LOSSLESS STRICHARTZ AND SPECTRAL PROJECTION ESTIMATES ON UNBOUNDED MANIFOLDS

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Abstract. We prove new lossless Strichartz and spectral projection estimates on asymptotically hyperbolic surfaces, and, in particular, on all convex cocompact hyperbolic surfaces. In order to do this, we also obtain log-scale lossless Strichartz and spectral projection estimates on manifolds of uniformly bounded geometry with non-positive and negative sectional curvatures, extending the recent works of the first two authors for compact manifolds. We are able to use these along with known L^2-local smoothing and new $L^2 \to L^q$ half-localized resolvent estimates to obtain our lossless bounds.

1. Introduction.

Two of the main goals of this paper are to prove lossless Strichartz and spectral projection estimates on negatively curved asymptotically hyperbolic surfaces. We also obtain frequency-dependent estimates on general manifolds of uniformly bounded geometry in all dimensions all of whose sectional curvatures are negative or nonpositive.

Our first result is the following Strichartz estimates for solutions $u = e^{-it\Delta_g}u_0$ of the Schrödinger equation

$$i\partial_t u(x,t) = \Delta_g u(x,t), \quad u(x,0) = u_0(x). \tag{1.1}$$

Theorem 1.1. *Let (M,g) be an even asymptotically hyperbolic surface with negative curvature. Then, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $p,q \geq 2$ and $(p,q) \neq (2,\infty)$, there exists $C_q = C_q(M)$ such that*

$$\left\| e^{-it\Delta_g} u_0 \right\|_{L^p_t L^q_x(M\times[0,1])} \leq C_q \|u_0\|_{L^2(M)}. \tag{1.2}$$

We shall review the hypotheses concerning (M,g) in the next section. We point out that any convex cocompact hyperbolic surface is an even asymptotically hyperbolic surface of (constant) negative curvature. See the figure below, and see [8] for more details.

Note that for convex cocompact hyperbolic manifolds with limit set dimension $\delta > \frac{n-1}{2}$, there always exists an eigenfunction ψ_δ of $-\Delta_g$ with eigenvalue $\delta(n-1-\delta)$ such that $\psi_\delta \in L^q(M)$ for all $q \geq 2$. Therefore, $[0,1]$ can not be replaced by $\mathbb{R}$ in (1.2) without imposing additional assumptions. See [17, Remark 1.3] for more details. Moreover, the pseudodifferential techniques that we employ in the proof of Theorem 1.1 also introduce errors that depend on the length of the time interval.

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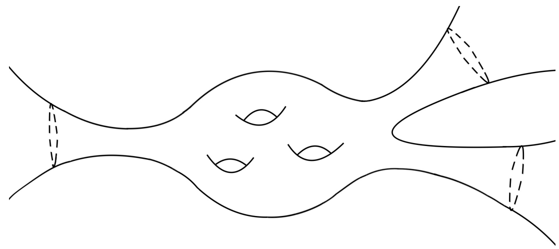

Figure 1. Convex cocompact hyperbolic surfaces

The estimates (1.2) are analogous to the standard Strichartz [46] and Keel-Tao [37] estimates for $\mathbb{R}^2$, in which case, by scaling, estimates as above over $t \in [0,1]$ are equivalent to ones over $t \in \mathbb{R}$. We are only able to treat the two-dimensional case of asymptotically hyperbolic manifolds here since some of the tools that we utilize, such as different types of L^2 local smoothing estimates for the Schrödinger propagators $e^{-it\Delta_g}$ seem to only be available in two-dimensions. As we shall see, though, the lossless log-scale estimates that we also require hold in all dimensions for manifolds of uniformly bounded geometry and nonpositive and negative sectional curvatures.

Besides the Euclidean estimates, there is a long history of Strichartz estimates for negatively curved asymptotically hyperbolic manifolds. On hyperbolic space $\mathbb{H}^n$, Anker and Pierfelice [2] and Ionescu and Staffilani [34] independently proved the mixed-norm Strichartz estimates via dispersive estimates that are unavailable for the manifolds that we are treating. Subsequently, Bouclet [10] proved these results on non-trapping asymptotically hyperbolic manifolds. Burq, Guillarmou and Hassell [17] then were able to handle certain manifolds with trapped geodesics, including n-dimensional convex cocompact hyperbolic manifolds whose limit set has Hausdorff dimension $< (n-1)/2$. Among these are hyperbolic cylinders ($n = 2$) whose central geodesic γ_0 is periodic and hence trapped. Burq, Guillarmou and Hassell [17] could obtain their Strichartz estimates for these convex cocompact hyperbolic manifolds via a logarithmic time dispersive estimate. Wang [49] proved Strichartz estimates for general (noncompact) convex cocompact hyperbolic surfaces with an ε loss of derivative. The results in Theorem 1.1 above seem to be the first lossless Strichartz estimates with no pressure condition, which seems to rule out the dispersive estimates that were used in these previous results. Note that if we replace $[0,1]$ by $\mathbb{R}$ in (1.2), the global in time Strichartz estimate was obtained by Chen [21] for all non-trapping asymptotically hyperbolic manifolds with no resonance at the bottom of spectrum.

Burq, Guillarmou and Hassell also proved more general results involving abstract hypotheses (cf. [17, Theorem 3.3]). We are able to adapt their proof to obtain our Theorem 1.1 using, as additional input, the local smoothing estimates following from the local resolvent estimates of Bourgain and Dyatlov [12] and the third author [48], as well as our new log-scale Strichartz estimates for manifolds of nonpositive curvature and bounded geometry that we shall describe shortly.

As mentioned above, another of our main results concerns spectral projection operators associated with the Laplace-Beltrami operator Δ_g. Before stating these, though, let us recall the universal estimates for compact manifolds of the second author [41] and the recent improvements by the first two authors [33]. If, for $q \in (2, \infty]$,

$$\mu(q) = \begin{cases} n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}, & q \geq q_c = \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{q}), & q \in (2, q_c], \end{cases} \tag{1.3}$$

and (M, g) is an n-dimensional compact manifold then the main result in [41] says that for $\lambda \gg 1$ and $q > 2$

$$\|\mathbf{1}_{[\lambda,\lambda+1]}(P)f\|_{L^q(M)} \leq C\lambda^{\mu(q)}\|f\|_{L^2(M)}, \quad P = \sqrt{-\Delta_g}, \tag{1.4}$$

with $\mathbf{1}_I(P)$ being the spectral projection operator associated with the spectral window $I \subset \mathbb{R}$. It was shown by one of us in [43] that the unit-band estimates (1.4) are always sharp. On the other hand, recently, the first two authors were able to obtain the following optimal bounds for compact manifolds all of whose sectional curvatures are negative

$$\|\mathbf{1}_{[\lambda,\lambda+\delta]}(P)f\|_{L^q(M)} \leq C_q\lambda^{\mu(q)}\delta^{1/2}\|f\|_{L^2(M)}, \ \delta \in [(\log\lambda)^{-1}, 1], \lambda \gg 1 \text{ and } \ q > 2. \tag{1.5}$$

Also, for later use, we note that, as was pointed out in [1], the proofs of the unit-band estimates (1.4) in [41] and [43] also can be used to show that (1.4) is valid for any manifold of uniformly bounded geometry.

One of our main results (stated below) is that (1.5) extends to all manifolds of uniformly bounded geometry and curvature pinched below zero. Using these log-scale results and certain $L^2 \to L^q$ localized resolvent estimates, we shall be able to adapt the proof of Theorem 1.1 to obtain the following results optimal for much smaller spectral windows.

Theorem 1.2. *Let (M, g) be an even asymptotically hyperbolic surface with negative curvature, and for $q > 2$, let $\mu(q)$ be as in* (1.3)*. Then for fixed $N_0 \in \mathbb{N}$ and $\lambda \gg 1$ we have the uniform bounds*

$$\|\mathbf{1}_{[\lambda,\lambda+\delta]}(P)f\|_{L^q(M)} \leq C_{q,N_0}\,\lambda^{\mu(q)}\delta^{1/2}\|f\|_{L^2(M)}, \ q \in (2, \infty], \ \textit{if} \ \delta \in [\lambda^{-N_0}, 1]. \tag{1.6}$$

As we pointed out before, there might be eigenfunctions of the Laplacian for (M, g), which means that the uniform bounds like those in (1.6) need not hold for all $\delta \in (0, 1]$. Besides this, the microlocal techniques that we shall employ require that, as in (1.6), $\delta \geq \lambda^{-N_0}$ for some $N_0 \in \mathbb{N}$.

As we pointed out earlier, a special case of our results is when (M, g) is a convex cocompact hyperbolic surface. Spectral projection estimates on these were studied in Anker, Germain and Léger [1], where somewhat weaker estimates were obtained with a λ^{ε}, $\forall\, \varepsilon > 0$, loss compared to our estimates. As was pointed out in [1], using arguments of one of us [43] mentioned before, one sees that the bounds in (1.6) are optimal.

Let us say a few more words about the proofs of the above two results. First, it will be relatively straightforward to use our generalizations in Theorem 1.3 below of the recent Strichartz estimates of two of us [32] and known local smoothing estimates for the Schrödinger propagator to obtain Theorem 1.1. We are able to do this by using an argument from [17], which we shall recall in the next section. Roughly speaking, near the compact trapping region in M we are able to obtain the needed dyadic results by gluing together uniform Strichartz estimates on intervals of length $\lambda^{-1} \cdot \log\lambda$ for solutions of (1.1) involving λ-frequency data using the known optimal log-loss local smoothing estimates associated with this region. This will allow us to show that the analog of the bounds in (1.2) are valid when the $L^p_t L^q_x$-norms are taken over x in a relatively compact neighborhood of the trapping region. The complement of this region then can easily be treated by the arguments in [17].

We shall employ a similar strategy to prove our new spectral projection estimates. As is standard, in order to prove the bounds in Theorem 1.2 for $\mathbf{1}_{[\lambda,\lambda+\delta]}(P)$, it is equivalent to prove the same bounds for the "approximate spectral projection operators"

$$\rho\big((\lambda\delta)^{-1}(-\Delta_g-\lambda^2)\big) = (2\pi)^{-1}\int_{-\infty}^{\infty} \lambda\delta\,\hat\rho(\lambda\delta t)\,e^{-it\Delta_g}\,e^{-it\lambda^2}\,dt, \tag{1.7}$$

with fixed $\rho\in\mathcal{S}(\mathbb{R})$ satisfying $\rho(0)=1$ and its Fourier transform, $\hat\rho$, supported in $[-1,1]$. Using this simple formula (also used in [1]), we can adapt the proof of Theorem 1.1, which seems to be a new approach. Near the trapping region we introduce spatial cutoffs, as well as t-cutoffs localizing to intervals of length $\lambda^{-1}\cdot\log\lambda$. We are able to naturally estimate some of the terms arising from the time cutoffs using Theorem 1.5 below for manifolds of uniformly bounded geometry along with the aforementioned local smoothing estimates for the Schrödinger propagator. Unfortunately, it is not as straightforward to handle all of the commutator terms that will arise in handling the complement of the trapping region. For, unlike in the proof of Theorem 1.1, we cannot appeal to the Christ-Kiselev lemma to handle the various "Duhamel terms" that arise in estimating (1.7), which, of course, involves a weighted superposition of the Schrödinger propagator, as opposed to the propagator itself occurring in the proof of the space-time estimates in Theorem 1.1. To deal with the problematic Duhamel terms that arise, we are led to a simple integration by parts argument, and the resulting boundary terms naturally give rise to half-localized $L^2\to L^q$ resolvent estimates paired with the available L^2 local smoothing estimates.

As was the case in [1], we shall handle the myriad issues that arise by constructing a "background manifold" $\tilde M$ that agrees with M near infinity. In the treatment of convex cocompact hyperbolic surfaces in [1], the background manifold was a finite union of hyperbolic cylinders on which optimal spectral projection estimates could be proved and utilized. In our case, $\tilde M$ is a simply connected asymptotically hyperbolic surface of negative curvature, which allows us to use the optimal spectral projection estimates of Chen and Hassell [22] for its Laplacian $\Delta_{\tilde g}$. As we alluded to before, to glue these together with estimates for the "trapping" compact region of M, we shall adapt the proof of the Strichartz estimates in Theorem 1.1. Since we cannot use the Christ-Kiselev lemma, certain half-localized L^q resolvent estimates involving the Laplacian on the background will arise. These uniform bounds appear to be new and are of the form

$$\big\|\,(\Delta_{\tilde g}+(\lambda+i\delta)^2)^{-1}(\chi h)\,\big\|_{L^q(\tilde M)}\le C_q\,\lambda^{\mu(q)-1}\|h\|_{L^2(\tilde M)},\;\;\delta\in(0,1),\;\;q\in(2,\infty),\;\;\chi\in C_0^\infty(\tilde M),$$

with $\mu(q)$ as in (1.3).

We are oversimplifying a bit here how we shall use the optimal estimates for the "background manifold" $\tilde M$ in our proof of spectral projection estimates in Theorem 1.2 for M. These are much more difficult to handle compared to the Strichartz estimates due to the "Duhamel terms" that seem to inevitably arise because we cannot use the Christ-Kiselev lemma.

Let us now describe the log-scale results on manifolds of uniformly bounded geometry that we mentioned above. These generalize recent joint work for compact manifolds of two of us [4], [7], [33], [31] and [32]. Recall that (M,g) is of uniformly bounded geometry if the injectivity radius $r_{\text{Inj}}(M)$ is *positive* and the Riemann curvature tensor R and all of its covariant derivatives are uniformly bounded. (See, e.g. Eldering [26, §2.1].)

We then have the following two results for general manifolds of bounded geometry with nonpositive sectional curvatures.

Theorem 1.3. *Suppose that (M,g) is a complete $(n-1)$-dimensional manifold of uniformly bounded geometry all of whose sectional curvatures are nonpositive. Then if $u=e^{-it\Delta_g}f$ denotes the solution of Schrödinger's equation*

$$i\partial_t u(t,x)=\Delta_g u(t,x),\ (t,x)\in\mathbb{R}\times M,\quad u|_{t=0}=f, \tag{1.8}$$

we have for fixed $\beta\in C_0^\infty((1/2,2))$ and all $\lambda\gg 1$ the uniform dyadic estimates

$$\big\|\,\beta(\sqrt{-\Delta_g}/\lambda)\,u\,\big\|_{L^p_tL^q_x(M\times[0,\lambda^{-1}\log\lambda])}\le C\|f\|_{L^2(M)} \tag{1.9}$$

for all exponents (p,q) satisfying the Keel-Tao condition

$$(n-1)(1/2-1/q)=2/p,\ p\in[2,\infty)\ \text{if}\ n-1\ge 3\ \text{and}\ \ p\in(2,\infty)\ \text{if}\ \ n-1=2. \tag{1.10}$$

The arguments in Burq, Gérard and Tzvetkov [16] yield the analog of (1.9) with $[0,\lambda^{-1}\log\lambda]$ replaced by $[0,\lambda^{-1}]$ for any complete manifold of uniformly bounded geometry. As in [4] and [32] we shall use the curvature assumption in order to obtain the above logarithmic improvements.

As is well known, typically the standard Littlewood-Paley estimates which are valid for $\mathbb{R}^n$ break down and must be modified for hyperbolic quotients; however, there are variants that allow one to use dyadic estimates like (1.9). See Bouclet [9]. Using these, we obtain from Theorem 1.3 the following improvements of the compact manifold estimates in [16].

Corollary 1.4. *Assume that (M,g) is as in Theorem 1.3. Then for (p,q) as in* (1.10) *we have*

$$\|(I+P)^{-1/p}\,(\log(2I+P))^{1/p}\,u\|_{L^p_tL^q_x(M\times[0,1])}\lesssim\|f\|_{L^2(M)}. \tag{1.11}$$

We shall postpone further discussion of the Littlewood-Paley estimates which can be used and the proof of this corollary in §4.

We shall also be able to obtain similar improvements of the universal estimates (1.4):

Theorem 1.5. *Suppose that (M,g) is a complete n-dimensional manifold of uniformly bounded geometry all of whose sectional curvatures are nonpositive. Then for $\lambda\gg1$*

$$\|\,\mathbf{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)\,\|_{2\to q}\lesssim\begin{cases}\lambda^{\mu(q)}(\log\lambda)^{-1/2}, & \text{if}\ \ q>q_c\\ \big(\lambda(\log\lambda)^{-1}\big)^{\mu(q)}, & \text{if}\ \ q\in(2,q_c].\end{cases} \tag{1.12}$$

Furthermore, if all of the sectional curvatures are pinched below $-\kappa_0^2$ with $\kappa_0>0$, then

$$\|\,\mathbf{1}_{[\lambda,\lambda+(\log\lambda)^{-1}]}(P)\,\|_{2\to q}\le C_q\lambda^{\mu(q)}(\log\lambda)^{-1/2},\ q\in(2,\infty]. \tag{1.13}$$

For $q=\infty$ and $q\in(q_c,\infty)$, the estimates in (1.12) for compact manifolds are due to Bérard [3] and Hassell and Tacy [28], respectively. Also, the bounds in (1.13) for hyperbolic space $\mathbb{H}^n$ were first proved by S. Huang and one of us [30] for exponents

$q \geq q_c$ and by Chen and Hassell [22] for $q \in (2, q_c)$. Additionally, it was shown in [33] and [31] that the bounds in (1.12) are sharp for flat compact manifolds, and, as noted in [33], those in (1.13) can never be improved since they yield (1.4). Also, for $q \in (2, q_c)$, the standard Knapp example implies that the bounds in (1.13) do not hold for the spectral projection operators associated with the Euclidean Laplacian in $\mathbb{R}^n$.

To prove the above results we shall need to make use of our assumption of (uniformly) bounded geometry. To be able to adapt the Euclidean bilinear harmonic techniques of Tao, Vargas and Vega [47] and Lee [38] that were used to prove analogous results for compact manifolds by two of us [33], [32], we shall make heavy use of the assumption regarding uniform bounds for derivatives of the curvature tensor. This will allow us to essentially reduce the local harmonic analysis step to individual coordinate charts. We shall also make heavy use of the assumption that M has positive injectivity radius in order to prove the global kernel estimates, which, along with the bilinear ones, will yield the above, just as was done earlier for compact manifolds.

Let us now present a simple counterexample showing that the above estimates break down without the assumption that $r_{\mathrm{Inj}}(M) > 0$, even for hyperbolic quotients. We shall use an argument in Appendix B of [1] which provided counterexamples for spectral projection bounds on hyperbolic surfaces with cusps. (See also [8, §5.3].)

To be more specific let us consider the n-dimensional parabolic cylinder having a cusp at one end. If we let $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_+$ be the upperhalf space model for hyperbolic space, this is

$$M = \mathbb{H}^n / \Gamma,$$

where Γ is translation of $\mathbb{R}^{n-1}$ by elements of $\mathbb{Z}^{n-1}$. So we identify M with $x + ix_n \in (-1/2 \times 1/2]^{n-1} \times \mathbb{R}_+$.

Recall that $\Delta_{\mathbb{H}^n} = (x_n)^2 \sum_j \partial_j^2 + (2-n)x_n\partial_n$. If we let $g(x) = x_n^{\frac{n-1}{2} - i\xi}$ a simple calculation shows that

$$-\Delta_{\mathbb{H}^n} g = \big((\tfrac{n-1}{2})^2 + \xi^2 \big) g.$$

Consider

$$\Phi_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\delta^{-1}(\lambda - \xi)))\, x_n^{\frac{n-1}{2} - i\xi}\, d\xi,$$

where ϕ is supported in $[-1/10, 1/10]$. Note that Φ_λ is independent of $(x_1, \dots, x_{n-1})$ and that the $\sqrt{-\Delta_{\mathbb{H}^n}}$ spectrum of Φ_λ is in $[\lambda - \delta, \lambda + \delta]$ if λ is large and $\delta \in (0, 1]$. Furthermore,

$$\Phi_\lambda(x) = \delta\, x_n^{\frac{n-1}{2} - i\lambda} \hat{\phi}(\delta \log x_n).$$

Using the change of coordinates $\omega = \log x_n$ we see that

$$\|\Phi_\lambda\|_{L^2(M)} = \delta \Big(\int_0^\infty x_n^{n-1}\, |\hat{\phi}(\delta \log x_n)|^2 \frac{dx_n}{x_n^n} \Big)^{1/2} = \delta \Big(\int_{-\infty}^{\infty} |\hat{\phi}(\delta\omega)|^2\, d\omega \Big)^{1/2}.$$

On the other hand

$$\begin{aligned} \|\Phi_\lambda\|_{L^q(M)} &= \delta \Big(\int_0^\infty x_n^{(n-1)\frac{q}{2}}\, |\hat{\phi}(\delta \log x_n)|^q\, \frac{dx_n}{x_n^n} \Big)^{1/q} \\ &= \delta \Big(\int_{-\infty}^{\infty} e^{(\frac{q}{2}-1)(n-1)\omega}\, |\hat{\phi}(\delta\omega)|^q\, d\omega \Big)^{1/q}. \end{aligned}$$

If we take $\phi(s) = a(s) \cdot \mathbf{1}_{[0,1]}(s)$ where $a \in C_0^\infty((-1/10, 1/10))$ satisfies $a(0) = 1$, then $|\hat{\phi}(\eta)| \approx |\eta|^{-1}$ for large $|\eta|$. In this case, by the preceding two identities, $\Phi_\lambda \in L^2(M)$ but $\Phi_\lambda \notin L^q(M)$ for any $q \in (2, \infty]$. Based on this, it is clear that the spectral projection operators $\mathbf{1}_{[\lambda,\lambda+\delta]}(\sqrt{-\Delta_{\mathbb{H}^n}})$ are unbounded between $L^2(M)$ and $L^q(M)$, and so the estimates in Theorem 1.5 cannot hold for this M, which has injectivity radius equal to zero.

One can similarly argue that the Strichartz estimates in Thereorm 1.3 also cannot hold for this hyperbolic quotient. Indeed the proof of (1.15) in [32] shows that, if the bounds in (1.9) were valid for a given pair (p, q), then we would have to have that for $\delta = \delta(\lambda) = (\log \lambda)^{-1}$ the spectral projection operators $\chi_{[\lambda,\lambda+\delta]}(\sqrt{-\Delta_{\mathbb{H}^n}})$ are bounded from L^2 to L^q with norm $O((\lambda/\log \lambda)^{1/q})$, which, by the above discussion, is impossible.

This paper is organized as follows. In the next section we shall prove Theorems 1.1 and 1.2 using the above estimates for manifolds of uniformly bounded geometry and known local smoothing estimates for Schrödinger propagators. In §3 we shall prove our log-scale estimates for manifolds of uniformly bounded geometry and appropriate curvature assumptions. For the sake of completeness, in §4, we shall also present the Littlewood-Paley estimates for manifolds of bounded geometry that we are using.

Throughout this paper, we write $X \gg Y$ (or $X \ll Y$) to mean $X \geq CY$ (or $X \leq Y/C$) for some large constant $C > 1$. Similarly, $X \gtrsim Y$ (or $X \lesssim Y$) denotes $X \geq CY$ (or $X \leq CY$) for some positive constant C.

2. Proofs of lossless estimates for asymptotically hyperbolic surfaces.

In this section, we shall see how we can apply Theorem 1.3 and 1.5 to prove lossless Strichartz and spectral projection estimates in Theorems 1.1 and 1.2. The proof of Theorem 1.3 and 1.5 will be given in the next section.

Throughout this section, let us assume that (M, g) is a (even) asymptotically hyperbolic manifold. This means there exists a compactification $\overline{M}$, which is a smooth manifold with boundary ∂M, and the metric near the boundary takes the form

$$g = \frac{dx_1^2 + g_1(x_1^2)}{x_1^2}, \quad x_1|_{\partial M} = 0, \quad dx_1|_{\partial M} \neq 0$$

where $g_1(x_1^2)$ is a smooth family of metrics on ∂M. Examples include convex cocompact hyperbolic manifolds and their metric perturbation. A convex cocompact hyperbolic manifold is a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ such that the convex core is compact. Intuitively, it is a hyperbolic manifold with finitely many funnel ends and no cusps.

