^2(S^d)} \leq C h^{\frac{d}{2} - \delta_\Gamma + \varepsilon_0 - 2(1-\rho)}.$$

Thus, $\Lambda(\Gamma)$ satisfies the fractal uncertainty principle with exponent $\beta = \frac{d}{2} - \delta_\Gamma + \varepsilon_0$ in the sense of [DZ16, Definition 1.1]. Applying [DZ16, Theorem 3], we conclude the Laplacian on M has only finitely many resonances in $\{\operatorname{Im}\lambda > \delta_\Gamma - \frac{d}{2} - \varepsilon_0 + \varepsilon\}$ for any $\varepsilon > 0$, proving Theorem 1.6.

5.3. **Computation of nonorthogonality constants.** The condition that $\Gamma \subset G$ being Zariski dense is qualitative, and so one needs to extract any quantitative condition, such as nonconcentration, from Zariski denseness by a compactness argument as in [SW21]. However, Qiuyu Ren has pointed out to us that for classical Schottky groups Γ in $SO(3, 1)_0 = PSL(2, \mathbf{C})$, there is a simple and effective way to compute the nonorthogonality constant in Definition 1.2. The key idea is to use the fact that Möbius transformations are conformal maps and preserve circles in order to derive (5.1).

We illustrate this by considering Schottky groups of genus 2. Let D_1, D_2, D_3, D_4 be four disjoint closed disks in $\mathbf{CP}^1 = \partial\mathbf{H}^3$, let $\gamma_1, \gamma_2 \in PSL(2, \mathbf{C})$ such that

$$\gamma_1(\overline{D_3^c}) = D_1, \quad \gamma_2(\overline{D_4^c}) = D_2, \quad \gamma_3 = \gamma_1^{-1}, \quad \gamma_4 = \gamma_2^{-1}.$$

Let $\Gamma = \langle\gamma_1, \gamma_2\rangle$ be the free group generated by γ_1 and γ_2. Thus, Γ is a Schottky group of genus 2.

Given vectors $v, w \in \mathbf{R}^2$, let $\angle(v, w)$ denote the angle between v, w. (We identify $\mathbf{CP}^1 \setminus \{\infty\}$ with $\mathbf{R}^2$, and we may assume that the D_i do not contain ∞.) We will choose the disks D_1, D_2, D_3, D_4 such that

$$\text{No circle (line) passes though all the four disks.} \tag{5.2}$$

The circle taken here is not necessarily a great circle.

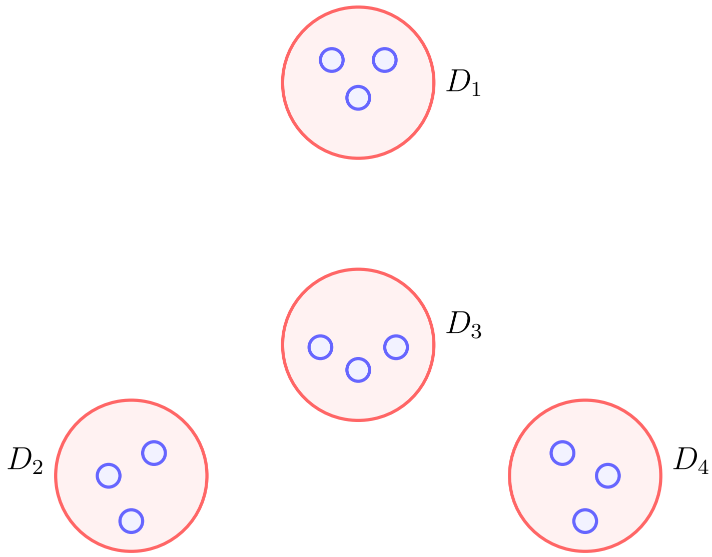

FIGURE 4. Iteration of disks under a Schottky group

Let $\bar{a} \equiv a + 2 \mod 4$ for $a \in \mathcal{A} = \{1, 2, 3, 4\}$, so that $\bar{1} = 3$, $\bar{2} = 4$. The limit set $\Lambda(\Gamma)$ is given by the Cantor-like procedure

$$\Lambda(\Gamma) = \bigcap_{n=1}^{\infty} \bigsqcup_{\mathbf{a} \in \mathcal{W}^n} D_{\mathbf{a}}, \quad \mathcal{W}^n = \{a_1 a_2 \cdots a_n \in \mathcal{A}^n : \overline{a_i} \neq a_{i+1}\}, \quad D_{\mathbf{a}} = \gamma_{a_1}(\gamma_{a_2} \cdots (\gamma_{a_{n-1}}(D_{a_n}))).$$