Let us also describe some dynamic properites of the geodesic flow e^{tH_p} on asymptotically hyperbolic manifolds. Let $S^*M = \{(x, \xi) \in T^*M : |\xi|_{g(x)} = 1\}$ be the cosphere bundle of M and $(x(t), \xi(t)) = e^{tH_p}(x, \xi)$. The outgoing set Γ_+ is defined as

$$\Gamma_+ := \{(x, \xi) \in S^*M : x(t) \not\to \infty \text{ as } t \to -\infty\}.$$

In other words, (x, ξ) does not escape to ∞ along the backward geodesic flow. Similarly, the incoming set Γ_- is defined as

$$\Gamma_- := \{(x, \xi) \in S^*M : x(t) \not\to \infty \text{ as } t \to +\infty\}.$$

The trapped set $K = \Gamma_+ \cap \Gamma_-$ is the intersection of the outgoing set and incoming set. In other words, $(x, \xi) \in K$ does not escape in either direction of the geodesic flow. For later use, let $\pi(K)$ be the projection of the trapped set K onto M.

For all asymptotically hyperbolic manifolds such as convex cocompact hyperbolic manifolds, it is known that $\Gamma_\pm$ are both closed and the trapped set K is compact, see e.g., Dyatlov–Zworski [25, Chapter 6] for more details. Moreover, by the convexity of the geodesic flow at infinity [25, Lemma 6.6], let $S \subset S^*M$ be a compact subset such that $S \cap \Gamma_- = \emptyset$ ($S \cap \Gamma_+ = \emptyset$, respectively), then for any compact set S', there exists a uniform constant $T = T(S, S') > 0$ such that

$$e^{tH_p}(x, \xi) \notin S', \quad (x, \xi) \in S \tag{2.1}$$

for any $t \geq T$ ($t \leq -T$, respectively).

2.1. Lossless Strichartz estimates.

To prove Theorem 1.1, we need three estimates for $\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$, $p, q \geq 2$ and $(p, q) \neq (2, \infty)$. Of course in the statement of Theorem 1.1, $n - 1 = 2$. We are letting n denote the space-time dimension of $M \times \mathbb{R}$ to be convention that we are using in Theorem 1.3 (and used before in [32]).

(a) Lossless Strichartz and local smoothing estimates in the nontrapping region. Let $\chi \in C_0^\infty(M)$ with $\chi = 1$ on $\pi(K)$,

$$\|(1-\chi)e^{-it\Delta_g}u_0\|_{L_t^p L_x^q(M\times[0,1])} \leq C\|u_0\|_{L^2(M)}. \tag{2.2}$$

One also needs a lossless local smoothing in the nontrapping region: Fix $\beta \in C_0^\infty((1/2, 2))$, for $\chi \in C_0^\infty(M)$ supported away from the trapped set $\pi(K)$, we have

$$\|\chi e^{-it\Delta_g}\beta(\sqrt{-\Delta_g}/\lambda)u_0\|_{L^2_{t,x}(M\times[0,1])} \leq C\lambda^{-1/2}\|u_0\|_{L^2(M)}. \tag{2.3}$$

(b) Local smoothing with logrithmic loss. Let $\chi \in C_0^\infty(M)$ with $\chi = 1$ on $\pi(K)$,

$$\|\chi e^{-it\Delta_g}\beta(\sqrt{-\Delta_g}/\lambda)u_0\|_{L^2_{t,x}(M\times[0,1])} \leq C\lambda^{-\frac{1}{2}}(\log\lambda)^{1/2}\|u_0\|_{L^2(M)}. \tag{2.4}$$

(c) Lossless Strichartz with log-scale gains compared to the universal estimates in [16]

$$\|e^{-it\Delta_g}\beta(\sqrt{-\Delta_g}/\lambda)u_0\|_{L_t^p L_x^q(M\times[0,\lambda^{-1}\log\lambda])} \leq C\|u_0\|_{L^2(M)}. \tag{2.5}$$

We recall a lemma from [17].

Lemma 2.1. *The estimates* (2.2)–(2.5) *imply the lossless Strichartz estimate*

$$\|e^{-it\Delta_g}u_0\|_{L_t^p L_x^q(M\times[0,1])} \leq C\|u_0\|_{L^2(M)}. \tag{2.6}$$

Proof. By the Littlewood Paley estimate in Lemma 4.1 and the remark below it, we may assume $u_0 = \beta(\sqrt{-\Delta_g}/\lambda)u_0$ with β as above. By (2.2), it suffices to show for any $\chi \in C_0^\infty(M)$ with $\chi = 1$ on $\pi(K)$, we have

$$\|\chi e^{-it\Delta_g}u_0\|_{L_t^p L_x^q(M\times[0,1])} \leq C\|u_0\|_{L^2(M)}.$$

Let $\alpha \in C_0^\infty((-1, 1))$ satisfying $\sum \alpha(t - j) = 1$, $t \in \mathbb{R}$. For $j \in \mathbb{Z}$ and $u(t) = e^{-it\Delta_g}u_0$, let us define $u_j = \alpha(t\,\lambda/\log\lambda - j)\chi u$. We have

$$(i\partial_t - \Delta_g)u_j = v_j + w_j$$

where

$$v_j = i\frac{\lambda}{\log\lambda}\alpha'(t\,\lambda/\log\lambda - j)\chi u, \quad w_j = -\alpha(t\,\lambda/\log\lambda - j)[\Delta_g, \chi]u. \tag{2.7}$$

Let $\chi_-, \chi_+ \in C_0^\infty$, satisfy $\chi_- = 1$ on suppχ and $\chi_+ = 0$ on the trapped set $\pi(K)$ and $\chi_+ = 1$ on supp$\nabla\chi$. Then

$$u_j = \chi_- u_j, \quad v_j = \chi_- v_j, \quad w_j = \chi_+ w_j.$$

Additionally, for any $\chi \in C_0^\infty(M)$, it is not hard show that $\beta(\sqrt{-\Delta_g}/\lambda)$ essentially commutes with χ. Specifically, if $\tilde\beta \in C_0^\infty((1/4,4))$ which equals one in a neighborhood of the support of β, we have

$$\chi\cdot\beta(\sqrt{-\Delta_g}/\lambda)f = \tilde\beta(\sqrt{-\Delta_g}/\lambda)\chi\beta(\sqrt{-\Delta_g}/\lambda)f + Rf, \tag{2.8}$$

where $\|Rf\|_{L^q(M)} \le C_N\lambda^{-N}\|f\|_{L^2(M)}$ for $q \ge 2$. For more details, see for instance, the proof of Lemma 4.2 below.

Thus, it suffices to estimate

$$u_j^{(1)} = \chi_- \int_{(j-1)\lambda^{-1}\log\lambda}^{t} e^{-i(t-s)\Delta_g}\tilde\beta(\sqrt{-\Delta_g}/\lambda)\chi_- v_j(s)ds,$$

and

$$u_j^{(2)} = \chi_- \int_{(j-1)\lambda^{-1}\log\lambda}^{t} e^{-i(t-s)\Delta_g}\tilde\beta(\sqrt{-\Delta_g}/\lambda)\chi_+ w_j(s)ds.$$

Let $\tilde u_j^{(1)}, \tilde u_j^{(2)}$ be the analog of $u_j^{(1)}, u_j^{(2)}$ with the upper bound of the integrals replaced by $(j+1)\lambda^{-1}\log\lambda$. Since $\chi_+ w_j$ is supported in the nontrapped region, by (2.3) and (2.5) we have

$$\|\tilde u_j^{(2)}\|_{L_t^pL_x^q} \lesssim \left\| \int_{(j-1)\lambda^{-1}\log\lambda}^{(j+1)\lambda^{-1}\log\lambda} e^{is\Delta_g}\tilde\beta(\sqrt{-\Delta_g}/\lambda)\chi_+ w_j(s)ds \right\|_{L^2} \lesssim \lambda^{-1/2}\|w_j\|_{L^2_{t,x}}.$$

The same estimate holds for $u_j^{(2)}$ by the Christ–Kiselev lemma. On the other hand, on the trapped region, by (2.5) and local smoothing (2.4), we have

$$\|\tilde u_j^{(1)}\|_{L_t^pL_x^q} \lesssim \left\| \int_{(j-1)\lambda^{-1}\log\lambda}^{(j+1)\lambda^{-1}\log\lambda} e^{is\Delta_g}\tilde\beta(\sqrt{-\Delta_g}/\lambda)\chi_- v_j(s)ds \right\|_{L^2} \lesssim \lambda^{-\frac12}(\log\lambda)^{1/2}\|v_j\|_{L^2_{t,x}}.$$

The same estimate holds for $u_j^{(1)}$ by the Christ–Kiselev Lemma.

Note that by using (2.8), it is not hard to verify that

$$\|w_j\|_{L^2_{t,x}} \lesssim \lambda\|\alpha(t\,\lambda/\log\lambda - j)\chi_+ u\|_{L^2_{t,x}}.$$

Thus by the local smoothing estimates (2.3) and (2.4), we have

$$\|\chi u\|^2_{L_t^pL_x^q} \le \sum_j \|u_j\|^2_{L_t^pL_x^q} \lesssim \sum_j \lambda^{-1}\|w_j\|^2_{L^2_{t,x}} + \lambda^{-1}\log\lambda\|v_j\|^2_{L^2_{t,x}} \lesssim \|u_0\|^2_{L^2}.$$

This completes the proof of (2.6). □

(2.2) is known for all asymptotically hyperbolic manifolds, see Bouclet [10, Theorem 1.2]. By [25, Theorem 7.2], the assumptions (2.3) and (2.4) follow from the following resolvent estimates. For $\chi \in C_0^\infty(M)$ supported away from the trapped set $\pi(K)$

$$\|\chi(-\Delta_g - (\lambda + i0)^2)^{-1}\chi\|_{L^2\to L^2} \le C\lambda^{-1}. \tag{2.9}$$

Additionally, if $\chi \in C_0^\infty(M)$ with $\chi = 1$ on $\pi(K)$,

$$\|\chi(-\Delta_g - (\lambda + i0)^2)^{-1}\chi\|_{L^2\to L^2} \le C\lambda^{-1}\log\lambda. \tag{2.10}$$

(2.9) is known to hold for asymptotically hyperbolic manifolds if we assume χ is supported sufficiently far away from $\pi(K)$ by Cardoso–Vodev [18], following the method of Carleman estimate in Burq [14]. Under the stronger condition that

$$\|\chi(-\Delta_g - (\lambda + i0)^2)^{-1}\chi\|_{L^2\to L^2} \le C\lambda^{N_0}, \quad \chi \in C_0^\infty(M), \tag{2.11}$$

the resolvent estimate (2.9) and the local smoothing estimate (2.3) follows from standard propagation estimates, see Datchev–Vasy [23] for the resolvent estimate in this case.

For convex cocompact hyperbolic surfaces, (2.10) follows from the result of Bourgain-Dyatlov [12, Theorem 2] and Burq [15, Lemma 4.5]. This was generalized to even asymptotically hyperbolic surfaces with negative curvature by the third author in [48]. In higher dimensions, (2.10) also hold under certain conditional trapping conditions, such as the pressure condition and normally hyperbolic trapping, see Nonnenmacher and Zworski [39, 40]. Finally, (2.5) follows from Theorem 1.3 which holds for all complete manifolds with bounded geometry and nonpositive sectional curvature.

Hence, (2.2)–(2.5) hold for all even asymptotically hyperbolic surfaces with negative curvature, which completes the proof of Theorem 1.1. Addtionally, for $\lambda \gg 1$, (2.3) and (2.4) remain valid when $[0,1]$ is replaced by $\mathbb{R}$ on all asymptotically hyperbolic surfaces with negative curvature. We will use this fact later in the proof of Theorem 1.2.

The lossless Strichartz estimate can be used to prove the following local well-posedness of the cubic nonlinear Schrödinger equation in the critical regularity.

Proposition 2.2. *Let $n-1=2$. Suppose the lossless Strichartz estimate* (2.6) *holds. Consider the Schrödinger equation*

$$i\partial_t u - \Delta_g u = F(u), \quad u(0,\cdot) = u_0(x) \in L^2(M) \tag{2.12}$$

where $F(u)$ is a homogeneous cubic polynomial of u and $\bar u$. Then there exists $T>0$ such that (2.12) *has a unique solution*

$$u(t,x) \in C([-T,T]; L^2(M)) \cap L^3([-T,T]; L^6(M)).$$

Moreover, if $u_0 \in H^s(M)$ for some $s>0$, then $u \in C([-T,T]; H^s(M))$.

Proof. Consider the map

$$G(u)(t,x) = e^{-it\Delta_g}u_0 - i\int_0^t e^{-i(t-t')\Delta_g}F(u)(t',x)dt'.$$

Let $0 < T \le 1$. Define the norm

$$\|u\|_{Y_T} := \sup_{t\in[-T,T]} \|u(t,\cdot)\|_{L^2(M)} + \|u\|_{L^3([-T,T];L^6(M))}.$$

Then

$$\|G(u)\|_{Y_T} \le C\|u_0\|_{L^2} + \int_{-T}^{T} \|F(u)\|_{L^2(M)} dt \le C\|u_0\|_{L^2} + C\|u\|^3_{L^3([-T,T];L^6(M))}$$

and

$$\begin{aligned} \|G(u) - G(v)\|_{Y_T} &\le \int_{-T}^{T} \|F(u) - F(v)\|_{L^2(M)} dt \\ &\le C(\|u\|_{L^3([-T,T];L^6(M))} + \|v\|_{L^3([-T,T];L^6(M))})^2 \|u - v\|_{Y_T}. \end{aligned} \tag{2.13}$$

Choose $T > 0$ such that $\|e^{-it\Delta_g} u_0\|_{L^3([-T,T];L^6(M))}$ is sufficiently small. Then G is a contraction map on

$$\{u \in Y_T : \|u\|_{L^3([-T,T];L^6(M))} \le \varepsilon\}.$$

This gives a unique fixed point of G, which is a solution to (2.12) in the space Y_T.

If $u_0 \in H^s(M)$, then the above proof works with the norm

$$\|u\|_{Y_T^s} := \sup_{t\in[-T,T]} \|u(t,\cdot)\|_{H^s(M)} + \|(1-\Delta)^{s/2}u\|_{L^3([-T,T];L^6(M))} + \|u\|_{L^3([-T,T];L^6(M))}.$$

If s is not an even integer, one needs to use the fact that

$$\|(1-\Delta)^{s/2}(u_1u_2u_3)\|_{L^2(M)} \lesssim \prod_{j=1}^{3} \Big(\|(1-\Delta)^{s/2}u_j\|_{L^6(M)} + \|u_j\|_{L^6(M)} + \|u_j\|_{L^2(M)} \Big).$$

This follows from the fractional Leibniz rule due to Kato–Ponce [36] near the diagonal, i.e. for the operator P with Schwartz kernel $K_s(x,y)\chi(d(x,y))$ where $K_s(x,y)$ is the Schwartz kernel of $(1-\Delta_g)^{s/2}$ and $\chi \in C_0^\infty(\mathbb{R})$ is a cutoff that $\chi(t) = 1$ for $|t| < 1$, we have

$$\|P(u_1u_2u_3)\|_{L^2} \le \prod_{j=1}^{3} (\|Pu_j\|_{L^6} + \|u_j\|_{L^6}).$$

On the other hand, the part of $(1-\Delta_g)^{s/2}$ away from the diagonal is a smoothing pseudodifferential operator, which is bounded from L^2 to L^q for $q \ge 2$. The uniqueness of the solution follows from (2.13). □

2.2. Lossless spectral projection estimates.

In this section we shall give the proof of Theorem 1.2. We may assume $\delta < (\log\lambda)^{-1}$ since the sharp estimates for $\delta = (\log\lambda)^{-1}$ follow from Theorem 1.5 which concerns a larger class of unbounded manifolds.

The proof of Theorem 1.2 relies on the construction of a "background" manifold $(\tilde{M}, \tilde{g})$ which agrees with M asymptotically and satisfies favorable spectral projection estimates. Specifically, we shall assume that $M = M_{tr} \cup M_\infty$ where $M_{tr} \subset M$ is compact and contains a neighborhood of the trapped set $\pi(K)$ defined at the beginning of this section. We shall construct $\tilde{M}$ such that the metric $\tilde{g}$ for $\tilde{M}$ agrees with the metric g on M_∞.

Recall that in the disk model D^2, in suitable coordinates, the metric of an asymptotically hyperbolic surface near the boundary is given by

$$4\,\frac{dr^2 + h(r,\theta)d\theta^2}{(1-r^2)^2}, \quad h \in C^\infty, \ \ h(1,\theta) = 1.$$

For example, if M is the hyperbolic plane, $h(r,\theta) = r^2$, while if M is the hyperbolic cylinder, $h(r,\theta) = \frac{1}{4}(1+r^2)^2$. See [25, Chapter 5] for more details. Let $\chi \in C_0^\infty((-1,1))$ with $\chi = 1$ in $(-1/2, 1/2)$, then we can define the metric on $\tilde{M}$ as

$$4\,\frac{dr^2 + r^2 d\theta^2}{(1-r^2)^2} + \chi(R(1-r))4\,\frac{(h(r,\theta) - r^2)d\theta^2}{(1-r^2)^2}, \tag{2.14}$$

where R is a fixed constant. Then we have the metric of $\tilde{M}$ agrees with M on the set $r \geq 1-(2R)^{-1}$. Furthermore, note that $|h(r,\theta)-r^2| \leq R^{-1}$ in the support of $\chi(R(1-r))$. By choosing R sufficiently large, it is straightforward to check that the Gaussian curvature K of $(\tilde{M}, g)$ satisfies the uniform bound $-\frac{3}{2} \leq K \leq -\frac{1}{2}$. Hence $\tilde{M}$ is a simply connected manifold with negative curvature and no conjugate points. Thus, as a consequence of Chen-Hassell [22, Theorem 6], we have the following sharp spectral projection estimates for $\tilde{M}$: If $\tilde{P} = \sqrt{-\Delta_{\tilde{g}}}$, for $\mu \approx \lambda \gg 1$,

$$\|\mathbf{1}_{[\mu,\mu+\delta]}(\tilde{P})h\|_{L^q(\tilde{M})} \lesssim \lambda^{\mu(q)}\delta^{\frac{1}{2}}\|h\|_{L^2(\tilde{M})}, \ \ \delta \in (0,1). \tag{2.15}$$

To prove Theorem 1.2, it suffices to prove the estimates in (1.6) for $q < \infty$ since the bounds for a given $q \in [6,\infty)$ imply those for larger q by a simple argument using Sobolev estimates. So, in what follows, we shall assume that $q \in (2,\infty)$. And if we fix $\beta \in C_0^\infty((1/2,2))$ with $\beta = 1$ in $(3/4, 5/4)$, it suffices to replace f in the left side of (1.6) with $f_\lambda = \beta(\sqrt{-\Delta_g}/\lambda)f$.

Let $\rho \in \mathcal{S}(\mathbb{R})$ satisfy $\rho(0) = 1$ and have Fourier transform vanishing outside of $[-1,1]$, and let $\chi_0 \in C_0^\infty(M)$ with $\chi_0 = 1$ on M_{tr} and $\chi_\infty = 1 - \chi_0$. To prove (1.6), it suffices to show that for δ in this inequality we have

$$\|\chi_\infty \rho((\lambda\delta)^{-1}(-\Delta_g - \lambda^2))f_\lambda\|_{L^q(M)} \lesssim \lambda^{\mu(q)}\delta^{\frac{1}{2}}\|f\|_{L^2(M)}, \tag{2.16}$$

as well as

$$\|\chi_0 \rho((\lambda\delta)^{-1}(-\Delta_g - \lambda^2))f_\lambda\|_{L^q(M)} \lesssim \lambda^{\mu(q)}\delta^{\frac{1}{2}}\|f\|_{L^2(M)}. \tag{2.17}$$

To prove (2.16), note that if $u = e^{-it\Delta_g}f_\lambda$ and we set $v = \chi_\infty u$, where χ_∞ is as above, then v solves the Cauchy problem on (M,g)

$$\begin{cases} (i\partial_t - \Delta_g)v = [\chi_\infty, \Delta_g]u \\ v|_{t=0} = \chi_\infty f_\lambda. \end{cases} \tag{2.18}$$

Since $\Delta_g = \Delta_{\tilde{g}}$ on supp χ_∞, v also solves the following Cauchy problem on the "background manifold" $(\tilde{M}, \tilde{g})$,

$$\begin{cases} (i\partial_t - \Delta_{\tilde{g}})v = [\chi_\infty, \Delta_g]u \\ v|_{t=0} = \chi_\infty f_\lambda. \end{cases} \tag{2.19}$$

Thus,

$$v = e^{-it\Delta_{\tilde{g}}}(\chi_\infty f) + i\int_0^t e^{-i(t-s)\Delta_{\tilde{g}}}\big([\Delta_g, \chi_\infty]u(s,\,\cdot\,)\big)\,ds. \tag{2.20}$$

By using the inverse Fourier transform, (2.20) implies

$$\chi_\infty\rho((\lambda\delta)^{-1}(-\Delta_g-\lambda^2))f_\lambda=\rho((\lambda\delta)^{-1}(-\Delta_{\tilde g}-\lambda^2))(\chi_\infty f_\lambda) + (2\pi)^{-1}i\int_{-\infty}^{\infty}\lambda\delta\,\hat\rho\big(\lambda\delta t\big)\,e^{-it\lambda^2}\Big(\int_0^t e^{-i(t-s)\Delta_{\tilde g}}\big([\Delta_g,\chi_\infty]u(s,\,\cdot\,)\big)\,ds\Big)\,dt. \tag{2.21}$$

By using the spectral projection estimates, (2.15), for $\tilde M$ it is not hard to check that we have the desired bounds for the first term in the right side of (2.21). So to prove (2.16) it suffices to show that

$$\|R_\lambda f\|_{L^q(M_\infty)}\lesssim \lambda^{\mu(q)}\delta^{1/2}\|f\|_{L^2(M)},\ 2<q<\infty, \tag{2.22}$$

where, if we set $\tilde\rho(t)=e^{-t}\hat\rho(t)$,

$$\begin{aligned} R_\lambda f&=\lambda\delta\int_{-\infty}^{\infty}e^{-it(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)}\tilde\rho(\lambda\delta t)\Big(\int_0^t e^{is\Delta_{\tilde g}}[\Delta_g,\chi_\infty]\big(e^{-is\Delta_g}f_\lambda\big)\,ds\Big)\,dt\\ &=-i(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)^{-1}(\lambda\delta)\int_{-\infty}^{\infty}e^{-it(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)}\frac{d}{dt}\big(\tilde\rho(\lambda\delta t)\big)\Big(\int_0^t e^{is\Delta_{\tilde g}}[\Delta_g,\chi_\infty]\big(e^{-is\Delta_g}f_\lambda\big)\,ds\Big)\,dt\\ &\quad-i(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)^{-1}\int_{-\infty}^{\infty}\lambda\delta[\Delta_g,\chi_\infty]\hat\rho(\lambda\delta t)e^{-it\lambda^2}e^{-it\Delta_g}f_\lambda\,dt\\ &=-i(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)^{-1}\big[R'_\lambda f+S_\lambda f\big], \end{aligned} \tag{2.23}$$

where R'_λ is the analog of R_λ with $\tilde\rho(\lambda\delta t)$ replaced by its derivative, and where S_λ is the last integral.

Note that by using Minkowski's integral inequality in the t-variable followed by a two-fold application of local smoothing as in the previous section, we have

$$\|R'_\lambda f\|_{L^2(M_\infty)}\lesssim(\lambda\delta)\cdot\lambda\cdot(\lambda^{-1/2})^2\|f\|_{L^2(M)}=\lambda\delta\|f\|_{L^2(M)}, \tag{2.24}$$

with the λ-factor arising due to the commutator. Also, it is not hard to use (2.15) along with the Cauchy-Schwarz inequality and L^2 orthogonality to prove that

$$\|(\Delta_{\tilde g}+\lambda^2+i\lambda\delta)^{-1}h\|_{L^q(\tilde M)}\lesssim\lambda^{\mu(q)-1}\delta^{-\frac12}\|h\|_{L^2(\tilde M)},\ \ q<\infty. \tag{2.25}$$

By (2.24) and (2.25) we know that the second to last term in (2.23) satisfies the desired bounds posited in (2.22).

To handle the other term in (2.23) involving S_λ, we will rely on the following key result concerning half-localized resolvent operators on the background manifold.

Proposition 2.3. *Let $(\tilde M,\tilde g)$ be defined as in* (2.14)*, which is asymptotically hyperbolic, simply connected and has negative curvature. If $\tilde\chi_0\in C_0^\infty(M_\infty)$ then for $\lambda\gg1$ and $\delta\in(\lambda^{-N_0},1/2)$ for some fixed $N_0>0$, we have*

$$\|(\Delta_{\tilde g}+\lambda^2+i\delta\lambda)^{-1}(\tilde\chi_0h)\|_{L^q(\tilde M)}\lesssim\lambda^{\mu(q)-1}\|h\|_{L^2(\tilde M)},\ 2<q<\infty. \tag{2.26}$$

The estimate in (2.26) shows a gain of $\delta^{\frac12}$ compared to the estimate in (2.25). This gain arises from the presence of the compact cutoff $\tilde\chi_0$, and such estimates do not hold on compact manifolds.

We shall postpone the proof of (2.26) until the end of the section. Let us first see how we can use it to handle the last term in (2.23). Since $S_\lambda f$ is compactly supported in M_∞, we find from (2.26) that

$$\|(\Delta_{\tilde g}+\lambda^2+i\delta\lambda)^{-1}S_\lambda f\|_{L^q(\tilde M)}\lesssim \lambda^{\mu(q)-1}\|S_\lambda f\|_{L^2(\tilde M)},\ 2<q<\infty. \tag{2.27}$$

On the other hand, if $\chi\in C_0^\infty(M_\infty)$ equals one on the support of $\nabla\chi_\infty$, we have

$$\|S_\lambda f\|_{L^2_x(\tilde M)}\lesssim (\lambda\delta)^{\frac12}\cdot\lambda\cdot\|\chi e^{-it\Delta_g}f_\lambda\|_{L^2([-(\lambda\delta)^{-1},\,(\lambda\delta)^{-1}]\times M)},$$

where the λ-factor comes from the commutator and we also used Schwarz's inequality here. We can use the lossless local smoothing estimates (2.3) on M to estimate the above $L^2_{t,x}$ norm and then obtain the desired bound as in (2.22) for the last term in (2.23). Here we require the local smoothing estimate in the range $[-(\lambda\delta)^{-1},(\lambda\delta)^{-1}]$.

Let us see how we can combine this argument along with the proof of the Strichartz estimates in the previous section to obtain (2.17). To this end, just as before, choose $\alpha\in C_0^\infty((-1,1))$ satisfying $\sum\alpha(t-j)=1$, $t\in\mathbb{R}$. Also, let

$$\alpha_j(t)=\alpha((\lambda/\log\lambda)t-j),$$

to obtain, like before a smooth partition of unity associated with $\lambda^{-1}\cdot\log\lambda$-intervals. Then, if ρ is as above and

$$u_j=\alpha_j(t)\chi_0e^{-it\Delta_g}f_\lambda,$$

as was previously done, split

$$\hat\rho(\lambda\delta t)u_j(t,x)=-i\hat\rho(\lambda\delta t)\int_0^t e^{-i(t-s)\Delta_g}v_j(s,x)ds-i\hat\rho(\lambda\delta t)\int_0^t e^{-i(t-s)\Delta_g}w_j(s,x)ds$$

where w_j and v_j are defined in (2.7). Let

$$I_j=[(j-1)\lambda^{-1}\log\lambda,(j+1)\lambda^{-1}\log\lambda],$$

then

$$\begin{aligned}\int\lambda\delta\hat\rho(\lambda\delta t)v_j(t)\,e^{-it\lambda^2}\,dt&=\lambda\delta\int_{I_j}e^{-it(\Delta_g+\lambda^2+i\lambda/\log\lambda)}e^{-t\lambda/\log\lambda}\hat\rho(\lambda\delta t)\\&\qquad\cdot\Big(\int_0^t\big(e^{is\Delta_g}[\partial_s,\alpha_j]\,\chi_0e^{-is\Delta_g}f_\lambda\big)\,ds\Big)\,dt\\&=-i(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}\big[R'_{j,v,\lambda}f+S_{j,v,\lambda}f\big],\end{aligned}$$

with

$$R'_{j,v,\lambda}f=\lambda\delta\int_{I_j}e^{-it(\Delta_g+\lambda^2+i\lambda/\log\lambda)}\frac{d}{dt}\big(e^{-t\lambda/\log\lambda}\hat\rho(\lambda\delta t)\big)\Big(\int_0^t\big(e^{is\Delta_g}[\partial_s,\alpha_j]\,\chi_0e^{-is\Delta_g}f_\lambda\big)\,ds\Big)\,dt,$$

and

$$S_{j,v,\lambda}f=\lambda\delta\int_{I_j}e^{-it\lambda^2}\hat\rho(\lambda\delta t)[\partial_t,\alpha_j]\chi_0e^{-it\Delta_g}f_\lambda\,dt.$$

Similarly, we set

$$\int\lambda\delta\hat\rho(\lambda\delta t)w_j(t)\,e^{it\lambda^2}\,dt=(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}\big[R'_{j,w,\lambda}f+S_{j,w,\lambda}f\big],$$

where

$$
\begin{aligned}
&R'_{j,w,\lambda}f \\
&= \lambda\delta \int_{I_j} e^{-it(\Delta_g+\lambda^2+i\lambda/\log\lambda)} \frac{d}{dt}\big(e^{-t\lambda/\log\lambda}\hat\rho(\lambda\delta t)\big)\Big(\int_0^t \big(e^{is\Delta_g}\alpha_j(s)[\Delta_g,\chi_0]\, e^{-is\Delta_g} f_\lambda\big)\, ds\Big)\, dt,
\end{aligned}
$$

and

$$
S_{j,w,\lambda}f = \lambda\delta \int_{I_j} e^{-it\lambda^2}\alpha_j(t)\hat\rho(\lambda\delta t)[\Delta_g,\chi_0]e^{-it\Delta_g} f_\lambda\, dt.
$$

Let us fix $\chi_1 \in C_0^\infty(M)$ such that $\chi_1 \equiv 1$ on the support of χ_0. Then, we have $u_j = \chi_1 u_j$, and since there are $O(\frac{1}{\delta\log\lambda})$ nonzero $\hat\rho v_j$ and $\hat\rho w_j$ summands, by the Cauchy-Schwarz inequality, we would obtain (2.17) if we could show

$$
\begin{aligned}
&\Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}R'_{j,v,\lambda}f\|_q^2\Big)^{1/2} \\
&\qquad + \Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}S_{j,v,\lambda}f\|_q^2\Big)^{1/2} \lesssim \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}\, \|f\|_2,
\end{aligned} \tag{2.28}
$$

and

$$
\begin{aligned}
&\Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}R'_{j,w,\lambda}f\|_q^2\Big)^{1/2} \\
&\qquad + \Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}S_{j,w,\lambda}f\|_q^2\Big)^{1/2} \lesssim \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}\, \|f\|_2.
\end{aligned} \tag{2.29}
$$

As we shall see later, the χ_1 cutoff function is only needed to handle the "$S-$term" in (2.29).

To prove the bounds for the "R'-terms" in (2.28) and (2.29) we shall make use of the following analog of (2.25)

$$
\|(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}h\|_{L^q(M)} \lesssim \lambda^{\mu(q)}(\log\lambda)^{-1/2}\,(\lambda/\log\lambda)^{-1}\,\|h\|_{L^2(M)}. \tag{2.30}
$$

This follows from the sharp spectral projection estimates in Theorem 1.5 and a simple argument using the Cauchy-Schwarz inequality and L^2 orthogonality.

Let I_j be as above. We then claim that

$$
\|R'_{j,v,\lambda}f\|_{L^2(M)} \lesssim \lambda\delta\,(\lambda/\log\lambda)^{1/2}\,\|\chi_0 e^{-is\Delta_g}f_\lambda\|_{L^2(I_j\times M)}. \tag{2.31}
$$

If so, by applying (2.30) and using local smoothing estimates for M, we would obtain

$$
\begin{aligned}
&\big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}R'_{j,v,\lambda}f\|_q^2\big)^{1/2} \\
&\qquad \lesssim \lambda^{\mu(q)}\lambda\delta(\log\lambda)^{-1/2}(\lambda/\log\lambda)^{-1/2}\|\chi_0 e^{-is\Delta_g}f_\lambda\|_{L^2_{s,x}(\mathbb{R}\times M)} \\
&\qquad \lesssim \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}\|f\|_2,
\end{aligned}
$$

which gives us the desired bounds for the first term in the left side of (2.28).

To prove (2.31), note that $\int_{I_j} e^{t\lambda/\log\lambda}|\frac{d}{dt}(e^{-t\lambda/\log\lambda}\hat\rho(\lambda\delta t))|\,dt = O(1)$. Thus, by Minkowski's integral inequality,

$$\|R'_{j,v,\lambda}f\|_2 \lesssim \lambda\delta \sup_{t\in I_j}\Big\| \int_0^t e^{is\Delta_g}[\partial_s,\alpha_j]\chi_0 e^{-is\Delta_g} f_\lambda\, ds\,\Big\|_{L^2(M)},$$

which leads to (2.31) by the arguments used to prove the Strichartz estimates for the v_j terms in the previous subsection.

Similarly, repeating arguments used before we obtain

$$\|R'_{j,w,\lambda}f\|_2 \lesssim \lambda\delta\lambda^{-1/2}\|[\Delta_g,\chi_0]e^{-is\Delta_g}f_\lambda\|_{L^2(I_j\times M)}.$$

Therefore, by (2.30)

$$\begin{aligned}&\Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}R'_{j,w,\lambda}f\|_q^2\Big)^{1/2}\\ &\qquad\lesssim \lambda^{\mu(q)}\lambda\delta(\log\lambda)^{-1/2}(\lambda/\log\lambda)^{-1}\lambda^{-1/2}\|[\Delta_g,\chi_0]e^{-is\Delta_g}f_\lambda\|_{L^2(\mathbb{R}\times M)}\\ &\qquad\lesssim \lambda^{\mu(q)}\lambda\delta(\log\lambda)^{-1/2}(\lambda/\log\lambda)^{-1}\lambda^{-1/2}\cdot\lambda\cdot\lambda^{-1/2}\|f\|_2,\end{aligned}$$

which means that we also have the desired bounds for the first term in the left side of (2.29).

It remains to estimate the second terms in the left sides of (2.28) and (2.29), i.e., the "S-terms". First, by (2.30) and Hölder's inequality, we have

$$\begin{aligned}\|(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}S_{j,v,\lambda}f\|_q &\lesssim \lambda^{\mu(q)-1}(\log\lambda)^{1/2}\|S_{j,v,\lambda}f\|_2\\ &\lesssim \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}(\lambda/\log\lambda)^{-1/2}\|\chi_0[\partial_s,\alpha_j]e^{-is\Delta_g}f_\lambda\|_{L^2(I_j\times M)}.\end{aligned}$$

Since $[\partial_s,\alpha_j]$ contributes $\lambda/\log\lambda$ to the estimates, if we square and sum over j and use local smoothing estimate (2.4) in M, we obtain that

$$\begin{aligned}&\Big(\sum_j \|\chi_1(\Delta_g+\lambda^2+i\lambda/\log\lambda)^{-1}S_{j,v,\lambda}f\|_q^2\Big)^{1/2}\\ &\qquad\lesssim \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}(\lambda/\log\lambda)^{-1/2}(\lambda/\log\lambda)^{1/2}\|f\|_2\\ &\qquad= \lambda^{\mu(q)}\delta(\log\lambda)^{1/2}\|f\|_2,\end{aligned}$$

as desired.

To estimate the "S-term" in (2.29), we shall need the following two-sided $L^2\to L^q$ localized resolvent estimate on M.

Proposition 2.4. *Let (M,g) be an asymptotically hyperbolic surface with negative curvature, $\chi_1\in C_0^\infty(M)$ with $\chi_1=1$ on M_{tr}, and $\tilde\chi_1\in C_0^\infty(M_\infty)$ supported away from the trapped set. Then, for $2<q<\infty$*

$$\|\chi_1(\Delta_g+\lambda^2+i(\log\lambda)^{-1}\lambda)^{-1}(\tilde\chi_1 h)\|_{L^q(M)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(M)}. \tag{2.32}$$

We shall postpone the proof to the end of the section. As will become evident in the proof, similar results also hold on asymptotically hyperbolic manifolds with nonpositive curvature in all dimensions.

To use (2.32), we first note that since $\chi_0 \equiv 1$ on M_{tr}, $\nabla\chi_0$, and thus $S_{j,w,\lambda}$ are supported away from the trapped set M_{tr}. So, by the local smoothing estimate (2.3) and the fact that $[\Delta_g, \chi_0]$ contributes a λ factor, we have

(2.33)
$$\Big(\sum_j \|S_{j,w,\lambda}f\|^2_{L^2(I_j\times M)}\Big)^{1/2} \lesssim \lambda\delta(\lambda/\log\lambda)^{-1/2}\Big(\sum_j \|[\Delta_g,\chi_0]e^{-is\Delta_g}f_\lambda\|^2_{L^2(I_j\times M)}\Big)^{1/2}$$
$$\lesssim \lambda\delta(\log\lambda)^{\frac12}\|f\|_2.$$

Therefore, if we use (2.32) and the above arguments, we see that the second term in the left side of (2.29) also satisfies the desired bound.

Thus, to finish the proof of (2.28) and (2.29), it remains to prove Propositions 2.3 and 2.4. To do so we shall make use of the following easy consequence of the lossless L^2-local smoothing bounds.

Lemma 2.5. *Let $\mu \in [\lambda/2, 2\lambda]$, $\lambda \gg 1$, $\delta \in (0, 1/2)$ and $\tilde\chi \in C_0^\infty(M_\infty)$. Then*

$$\|\mathbf{1}_{[\mu,\mu+\delta)}(\tilde P)(\tilde\chi h)\|_{L^2(\tilde M)} \lesssim \delta^{1/2}\|h\|_{L^2(\tilde M)}. \tag{2.34}$$

and

$$\|\mathbf{1}_{[\mu,\mu+\delta)}(P)(\tilde\chi h)\|_{L^2(M)} \lesssim \delta^{1/2}\|h\|_{L^2(M)}. \tag{2.35}$$

Proof. Choose $a \in \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp}\hat a \subset (-1,1)$ and $a(t) \ge 1$ on $[-20, 20]$ and let $\tilde\beta = 1$ on $[1/10, 10]$ and supported in $[1/20, 20]$. Then, by orthogonality and duality, for $\mu \in [\lambda/2, 2\lambda]$

$$\|\mathbf{1}_{[\mu,\mu+\delta)}(\tilde P)\tilde\chi\|_{L^2\to L^2} \le \|a((\lambda\delta)^{-1}(-\Delta_{\tilde g}-\mu^2))\,\tilde\beta(\sqrt{-\Delta_{\tilde g}}/\lambda)\chi\|_{L^2\to L^2}$$
$$= \big\|\tilde\chi\,\tilde\beta(\sqrt{-\Delta_{\tilde g}}/\lambda)a((\lambda\delta)^{-1}(-\Delta_{\tilde g}-\mu^2))\,\big\|_{L^2\to L^2}.$$

By using the Cauchy-Schwarz inequality and the lossless local smoothing estimates, we have

$$\|\tilde\chi\,\tilde\beta(\sqrt{-\Delta_{\tilde g}}/\lambda)a((\lambda\delta)^{-1}(-\Delta_{\tilde g}-\mu^2))\,h\|_{L^2(\tilde M)}$$
$$= \Big\|\int_{-(\lambda\delta)^{-1}}^{(\lambda\delta)^{-1}} (\lambda\delta)\hat a(\lambda\delta t)e^{-it\mu^2}\tilde\chi e^{-it\Delta_{\tilde g}}\tilde\beta(-\Delta_{\tilde g}/\lambda^2)h\,dt\Big\|_{L^2(\tilde M)}$$
$$\lesssim (\lambda\delta)\cdot(\lambda\delta)^{-1/2}\lambda^{-1/2}\|h\|_{L^2(\tilde M)}.$$

Here we require the variant of (2.3) with $[0,1]$ replaced by $\mathbb{R}$. This leads to (2.34). A similar argument yields (2.35). □

Proof of Proposition 2.3. We first note that, by adjusting the values of δ slightly if necessary, proving (2.26) is equivalent to showing that for all $\lambda \gg 1$, $\delta \in (\lambda^{-N_0}, 1/2)$,

$$\|(\Delta_{\tilde g} + (\lambda+i\delta)^2)^{-1}(\tilde\chi_0 h)\|_{L^q(\tilde M)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(\tilde M)},\ 2 < q < \infty. \tag{2.36}$$

Recall that if $\tilde P = \sqrt{-\Delta_{\tilde g}}$, we have the following identity (see e.g., [13])

$$\big(\Delta_{\tilde g} + (\lambda+i\delta)^2\big)^{-1} = \frac{1}{i(\lambda+i\delta)}\int_0^\infty e^{it\lambda - t\delta}\cos(t\tilde P)\,dt. \tag{2.37}$$

Let us fix $\beta \in C_0^\infty((1/2,2))$ satisfying $\sum_{j=-\infty}^{\infty} \beta(s/2^j) = 1$, and define

$$T_j f = \frac{1}{i(\lambda+i\delta)} \int_0^\infty \beta(2^{-j}t) e^{it\lambda - t\delta} \cos(t\tilde{P}) f \, dt. \tag{2.38}$$

Then it suffices to obtain suitable bounds for the T_j operators. It is also straightforward to check that the symbol of T_j is

$$T_j(\tau) = \frac{1}{i(\lambda+i\delta)} \int_0^\infty \beta(2^{-j}t) e^{it\lambda - t\delta} \cos(t\tau) f \, dt = O(\lambda^{-1} 2^j (1 + 2^j |\tau - \lambda|)^{-N}). \tag{2.39}$$

Note that by (2.37) we have $(\Delta_{\tilde{g}} + (\lambda+i\delta)^2)^{-1} = \sum_{-\infty}^{\infty} T_j$. To prove the half-resolvent estimates (2.36), it will be natural to separately consider the contribution of the terms with $2^j \le 1$, $1 \le 2^j \lesssim \log\lambda$ and $\log\lambda \lesssim 2^j$.

(i) $2^j \le 1$.

This case can be handled using the local arguments in Bourgain, Shao, Sogge and Yao [13], as well as related earlier work of Dos Santos Ferreira, Kenig and Salo [24], where resolvent estimates on compact manifolds was considered. The $\tilde{\chi}_0$ cutoff function is not needed in this case.

First, if $2^j \in [\lambda^{-1}, 1]$, we will show that

$$\|T_j\|_{L^2 \to L^q} \lesssim \lambda^{\mu(q)-1} 2^{j/2}, \;\; q \ge 6. \tag{2.40}$$

This would yield the desired result $\|\sum_{\lambda^{-1}<2^j\le 1} T_j\|_{2\to q} = O(\lambda^{\mu(q)-1})$ by interpolating with the trivial $L^2 \to L^2$ bound and summing over j.

To prove (2.40), as in [13], by using the Hadamard parametrix for $\cos(t\tilde{P})$, it is not hard to show that if $\lambda^{-1} \le 2^j \le 1$, the kernel of T_j operators satisfies

$$T_j(x,y) = \begin{cases} \lambda^{-1/2} 2^{-j/2} e^{i\lambda d_{\tilde{g}}(x,y)} a_\lambda(x,y), \;\; d_{\tilde{g}}(x,y) \in [2^{j-2}, 2^{j+2}] \\ O(\lambda^{-1} 2^{-j}), \;\; d_{\tilde{g}}(x,y) \le 2^{j-2}, \end{cases} \tag{2.41}$$

where $|\nabla_{x,y}^\alpha a_\lambda(x,y)| \le C_\alpha d_{\tilde{g}}(x,y)^{-\alpha}$. Additionally, by the finite propagation speed property of the wave propagator, $T_j(x,y) = 0$ if $d_{\tilde{g}}(x,y) \ge 2^{j+2}$. Thus, if $d_{\tilde{g}}(x,y) \in [2^{j-2}, 2^{j+2}]$, the bound in (2.40) follows from the oscillatory integral bounds of Hörmander [29] and Stein [45], combined with a scaling argument. And the other case when $d_{\tilde{g}}(x,y) \le 2^{j-2}$ follows from Young's inequality.

On the other hand, if $2^j \le \lambda^{-1}$, by integration by parts in t-variable once, one can show that the symbol of the operator $\sum_{\{j:2^j\le\lambda^{-1}\}} T_j(\tilde{P})$ satisfies

$$\sum_{\{j:2^j\le\lambda^{-1}\}} T_j(\tau) = O(\lambda^{-1}(\lambda + |\tau|)^{-1}).$$

Since we are assuming that $q < \infty$, by Sobolev estimates we have

$$\|\sum_{\{j:2^j\le\lambda^{-1}\}} T_j(\tilde{P})(\tilde{\chi}_0 h)\|_{L^q(\tilde{M})} \lesssim \lambda^{-1} \|\tilde{\chi}_0 h\|_{L^2(\tilde{M})} \lesssim \lambda^{-1} \|h\|_{L^2(\tilde{M})}.$$

To deal with the two remaining cases corresponding to sums over $2^j \ge 1$, we shall require the following lemma.

Lemma 2.6. *Let $(\tilde{M}, \tilde{g})$ be defined as in* (2.14)*, which is asymptotically hyperbolic, simply connected and has negative curvature. For $2^j \geq 1$, if T_j is defined as in* (2.38)*, we have*

$$\|T_j f\|_{L^\infty(\tilde{M})} \lesssim_N (\lambda^{-1/2} e^{-2^{j-3}} + (2^j\lambda)^{-N})\|f\|_{L^1(\tilde{M})}. \tag{2.42}$$

Proof of Lemma 2.6. The proof of (2.42) relies heavily on the kernel estimate for the spectral measure of $\tilde{P}$ established in the work of Chen and Hassell [22]. Recall that if $dE_{\tilde{P}}(\mu)$ denote the spectral measure for $\tilde{P}$, we have

$$T_j f = \frac{1}{i(\lambda + i\delta)} \int_0^\infty \int_0^\infty \beta(2^{-j}t) e^{it\lambda - t\delta} \cos(t\mu) dE_{\tilde{P}}(\mu) f \, d\mu dt. \tag{2.43}$$

Let us collect several useful facts about the spectral measure $\tilde{P}$. For high energies $\lambda \gg 1$, by [22, Theorem 3], we have

$$dE_{\tilde{P}}(\lambda)(x,y) = \sum_{\pm} \lambda e^{\pm i\lambda d_{\tilde{g}}(x,y)} b_\pm(\lambda, x, y) + a(\lambda, x, y), \tag{2.44}$$

where $d_{\tilde{g}}(x,y)$ denotes the distance function on $\tilde{M}$,

$$\Big|\Big(\frac{d}{d\lambda}\Big)^j b_\pm(\lambda, x, y)\Big| \lesssim_j \begin{cases} \lambda^{-j}(1 + \lambda d_{\tilde{g}}(x,y))^{-\frac{1}{2}}, \ d_{\tilde{g}}(x,y) \leq 1 \\ \lambda^{-\frac{1}{2}-j} e^{-\frac{d_{\tilde{g}}(x,y)}{2}}, \ d_{\tilde{g}}(x,y) \geq 1, \end{cases} \tag{2.45}$$

and

$$\Big|\Big(\frac{d}{d\lambda}\Big)^j a(\lambda, x, y)\Big| \lesssim_{j,N} \lambda^{-N}. \tag{2.46}$$

If we fix $\rho \in C_0^\infty(1/4, 4)$ with $\rho = 1$ in $(1/2, 2)$, and define

$$\tilde{T}_j f = \frac{1}{i(\lambda + i\delta)} \int_0^\infty \int_0^\infty \beta(2^{-j}t) e^{it\lambda - t\delta} \cos(t\mu)\rho(\mu/\lambda) dE_{\tilde{P}}(\mu) f \, d\mu dt, \tag{2.47}$$

then, by integrating by parts in the t variable, we see that the symbol of the operator $T_j - \tilde{T}_j$ is $O\big((2^j(|\tau| + \lambda))^{-N}\big)$. Thus, by using dyadic Sobolev estimates, it is not hard to show

$$\|(T_j - \tilde{T}_j)f\|_{L^\infty(\tilde{M})} \lesssim_N (2^j\lambda)^{-N}\|f\|_{L^1(\tilde{M})}. \tag{2.48}$$

Consequently, it suffices to show that the operators $\tilde{T}_j$ satisfiy the desired $L^1 \to L^\infty$ bound in (2.42). If $d_{\tilde{g}}(x,y) \notin [2^{j-2}, 2^{j+2}]$, then by using (2.45) and (2.46), and performing integration by parts in both t, μ variables enough times, we have

$$|\tilde{T}_j(x,y)| \leq C_{N_1,N_2}(2^j\delta)^{-N_1}\lambda^{-N_2}. \tag{2.49}$$

Since we are assuming $\delta \geq \lambda^{-N_0}$, by choosing $N_2 \gg N_0 N_1$, this bound is controlled by the second term in the right side of (2.42). On the other hand, if $d_{\tilde{g}}(x,y) \in [2^{j-2}, 2^{j+2}]$, (2.42) follows directly from the pointwise bound in (2.45) and (2.46). This completes the proof of (2.42). □

Using this lemma we can handle the contribution of the T_j terms with:
(ii) $2^j \geq C \log \lambda$.