The nonorthogonality condition (1.3) follows from the nonconcentration property (5.1). Thus it suffices to find absolute constants $0 < c_1 < 1$ and $\kappa = \kappa(\Gamma) > 0$ such that for each $x \in \Lambda(\Gamma)$, $\epsilon > 0$, and unit vector $w \in \mathbb{R}^2$, an element $y \in \Lambda(\Gamma) \cap B(x, \epsilon) \setminus B(x, c_1\epsilon)$ such that

$$|\cos \angle(x - y, w)| \geq \kappa.$$

Suppose $x \in D_{\mathbf{a}} = D_{\mathbf{a}_0 b}$ and $B(x, \epsilon)$ is roughly of the size of $D_{\mathbf{a}_0}$. Then there are two other disks in $D_{\mathbf{a}_0}$, which we call $D_{\mathbf{a}_0 c}$ and $D_{\mathbf{a}_0 d}$. By condition (5.2) and conformal invariance of the action of Γ, we know that for any $y_c \in D_{\mathbf{a}_0 c} \cap \Lambda(\Gamma)$ and $y_d \in D_{\mathbf{a}_0 d} \cap \Lambda(\Gamma)$,

$$\text{the circle passing through } x, y_c, y_d \text{ lies inside } D_{\mathbf{a}_0}. \tag{5.3}$$

A Möbius transformation preserving the unit disk is a composition of rotation and the map

$$z \mapsto \frac{a - z}{1 - \bar{a}z}$$

A simple computation shows the angles of the triangle $\Delta(x, y_c, y_d)$ is uniformly lower bounded under conformal maps preserving $D_{\mathbf{a}_0}$ if we assume (5.3). This implies that

$$\theta < \angle(y_c - x, y_d - x) < \pi - \theta$$

for some constant θ depending on the initial angles between $\gamma_a(D_b), a \neq \bar{b}$. Thus, by the pigeonhole principle,

$$\max(|\cos \angle(y_c - x, w)|, |\cos \angle(y_d - x, w)|) \geq \cos\left(\frac{\pi - \theta}{2}\right).$$

If we assume moreover

$$\begin{aligned}&\text{For any } b \neq \bar{a} \neq c, \text{ there exists } a' \neq a, b' \neq \overline{a'} \text{ such that}\\ &\text{no circle passes through } \gamma_a(D_b), \gamma_a(D_c), \gamma_{a'}(D_{b'}) \text{ and } D_{\bar{a}}\end{aligned} \tag{5.4}$$

(which can be achieved if we choose the disks D_a to be small and with generic centers), then we can derive a lower bound on c_1 in a similar way. To be more precise, let $x \in D_{\mathbf{a}} = D_{\mathbf{a}_0 b} = S_{\mathbf{a}_1 ab}$ as before, then by assumption 5.4, there exists $a' \neq a$ and $b' \neq \overline{a'}$ such that

$$\text{The circle passing through } D_{\mathbf{a}_0 b}, D_{\mathbf{a}_0 c} \text{ and } D_{\mathbf{a_1} a'b'} \text{ lies inside } D_{\mathbf{a}_1}. \tag{5.5}$$

In particular, for any $y_{a'b'} \in D_{\mathbf{a_1} a'b'}$, the angles of the triangle $\Delta(x, y_c, y_{a'b'})$ are lower bounded. This in particular implies that the length of $\overline{xy_c}$ is comparable to the length of $\overline{y_c y_{a'b'}}$, which by the previous step is comparable with the size of $D_{\mathbf{a}_0}$. This allows us to compute a lower bound of c_1.

If one runs this procedure carefully, then it would be possible to compute an explicit nonorthogonality constant in terms of the angles between the disks $\gamma_a(D_b)$ in the initial step and the uniform constants in doing conformal transformations.

We do not bother to do the computation here, but we include Figure 4 to indicate how the procedure works. Conformal invariance ensures us that the small blue disks always have an angle that lies in $[\theta, \pi - \theta]$.

While one needs to compute the above parameters κ, θ for any given Zariski dense classical Schottky group Γ, we claim that this is always possible in principle, at least after passing to a finer scale. We say that a pair of words $\mathbf{a}, \mathbf{b} \in \mathcal{W}^n$, $n \in \mathbf{N} \cup \{+\infty\}$, is *$\varepsilon$-separated* if their weighted Hamming distance satisfies

$$\sum_{i=1}^{n} \frac{1_{a_i \neq b_i}}{2^i} \geq \varepsilon.$$

Lemma 5.5. *Let Γ be a classical Schottky group which is Zariski dense in $PSL(2, \mathbf{C})$. For every $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that for every $n \geq N$ and every triple of words $\mathbf{a}^n, \mathbf{b}^n, \mathbf{c}^n \in \mathcal{W}^n$ which are pairwise ε-separated, there exists $\mathbf{d}^n \in \mathcal{W}^n$ such that for every circle X which meets all three disks $D_{\mathbf{a}^n}, D_{\mathbf{b}^n}, D_{\mathbf{c}^n}$, X does not meet $D_{\mathbf{d}^n}$.*

Proof. We first prove an analogous result for the set of infinite words $\mathcal{W}^\infty$, and then reduce the finite case to the infinite case. To formulate it, let $x_{\mathbf{a}}$ be the unique point in $\lim_n D_{a_1 \cdots a_n}$ (so $\mathbf{a} \mapsto x_{\mathbf{a}}$ is a homeomorphism $\mathcal{W}^\infty \to \Lambda(\Gamma)$ where $\mathcal{W}^\infty$ is given the product topology).