First we note that, by spectral theorem,

$$\|T_j\chi_0 f\|_2 \lesssim \lambda^{-1}2^j\|\chi_0 f\|_2 \lesssim \lambda^{-1}2^j\|f\|_2. \tag{2.50}$$

And by (2.42) and Schwarz's inequality, we also have

$$\begin{aligned}\|T_j\chi_0 f\|_\infty &\lesssim_N (\lambda^{-1/2}e^{-2^{j-3}}+(2^j\lambda)^{-N})\|\chi_0 f\|_1 \\ &\lesssim_N (\lambda^{-1/2}e^{-2^{j-3}}+(2^j\lambda)^{-N})\|f\|_2.\end{aligned} \tag{2.51}$$

Thus if we choose C large enough which may depend on q, by (2.50), (2.51) and Hölder's inequality, we have

$$\|T_j\chi_0 f\|_q \lesssim \lambda^{-1}2^{-j}\|f\|_2, \quad 2^j \ge C\log\lambda. \tag{2.52}$$

Summing over j gives us the desired bound, $\|\sum_{2^j\ge C\log\lambda} T_j\|_{2\to q} = O(\lambda^{\mu(q)-1})$.

Our final case involves the T_j with:
(iii) $1 \le 2^j \le C\log\lambda$.

To handle the contribution of these terms, we shall first prove that for each fixed j with $2^j \ge 1$, we have the uniform bounds

$$\|T_j\chi_0 f\|_{L^q(\tilde M)} \lesssim \lambda^{\mu(q)-1}\|f\|_{L^2(\tilde M)}. \tag{2.53}$$

To see this, let us define

$$E_{\lambda,j,k} = \mathbf{1}_{[\lambda+2^{-j}k,\lambda+(k+1)2^{-j})}(\tilde P). \tag{2.54}$$

By using (2.15) along with (2.34) for $\delta = 2^{-j}$, we have

$$\begin{aligned}&\|\mathbf{1}_{[\lambda/2,2\lambda]}(\tilde P)\,T_j\chi_0 f\|_{L^q(\tilde M)} \\ &\qquad\le \sum_{|k|\lesssim\lambda 2^j}\|\mathbf{1}_{[\lambda/2,2\lambda]}(\tilde P)\,E_{\lambda,j,k}T_j\chi_0 f\|_{L^q(\tilde M)} \\ &\qquad\le \lambda^{\mu(q)}2^{-j/2}\sum_{|k|\lesssim\lambda 2^j}\|\mathbf{1}_{[\lambda/2,2\lambda]}(\tilde P)E_{\lambda,j,k}T_j\chi_0 f\|_{L^2(\tilde M)} \\ &\qquad\lesssim \lambda^{\mu(q)}2^{-j/2}\sum_{|k|\lesssim\lambda 2^j}(1+|k|)^{-N}\lambda^{-1}2^j\|\mathbf{1}_{[\lambda/2,2\lambda]}(\tilde P)E_{\lambda,j,k}\chi_0 f\|_{L^2(\tilde M)} \\ &\qquad\lesssim \lambda^{\mu(q)-1}\|f\|_{L^2(\tilde M)},\end{aligned}$$

using (2.34) in the last step. The case when the spectrum is outside $[\lambda/2,2\lambda]$ can be handled using Sobolev estimates, as in case (i). Thus, the proof of (2.53) is complete.

In view of (2.53), it suffices to consider the values of j such that $C_0 \le 2^j \le c_0\log\lambda$ where C_0 is sufficiently large and c_0 is sufficiently small. We shall specify the choices of C_0 and c_0 later in the proof. Furthermore, as shown in the proof of (2.42), $|T_j(x,y)| = O(\lambda^{-N})$ if $d_{\tilde g}(x,y) \notin [2^{j-2},2^{j+2}]$. Hence, if, as we may, we assume $\tilde\chi_0$ is supported in a small neighborhood of some point y_0, it suffices to show that

$$\Big\|\sum_{\{j:C_0\le 2^j\le c_0\log\lambda\}} T_j(\tilde\chi_0 h)\Big\|_{L^q(S)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(\tilde M)}, \tag{2.55}$$

$$\text{where } S = \{x\in\tilde M : \frac{C_0}{4} \le d_{\tilde g}(x,y_0) \le 4c_0\log\lambda\}.$$

To proceed, first note that by (2.39), if we fix $\beta \in C_0^\infty((1/4,4))$ with $\beta = 1$ on (1/2,2), it suffices to show

$$\| \sum_{\{j: C_0 \le 2^j \le c_0 \log \lambda\}} \beta(\tilde{P}/\lambda) T_j(\tilde{\chi}_0 h) \|_{L^q(S)} \lesssim \lambda^{\mu(q)-1} \|h\|_{L^2(\tilde{M})}. \tag{2.56}$$

To prove (2.56), we need to introduce microlocal cutoffs involving pseudodifferential operators. If we fix δ_0 with $0 < \delta_0 \ll 1$, then since $\tilde{M}$ has bounded geometry, we can cover the set S by a partition of unity $\{\psi_k\}$, which satisfies

$$1 = \sum_k \psi_k(x), \quad \operatorname{supp} \psi_k \subset B(x_k, \delta_0), \tag{2.57}$$

with $|\partial_x^j \psi| \lesssim 1$ uniformly in the normal coordinates around x_k for different k. Here $B(x_k, \delta_0)$ denotes geodesic balls of radius δ_0 with $d_{\tilde{g}}(x_k, x_\ell) \ge \delta_0$ if $k \ne \ell$, and the balls $B(x_k, 2\delta_0)$ have finite overlap. By a simple volume counting argument, the number of values of k for which $\operatorname{supp} \psi_k \cap S \ne \emptyset$ is $O(\lambda^{Cc_0})$ for some fixed constant C. See (3.1)–(3.3) in the next section for more discussions about the properties of manifolds with bounded geometry.

If we extend $\beta \in C_0^\infty((1/4,4))$ to an even function by letting $\beta(s) = \beta(|s|)$, then we can choose an even function $\rho \in C_0^\infty$ satisfying $\rho(t) = 1$, $|t| \le \delta_0/4$ and $\rho(t) = 0$, $|t| \ge \delta_0/2$ such that

$$\begin{aligned} \beta(\tilde{P}/\lambda) =& (2\pi)^{-1} \int_{\mathbb{R}} \lambda \hat{\beta}(\lambda t) \cos t\tilde{P} dt \\ =& (2\pi)^{-1} \int \rho(t) \lambda \hat{\beta}(\lambda t) \cos t\tilde{P} dt + (2\pi)^{-1} \int (1-\rho(t)) \lambda \hat{\beta}(\lambda t) \cos t\tilde{P} dt \\ =& B + C. \end{aligned} \tag{2.58}$$

It is not hard to check that the symbol of the operator C is $O((1+|\tau|+\lambda)^{-N})$. Therefore, by Sobolev estimates, we have $\|C\|_{L^2 \to L^q} \lesssim_N \lambda^{-N}$. On the other hand, by using the finite propagation speed property of the wave propagator, one can argue as in the compact manifold case to show that B is a pseudodifferential operator with principal symbol $\beta(p(x,\xi))$, with $p(x,\xi)$ here being the principal symbol of $\tilde{P}$. See Theorem 4.3.1 in [43] for more details.

Next, choose $\tilde{\psi}_k \in C_0^\infty$ with $\tilde{\psi}_k(y) = 1$ for $y \in B(x_k, \frac{5}{4}\delta_0)$ and $\tilde{\psi}_k(y) = 0$ for $y \notin B(x_k, \frac{3}{2}\delta_0)$. As with the ψ_k we may assume that the $\tilde{\psi}_k$ have bounded derivatives in the normal coordinates about x_k by taking $\delta_0 > 0$ small enough given that $\tilde{M}$ is of bounded geometry. Then, if $B(x,y)$ is the kernel of B, we have $\psi_k(x)B(x,y) = \psi_k(x)B(x,y)\tilde{\psi}_k(y) + O(\lambda^{-N})$, and so

$$\begin{aligned} &\psi_k(x)B(x,y) \\ &= (2\pi)^{-n} \lambda^n \int e^{i\lambda\langle x-y,\xi\rangle} \psi_k(x) \beta(p(x,\xi)) \tilde{\psi}_k(y) d\xi + R_k(x,y) \\ &= A_k(x,y) + R_k(x,y). \end{aligned} \tag{2.59}$$

R_k is a lower order pseudodifferential operator which satisfies

$$\|R_k\|_{L^2 \to L^q} \lesssim \lambda^{-1+2(\frac{1}{2}-\frac{1}{q})}, \ q \ge 2. \tag{2.60}$$

Moreover,

$$R_k(x,y)=0, \text{ if } x\notin B(x_j,\delta_0) \text{ or } y\notin B(x_j,3\delta_0/2).$$

Let

$$H_p=\frac{\partial p}{\partial \xi}\frac{\partial}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial}{\partial \xi}$$

denote the Hamilton vector field associated with the principal symbol $p(x,\xi)$ of $\tilde{P}$. Let $\Phi_t=e^{tH_p}: T^*M\setminus 0\to T^*M\setminus 0$ denote the geodesic flow on the cotangent bundle generated by H_p. For each x_k, let ω_k be the unit covector such that $\Phi_{-t}(x_k,\omega_k)=(y_0,\eta_0)$ for some η_0 and $t=d_{\tilde{g}}(x_k,y_0)$, with y_0 as in (2.55). We define $a_k(x,\xi)\in C^\infty$ such that in the normal coordinate around x_k,

$$a_k(x,\xi)=0 \text{ if } \Big|\tfrac{\xi}{|\xi|_{\tilde{g}(x)}}-\omega_k\Big|\geq 2\delta_1, \text{ and } a_k(x,\xi)=1 \text{ if } \Big|\tfrac{\xi}{|\xi|_{\tilde{g}(x)}}-\omega_k\Big|\leq \delta_1. \tag{2.61}$$

Here $|\xi|_{\tilde{g}(x)}=p(x,\xi)$, $\delta_1\ll 1$ is a fixed small constant that will be chosen later. By the proof of Lemma 2.8 below, we may assume that $\partial_x^\alpha\partial_\xi^\gamma a_k=O(1)$ if $p(x,\xi)=1$, independent of k, with ∂_x denoting derivatives in the normal coordinate system about x_k.

We finally define the kernel of the microlocal cutoffs $A_{k,0}$ and $A_{k,1}$ as

$$\begin{aligned}A_k(x,y)&=A_{k,0}(x,y)+A_{k,1}(x,y)\\&=(2\pi)^{-n}\lambda^n\int e^{i\lambda\langle x-y,\xi\rangle}\psi_k(x)a_k(x,\xi)\beta((p(x,\xi))\tilde{\psi}_k(y)d\xi\\&\quad+(2\pi)^{-n}\lambda^n\int e^{i\lambda\langle x-y,\xi\rangle}\psi_k(x)(1-a_k(x,\xi))\beta((p(x,\xi))\tilde{\psi}_k(y)d\xi.\end{aligned} \tag{2.62}$$

The above operators satisfy

$$\|A_{k,\ell}\|_{L^p(\tilde{M})\to L^p(\tilde{M})}=O(1),\ 1\leq p\leq\infty,\ \ell=0,1. \tag{2.63}$$

This, combined with the support properties of $A_{k,\ell}$ implies that

$$\|\sum_k A_{k,\ell}h\|_{L^p(\tilde{M})}\lesssim\|h\|_{L^p(\tilde{M})},\ 1\leq p\leq\infty,\ \ell=0,1. \tag{2.64}$$

By (2.57), (2.59) and (2.62), to prove (2.56), it suffices to show

$$\|\sum_{\{j:C_0\leq 2^j\leq c_0\log\lambda\}}\sum_k A_{k,0}T_j(\tilde{\chi}_0h)\|_{L^q(S)}\lesssim\lambda^{\mu(q)-1}\|h\|_{L^2(\tilde{M})}, \tag{2.65}$$

as well as

$$\|\sum_{\{j:C_0\leq 2^j\leq c_0\log\lambda\}}\sum_k A_{k,1}T_j(\tilde{\chi}_0h)\|_{L^q(S)}\lesssim_N\lambda^{-N}\|h\|_{L^2(\tilde{M})}. \tag{2.66}$$

The other term involving R_k is more straightforward to handle. More explicitly, by (2.60), the support property of R_k, and (2.50), we have

$$\begin{aligned}&\Big\|\sum_{\{j:C_0\le 2^j\le c_0\log\lambda\}}\sum_k R_kT_j(\tilde\chi_0 h)\Big\|_{L^q(S)}\\ &\lesssim \lambda^{-1+2(\frac12-\frac1q)}\Big\|\sum_{\{j:C_0\le 2^j\le c_0\log\lambda\}} T_j(\tilde\chi_0 h)\Big\|_{L^2(\tilde M)}\\ &\lesssim \lambda^{-1+2(\frac12-\frac1q)}\sum_{\{j:C_0\le 2^j\le c_0\log\lambda\}}\lambda^{-1}2^j\|\tilde\chi_0 h\|_{L^2(\tilde M)}.\end{aligned}\tag{2.67}$$

Note that $-1+2(\frac12-\frac1q)<\mu(q)-\frac12$ for all $q\ge 2$. Therefore, by choosing c_0 sufficiently small, the bound in (2.67) is better than the estimate in (2.56).

The main reason that (2.66) holds is that, by (2.61), the microlocal support of the operator $A_{k,1}$ does not propagate to the support of $\tilde\chi_0$ along the backward geodesic flow, which leads to the rapidly decaying term $O(\lambda^{-N})$. Similarly, the analog of (2.66) holds for the operator $A_{k,0}$ if we replace T_j with its adjoint T_j^*, as the microlocal support of the operator $A_{k,0}$ does not propagate to the support of $\tilde\chi_0$ along the forward geodesic flow. Therefore, one can replace T_j in (2.65) with $T_j-T_j^*$ by introducing a rapidly decaying term.

On the other hand, one can show that the half localized operator, involving the difference of the resolvent operator $(\Delta_{\tilde g}+(\lambda+i\delta)^2)^{-1}$ and its adjoint, satisfies the desired bound by a simple argument using the sharp spectral projection bound (2.15) along with Lemma 2.5 (for a precise statement, see (2.85)). This allows us to obtain the estimate (2.65) with T_j replaced with $T_j-T_j^*$, which will conclude the proof combined with the other parts.

More explicitly, to prove (2.66), we shall require the following two lemmas.

Lemma 2.7. *Let ψ_k and $\tilde\psi_k$ be defined as in* (2.57) *and* (2.59)*. Fix $x\in\operatorname{supp}\tilde\psi_k$, let $\omega_0(x)$ be the unit covector such that, if y_0 is as in* (2.55)*,*

$$\Phi_{-t_0}(x,\omega_0(x))=(y_0,\eta_0)\ \text{for some unit covector }\eta_0\ \text{ and }\ t_0=d_{\tilde g}(x,y_0).\tag{2.68}$$

If $\operatorname{supp}\psi_k\cap S\neq\emptyset$, we have $\omega_0(x)$ is uniformly continuous in x over $\operatorname{supp}\tilde\psi_k$.

Proof. Note that the covector field $\omega_0(x)=\nabla_x d_{\tilde g}(x,y_0)$ is a 1-form where ∇_x is the covariant derivative in x (which is the ordinary differential when it acts on functions). Let $\gamma_{y_0,x}(t)$ be the unit speed geodesic emanating from y_0 with $\gamma_{y_0,x}(d_{\tilde g}(x,y_0))=x$. By the second variation formula [20, (1.17)], for normalized vector fields X and Y perpendicular to $\gamma'_{y_0,x}(t)$, we have

$$\nabla_x\omega_0(x)(X,Y)=\nabla_x^2 d_{\tilde g}(x,y_0)(X,Y)=\langle J_X'(t),Y\rangle|_{t=d_{\tilde g}(x,y_0)}$$

where $J_X(t)$ is the Jacobi field along $\gamma_{y_0,x}(t)$ such that

$$J_X(0)=0,\quad J_X(d_{\tilde g}(x,y_0))=X.$$

Since $\tilde M$ has negative curvature, $J_X(t)$ is well defined by the above conditions.

By the Hessian comparison theorem [27, Theorem A], since the curvature K of $\tilde M$ is bounded below by -2, $\nabla_x^2 d_{\tilde g}(x,y_0)$ is controlled by $\nabla_{\tilde x}^2\tilde d(\tilde x,\tilde y_0)$ if $\tilde d(\tilde x,\tilde y_0)$ is the distance function on the space form with constant curvature -2 and $d_{\tilde{g}}(x, y_0) = \tilde{d}(\tilde{x}, \tilde{y}_0)$. More precisely, for any normalized vector fields X on $\tilde{M}$ perpendicular to $\gamma_{y_0,x}$ and $\tilde{X}$ on the space form with constant curvature -2 perpendicular to the geodesic connecting $\tilde{x}$ and $\tilde{y}_0$, we have

$$\nabla_x^2 d_{\tilde{g}}(x, y_0)(X, X) \leq \nabla_{\tilde{x}}^2 \tilde{d}(\tilde{x}, \tilde{y}_0)(\tilde{X}, \tilde{X}).$$

By using the second variation formula again, the Hessian of the distance function $\tilde{d}(\tilde{x}, \tilde{y}_0)$ on the space form can be computed directly with

$$\tilde{J}_{\tilde{X}}(t) = \frac{\sinh \sqrt{2}t}{\sinh \sqrt{2}\tilde{d}(\tilde{x}, \tilde{y}_0)} \tilde{X}, \quad \tilde{J}'_{\tilde{X}}(\tilde{d}(\tilde{x}, \tilde{y}_0)) = \sqrt{2}\, \frac{\cosh \sqrt{2}\tilde{d}(\tilde{x}, \tilde{y}_0)}{\sinh \sqrt{2}\tilde{d}(\tilde{x}, \tilde{y}_0)} \tilde{X}.$$

Since $\tilde{J}'_{\tilde{X}}(\tilde{d}(\tilde{x}, \tilde{y}_0))$ is uniformly bounded for $\tilde{d}(\tilde{x}, \tilde{y}_0) > 1$, the Hessian matrix $\nabla_x \omega_0(x)$ is uniformly bounded above. On the other hand, a similar comparison with the space form of curvature $-1/2$ or the Euclidean space shows the Hessian matrix $\nabla_x \omega_0(x)$ is uniformly bounded below. Therefore, we conclude $\nabla_x \omega_0(x) = \nabla_x^2 d_{\tilde{g}}(x, y_0)$ is uniformly bounded when $x \in \operatorname{supp} \tilde{\psi}_k$. By definition, if Γ_{ij}^k are the Christoffel symbols, we have

$$\nabla_x^2 f(\partial_i, \partial_j) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k}.$$

Since $\tilde{M}$ has bounded geometry, in the normal coordinates around x_k, $\left|\frac{\partial\, d_{\tilde{g}}(x,y_0)}{\partial x_k}\right| + |\Gamma_{ij}^k| \lesssim 1$ if $|x - x_k| \lesssim 1$. Thus, the standard coordinate derivative of $\omega_0(x)$ is uniformly bounded. This completes the proof of the lemma. $\square$

Lemma 2.8. *Let C_0 be defined as in* (2.55)*, assume that* $\operatorname{supp} \psi_k \cap S \neq \emptyset$ *and that ω_k is as in* (2.61)*. For any $\delta_1 > 0$, we can fix C_0 large enough and choose δ_0 in* (2.57) *sufficiently small such that for any $x \in \operatorname{supp} \tilde{\psi}_k$, if in the normal coordinate around x_k, $\left|\frac{\xi}{|\xi|_{\tilde{g}(x)}} - \omega_k\right| > \delta_1/10$, then $(x(t), \xi(t)) = \Phi_{-t}(x, \xi)$ satisfies*

$$d_{\tilde{g}}(x(t), y_0) \geq 1, \;\; \forall\, t \geq 0, \tag{2.69}$$

y_0 as in (2.55)*. Moreover, if we choose δ_1 to be sufficiently small, then $\left|\frac{\xi}{|\xi|_{\tilde{g}(x)}} - \omega_k\right| < 10\delta_1$ implies*

$$d_{\tilde{g}}(x(t), y_0) \geq 1, \;\; \forall\, t \leq 0. \tag{2.70}$$

Proof. We shall first prove (2.69) by contradiction, and then give the proof of (2.70) by using (2.69). Fix $x \in \operatorname{supp} \tilde{\psi}_k$, let $\omega_0(x)$ be defined as in (2.68). Suppose there exists some point y_1 such that $d_{\tilde{g}}(y_1, y_0) < 1$ and $\Phi_{-t_1}(x, \omega_1(x)) = (y_1, \eta_1)$ for some unit covectors $\omega_1(x)$ and η_1 with $|\omega_1(x) - \omega_k| > \delta_1/10$ and $t_1 = d_{\tilde{g}}(x, y_1)$. By Lemma 2.7, we can choose δ_0 in the definition of $\operatorname{supp} \tilde{\psi}_k$ above sufficiently small such that $|\omega_0(x) - \omega_k| = |\omega_0(x) - \omega_0(x_k)| \leq \delta_1/20$. This implies that $|\omega_1(x) - \omega_0(x)| > \delta_1/20$.

Since $\operatorname{supp} \psi_k \cap S \neq \emptyset$ and $x \in \operatorname{supp} \tilde{\psi}_k$, if t_0, t_1 are defined as above, we have $\frac{C_0}{8} \leq t_0, t_1 \leq 8c_0 \log \lambda$, and $|t_0 - t_1| \leq 1$ by the triangle inequality. If we denote $(y_2, \eta_2) = \Phi_{-t_0}(x, \omega_1(x))$, we have $d_{\tilde{g}}(y_2, y_0) < d_{\tilde{g}}(y_2, y_1) + d_{\tilde{g}}(y_1, y_0) < 2$. In normal coordinates around x_k, we have $|\omega_0(x)|, |\omega_1(x)| \approx 1$. Since the curvature K of $\tilde{M}$ is bounded above by

$-1/2$, by using the Aleksandrov–Toponogov triangle comparison theorem [35, Theorem 4.1], it is not hard to show

$$|\omega_1(x)-\omega_0(x)| \lesssim \frac{\sinh(d_{\tilde g}(y_2,y_0)/\sqrt{2})}{\sinh(t_1/\sqrt{2})} \lesssim e^{-t_1/\sqrt{2}} \lesssim e^{-\frac{1}{16}C_0}. \tag{2.71}$$

By choosing C_0 sufficiently large, this contradicts the fact that $|\omega_1(x)-\omega_0(x)|>\delta_1/20$.

To prove (2.70), note that by choosing δ_1 small enough, $\left|\frac{\xi}{|\xi|_{\tilde g(x)}}-\omega_k\right|<10\delta_1$ implies that $\left|\frac{\xi}{|\xi|_{\tilde g(x)}}+\omega_k\right|>\delta_1/10$. Thus, (2.70) follows directly from (2.69). □

Now we shall give the proof of (2.66). We may assume $\operatorname{supp}\tilde\chi_0$ has diameter less than 1 so Lemma 2.8 tells us when $x(t)$ intersects $\operatorname{supp}\tilde\chi_0$. By triangle inequality, it suffices to show

$$\|A_{k,1}T_j(\tilde\chi_0 h)\|_{L^q(S)} \lesssim_N \lambda^{-N}\|h\|_{L^2(\tilde M)},\ C_0\le 2^j\le c_0\log\lambda. \tag{2.72}$$

Note that the volume of the set S is $O(\lambda^{Cc_0})$ for some constant C. To prove (2.72), it suffices to show the following pointwise bound

$$\int_0^\infty \beta(2^{-j}t)e^{it\lambda-t\delta}(A_{k,1}\circ\cos(t\tilde P))(x,y)\tilde\chi_0(y)\,dt \lesssim_N \lambda^{-N}. \tag{2.73}$$

We shall give the proof of (2.73) using the Hadamard parametrix, as this approach is more easily adaptable to the proof of Proposition 2.4. Alternatively, one could also prove (2.73) using kernel estimates for the spectral measure, as in the proof of Lemma 2.6.

Since $\tilde M$ is simply connected with negative curvature, we can use the Hadamard parametrix to express the solution $\cos t\tilde P$ in normal coordinates around x_k as follows:

$$\cos t\tilde P(x,y)=\sum_{\nu=0}^{N} w_\nu(x,y)W_\nu(t,x,y)+R_N(t,x,y), \tag{2.74}$$

where $w_\nu\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$,

$$W_0(t,x,y)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{id_{\tilde g}(x,y)\xi_1}\cos t|\xi|\,d\xi, \tag{2.75}$$

while for $\nu=1,2,\dots$, $W_\nu(t,x,y)$ is a finite linear combination of Fourier integrals of the form

$$\int_{\mathbb{R}^n} e^{id_{\tilde g}(x,y)\xi_1}e^{\pm it|\xi|}\alpha_\nu(|\xi|)\,d\xi,\ \text{with}\ \ \alpha_\nu(\tau)=0,\ \text{for}\ \tau\le 1\ \text{and}\ \ \partial_\tau^j\alpha_\nu(\tau)\lesssim\tau^{-\nu-j}, \tag{2.76}$$

and, if N_0 is given, then if N is large enough,

$$|\partial_t^j R_N(t,x,y)|\le C\exp(Ct),\quad 0\le j\le N_0, \tag{2.77}$$

for a fixed constant C. And the coefficients $w_\nu(x,y)$ satisfy

$$w_0(\tilde x,\tilde y)\le 1, \tag{2.78}$$

as well as

$$|\partial_x^\beta w_\nu(x,y)|\le C\exp(Cr),\ |\beta|,\nu\le N_0,\quad r=d_{\tilde g}(x,y), \tag{2.79}$$

for some uniform constant C (depending on $\tilde{g}$ and N_0). We also have the similar bound for the distance function

$$|\partial^\beta_{x,y} d_{\tilde{g}}(x,y)| \le C\exp(Cr),\ |\beta|\le N_0, \quad r = d_{\tilde{g}}(x,y). \tag{2.80}$$

The facts that we have just recited are well known. One can see, for instance, [3] or [42, §1.1, §3.6] for background regarding the Hadamard parametrix, and [44] for a discussion of properties of w_0.

Let us focus on the $\nu = 0$ term. The higher order terms can be treated similarly and satisfy better bounds, and the error term involving R_N certainly satisfies desired bound by using (2.63),(2.77), and an integration by parts argument in the t variable. The kernel of the main term in (2.103) is

$$\begin{aligned}K(x,y) = (2\pi)^{-2n}\lambda^{2n} \iiiint \beta(2^{-j}t) e^{it\lambda - t\delta} e^{i\lambda\langle x-z,\xi\rangle}\psi_k(x)(1-a_k(x,\xi))\beta((p(x,\xi)) \\ \cdot \tilde{\psi}_k(z)\tilde{\chi}_0(y)\cdot w_0(z,y)e^{i\lambda d_{\tilde{g}}(z,y)\eta_1}\cos(t\lambda|\eta|)dzd\xi d\eta dt.\end{aligned} \tag{2.81}$$

We can replace $\cos(t\lambda|\eta|)$ by $e^{-it\lambda|\eta|}$ since the term involving $e^{it\lambda|\eta|}$ is rapidly decreasing through integration by part in the t-variable. A similar integration by parts argument in the z,η variables also shows that we may assume $\eta_1/p(x,\xi)\in[1-\delta_2, 1+\delta_2]$ for some sufficiently small δ_2.

We claim that we have for $z\in \operatorname{supp}\tilde{\psi}_k$ and $y\in\operatorname{supp}\tilde{\chi}_0$,

$$|\nabla_z d_{\tilde{g}}(z,y) - \omega_k| \le \frac{\delta_1}{10}. \tag{2.82}$$

This is because $\Phi_{-t}(z,\nabla_z d_{\tilde{g}}(z,y)) = (y,\xi_0)$ for some ξ_0 and $t = d_{\tilde{g}}(z,y)$, which by (2.69) implies $y\notin\operatorname{supp}\tilde{\chi}_0$ if $|\nabla_z d_{\tilde{g}}(z,y)-\omega_k| > \delta_1/10$.

Recall that $\eta_1/p(x,\xi)\in[1-\delta_2,1+\delta_2]$. By choosing δ_2 small enough, (2.82) implies that for any $(x,\xi)\in\operatorname{supp} a_{k,1}(x,\xi)$, we have

$$|\eta_1\nabla_z d_{\tilde{g}}(z,y) - \xi| \ge \frac{\delta_1}{2}.$$

Hence a simple integration by parts argument in the z variable yields that the kernel in (2.81) is $O(\lambda^{-N})$, which completes the proof.

Now we give the proof of (2.65). By (2.64) and our previous results for the operators $T_j\tilde{\chi}_0$ when $2^j\le C_0$ and $2^j\ge c_0\log\lambda$, proving (2.65) is equivalent to showing that

$$\|\sum_k A_{k,0}(\Delta_{\tilde{g}} + (\lambda+i\delta)^2)^{-1}(\tilde{\chi}_0 h)\|_{L^q(S)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(\tilde{M})}. \tag{2.83}$$

To prove (2.83), it suffices to show

$$\|\sum_k A_{k,0}(\Delta_{\tilde{g}} + (\lambda-i\delta)^2)^{-1}(\tilde{\chi}_0 h)\|_{L^q(S)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(\tilde{M})}, \tag{2.84}$$

as well as

$$\|\sum_k A_{k,0}\big((\Delta_{\tilde{g}}+(\lambda+i\delta)^2)^{-1} - (\Delta_{\tilde{g}}+(\lambda-i\delta)^2)^{-1}\big)(\tilde{\chi}_0 h)\|_{L^q(S)} \lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(\tilde{M})}. \tag{2.85}$$

Note that if we define $E_{\lambda,k} = \mathbf{1}_{[\lambda+k\delta,\lambda+(k+1)\delta)}(\tilde{P})$, then the symbol of the operator

$$E_{\lambda,k}\big((\Delta_{\tilde{g}} + (\lambda+i\delta)^2)^{-1} - (\Delta_{\tilde{g}} + (\lambda-i\delta)^2)^{-1}\big)$$

is $O\left((\lambda\delta)^{-1}(1+|k|)^{-2}\right)$. Thus (2.85) can be proved using the same arguments as in the proof of (2.53).

To prove (2.84), note that by taking the complex conjugate of both side of (2.37), we have

$$\left(\Delta_{\tilde{g}} + (\lambda - i\delta)^2\right)^{-1} = \frac{i}{(\lambda - i\delta)} \int_0^\infty e^{-it\lambda - t\delta} \cos(t\tilde{P})\, dt. \tag{2.86}$$

As in (2.38), if we define

$$\bar{T}_j f = \frac{i}{(\lambda - i\delta)} \int_0^\infty \beta(2^{-j}t) e^{-it\lambda - t\delta} \cos(t\tilde{P}) f\, dt, \tag{2.87}$$

then the above arguments implies that the analog of (2.84), involving the operators $\bar{T}_j\tilde{\chi}_0$ for $2^j \le C_0$ and $2^j \ge c_0 \log\lambda$, satisfies the desired bound. Thus, it suffices to show

$$\| \sum_{\{j: C_0 \le 2^j \le c_0 \log\lambda\}} \sum_k A_{k,0}\bar{T}_j(\tilde{\chi}_0 h)\|_{L^q(S)} \lesssim_N \lambda^{-N} \|h\|_{L^2(\tilde{M})}. \tag{2.88}$$

By applying the Hadamard parametrix and using (2.70), the proof of (2.88) follows similarly to that of (2.66). Hence, we omit the details here. □

Proof of Proposition 2.4. As in the proof of Proposition 2.3, it suffices to show the following equivalent version of (2.32),

$$\left\|\chi_1 \left(\Delta_g + (\lambda + i(\log\lambda)^{-1})^2\right)^{-1} \tilde{\chi}_1\right\|_{L^2\to L^q} \lesssim \lambda^{\mu(q)-1}. \tag{2.89}$$

And if we fix $\beta \in C_0^\infty((1/4,4))$ with $\beta = 1$ in (1/2,2), it suffices to show

$$\left\|\chi_1 \left(\Delta_g + (\lambda + i(\log\lambda)^{-1})^2\right)^{-1} \beta(P/\lambda)\tilde{\chi}_1\right\|_{L^2\to L^q} \lesssim \lambda^{\mu(q)-1}, \tag{2.90}$$

since the analogous estimate involving $(I - \beta(P/\lambda))$ follows from Sobolev estimates and does not require the χ_1 and $\tilde{\chi}_1$ cutoff functions.

We shall need the following lemma which is analogous to Lemma 2.8:

Lemma 2.9. *There exist zero-order pseudodifferential operators $A_\pm$ with compactly supported Schwartz kernel such that*

$$\beta(P/\lambda)\tilde{\chi}_1 = A_+ + A_- + R, \tag{2.91}$$

with $\|R\|_{L^2\to L^2} \lesssim \lambda^{-1}$. In local coordinates, $A_\pm$ is of the form

$$A_\pm u(x) = (2\pi)^{-n}\lambda^n \int\int e^{i\lambda\langle x-y,\xi\rangle} A_\pm(x,y,\xi) u(y) dy d\xi, \quad A_\pm(x,y,\xi) \in C_0^\infty(T^*M). \tag{2.92}$$

For all $(x,y,\xi) \in \operatorname{supp} A_+(x,y,\xi)$, if $(x(t),\xi(t)) = \Phi_t(x,\xi)$, we have

$$\operatorname{dist}(x(t), \operatorname{supp}\chi_1) \ge 1 \ \textit{for}\ t \ge C, \tag{2.93}$$

for some sufficiently large constant C. Similarly, for all $(x,y,\xi) \in \operatorname{supp} A_-(x,y,\xi)$, we have

$$\operatorname{dist}(x(t), \operatorname{supp}\chi_1) \ge 1 \ \textit{as}\ t \le -C. \tag{2.94}$$

Proof. As in (2.58), if we extend β to be an even function, then we can write $\beta(P/\lambda) = B + C$ where $\|C\|_{L^2\to L^2} \lesssim_N \lambda^{-N}$, and B is a pseudodifferential operator with principal symbol $\beta(p(x,\xi))$, with $p(x,\xi)$ here now being the principal symbol of P.

Next, choose $\psi \in C_0^\infty(M)$ with $\psi = 1$ in a neighborhood of the support of $\tilde{\chi}_1$ and $\psi = 0$ on M_{tr}. Without loss of generality, we may assume both ψ and $\tilde{\chi}_1$ are supported in a sufficiently small neighborhood of some fixed point y_0. Then, in normal coordinates around y_0, if $B(x,y)$ is the Schwartz kernel of B, we have $B(x,y)\tilde{\chi}_1(y) = \psi(x)B(x,y)\tilde{\chi}_1(y) + O(\lambda^{-N})$. Since B has principal symbol $\beta(p(x,\xi))$,

$$B(x,y)\tilde{\chi}_1(y) = (2\pi)^{-n}\lambda^n \int e^{i\lambda\langle x-y,\xi\rangle}\psi(x)\beta(p(x,\xi))\tilde{\chi}_1(y)d\xi + R(x,y), \tag{2.95}$$

where R is a lower order pseudodifferential operator which satisfies $\|R\|_{L^2\to L^2} = O(\lambda^{-1})$. Let $S = \{(x,\xi) \in S^*M : x \in \operatorname{supp}\psi(x)\}$. Since $\psi(x) = 0$ on M_{tr} and $\Gamma_+ \cap \Gamma_- \subset M_{tr}$, the two sets $\Gamma_+ \cap S$ and $\Gamma_- \cap S$ are disjoint. Since $\Gamma_\pm$ are closed and S is compact, there exists $\phi_\pm \in C_0^\infty(S^*M)$ subordinate to the open cover $S \subset (U \setminus \Gamma_-) \cup (U \setminus \Gamma_+)$ where U is a small neighbourhood of S, such that

$$\phi_+(x,\xi) + \phi_-(x,\xi) = 1, \quad (x,\xi) \in S, \tag{2.96}$$

with $\operatorname{supp}\phi_+ \cap \Gamma_- = \emptyset$ and $\operatorname{supp}\phi_- \cap \Gamma_+ = \emptyset$. If we define the operators $A_\pm$ by

$$A_\pm f(x) = (2\pi)^{-n}\lambda^n \int e^{i\lambda\langle x-y,\xi\rangle}\phi_\pm(x,\xi/|\xi|_g)\psi(x)\beta(p(x,\xi))\tilde{\chi}_1(y)f(y)d\xi dy, \tag{2.97}$$

then we obtain (2.91). Moreover, by (2.1), $(x,\xi) \in \operatorname{supp}\phi_\pm$ satisfies (2.93) and (2.94) for sufficiently large C, respectively. ∎

For later use, it is straightforward to check that the $A_\pm$ operators satisfy

$$\|A_\pm\|_{L^p\to L^p} = O(1), \ \forall \ 1 \le p \le \infty. \tag{2.98}$$

To prove (2.90) it suffices to show

$$\left\|\chi_1\left(\Delta_g + (\lambda + i(\log\lambda)^{-1})^2\right)^{-1} \circ A\right\|_{L^2\to L^q} \lesssim \lambda^{\mu(q)-1}, \quad A = A_+, A_-. \tag{2.99}$$

The other term involving R is more straightforward to handle and does not require the χ_1 cut-off function on the left. More explicitly, by (2.30) along with the fact that $\|R\|_{L^2\to L^2} \lesssim \lambda^{-1}$, we have

$$\begin{aligned}\left\|\left(\Delta_g + (\lambda + i(\log\lambda)^{-1})^2\right)^{-1} R(h)\right\|_{L^q(M)} &\lesssim \lambda^{\mu(q)-1}(\log\lambda)^{\frac12}\|R(h)\|_{L^2(M)} \\ &\lesssim \lambda^{\mu(q)-2}(\log\lambda)^{\frac12}\|h\|_{L^2(M)},\end{aligned} \tag{2.100}$$

which is better than the required estimate in (2.90).

Let us first prove (2.99) for $A = A_+$. The main strategy in the proof is similar to what was used in the proof of Proposition 2.3. If we define T_j as in (2.38), it is natural to separately consider the contribution of the terms with $2^j \lesssim 1$, $1 \lesssim 2^j \lesssim \log\lambda$ and $\log\lambda \lesssim 2^j$.

(i) $2^j \le 10C$ for C as in (2.94).

One can directly apply the arguments from case (i) in the proof of Proposition 2.3 to handle this case. There is no need to make use of χ_1 and A operator here.

(ii) $2^j \geq c_0 \log \lambda$ for some small enough c_0.

Let $E_{\lambda,k} = \mathbf{1}_{[\lambda+k/\log\lambda, \lambda+(k+1)/\log\lambda)}(P)$. Then by integration by parts in t-variable, the symbol of

$$S_k = \frac{1}{i(\lambda + i(\log\lambda)^{-1})} E_{\lambda,k} \int_0^\infty e^{it\lambda - t/\log\lambda} \cos(tP) \sum_{2^j \geq c_0 \log\lambda} \beta(2^{-j}t)\, dt$$

is $O\left(\lambda^{-1}\log\lambda(1+|k|)^{-N}\right)$. Thus, by the sharp spectral projection bound in (1.13) and (2.35) with $\delta = (\log\lambda)^{-1}$

$$\begin{aligned}
&\|\mathbf{1}_{[\lambda/2,2\lambda]}(P) \sum_{|k|\lesssim\lambda\log\lambda} (S_k \circ Ah)\|_{L^q(M)} \\
&\leq \sum_{|k|\lesssim\lambda\log\lambda} \|\mathbf{1}_{[\lambda/2,2\lambda]}(P)\,(S_k\circ Ah)\|_{L^q(M)} \\
&\leq \lambda^{\mu(q)}(\log\lambda)^{-1/2} \sum_{|k|\lesssim\lambda\log\lambda} \|\mathbf{1}_{[\lambda/2,2\lambda]}(P)\,(S_k\circ Ah)\|_{L^2(M)} \\
&\lesssim \lambda^{\mu(q)}(\log\lambda)^{-1/2} \sum_{|k|\lesssim\lambda\log\lambda} (1+|k|)^{-N}\lambda^{-1}\log\lambda \|\mathbf{1}_{[\lambda/2,2\lambda]}(P) E_{\lambda,k}\circ(Ah)\|_{L^2(M)} \\
&\lesssim \lambda^{\mu(q)-1}\|h\|_{L^2(M)},
\end{aligned}$$

using (2.35), (2.97) and (2.98) in the last step. The case when the spectrum is outside $[\lambda/2, 2\lambda]$ can be handled using Sobolev estimates and satisfies better bounds.

(iii) $10C \leq 2^j \leq c_0 \log\lambda$ for C as in (2.94).

This is the case where we require compact cutoff functions on both sides. By duality, it suffices to show that the operator

$$T = \sum_{\{j: 10C \leq 2^j \leq c_0\log\lambda\}} \frac{i}{(\lambda - i(\log\lambda)^{-1})} \int_0^\infty \beta(2^{-j}t) e^{-it\lambda - t/\log\lambda} A\circ\cos(tP)\chi_1\, dt \tag{2.101}$$

satisfies the same $L^{q'} \to L^2$ mapping bound as in (2.99).

To proceed, since (M,g) has nonpositive sectional curvature, we can use the Cartan-Hadamard theorem to lift the calculation up to the universal cover of (M,g) using the formula (see e.g., [42, (3.6.4)])

$$(\cos t\sqrt{-\Delta_g})(x,y) = \sum_{\alpha\in\Gamma} (\cos t\sqrt{-\Delta_{\tilde g}})(\tilde x, \alpha(\tilde y)). \tag{2.102}$$

Here $(\mathbb{R}^n, \tilde g)$ is the universal cover of (M,g), with $\tilde g$ now being the Riemannian metric on $\mathbb{R}^n$ obtained by pulling back the metric g via the covering map. Also, $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ are the deck transformations, and $\tilde x, \tilde y \in D$ with $D \simeq M$ being a Dirichlet fundamental domain.

Note that by finite propagation speed, $\cos(t\sqrt{-\Delta_{\tilde g}})(\tilde x, \alpha(\tilde y)) = 0$ if $d_g(\tilde x, \alpha(\tilde y)) > |t|$. Thus, for each fixed $\tilde x$, by using a simple volume counting argument using the fact that the injectivity radius is positive along with the bounded geometry of (M,g), the number of deck transformations α such that $d_g(\tilde x, \alpha(\tilde y)) \lesssim c_0 \log\lambda$ is $O(\lambda^{C_M c_0})$. The proof of this, along with additional properties of manifolds with bounded geometry, will be provided in the next section (see (3.1)–(3.3) and the discussion following (3.161)).

Therefore, it suffices to show that for each fixed α, we have

$$\int_0^\infty \beta(2^{-j}t)e^{-it\lambda - t/\log\lambda}(A\circ\cos(t\sqrt{-\Delta_{\tilde g}}))(\tilde x,\alpha(\tilde y))\chi_1(\tilde y)\,dt \lesssim_N \lambda^{-N}. \tag{2.103}$$

Here, we slightly abuse the notation by identifying χ_1 with a compactly supported function on the fundamental domain D, and A here denotes the lift of the operator on (M,g) to $(\mathbb{R}^n,\tilde g)$ via the covering map. By applying the Hadamard parametrix and using (2.93), (2.103) follows from the same arguments as in the proof of (2.73). We omit the details here.

This finishes the proof of (2.99) if $A=A_+$.

Similarly, if $A=A_-$, we can use the arguments in the proof of Proposition 2.3 to show that

$$\left\| \chi_1\left(\Delta_g+(\lambda-i(\log\lambda)^{-1})^2\right)^{-1}\circ A\right\|_{L^2\to L^q} \lesssim \lambda^{\mu(q)-1}, \tag{2.104}$$

as well as

$$\left\|\chi_1\left(\left(\Delta_g+(\lambda+i(\log\lambda)^{-1})^2\right)^{-1}-\left(\Delta_g+(\lambda-i(\log\lambda)^{-1})^2\right)^{-1}\right)\circ A\right\|_{L^2\to L^q} \lesssim \lambda^{\mu(q)-1}. \tag{2.105}$$

These two inequalities yield (2.99) with $A=A_-$.

By repeating the above arguments, (2.104) is a consequence of

$$\int_0^\infty \beta(2^{-j}t)e^{it\lambda - t/\log\lambda}(A\circ\cos(t\sqrt{-\Delta_{\tilde g}}))(\tilde x,\alpha(\tilde y))\chi_1(\tilde y)\,dt \lesssim_N \lambda^{-N}. \tag{2.106}$$

This follows from the same arguments as in the proof of (2.73), utilizing Hadamard parametrix and (2.94).

On the other hand, if we define $E_{\lambda,k}=\mathbf{1}_{[\lambda+k/\log\lambda,\lambda+(k+1)/\log\lambda)}(P)$, then the symbol of the operator

$$E_{\lambda,k}\left(\left(\Delta_g+(\lambda+i(\log\lambda)^{-1})^2\right)^{-1}-\left(\Delta_g+(\lambda-i(\log\lambda)^{-1})^2\right)^{-1}\right)$$

is $O\left(\lambda^{-1}\log\lambda(1+|k|)^{-2}\right)$. Thus, (2.105) can be proved using the same arguments as in case (ii) of the proof of Proposition 2.4. ☐

3. **Manifolds of bounded geometry.**

As we mentioned before, using the assumption of bounded geometry, we shall be able to modify the arguments that were used in [4], [7], [33] and [32] to obtain (1.9), (1.12) and (1.13) in the special case where (M,g) was a *compact* Riemannian manifold. The basic facts that will allow us to carry out the local harmonic analysis for general manifolds of bounded geometry can be found, for instance, in Chapter 2 of Eldering [26]. In addition to extending the local harmonic analysis that was used in these earlier works, we shall also need to show that the global kernel estimates in [6] and the aforementioned earlier works hold for manifolds of bounded geometry and nonpositive curvature. As we shall see in the end of the section, like in the earlier works, we can do this by lifting the calculations up to the universal cover and exploiting the fact that $r_{\mathrm{Inj}}(M)>0$ if M is of bounded geometry. After possibly multiplying the metric by a constant, we may also assume that $r_{\mathrm{Inj}}(M)>1$, as we shall do throughout this section.

Let us quickly review facts that we shall require for our arguments. First, there is a $\delta = \delta(M) > 0$ so that the coordinate charts given by the exponential map are defined on all geodesic balls $B(x, 2\delta)$, $x \in M$, of radius $2\delta > 0$. Furthermore, in the resulting normal coordinates, the Riemannian distance, $d_g(x_1, x_2)$, is comparable to $|\exp_x^{-1}(x_1) - \exp_x^{-1}(x_2)|$, independent of $x \in M$. Additionally, derivatives of the transition maps from these coordinates are also uniformly bounded. (See Proposition 2.5 and Lemma 2.6 in [26].)

Furthermore, there is a uniformly locally finite cover by geodesic balls. By this we mean that there is a $\delta(M) > 0$ so that whenever $\delta \in (0, \delta(M)]$ there is a countable covering by geodesic balls

$$M = \bigcup_j B(x_j, \delta) \quad \text{with } d_g(x_j, x_k) \ge \delta \text{ if } j \ne k. \tag{3.1}$$

Furthermore, assuming that $\delta(M)$ is small enough, for δ as above, we can assume that the covering also satisfies

$$\text{Card}\big\{j : B(x_j, 2\delta) \cap B(x, 2\delta) \ne \emptyset\big\} \le C_0, \ \ \forall\, x \in M, \tag{3.2}$$

for a uniform constant $C_0 = C_0(M) < \infty$. (See [8, Lemma 2.16].)

From this one also sees that we can also choose $\delta = \delta(M) > 0$ small enough so that there is a C^∞ partition of unity $\{\psi_i\}$,

$$1 = \sum_j \psi_j(x), \quad \text{supp}\psi_j \subset B(x_j, \delta), \tag{3.3}$$

with uniform control of each derivative all of the $\{\psi_j\}$ in the normal coordinates described above. (See Lemma 2.17 and Definition 2.9 in [26].)