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{W}^\infty$ be distinct. Then there is a unique circle $X_{\mathbf{abc}} \subset \mathbf{CP}^1$ passing through $x_{\mathbf{a}}, x_{\mathbf{b}}, x_{\mathbf{c}}$. We claim that there exists $\mathbf{d} \in \mathcal{W}^\infty$ such that $x_{\mathbf{d}} \notin X_{\mathbf{abc}}$. To see this, suppose not; then $\Lambda(\Gamma)$ is contained in a circle, which up to a Möbius transformation we may assume to be $\mathbf{RP}^1$. Moreover we may assume that $\infty \notin \Lambda(\Gamma)$ since $\Lambda(\Gamma)$ is topologically a Cantor set and so cannot be equal to $\mathbf{RP}^1$. Thus, $\Lambda(\Gamma)$ is contained in $\mathbf{R} = \{z \in \mathbf{C} : \operatorname{Im} z = 0\}$, so if we take $w = i$ in Proposition 5.3, we reach a contradiction.

We now address the finite case. Suppose that the lemma fails on some $\mathbf{a}^n, \mathbf{b}^n, \mathbf{c}^n \in \mathcal{W}^n$ for each $n \in \mathbf{N}$ which are ε-separated, so for every $\mathbf{d}^n \in \mathcal{W}^n$ there exists a circle $X(\mathbf{d}^n)$ which meets all disks $D_{\mathbf{a}^n}, D_{\mathbf{b}^n}, D_{\mathbf{c}^n}, D_{\mathbf{d}^n}$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{W}^\infty$ be the limits of $\mathbf{a}^n$, et cetra, and let $\mathbf{d} \in \mathcal{W}^\infty$ be given. Then $\mathbf{d} = \lim_n \mathbf{d}^n$ for some sequence $\mathbf{d}^n \in \mathcal{W}^n$, and we can define $X := \lim_n X(\mathbf{d}^n)$ in Hausdorff distance. Then, $x_{\mathbf{a}}, x_{\mathbf{b}}, x_{\mathbf{c}}, x_{\mathbf{d}} \in X$, and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are ε-separated, hence distinct. Moreover, X is the limit of circles in $\mathbf{CP}^1$ whose radii are bounded from below (by ε-separation), so X is a circle, hence $X = X_{\mathbf{abc}}$. This contradicts the infinite case. $\square$

Assuming Lemma 5.5, for $D_{\mathbf{a}} = D_{a_1\cdots a_{2n}}$, we can find $\mathbf{b}, \mathbf{c} \in \mathcal{W}^{2n}$ such that any circle passing through $D_{\mathbf{a}}, D_{\mathbf{b}}$ and $D_{\mathbf{c}}$ lies in the disk D_{a_1}. This is because given $D_{a_1\cdots a_{2n}}$ and $\overline{D^c_{a_1}}$, we have

$$\gamma_{\bar{a}_n}\cdots\gamma_{\bar{a}_2}\gamma_{\bar{a}_1}(D_{a_1\cdots a_{2n}}) = D_{a_{n+1}\cdots a_{2n}}, \quad \gamma_{\bar{a}_n}\cdots\gamma_{\bar{a}_2}\gamma_{\bar{a}_1}(\overline{D^c_{a_1}}) = D_{\bar{a}_n\cdots\bar{a}_2\bar{a}_1}.$$

By Lemma 5.5, there exists $\mathbf{b}_0, \mathbf{c}_0 \in \mathcal{W}^n$ such that no circle passes through $D_{a_{n+1}\cdots a_{2n}}$, $D_{\bar{a}_n\cdots\bar{a}_2\bar{a}_1}$, $D_{\mathbf{b}_0}$ and $D_{\mathbf{c}_0}$. Applying $\gamma_{a_1}\cdots\gamma_{a_n}$, we conclude any circle passing through

$$D_{a_1\cdots a_{2n}}, D_{a_1\cdots a_n\mathbf{b}_0}, D_{a_1\cdots a_n\mathbf{c}_0}$$

lies inside D_{a_1} (there might be cancellations for the words $a_1\cdots a_n\mathbf{b}_0$ and $a_1\cdots a_n\mathbf{c}_0$ but one can always pass to a smaller disk). This allows us to compute the angle θ as before for general Zariski dense classical Schottky groups.

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