Using our assumption of bounded geometry, as we shall describe shortly, we can also construct a microlocal partition of unity involving pseudodifferential operators supported in the δ-balls in the above covering. It will be convenient, as in the compact manifold case, to use such microlocal cutoffs for the local harmonic analysis that we shall require in the proof of Theorems 1.3 and 1.5. These operators will, in effect, give us a second microlocalization needed to apply bilinear harmonic analysis.

3.1. Log-scale spectral projection estimates

Let us first show how we can adapt the proofs for the compact manifold case treated in [33] to prove Theorem 1.5 since the second microlocalization and the resulting arguments is a bit more straightforward than what is needed for the Strichartz estimates in Theorem 1.3.

Before describing these pseudodifferential cutoffs, let us introduce another local operator which we shall require. To do so, let us fix, following [33], $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\begin{aligned} \rho(0) = 1, \quad \hat\rho(t) = 0, \ t \notin \delta_1 \cdot [1-\delta_2, 1+\delta_2] = [\delta_1 - \delta_1\delta_2,\, \delta_1 + \delta_1\delta_2], \\ \text{with } 0 < \delta_1, \delta_2 < \tfrac{1}{2}\min(r_{\text{Inj}}(M), 1) \end{aligned} \tag{3.4}$$

as above, with δ_1, δ_2 to be specified later when in order to apply bilinear oscillatory integral results from [38] and [47], just as was done in [33]. We then define the "local" operators

$$\sigma_\lambda = \rho(\lambda - P) + \rho(\lambda + P). \tag{3.5}$$

We call these local since their kernels satisfy

$$\sigma_\lambda(x, y) = 0 \quad \text{if} \;\; d_g(x, y) > r, \;\; r = \delta_1(1 + \delta_2) < \delta/2. \tag{3.6}$$

This follows from the fact that, since $\hat{\rho}$ has support as in (3.4) we have, by Euler's formula,

$$\sigma_\lambda = \pi^{-1} \int_0^{\delta_1(1+\delta_2)} \hat{\rho}(t)\, e^{i\lambda t} \, \cos(tP)\, dt. \tag{3.7}$$

Finally, by finite propagation speed, $(\cos(tP))(x, y) = 0$ if $d_g(x, y) > |t|$, which, along with the preceding identity, yields (3.6).

We also note that by (1.4) and orthogonality we have

$$\|\sigma_\lambda\|_{2\to q} = O(\lambda^{\mu(q)}), \tag{3.8}$$

with $\mu(q)$ as in (1.3). We shall also consider the "global" smoothed out spectral projection operators

$$\rho_\lambda = \rho(T(\lambda - P)), \quad T = c_0 \log \lambda, \tag{3.9}$$

where $c_0 > 0$ shall be fixed later. We then conclude that, in order to prove Theorem 1.5 it suffices to show that if all of the sectional curvatures of M are nonpositive then we have

$$\|\rho_\lambda\|_{2\to q} \lesssim \begin{cases} \lambda^{\mu(q)}(\log \lambda)^{-1/2}, & \text{if} \;\; q > q_c \\ \big(\lambda(\log \lambda)^{-1}\big)^{\mu(q)}, & \text{if} \;\; q \in (2, q_c], \end{cases} \tag{3.10}$$

while, if all the sectional curvatures are all pinched below $-\kappa_0^2$ with $\kappa_0 > 0$, we have the stronger estimates

$$\|\rho_\lambda\|_{2\to q} \le C_q \, \lambda^{\mu(q)} (\log \lambda)^{-1/2}, \;\; q \in (2, \infty]. \tag{3.11}$$

For later use, note that by (1.4) and a simple orthogonality argument we have

$$\|(I - \sigma_\lambda) \circ \rho_\lambda\|_{2\to q} \le C\lambda^{\mu(q)} \, (\log \lambda)^{-1}, \;\; q \in (2, \infty]. \tag{3.12}$$

As in earlier works, the task of obtaining the bounds in (3.10) and (3.11) naturally splits into three cases: $q = q_c$, $q \in (q_c, \infty]$ and $q \in (2, q_c)$. Handling the critical exponent is the most difficult. So, we shall first prove the special case of these to bounds for this exponent:

$$\|\rho_\lambda\|_{2\to q_c} \lesssim \big(\lambda(\log \lambda)^{-1}\big)^{\mu(q_c)}, \quad \text{if all of the sectional curvatures of } M \text{ are nonpositive,} \tag{3.13}$$

and the stronger results

$$\|\rho_\lambda\|_{2\to q_c} \lesssim \lambda^{\mu(q_c)} (\log \lambda)^{-1/2}, \quad \text{if all of the sectional curvatures of } M \text{ are } \le -\kappa_0^2, \text{ some } \kappa_0 > 0. \tag{3.14}$$

Let us now describe the microlocal operators which will be utilized to give us our very useful second microlocalization. We shall use the fact that for each fixed j, we can write

$$\psi_j(x)(\sigma_\lambda h)(x) = \sum_{\ell=1}^{K} \big(A_{j,\ell} \circ \sigma_\lambda\big)(h)(x) + R_j h(x), \tag{3.15}$$

where the $A_{j,l}$ are pseudodifferential operators in a bounded subset of $S^0_{1,0}$ and the kernels of the above operators satisfy

$$A_{j,\ell}(x,y),\ R_j(x,y) = 0, \quad \text{if } \ x \notin B(x_j,\delta) \text{ or } \ y \notin B(x_j, 3\delta/2). \tag{3.16}$$

Furthermore, in the normal coordinate system about x_j described above we may assume that

$$A_{j,\ell}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a^\lambda_{j,\ell}(x,y,\xi)\, d\xi \tag{3.17}$$

with $a^\lambda_{j,\ell}$ supported in an $O(K^{-1/(n-1)})$ conic neighborhood of some $\xi_{j,\ell} \in S^{n-1}$, $a^\lambda_{j,\ell}(\xi) = 0$ if $|\xi| \notin [c_1\lambda, \lambda/c_1]$, with $c_1 \in (0,1)$ independent of j. So, if K is large enough, we may assume that

$$a^\lambda_{j,\ell}(x,y,\xi) = 0 \text{ when } \ x \notin B(x_j,\delta),\ y \notin B(x_j, 2\delta), \ \text{or } \ \Big|\tfrac{\xi_{j,\ell}}{|\xi_{j,\ell}|} - \tfrac{\xi}{|\xi|}\Big| \geq \delta, \tag{3.18}$$

and also that this symbol satisfies the natural size estimates corresponding to these support properties

$$\partial^{\alpha_1}_{x,y} \partial^{\alpha_2}_{\xi} a^\lambda_{j,\ell} = O_\delta(\lambda^{-|\alpha_2|}).$$

In addition to (3.16), we may assume by fixing $c_1 > 0$ small enough that we have the uniform bounds

$$R_j(x,y) = O(\lambda^{-N}), \quad N = 1,2,\dots. \tag{3.19}$$

The above dyadic pseudodifferential operators satisfy

$$\|A_{j,\ell}\|_{L^p(M)\to L^p(M)} = O(1), \quad 1 \le p \le \infty. \tag{3.20}$$

One constructs the above microlocal cutoffs $A_{j,\ell}$ using standard arguments from the theory of pseudodifferential operators. The resulting symbols can have dyadic support $|\xi| \approx \lambda$, just as in the case of compact manifolds treated in [33], since the left side of (3.15) involves $\sigma_\lambda = \sigma(\lambda - P)$. Furthermore, using the fact that we are assuming that M is of bounded geometry, by the above discussion, the implicit constants in the above description of this second microlocalization can be chosen to be independent of j if $\delta \le \delta(M)$ and K in (3.15) are fixed.

We also note that, due to our assumptions, we can assume that the symbols $a^\lambda_{j,\ell}$ vanish outside of a small conic neighborhood of $(x_j, \xi_{j,\ell})$ by choosing $\delta \le \delta(M)$ to be small and K to large. As in the compact manifold case, this will be useful when we need to use our local harmonic analysis.

One consequence of this, (3.2), (3.16) and (3.20) is that if we fix $\ell_0 \in \{1,\dots,K\}$, then these dyadic operators satisfy

$$A = A_{\ell_0} = \sum_j A_{j,\ell_0} \in S^0_{1,0} \quad \text{and} \ \ \|A\|_{L^p(M)\to L^p(M)} = O(1),\ 1 \le p \le \infty. \tag{3.21}$$

These microlocal cutoffs will play the role of the "B" operators that were used for compact manifolds in in [4], [7] and [33]. Note also that by (3.2) and (3.16) we also have

$$\|Rh\|_{L^q(M)} \le C_{p,q,M}\, \|h\|_{L^p(M)},\ 1 \le p \le q \le \infty,\ \text{if}\ ,\ R = \sum_j R_j. \tag{3.22}$$

Note that, in view of (3.3), (3.12), (3.15), (3.21) and (3.22), in order to prove (3.13) and (3.14), it suffices to prove that if all the sectional curvatures of M are nonpositive

$$\|A\sigma_\lambda \rho_\lambda\|_{2\to q_c} \lesssim \big(\lambda (\log \lambda)^{-1}\big)^{\mu(q_c)}, \tag{3.23}$$

while, if all the sectional curvatures are all pinched below $-\kappa_0^2$ with $\kappa_0 > 0$, we have

$$\|A\sigma_\lambda \rho_\lambda\|_{2\to q_c} \lesssim \lambda^{\mu(q_c)} (\log \lambda)^{-1/2}. \tag{3.24}$$

Using (3.2) and (3.16), we have for $q \ge 2$

$$|A\sigma_\lambda \rho_\lambda f(x)| \le C \|A_{j,\ell_0}(\sigma_\lambda \rho_\lambda f)(x)\|_{\ell^q_j}.$$

Consequently, if we consider the vector-valued operators

$$\mathcal{A}h = (A_{1,\ell_0} h,\, A_{2,\ell_0} h, \dots) \tag{3.25}$$

we have

$$\|A\sigma_\lambda \rho_\lambda f\|_{L^q(M)} \lesssim \|\mathcal{A}(\sigma_\lambda \rho_\lambda f)\|_{L^q_x \ell^q_j (M\times\mathbb{N})}, \quad q \in [2,\infty]. \tag{3.26}$$

Note for later use that by (3.2), (3.20) and (3.21) we also have

$$\|\mathcal{A}h\|_{L^p_x \ell^p_j} \le C \|h\|_{L^p(M)}, \quad 1 \le p \le \infty. \tag{3.27}$$

In view of (3.26), in order to prove (3.23) and (3.24), it suffices to show that when all of the sectional curvatures of M are nonpositive we have

$$\|\mathcal{A}\sigma_\lambda \rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j} \lesssim \big(\lambda (\log \lambda)^{-1}\big)^{\mu(q_c)} \|f\|_{L^2(M)}, \tag{3.28}$$

while, if all the sectional curvatures are all pinched below $-\kappa_0^2$ with $\kappa_0 > 0$, we have

$$\|\mathcal{A}\sigma_\lambda \rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j} \lesssim \lambda^{\mu(q_c)} (\log \lambda)^{-1/2} \|f\|_{L^2(M)}. \tag{3.29}$$

The operators $\mathcal{A}\sigma_\lambda \rho_\lambda$ play the role of the $\tilde\rho_\lambda$ operators in [7] and [33]. We are introducing this vector-valued approach to easily allow us to only have to carry out the local bilinear harmonic analysis in individual coordinate patches coming from the geodesic normal coordinates in the balls $B(x_j, 2\delta)$ mentioned before. In the compact case treated by two of us and coauthors, this was not necessary since M could be covered by finitely many balls of sufficiently small radius on which the bilinear analysis could be carried out.

In proving these two estimates we may, of course, assume, as we shall throughout this section, that

$$\|f\|_{L^2(M)} = 1. \tag{3.30}$$

Then, similar to the compact manifold case, let us define vector-valued sets

$$\begin{aligned} A_+ &= \{(x,j) :\ |(\mathcal{A}\sigma_\lambda \rho_\lambda f)(x,j)| \ge \lambda^{\frac{n-1}{4}+\frac18}\} \\ A_- &= \{(x,j) :\ |(\mathcal{A}\sigma_\lambda \rho_\lambda f)(x,j)| < \lambda^{\frac{n-1}{4}+\frac18}\}. \end{aligned} \tag{3.31}$$

Recall here that

$$(\mathcal{A}\sigma_\lambda\rho_\lambda f)(x,j) = A_{j,\ell_0}\sigma_\lambda\rho_\lambda f(x). \tag{3.32}$$

In order to prove (3.28) and (3.29), it suffices to show that we have the following two results. First, for all complete manifolds of bounded geometry and nonpositive sectional curvatures, we have for $\lambda \gg 1$ the large height estimates

$$\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)} \lesssim \lambda^{\mu(q_c)} T^{-1/2}, \quad \text{if } T = c_0 \log\lambda, \text{ with } c_0 = c_0(M) > 0 \text{ sufficiently small.} \tag{3.33}$$

The remaining estimate, for small heights, which would yield the above desired bounds for $q = q_c$ then would be the following for T as above

$$\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_-)} \lesssim \begin{cases} \left(\lambda T^{-1}\right)^{\mu(q_c)}, & \text{if all the sectional curvatures of } M \text{ are nonpositive} \\ \lambda^{\mu(q_c)} T^{-1/2}, & \text{if all the sectional curvatures of } M \text{ are } \le -\kappa_0^2, \text{ some } \kappa_0 > 0, \end{cases} \tag{3.34}$$

with T in (3.9) as in the preceding inequality.

In order to prove (3.33) and also the estimates in Theorem 1.5 for $q > q_c$ we shall require the following lemma.

Lemma 3.1. *Let $\Psi = |\rho|^2$ and fix*

$$a \in C_0^\infty((-1,1)) \text{ satisfying } a(t) = 1, \ |t| \le 1/2.$$

Then, if $G_\lambda = G_{\lambda,T}$ is defined by

$$G_\lambda = G_\lambda(P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1 - a(t))\, T^{-1}\hat{\Psi}(t/T)\, e^{it\lambda}\, e^{-itP}\, dt, \tag{3.35}$$

we have for $c_0 = c_0(M) > 0$ sufficiently small and $\lambda \gg 1$

$$\|G_\lambda\|_{L^1(M)\to L^\infty(M)} = O(\lambda^{\frac{n-1}{2}} \exp(C_M T)), \ 1 \le T \le c_0 \log\lambda, \tag{3.36}$$

assuming that M is of bounded geometry and that all of its sectional curvatures are nonpositive.

Note that if $L_\lambda = L_{\lambda,T}$ is given by

$$L_\lambda = L_\lambda(P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)\, T^{-1}\hat{\Psi}(t/T)\, e^{it\lambda} e^{-itP}\, dt, \tag{3.37}$$

then

$$G_\lambda + L_\lambda = \Psi(T(\lambda - P)) = \rho_\lambda\rho_\lambda^*. \tag{3.38}$$

Furthermore, it is simple to use (1.4) and a simple orthogonality argument to see that if q_c' is the dual exponent for q_c then

$$\|L_\lambda\|_{L^{q_c'}(M)\to L^{q_c}(M)} = O(T^{-1}\lambda^{2\mu(q_c)}) = O(T^{-1}\lambda^{2/q_c}). \tag{3.39}$$

We shall postpone the proof of this lemma until the end of this section. Let us now see how we can use it along with the local estimate (3.8) to prove the large height estimates. As two of us did for compact manifolds, we shall rely on a variant of an argument of Bourgain [11], along with (3.36) and (3.39).

Proof of (3.33). This just follows from the proof of (2.18) in [33]; however, we shall give the argument here for the sake of completeness. We shall be assuming here that, as in (3.33), $T = c_0 \log\lambda$, with $c_0 > 0$ to be specified in a moment.

We first note that, by (3.12), (3.25) and (3.30)

$$\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)} \le \|\mathcal{A}\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)} + C\lambda^{1/q_c}/\log\lambda, \tag{3.40}$$

since, by (1.3),

$$\mu(q_c) = 1/q_c.$$

As a result, we would obtain (3.33) if we could show that

$$\|\mathcal{A}\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)} \le C\lambda^{1/q_c}(\log\lambda)^{-1/2} + \tfrac{1}{2}\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)}. \tag{3.41}$$

To prove this, similar to what was done in [33], choose $g = g(x,j)$ vanishing outside A_+ so that

$$\|g\|_{L^{q_c'}_x \ell^{q_c'}_j(A_+)} = 1 \text{ and } \|\mathcal{A}\rho_\lambda f\|_{L^{q_c}_x \ell^{q_c}_j(A_+)} = \sum_j \int (\mathcal{A}\rho_\lambda f)(x,j)\, \overline{\mathbf{1}_{A_+}(x,j)\cdot g(x,j)}\, dx. \tag{3.42}$$

Then, similar to (3.4) in [33], using (3.30) and (3.38) we find that

$$\begin{aligned}
\|\mathcal{A}\rho_\lambda f\|^2_{L^{q_c}_x \ell^{q_c}_j(A_+)} &= \Big(\int_M f(x)\cdot \overline{\big(\rho_\lambda^*\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\big)(x)}\, dx\Big)^2 \\
&\le \int_M \big|\big(\rho_\lambda^*\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\big)(x)\big|^2\, dx \\
&= \sum_j \int_M \big(\big(\mathcal{A}\circ\Psi(T(\lambda-P))\circ\mathcal{A}^*\big)(\mathbf{1}\cdot g)\big)(x,j)\cdot\overline{(\mathbf{1}_{A_+}\cdot g)(x,j)}\, dx \\
&= \sum_j \int_M \big(\big(\mathcal{A}\circ L_\lambda\circ\mathcal{A}^*\big)(\mathbf{1}\cdot g)\big)(x,j)\cdot\overline{(\mathbf{1}_{A_+}\cdot g)(x,j)}\, dx \\
&\quad+ \sum_j \int_M \big(\big(\mathcal{A}\circ G_\lambda\circ\mathcal{A}^*\big)(\mathbf{1}\cdot g)\big)(x,j)\cdot\overline{(\mathbf{1}_{A_+}\cdot g)(x,j)}\, dx \\
&\quad= I + II.
\end{aligned}$$

By (3.27), (3.39) and (3.42) and Hölder's inequality, we have

$$\begin{aligned}
|I| &\le \|(\mathcal{A}L_\lambda\mathcal{A}^*)(\mathbf{1}_{A_+}\cdot g)\|_{L^{q_c}_x\ell^{q_c}_j}\cdot\|\mathbf{1}_{A_+}\cdot g\|_{L^{q_c'}_x\ell^{q_c'}_j} \\
&\lesssim \|L_\lambda\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^{q_c}_x}\cdot 1 \\
&\lesssim T^{-1}\lambda^{2/q_c}\|\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^{q_c'}_x} \\
&\lesssim T^{-1}\lambda^{2/q_c}\|\mathbf{1}_{A_+}\cdot g\|_{L^{q_c'}_x\ell^{q_c'}_j} = T^{-1}\lambda^{2/q_c}.
\end{aligned}$$

To estimate II, note that, by (3.27), $\|\mathcal{A}^*\|_{L^1_x\ell^1_j\to L^1_x}$, $\|\mathcal{A}\|_{L^\infty_x\to L^\infty_x\ell^\infty_j} = O(1)$. Also, if $c_0 > 0$ is chosen small enough, then, by (3.36) we have

$$\|G_\lambda\|_{L^1(M)\to L^\infty(M)} = O(\lambda^{\frac{n-1}{2}+\frac{1}{8}}).$$

Therefore, we have, by the above argument,

$$\begin{aligned}|II| &\le \|(\mathcal{A}G_\lambda\mathcal{A}^*)(\mathbf{1}_{A_+}\cdot g)\|_{L^\infty_x\ell^\infty_j}\|\mathbf{1}_{A_+}\cdot g\|_{L^1_x\ell^1_j}\\ &\lesssim \|(G_\lambda\mathcal{A}^*)(\mathbf{1}_{A_+}\cdot g)\|_{L^\infty_x}\|\mathbf{1}_{A_+}\cdot g\|_{L^1_x\ell^1_j}\\ &\le C\lambda^{\frac{n-1}{2}+\frac18}\|\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^1_x}\|\mathbf{1}_{A_+}\cdot g\|_{L^1_x\ell^1_j}\\ &\le C'\lambda^{\frac{n-1}{2}+\frac18}\|\mathbf{1}_{A_+}\cdot g\|^2_{L^1_x\ell^1_j}\\ &\le C'\lambda^{\frac{n-1}{2}+\frac18}\|g\|^2_{L^{q_c'}_x\ell^{q_c'}_j(A_+)}\cdot\|\mathbf{1}_{A_+}\|^2_{L^{q_c}_x\ell^{q_c}_j}\\ &= C'\lambda^{\frac{n-1}{2}+\frac18}\|\mathbf{1}_{A_+}\|^2_{L^{q_c}_x\ell^{q_c}_j}.\end{aligned}$$

But, by the definition of A_+ in (3.31)

$$\|\mathbf{1}_{A_+}\|^2_{L^{q_c}_x\ell^{q_c}_j}\le\big(\lambda^{\frac{n-1}{4}+\frac18}\big)^{-2}\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|^2_{L^{q_c}_x\ell^{q_c}_j(A_+)}.$$

So, assuming, as we may that $\lambda\gg1$ is large, we have

$$|II|\le C\lambda^{-1/8}\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|^2_{L^{q_c}_x\ell^{q_c}_j(A_+)}\le\tfrac12\|\mathcal{A}\sigma_\lambda\rho_\lambda f\|^2_{L^{q_c}_x\ell^{q_c}_j(A_+)}.$$

The estimates for I and II yield (3.33). □

Let us also see how we can use Lemma 3.1 to obtain the bounds in Theorem 1.5 for $q>q_c$, which extend the results for compact manifolds of Hassell and Tacy [28].

Proof of $q>q_c$ bounds. Let us now prove the estimates in (1.12) for $q>q_c$. For a given such q, it suffices to show that

$$\|\rho_\lambda\|_{2\to q}\lesssim T^{-1/2}\lambda^{\mu(q)},$$

with $T=c_q\log\lambda$, $c_q=c_q(M)>0$ sufficiently small. This in turn is equivalent to showing that

$$\|\Psi(T(\lambda-P))\|_{q'\to q}\lesssim T^{-1}\lambda^{2\mu(q)},\ \ q>q_c, \tag{3.43}$$

with $\Psi=|\rho|^2$, as above.

If, as in (3.38), $\Psi(T(\lambda-P))=L_\lambda+G_\lambda$, it is straightforward to check that (1.4) yields

$$\|L_\lambda\|_{q'\to q}\lesssim T^{-1}\lambda^{2\mu(q)}.$$

Furthermore, by (3.36) and orthogonality, we have

$$\|G_\lambda\|_{2\to2}=O(1).$$

If we interpolate between this estimate and (3.36) we obtain for $T\lesssim c_0\log\lambda$ as above

$$\|G_\lambda\|_{q'\to q}=O(\lambda^{\frac{(n-1)(q-2)}{2q}}\exp(C_MT)),\quad q>2.$$

Once checks that $\frac{(n-1)(q-2)}{2q}<2\mu(q)=2n(\frac12-\frac1q)-1$ if $q>q_c=\frac{2(n+1)}{n-1}$. As a result, for such an exponent, we have, for such q, $\|G_\lambda\|_{q'\to q}=O(\lambda^{2\mu(q)-\varepsilon_q})$, some $\varepsilon_q>0$, if, as we may assume $\lambda\gg1$ and $T=c_q\log\lambda$ with $c_q>0$ sufficiently small.

Since this and the above bound for L_λ yields (3.43), the proof of the spectral projection estimates in Theorem 1.5 for $q>q_c$ is complete. □

Next, we note that we would complete the proof of the spectral projection estimates in Theorem 1.5 for $q = q_c$ if we could prove the low height estimates (3.34) which, unlike (3.33), differ depending on the curvature assumptions. For this we shall need to use local bilinear harmonic analysis which is a variable coefficient variant of that in Tao, Vargas and Vega [47] and relies on bilinear oscillatory integral estimates of Lee [38]. Since the microlocal cutoffs in (3.21) arise from the partition of unity in (3.15) corresponding to the balls $\{B(x_j, \delta)\}$ whose doubles have finite overlap, we shall be able carry out this analysis in each ball $B(x_j, 2\delta)$ using geodesic normal coordinates about the center. Since, as we pointed out earlier, our assumption of bounded geometry ensures bounded transition maps and uniform bounds on derivatives of the metric, we shall be able to localize to individual balls. As a result, we just need to repeat the arguments in the earlier work of two of use [33] for compact manifolds, which also reduced to bilinear analysis in a fixed coordinate chart.

Just as in the earlier works for compact manifolds, [4], [7], [33], to prove (3.34), besides (3.15), we shall need to use a second microlocalization, which involves localizing in $\theta \geq \lambda^{-1/8}$ neighborhoods of geodesics in a fixed coordinate chart. To describe this, let us fix j in (3.15), as well as $\ell_0 \in \{1, \dots, K\}$ and consider the resulting pseudodifferential cutoff, A_{j,ℓ_0}, which is a summand in (3.21). Its symbol then satisfies the conditions in (3.18). The resulting geodesic normal coordinates on $B(x_j, 2\delta)$ vanish at x_j. We then have that the metric g_{jk} satisfies $g_{jk}(y) = \delta_j^k + O((d_g(x_j, y))^2)$. We may also assume that $\xi_{j,\ell_0} = (0, \dots, 0, 1)$. Since we are fixing j and ℓ_0 for now, analogous [33], let us simplify the notation a bit by letting

$$\tilde{\sigma}_\lambda = A_{j,\ell_0} \sigma_\lambda, \tag{3.44}$$

which is analogous to (2.10) in [33].

For dyadic $\theta \geq \lambda^{-1/8}$, the additional microlocal cutoffs that we require correspond to θ-nets of geodesics, $\{\gamma_\nu\}$, in S^*M passing through points (y, η) near $(0, (0, \dots, 0, 1))$. To define them, fix a function $q \in C_0^\infty(\mathbb{R}^{2(n-1)})$ supported in $\{z : |z_i| \leq 1, \, 1 \leq i \leq 2(n-1)\}$ satisfying

$$\sum_{k \in \mathbb{Z}^{2(n-1)}} q(z - k) \equiv 1. \tag{3.45}$$

To use this, let

$$\mathit{\Pi} = \{y : y_n = 0\}$$

be the points in $\Omega = B(x_j, 2\delta)$ whose last coordinate vanishes. Also let $y' = (y_1, \dots, y_{n-1})$ and $\eta' = (\eta_1, \dots, \eta_{n-1})$ denote the first $(n-1)$ coordinates of y and η, respectively, with $(y, \eta) \in S^*\Omega$. We shall always have $\theta \in [\lambda^{-1/8}, 1]$, and, $\lambda^{-1/8}$, here, of course, is related to the height decomposition (3.31).

We then can extend the definition of our cutoffs to a neighborhood of $(0, (0, \dots, , 1))$ by setting for $(x, \xi) \in S^*\Omega$ in this neighborhood

$$q_k^\theta(x, \xi) = q(\theta^{-1}(y', \eta') - k) \text{ if } \Phi_s(x, \xi) = (y', 0, \eta', \eta_n) \text{ with } s = d_g(x, \mathit{\Pi}). \tag{3.46}$$

Here Φ_s denotes the geodesic flow in $S^*\Omega$ and $d_g(\,\cdot\,,\,\cdot\,)$ is geodesic distance. Consequently, $q_k^\theta(x, \xi)$ is constant on all geodesics $(x(s), \xi(s)) \in S^*\Omega$ with $x(0) \in \mathit{\Pi}$ near 0 and $\xi(0)$ near $(0, \dots, 0, 1)$. Therefore,

$$q_k^\theta(\Phi_s(x, \xi)) = q_k^\theta(x, \xi). \tag{3.47}$$

We then extend the definition of the cutoffs to a conic neighborhood of $(0,(0,\dots,0,1))$ in $T^*\Omega\backslash 0$ by setting

$$q_k^\theta(x,\xi)=q_k^\theta(x,\xi/p(x,\xi)), \tag{3.48}$$

with $p(x,\xi)$ being the principal symbol of $P=\sqrt{-\Delta_g}$.

Note also that if $(y'_\nu,\eta'_\nu)=\theta k=\nu$ and γ_ν is the geodesic in $S^*\Omega$ passing through $(y'_\nu,0,\eta_\nu)\in S^*\Omega$ with $\eta_\nu\in S^*_{(y',0)}\Omega$ having η'_ν as its first $(n-1)$ coordinates and $\eta_n>0$ then

$$q_k^\theta(x,\xi)=0 \text{ if } \operatorname{dist}\,((x,\xi),\gamma_\nu)\ge C_0\theta,\ \nu=\theta k \tag{3.49}$$

for a uniform constant C_0. Also, q_k^θ satisfies the estimates

$$|\partial_x^\sigma\partial_\xi^\gamma q_k^\theta(x,\xi)|\lesssim \theta^{-|\sigma|-|\gamma|},\ (x,\xi)\in S^*\Omega \tag{3.50}$$

related to this support property.

Next, fix $\tilde\psi\in C_0^\infty$ supported in $|x|<3\delta/2$ which equals one when $|x|\le 5\delta/4$. Additinally, fix $\tilde{\tilde\psi}\in C_0^\infty$ supported in $|x|<2\delta$ which equals one when $|x|\le 3\delta/2$. Also, fix $\tilde\beta\in C_0^\infty((0,\infty))$ so that $\tilde\beta(p(x,\xi)/\lambda)$ equals one in a neighborhood of the ξ support of a^λ_{j,ℓ_0}. We then define the compound symbols $Q^\theta_\nu=Q^\theta_{j,\ell_0,\nu}$ and associated operators by

$$\begin{aligned} Q_\nu^\theta(x,y,\xi)&=\tilde\psi(x)\tilde{\tilde\psi}(y)q_k^\theta(x,\xi)\,\tilde\beta((x,\xi)/\lambda),\ \nu=\theta k\in\theta\cdot\mathbb{Z}^{2(n-1)}, \text{ and}\\ Q_\nu^\theta h(x)&=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}Q_\nu^\theta(x,y,\xi)\,h(y)\,d\xi dy.\end{aligned} \tag{3.51}$$

It follows that these dyadic pseudodifferential operators belong to a bounded subset of $S^0_{7/8,1/8}$ due to our assumption that $\theta\in[\lambda^{-1/8},1]$. We have constructed these operators so that for small enough $\delta_0>0$ we have

$$\begin{aligned} Q_\nu^\theta(x,y,\xi)&=Q_\nu^\theta(z,y,\eta),\ (z,\eta)=\Phi_t(x,\xi),\\ &\text{if } \operatorname{dist}\,((x,\xi),\,\operatorname{supp}A_{j,\ell_0})\le\delta_0 \text{ and } |t|\le 2\delta_0.\end{aligned} \tag{3.52}$$

The compound symbol involves the cutoff $\tilde{\tilde\psi}(y)$ which equals one on a neighborhood of the x-support of A_{j,ℓ_0} as well as the support of $\tilde\psi$. We use cutoffs in both variables since M is not assumed to be compact and we want to avoid issues at infinity. This symbol in (3.52) vanishes when either x or y is outside the 2δ-ball about the origin in our coordinates for Ω. By (3.7) (3.16) and (3.44), we can fix δ_1 in (3.4) small enough so that we also have, analogous to (2.41) in [33],

$$\begin{aligned}&\tilde\sigma_\lambda=\sum_\nu\tilde\sigma_\lambda Q_\nu^{\theta_0}+R,\quad R=R_{\lambda,j,\ell_0},\ \theta_0=\lambda^{-1/8},\ \tilde\sigma_\lambda=A_{j,\ell_0}\sigma_\lambda,\\ &\text{where } R(x,y)=O(\lambda^{-N}),\forall N \text{ and } R(x,y)=0, \text{ if } x\notin B(x_j,2\delta) \text{ or } y\notin B(x_j,2\delta),\end{aligned} \tag{3.53}$$

with bounds for the remainder kernel independent of j.

Let us now point out straightforward but useful properties of our operators. First, by (3.16), (3.53) and the support properties of $\tilde\psi$, $\tilde{\tilde\psi}$, we have

$$\tilde\sigma_\lambda Q^{\theta_0}_\nu h = \mathbf{1}_{B(x_j,2\delta)}\cdot\tilde\sigma_\lambda Q^{\theta_0}_\nu\big(\mathbf{1}_{B(x_j,2\delta)}\cdot h\big),\ Q^{\theta_0}_\nu = Q^{\theta_0}_{j,\ell_0,\nu} \quad \text{and}\quad Rh=\mathbf{1}_{B(x_j,2\delta)}\cdot R\big(\mathbf{1}_{B(x_j,2\delta)}\cdot h\big),\ R=R_{\lambda,j,\ell_0}. \tag{3.54}$$

Also, we have the uniform bounds

$$\begin{aligned}\|Q^{\theta_0}_\nu h\|_{\ell^q_\nu L^q(M)} &\lesssim \|h\|_{L^q(M)},\ 2\le q\le\infty\\ \big\|\sum_{\nu'}(Q^{\theta_0}_\nu)^* H(\nu',\,\cdot\,)\|_{L^p(M)} &\lesssim \|H\|_{\ell^p_{\nu'}L^p(M)},\ 1\le p\le 2.\end{aligned} \tag{3.55}$$

The second estimate follows via duality from the first. The first one is (2.33) in Lemma 2.2 of [33]. By interpolation, one just needs to verify that the estimate holds for the two endpoints, $p=2$ and $p=\infty$. The former follows via an almost orthogonality argument, and the latter from the fact that we have the uniform bounds

$$\sup_{x\in B(x_j,2\delta)}\int_{B(x_j,2\delta)}|Q^{\theta_0}_\nu(x,y)|\,dy\le C.$$

See [33] for more details.

Note that if we use (3.55) along with (3.51) and the finite overlap of the balls $\{B(x_j,2\delta)\}$ we obtain for our fixed $\ell_0=1,\dots,K$

$$\begin{aligned}\big(\sum_{j,\nu}\|Q^{\theta_0}_{j,\ell_0,\nu}h\|^q_{L^q(M)}\big)^{1/q} &\lesssim \|h\|_{L^q(M)},\ 2\le q\le\infty\\ \big\|\sum_{j',\nu'}(Q^{\theta_0}_{j',\ell_0,\nu'})^* H(\nu',j',\,\cdot\,)\|_{L^p(M)} &\lesssim \|H\|_{\ell^p_{\nu'}L^p(M)},\ 1\le p\le 2.\end{aligned} \tag{3.56}$$

In addition to this inequality and (3.12) we shall require another that follows almost directly from a result in [33]. Specifically, we require the following commutator bounds

$$\big\|(A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}-A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda)h\|_{L^q(M)}\le C_q\lambda^{\mu(q)-1/4}\|h\|_{L^2(B(x_j,2\delta))}, \tag{3.57}$$

with $\mu(q)$ is as in (1.3), assuming that δ, as well as δ_1 in (3.4) are fixed small enough.

To see this, let $\tilde A_{j,\ell_0}$ be a 0-order pseudo-differential operator with symbol $\tilde a^\lambda_{j,\ell_0}(x,y,\xi)$ supported in $|\xi|\approx\lambda$ and equals one in the support of the symbol $a^\lambda_{j,\ell_0}(x,y,\xi)$ of the A_{j,ℓ_0} operator, then it is not hard to see that

$$\|A_{j,\ell_0}-\tilde A_{j,\ell_0}A_{j,\ell_0}\|_{L^p_x\to L^p_x}=O(\lambda^{-N})\ \forall N \text{ if } \ 1\le p\le\infty. \tag{3.58}$$

And by using the fact that the kernel $\tilde A_{j,\ell_0}(x,y)$ is $O(\lambda^n(1+\lambda|x-y|)^{-N})$ and Young's inequality, we also have

$$\|\tilde A_{j,\ell_0}\|_{L^2_x\to L^p_x}=O(\lambda^{n(\frac12-\frac1p)}) \text{ if } \ 2\le p\le\infty. \tag{3.59}$$

Thus to prove (3.57) it suffices to show

$$\big\|(A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}-A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda)h\|_{L^2(M)}\le C_q\lambda^{-3/4}\|h\|_{L^2(B(x_j,2\delta))} \tag{3.60}$$

since $n(\frac12-\frac1p)-\frac34\le\mu(q)-\frac14$ for $q\le q_c$. This follows from the proof of (2.39) in [33] since, by (3.18), $A_{j,\ell_0}f$ vanishes outside $B(x_j,2\delta)$ and the two operators in (3.60) vanish when acting on functions vanishing on $B(x_j,2\delta)$. This, just as in [33], allows one to prove

(3.60), exactly as in [33], by just working in a coordinate chart ($B(x_j, 2\delta)$ here) and, to obtain the inequality using (3.47), (3.52) and Egorov's theorem related to the properties of the half wave operator e^{itP} in this local coordinate.

Next, as in [4] and [33], we note that we can write for θ_0 and $\tilde\sigma_\lambda$ as in (3.44)

$$\big(\tilde\sigma_\lambda h\big)^2 = \sum_{\nu,\nu'} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0} h\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h\big) + O(\lambda^{-N}\|h\|^2_{L^2(B(x_j,2\delta))}),\ \forall N. \tag{3.61}$$

Note that the $\nu = \theta_0\cdot\mathbb{Z}^{2(n-1)}$ index a $\lambda^{-1/8}$-separated lattice in $\mathbb{R}^{2(n-1)}$. As in earlier works, to be able to apply bilinear oscillatory integral results, we need to organize the pairs (ν,ν') in the above sum. As in [47], we first consider dyadic cubes τ^θ_μ in $\mathbb{R}^{2(n-1)}$ of sidelength $\theta = 2^k\theta_0 = 2^k\lambda^{-1/8}$, with τ^θ_μ denoting translation of the cube $[0,\theta)^{2(n-1)}$ by $\mu = \theta\cdot\mathbb{Z}^{2(n-1)}$. We then say two such cubes are *close* if they are not adjacent but have adjacent parents of sidelength 2θ. In this case we write $\tau^\theta_\mu \sim \tau^\theta_{\mu'}$. Note that close cubes satisfy dist $(\tau^\theta_\mu, \tau^\theta_\mu) \approx \theta$ and also that each fixed cube has $O(1)$ cubes that are "close" to it. Moreover, as was noted in [47], any distinct points $\nu,\nu' \in \mathbb{R}^{2(n-1)}$ must lie in a unique pair of close in this Whitney decomposition of $\mathbb{R}^{2(n-1)}$. Consequently, there must be a unique triple $(\theta = 2^k\theta_0, \mu, \mu')$ such that $(\mu,\mu') \in \tau^\theta_\mu \times \tau^\theta_{\mu'}$ and $\tau^\theta_\mu \sim \tau^\theta_{\mu'}$.

We also note that if, as we shall, we fix the δ occurring in the construction of the $\{A_{j,\ell}\}$ to be small enough then we only need to consider $\theta = 2^k\theta_0 \ll 1$ when dealing with the bilinear sum in (3.61).

Based on these observations, we can organize the sum in (3.61) as follows

$$\begin{aligned}\sum_{\{k\in\mathbb{N}:\, k\ge 10 \text{ and } \theta=2^k\theta_0\ll 1\}} \ \sum_{\{(\mu,\mu'):\, \tau^\theta_\mu\sim\tau^\theta_{\mu'}\}} \ \sum_{\{(\nu,\nu')\in\tau^\theta_\mu\times\tau^\theta_{\mu'}\}} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0} h\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h\big)\\ + \sum_{(\nu,\nu')\in\Xi_{\theta_0}} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0} h\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h\big),\end{aligned} \tag{3.62}$$

where Ξ_0 indexes the remaining pairs such that $|\nu-\nu'| \lesssim \theta_0 = \lambda^{-1/8}$, including the diagonal ones where $\nu = \nu'$.

Let us then set for our fixed (j,ℓ_0) and $\tilde\sigma_\lambda = A_{j,\ell_0}\sigma_\lambda$

$$\Upsilon^{\mathrm{diag}}_{j,\ell_0}(h) = \Upsilon^{\mathrm{diag}}(h) = \sum_{(\nu,\nu')\in\Xi_{\theta_0}} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0} h\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h\big) \tag{3.63}$$

and

$$\Upsilon^{\mathrm{far}}_{j,\ell_0}(h) = \Upsilon^{\mathrm{far}}(h) = \sum_{(\nu,\nu')\notin\Xi_{\theta_0}} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0} h\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h\big) + O(\lambda^{-N}\|h\|^2_{L^2(B(x_j,2\delta))}), \tag{3.64}$$

with the last term being the error term in (3.61). Due to this splitting we have the analog of (5.5) in [33]

$$(\tilde\sigma_\lambda h)^2 = \Upsilon^{\mathrm{diag}}(h) + \Upsilon^{\mathrm{far}}(h). \tag{3.65}$$

We shall use this decomposition when $n \ge 3$, since then $q_c \le 4$, which allows us to use bilinear ideas from [47], exploiting the fact that $q_c/2 \in [1,2]$. When the dimension n

of M equals 2, though, the critical exponent $q_c = 6$, which, as in [4] and [33], requires a slight modification of the above splitting.

Specifically, for $n = 2$, we first, as in these two earlier works, set

$$T_\nu h = \sum_{\nu' :\, (\nu,\nu')\in \Xi_{\theta_0}} (\tilde\sigma_\lambda Q_\nu^{\theta_0} h)(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0} h), \tag{3.66}$$

and write

$$(\Upsilon^{\rm diag}(h))^2 = \big(\sum_\nu T_\nu h\big)^2 = \sum_{\nu_1,\nu_2} T_{\nu_1} h T_{\nu_2} h. \tag{3.67}$$

If, as above, we fix δ small enough then the sum in (3.67) can be organized as

$$\begin{aligned}(\sum_{\{k\in\mathbb{N}:\, k\ge 20 \text{ and } \theta=2^k\theta_0\ll 1\}} \sum_{\{(\mu_1,\mu_2):\, \tau^\theta_{\mu_1}\sim\tau^\theta_{\mu_2}\}} \sum_{\{(\nu_1,\nu_2)\in \tau^\theta_{\mu_1}\times\tau^\theta_{\mu_2}\}} + \sum_{(\nu_1,\nu_2)\in\overline{\Xi}_{\theta_0}}) T_{\nu_1} h T_{\nu_2} h, \\ = \overline{\Upsilon}^{\rm far}(h) + \overline{\Upsilon}^{\rm diag}(h).\end{aligned} \tag{3.68}$$

Here $\overline{\Xi}_{\theta_0}$ indexes the near diagonal pairs. This is another Whitney decomposition similar to the one in (3.62), but the diagonal set $\overline{\Xi}_{\theta_0}$ is much larger than the set Ξ_{θ_0} in (3.62). More explicitly, when $n = 2$, it is not hard to check that $|\nu-\nu'| \le 2^{11}\theta_0$ if $(\nu,\nu')\in\Xi_{\theta_0}$ while $|\nu_1-\nu_2| \le 2^{21}\theta_0$ if $(\nu_1,\nu_2)\in\overline{\Xi}_{\theta_0}$. As noted in [33], this helps to simplify the calculations needed for $\overline{\Upsilon}^{\rm far}(h)$. Note that for our fixed (j,ℓ_0), $\overline{\Upsilon}^{\rm diag}(h) = \overline{\Upsilon}^{\rm diag}_{j,\ell_0}(h)$, $\overline{\Upsilon}^{\rm far}(h) = \overline{\Upsilon}^{\rm far}_{j,\ell_0}(h)$, and $\Upsilon^{\rm far}_{j,\ell_0}(h) = \Upsilon^{\rm far}(h)$ as in (3.64), we then have

$$(\tilde\sigma_\lambda h)^4 \le 2(\Upsilon^{\rm diag} h)^2 + 2(\Upsilon^{\rm far} h)^2 = 2\overline{\Upsilon}^{\rm diag}(h) + 2\overline{\Upsilon}^{\rm far}(h) + 2(\Upsilon^{\rm far}(h))^2, \text{ if } n=2. \tag{3.69}$$

We have organized the sums expanding the left side of (3.61) exactly as in [33]. In view of (3.54) each of the summands in the above decompositions is localized to our coordinate chart $\Omega = B(x_j, 2\delta)$ on which we are using geodesic normal coordinates about the center. Since our bounded geometry assumptions ensures we have uniform control of the metric and its derivatives, for $\delta > 0$ fixed small enough, we can simply repeat the proof of Lemma 5.1 in [33] to obtain the following variable coefficient variant of Lemma 6.1 in Tao, Vargas and Vega [47].

Lemma 3.2. *Let $\theta_0 = \lambda^{-1/8}$ with $\lambda \gg 1$. If $n \ge 3$ there is a uniform constant $C = C_M$ independent of (j,ℓ_0) so that if, as in (3.51), $Q_\nu^{\theta_0} = Q_{j,\ell_0,\nu}^{\theta_0}$*

$$\begin{aligned}\big\|\Upsilon^{diag}_{j,\ell_0}(h)\big\|_{L^{q_c/2}(M)} \le C\big(\sum_\nu \big\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} h\big\|^{q_c}_{L^{q_c}(B(x_j,2\delta))}\big)^{2/q_c} \\ + O(\lambda^{\frac{2}{q_c}-}\|h\|^2_{L^2(B(x_j,2\delta))}).\end{aligned} \tag{3.70}$$

Also, for all $n \ge 2$, if $q \in (2, \frac{2(n+2)}{n}]$ and $\mu(q)$ as in (1.3), there is a uniform constant $C_q = C(q,M)$ so that

$$\begin{aligned}\big\|\Upsilon^{diag}_{j,\ell_0}(h)\big\|_{L^{q/2}(M)} \le C_q\big(\sum_\nu \big\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} h\big\|^{q}_{L^{q}(B(x_j,2\delta))}\big)^{2/q} \\ + O(\lambda^{2\mu(q)-}\|h\|^2_{L^2(B(x_j,2\delta))}).\end{aligned} \tag{3.71}$$

Additionally, for $n=2$ there is a uniform constant $C=C(M)$ so that

$$\tag{3.72} \big\|\overline{\Upsilon}^{diag}_{j,\ell_0}(h)\big\|_{L^{3/2}(M)} \le C\big(\sum_\nu \big\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}h\|^6_{L^6(B(x_j,2\delta))}\big)^{2/3}+O(\lambda^{\frac23-}\|h\|^4_{L^2(B(x_j,2\delta))}).$$

In the above and what follows $O(\lambda^{\mu-})$ denotes $O(\lambda^{\mu-\varepsilon_0})$ for some $\varepsilon_0>0$.

If we fix δ as well as δ_1,δ_2 in (3.4) small enough, then we can use Lee's [38] bilinear oscillatory integral theorem and repeat the proof of Lemma 5.2 in [33] to obtain the following.

Lemma 3.3. *Let $n\ge 2$ and $\Upsilon^{far}(h)=\Upsilon^{far}_{j,\ell_0}(h)$ be as above with $\theta_0=\lambda^{-1/8}$. Then for all $\varepsilon>0$ there is a $C_\varepsilon=C(\varepsilon,M)$ so that*

$$\tag{3.73} \int_M |\Upsilon^{far}_{j,\ell_0}(h)|^{q/2}\,dx \le C_\varepsilon\lambda^{1+\varepsilon}\left(\lambda^{7/8}\right)^{\frac{n-1}{2}(q-q_c)}\|h\|^q_{L^2(B(x_j,2\delta))}, \quad q=\tfrac{2(n+2)}{n}.$$

Similarly, for all $n\ge 2$, there is a constant $C_q=C(q,M)$ so that

$$\tag{3.74} \int_M |\Upsilon^{far}_{j,\ell_0}(h)|^{q/2}\,dx \le C_q\,\lambda^{q\cdot\mu(q)-}\|h\|^q_{L^2(B(x_j,2\delta))}, \quad 2<q<\tfrac{2(n+2)}{n},$$

and, if $n=2$ and $\overline{\Upsilon}^{far}(h)=\overline{\Upsilon}^{far}_{j,\ell_0}(h)$ as in (3.68),

$$\tag{3.75} \int_M |\overline{\Upsilon}^{far}_{j,\ell_0}(h)|\,dx \le C_\varepsilon\lambda^{1+\varepsilon}\lambda^{-7/8}\|h\|^4_{L^2(B(x_j,2\delta))},\ \forall\,\varepsilon>0,$$

with $C_\varepsilon=C(\varepsilon,M)$.

We now have collected the main ingredients that we need to prove the critical low height estimates.

Proof of (3.34). Let us assume that $n\ge 3$. A main step in the proof of the A_- estimates then is to obtain the analog of (2.44) in [33]. We shall do so largely by repeating its proof, which we do so for the sake of completeness in order to note the small changes needed to take into account that, unlike (2.44) in [33], (3.34) here is a vector valued inequality. As noted before, we have taken this framework to help us exploit our assumption of bounded geometry, and, in particular, the fact that the doubles of the balls in our covering of M, $\{B(x_j,2\delta)\}$, have uniformly finite overlap.

We first note that if $q=\frac{2(n+2)}{n}<q_c$, then by (3.32) and (3.65) for our fixed ℓ_0 we have

$$\begin{aligned}&\big|\big(\mathcal{A}\sigma_\lambda(\rho_\lambda f)(x,j)\big)^2\big|^{q_c/2}=\big|\big(A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\big)^2\big|^{q_c/2}\\ &\quad=|A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\cdot A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{\frac{q_c-q}{2}}\big|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(\rho_\lambda f)(x)+\Upsilon^{\mathrm{far}}_{j,\ell_0}(\rho_\lambda f)(x)\big|^{q/2}\\ &\quad\le |A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\cdot A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{\frac{q_c-q}{2}}2^{q/2}\big(|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(\rho_\lambda f)(x)|^{q/2}+|\Upsilon^{\mathrm{far}}_{j,\ell_0}(\rho_\lambda f)(x)|^{q/2}\big).\end{aligned}$$

Also, if A_- is as in (3.31),

$$\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{q_c}_{L^{q_c}_x\ell^{q_c}_j(A_-)}=\int_M\sum_j \mathbf{1}_{A_-}(x,j)\,|A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{q_c}\,dx.$$

Thus,

$$
\begin{aligned}
\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{q_c}_{L^{q_c}_x\ell^{q_c}_j(A_-)} &= \sum_j \int \mathbf{1}_{A_-}(x,j)\,|A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\cdot A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{q_c/2}\,dx \\
\le C\sum_j\int \big[\mathbf{1}_{A_-}(x,j)\,&|A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\cdot A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{\frac{q_c-q}{2}}\big]\,|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(\rho_\lambda f)(x)|^{q/2}\,dx \\
+ C\sum_j\int \big[\mathbf{1}_{A_-}(x,j)\,&|A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)\cdot A_{j,\ell_0}(\sigma_\lambda\rho_\lambda f)(x)|^{\frac{q_c-q}{2}}\big]\,|\Upsilon^{\mathrm{far}}_{j,\ell_0}(\rho_\lambda f)(x)|^{q/2}\,dx \\
&= C(I+II).
\end{aligned}
$$

To handle II we recall that by (3.31), (3.32) and (3.54)

$$
|\mathbf{1}_{A_-}(x,j)A_{j,\ell_0}\sigma_\lambda\rho_\lambda f(x)| \le \lambda^{\frac{n-1}{4}+\frac{1}{8}}.
$$

Thus, by (3.73),

$$
\begin{aligned}
CII &\lesssim \lambda^{(\frac{n-1}{4}+\frac18)(q_c-q)}\cdot\lambda^{1+\varepsilon}\,(\lambda^{\frac78})^{\frac{n-1}{2}(q-q_c)}\sum_j \|\rho_\lambda f\|^q_{L^2(B(x_j,2\delta)} \\
&\lesssim \lambda^{1-\delta_n+\varepsilon}\|\rho_\lambda f\|^q_{L^2(M)} \le \lambda^{1-\delta_n+\varepsilon}\|f\|^q_{L^2(M)} = \lambda^{1-\delta_n+\varepsilon},
\end{aligned}
$$

using also, in the second inequality, the bounded overlap of the $\{B(x_j,2\delta)\}$. Also, a simple calculation shows that $\delta_n>0$.

To control CI, as in [4] and [33], we use Hölder's inequality and Young's inequality along with (3.70) to get

$$
\begin{aligned}
CI &\le \|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\cdot\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{\frac{q_c-q}{2}}_{L^{q_c/2}_x\ell^{q_c/2}_j(A_-)}\cdot C\|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(\rho_\lambda f)\|^{q/2}_{L^{q_c/2}_x\ell^{q_c/2}_j} \\
&\le \tfrac{q_c-q}{q_c}\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\cdot\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{q_c/2}_{L^{q_c/2}_x\ell^{q_c/2}_j(A_-)} + \tfrac{q}{q_c}C\|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(\rho_\lambda f)\|^{q_c/2}_{L^{q_c/2}_x\ell^{q_c/2}_j} \\
&\le \tfrac{q_c-q}{q_c}\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{q_c}_{L^{q_c}_x\ell^{q_c}_j(A_-)} \\
&\qquad + C'\tfrac{q}{q_c}\Big[\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(B(x_j,2\delta))} + \lambda^{1-}\big(\sum_j\|\rho_\lambda f\|_{L^2(B(x_j,2\delta))}\big)^{q_c/2}\Big] \\
&\le \tfrac{q_c-q}{q_c}\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|^{q_c}_{L^{q_c}_x\ell^{q_c}_j(A_-)} + C''\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(B(x_j,2\delta))} + \lambda^{1-},
\end{aligned}
$$

again using the finite overlap of the $\{B(x_j,2\delta)\}$.

Since $\frac{q_c-q}{q_c}<1$ we can use the bounds for CI and CII to obtain the key inequality

$$
\|\mathcal{A}(\sigma_\lambda\rho_\lambda f)\|_{L^{q_c}_x\ell^{q_c}_j(A_-)} \lesssim \big(\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(M)}\big)^{1/q_c} + \lambda^{\frac{1}{q_c}-},
$$

which is the analog of the estimate (2.44) in Proposition 2.3 in [33] for $n\ge3$. One can similarly use (3.72) and (3.75) and modify the arguments in [33] to handle the two-dimensional case. Similarly, if one also uses (3.74) one can obtain an analog of the preceding estimate for subcritical exponents, $2<q<q_c$, which is more straightforward and does not require the norm in the left to be taken over A_-.

Thus, just as the preceding inequality followed from straightforward modifications of the arguments in [33], so do the other estimates in the following result coming from variable coefficient variants of the bilinear harmonic analysis in [47].

Proposition 3.4. *Fix a complete $n \ge 2$ Riemannian manifold (M,g) of bounded geometry and assume that (3.30) is valid. Then $\lambda \gg 1$ and $\theta_0 = \lambda^{-1/8}$*

$$\|\mathcal{A}(\sigma_\lambda \rho_\lambda f)\|_{L^{q_c}_x \ell^{q_c}_j(A_-)} \lesssim \big(\sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(M)}\big)^{1/q_c} + \lambda^{\frac{1}{q_c}-}, \tag{3.76}$$

assuming that δ and δ_1 above are small enough. Additionally, for $2 < q \le \frac{2(n+2)}{n}$,

$$\big(\sum_j \|A_{j,\ell_0}(\sigma_\lambda \rho_\lambda f)\|^q_{L^q_x(M)}\big)^{1/q} \lesssim \big(\sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^q_{L^q(M)}\big)^{1/q} + \lambda^{\mu(q)-}. \tag{3.77}$$

Due to (3.76), in order to prove (3.34) and finish the proof of the q_c-estimates in Theorem 1.5, it suffices to show that if, as above, we take $T = c_0 \log\lambda$ as in (3.33), then

$$\big(\sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(M)}\big)^{1/q_c} \lesssim \lambda^{\mu(q_c)} T^{-1/2}, \tag{3.78}$$

if all the sectional curvatures of M are $\le -\kappa_0^2$, some $\kappa_0 > 0$,

and

$$\big(\sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{L^{q_c}(M)}\big)^{1/q_c} \lesssim \big(\lambda T^{-1}\big)^{\mu(q_c)}, \tag{3.79}$$

if all the sectional curvatures of M are nonpositive.

To prove these two estimates we shall argue as in the proof of (2.56) in [33] and use (3.8), (3.20) and (3.57) to obtain

$$\begin{aligned}
\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c} &= \sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_{q_c} \cdot \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c-2}_{q_c} \\
&\le \sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_{q_c} \cdot \|A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c-2}_{q_c} \\
&\quad + \sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_{q_c} \cdot \|\,(A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} - A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda)\rho_\lambda f\|^{q_c-2}_{q_c} \\
&\lesssim \sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_{q_c} \cdot \|A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c-2}_{q_c} \\
&\quad + \sum_{j,\nu} \lambda^{2/q_c}\|Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_2 \cdot \lambda^{(\frac{1}{q_c}-\frac14)(q_c-2)}\|\rho_\lambda f\|^{q_c-2}_2 \\
&\lesssim \sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^2_{q_c} \cdot \|A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c-2}_{q_c} + \lambda^{1-\frac14(q_c-2)} \\
&\le C\big(\sum_{j,\nu} \|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}\big)^{\frac{2}{q_c}} \big(\sum_{j,\nu} \|A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c}_{q_c}\big)^{\frac{q_c-2}{q_c}} + C\lambda^{1-\frac14(q_c-2)}.
\end{aligned}$$

By Young's inequality, the second to last term is bounded for any $\kappa > 0$ by

$$C\Big[\,\frac{2}{q_c}\kappa^{\frac{q_c}{2}}\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}+\frac{q_c-2}{q_c}\kappa^{-\frac{q_c}{q_c-2}}\sum_{j,\nu}\|A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c}_{q_c}\Big].$$

If κ is small enough the first term here is smaller than half of the left side of the preceding inequality. So, by an absorbing argument and the fact that $q_c > 2$, we conclude that

$$\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}\lesssim\sum_{j,\nu}\|A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c}_{q_c}+\lambda^{1-}.$$

If we next use (3.20), (3.55), followed by (3.12) we find that we can control the first term in the right as follows

$$\begin{aligned}\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}&\lesssim\sum_{j,\nu}\|Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda\rho_\lambda f\|^{q_c}_{q_c}\\&\lesssim\sum_{j,\nu}\Big[\,\|Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}+\|Q^{\theta_0}_{j,\ell_0,\nu}(I-\sigma_\lambda)\rho_\lambda f\|^{q_c}_{q_c}\Big]\\&\lesssim\sum_{j,\nu}\|Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}+\|(I-\sigma_\lambda)\rho_\lambda f\|^{q_c}_{q_c}\\&\lesssim\sum_{j,\nu}\|Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f\|^{q_c}_{q_c}+\lambda\cdot(\log\lambda)^{-q_c}.\end{aligned}$$

If we combine (3.76) and the preceding two inequalities we conclude that we would obtain (3.33) and consequently finish the proof of the estimates in Theorem 1.5 if, for T as in (3.9), we could show that

$$Uf(x,j,\nu)=(Q^{\theta_0}_{j,\ell_0,\nu}\rho_\lambda f)(x),\tag{3.80}$$

satisfies

$$\|Uf\|_{\ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x}\lesssim\lambda^{\frac{1}{q_c}}T^{-1/2}\|f\|_{L^2(M)}$$

if all the sectional curvatures of (M,g) are $\le -\kappa_0^2$ for some $\kappa_0>0$ as well as

$$\|Uf\|_{\ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x}\lesssim\big(\lambda T^{-1}\big)^{\frac{1}{q_c}}\|f\|_{L^2(M)}$$

if all the sectional curvatures of (M,g) are nonpositive. Equivalently, this would be a consequence of the following

$$\begin{aligned}&\|UU^*\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x\to\ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x}\\&\quad\lesssim\begin{cases}\lambda^{2/q_c}T^{-1}, &\text{if all the sectional curvatures of } M \text{ are } \le -\kappa_0^2,\text{ some }\kappa_0>0,\\ \big(\lambda T^{-1}\big)^{2/q_c} &\text{if all the sectional curvatures of } M \text{ are nonpositive.}\end{cases}\end{aligned}\tag{3.81}$$

To prove the large height A_+ estimates (3.33) we split $\rho_\lambda\rho_\lambda^*=L_\lambda+G_\lambda$ as in (3.38). To prove (3.81), we require an additional dyadic decomposition, as well as taking into account the second microlocal decomposition afforded by the $\{Q^{\theta_0}_{j,\ell,\nu}\}$. To obtain this dyadic decomposition, we fix a Littlewood-Paley bump function $\beta\in C_0^\infty((1/2,2))$ satisfying $\sum_{k=-\infty}^{\infty}\beta(s/2^k)=1,\ s>0$. If we let $\beta_0(t)=1-\sum_{k=1}^{\infty}\beta(|t|/2^k)$, then $\beta_0\in C_0^\infty(\mathbb{R})$

equals one near the origin, and so plays the role of $a(t)$ in (3.37). So, analogous to (3.37), we set

$$L_{\lambda,T} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \beta_0(t)\, T^{-1}\hat{\Psi}(t/T)\, e^{it\lambda}\, e^{-itP}\, dt,$$

with, as in Lemma 3.1, $\Psi = |\rho|^2$.

If then $G_\lambda = G_{\lambda,T}$ is as in (3.35) with $a = \beta_0$, we use the dyadic decomposition given by

$$G_{\lambda,T,N} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \beta(|t|/N)\, T^{-1}\hat{\Psi}(t/T)\, e^{it\lambda}\, e^{-itP}\, dt, \tag{3.82}$$

so that, if we consider the resulting dyadic sum, we have

$$G_\lambda = \sum_{1\le 2^k = N \lesssim T} G_{\lambda,T,N}. \tag{3.83}$$

Then, if we set,

$$\big(W_N F\big)(x,j,\nu) = \sum_{j',\nu'} Q^{\theta_0}_{j,\ell_0,\nu} \circ G_{\lambda,T,N} \circ \big(Q^{\theta_0}_{j',\ell_0,\nu'}\big)^* F(x,j',\nu') \tag{3.84}$$

by (3.38), (3.80) and (3.83) we have

$$\begin{aligned}\big(UU^* F\big)(x,j,\nu) = \sum_{j',\nu'} \big(\, Q^{\theta_0}_{j,\ell_0,\nu} \circ L_{\lambda,T} \circ (Q^{\theta_0}_{j',\ell_0,\nu'})^*\big) F(x,j',\nu') \\ + \sum_{1\le N = 2^k \lesssim T} \big(W_N F\big)(x,j,\nu).\end{aligned} \tag{3.85}$$

The operator $L_{\lambda,T}$ satisfies the bounds in (3.39). If we use this along with the first inequality in (3.56) for $q = q_c$ followed by this bound and then the second inequality for $p = q_c'$ in (3.55) we obtain

$$\Big\| \sum_{j',\nu'} \big(\, Q^{\theta_0}_{j,\ell_0,\nu} \circ L_{\lambda,T} \circ (Q^{\theta_0}_{j',\ell_0,\nu'})^*\big) F(\,\cdot\,,j',\nu')\Big\|_{\ell^{q_c}_j \ell^{q_c}_\nu L^{q_c}_x} \le C\lambda^{2/q_c} T^{-1} \|F\|_{\ell^{q_c'}_{j'} \ell^{q_c'}_{\nu'} L^{q_c'}_x},$$

which agrees with the bounds in (3.81) for strictly negative sectional curvatures and is better than the bounds posited for nonpositive curvature.

To obtain the desired bounds for the last term in (3.85), we shall require the following result which plays here the role of Lemma 3.1.

Lemma 3.5. *Let $G_{\lambda,T,N}$ be as in* (3.82)*. Then for $\lambda \gg 1$ and $\theta_0 = \lambda^{-1/8}$ we have the uniform bounds*

$$\big\| Q^{\theta_0}_{j,\ell_0,\nu} G_{\lambda,T,N} \big\|_{L^1(M)\to L^\infty(M)} \le C_M T^{-1} \lambda^{\frac{n-1}{2}} N^{1-\frac{n-1}{2}},\ N \ge 1, \tag{3.86}$$

if (M,g) is a complete manifold of bounded geometry all of whose sectional curvatures are nonpositive and $T = c_0 \log\lambda$ is fixed with $c_0 = c_0(M) > 0$ sufficiently small Moreover, if we assume that all of the sectional curvatures are $\le -\kappa_0^2$, some $\kappa_0 > 0$, then we have the uniform bounds

$$\big\| Q^{\theta_0}_{j,\ell_0,\nu} G_{\lambda,T,N} \big\|_{L^1(M)\to L^\infty(M)} \le C_M T^{-1} \lambda^{\frac{n-1}{2}} N^{-m},\ \forall\, m \in \mathbb{N}. \tag{3.87}$$

Like those in Lemma 3.1, these two bounds follow from kernel estimates which we shall obtain at the end of this section.

Let us now see how we can use this lemma to see that the last term in the right side of (3.85) satisfies the bounds in (3.81).

We first notice that, by (3.82), the operators in (3.82) have $O(T^{-1}N)$ $L^2(M) \to L^2(M)$ operator norms. Thus, by (3.56) for $q = p = 2$ (almost orthogonality), we have, by (3.84)

$$\|W_N\|_{\ell^2_{j'}\ell^2_{\nu'}L^2_x \to \ell^2_j\ell^2_\nu L^2_x} = O(T^{-1}N). \tag{3.88}$$

If we use the second inequality in (3.56) for $p = 1$ along with (3.84) and (3.86), we also obtain that for T as above and $N = 2^k \geq 1$

$$\|W_N\|_{\ell^1_{j'}\ell^1_{\nu'}L^1_x \to \ell^\infty_j\ell^\infty_\nu L^\infty_x} = O(T^{-1}\lambda^{\frac{n-1}{2}}N^{1-\frac{n-1}{2}}) \tag{3.89}$$

if all of the sectional curvatures of M are nonpositive, as well as

$$\|W_N\|_{\ell^1_{j'}\ell^1_{\nu'}L^1_x \to \ell^\infty_j\ell^\infty_\nu L^\infty_x} = O(T^{-1}\lambda^{\frac{n-1}{2}}N^{-m}), \ \forall m \in \mathbb{N}, \tag{3.90}$$

if all of the principal curvatures are pinched below zero as in (3.87).

If we interpolate between (3.88) and (3.90) we obtain

$$\|W_N\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x \to \ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x} = O(T^{-1}\lambda^{2/q_c}N^{1-m}), \ \forall m \in \mathbb{N}, \tag{3.91}$$

if all of the sectional curvatures of M are $\leq -\kappa_0^2$, some $\kappa_0 > 0$. As a result, we can estimate the last term in (3.85) as follows

$$\begin{aligned}\Big\|\sum_{1\leq N=2^k\lesssim T} W_N F\Big\|_{\ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x} &\lesssim T^{-1}\lambda^{2/q_c}\sum_{1=N\lesssim T} N^{-1}\|F\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x} \\ &\lesssim T^{-1}\lambda^{2/q_c}\|F\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x},\end{aligned} \tag{3.92}$$

and so this term also satisfies the bounds in (3.81).

If we merely assume that all of the sectional curvatures are nonpositive, then (3.88), (3.89) and interpolation yield

$$\|W_N\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x \to \ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x} = O(T^{-1}\lambda^{2/q_c}N^{1-\frac{n-1}{n+1}}), \ \forall m \in \mathbb{N},$$

Since $\frac{n-1}{n+1} = \frac{2}{q_c}$, we therefore obtain

$$\Big\|\sum_{1\leq N=2^k\lesssim T} W_N F\Big\|_{\ell^{q_c}_j\ell^{q_c}_\nu L^{q_c}_x} \lesssim (\lambda T^{-1})^{2/q_c}\|F\|_{\ell^{q_c'}_{j'}\ell^{q_c'}_{\nu'}L^{q_c'}_x}, \tag{3.93}$$

as desired under this curvature assumption.

Inequalities (3.92), (3.93) along with the earlier bounds for the first term in (3.85) yield (3.81). As a result, except for needing to prove Lemmas 3.1 and 3.5, the proof of the bounds in Theorem 1.5 for $q = q_c$ is complete.

Since we also earlier obtained the bounds for $q \in (q_c, \infty]$, it only remains to obtain the bounds for $q \in (2, q_c)$. If the curvatures are assumed to be nonpositive, then the bounds in (1.12) for these exponents just follow from interpolating between the bounds for $q = q_c$ and the trivial L^2-estimate. So, to complete the proof of the Theorem, by (3.12) it suffices to show that for T as above we have

$$\|\sigma_\lambda \rho_\lambda\|_{L^2(M)\to L^q(M)} = O(T^{-1/2}\lambda^{\mu(q)}),\ q \in (2, q_c)$$

when all the sectional curvatures of M are pinched below zero and T is as above. By interpolating with the $q = q_c$ estimate that we just obtained, it suffices to show that, under these assumptions, we have

$$\|\sigma_\lambda \rho_\lambda\|_{L^2(M)\to L^q(M)} = O(T^{-1/2}\lambda^{\mu(q)}),\ q \in (2, \tfrac{2(n+2)}{n}]. \tag{3.94}$$

This just follows from the above arguments which gave us the bounds in (3.34) for $q = q_c$ under this curvature assumption if we use (3.77) in place of (3.76). The argument is a bit simpler since the norms in the left side of (3.77) are over M. So, we do not need for this case to split $M = A_- \cup A_+$ to handle the exponents in (3.94). □

3.2. Log-scale Strichartz estimates

In this section, let us see how we can follow the ideas in the previous section to adapt the proofs for the compact manifold case treated in [5, 32] to prove Theorem 1.3.

To align with the numerology in the previous section on the spectral projection estimates, throughout this section, we shall always assume the manifold is $(n-1)$-dimensional. Additionally, we will repeatedly use symbols such as σ_λ and Q_ν^θ; however, it is important to note that they represent different operators in this section.

To start, let us fix

$$\eta \in C_0^\infty((-1,1)) \quad \text{with} \quad \eta(t) = 1,\ \ |t| \le 1/2. \tag{3.95}$$

We shall consider the dyadic time-localized dilated Schrödinger operators

$$S_\lambda = \eta(t/T)e^{-it\lambda^{-1}\Delta_g}\beta(P/\lambda), \tag{3.96}$$

where $\beta \in C_0^\infty((1/2,2))$ as in (1.9) and $T = c_0 \log\lambda$ for some small constant c_0 we shall specify later. By changing scale in time, to prove Theorem 1.3 it suffices to show that if all of the sectional curvatures of M are nonpositive then for (p,q) satisfying (1.10) we have

$$\|S_\lambda f\|_{L_t^p L_x^q(M\times[0,T])} \le C\lambda^{\frac{1}{p}}\, \|f\|_{L^2(M)}, \quad \text{if} \ \ T = c_0\log\lambda. \tag{3.97}$$

Note that if we replace $[0,T]$ by $[0,1]$, then by using the analog of (1.9) for intervals $[0,\lambda^{-1}]$ along with a rescaling argument, we have for any complete manifold of bounded geometry

$$\|S_\lambda f\|_{L_t^p L_x^q(M\times[0,1])} \le C\lambda^{\frac{1}{p}}\, \|f\|_{L^2(M)}. \tag{3.98}$$

Now we shall introduce the auxiliary operators that allow us to use bilinear techniques. Let $\rho \in \mathcal{S}(\mathbb{R})$ satisfying (3.4), we define the local operators

$$\sigma_\lambda = \Big(\rho\big(\lambda^{1/2}|D_t|^{1/2} - P\big) + \rho\big(\lambda^{1/2}|D_t|^{1/2} + P\big)\Big)\ \tilde\beta(D_t/\lambda), \tag{3.99}$$

where

$$\tilde\beta \in C_0^\infty((1/8,8)) \quad \text{satisfies} \ \ \tilde\beta = 1 \ \text{on} \ \ [1/6,6]. \tag{3.100}$$

Note that by by Euler's formula,

$$\sigma_\lambda(x,t;y,s) = \frac{1}{2\pi^2}\iint e^{i(t-s)\tau} e^{ir\lambda^{1/2}\tau^{1/2}}\,\tilde\beta(\tau/\lambda)\,\hat\rho(r)\,\cos(rP)(x,y)\,drd\tau. \tag{3.101}$$

Thus by the support properties of $\hat\rho$ as in (3.4) and finite propagation speed, we have

$$\sigma_\lambda(x,t;y,s)=0 \quad \text{if} \;\; d_g(x,y)>r,\; r=\delta_1(1+\delta_2)<1. \tag{3.102}$$

For admissible pairs (p,q) as in (1.10), the local operators satisfy

$$\|(I-\sigma_\lambda)\circ S_\lambda f\|_{L^p_tL^q_x(M\times[0,T])} \le CT^{\frac1p-\frac12}\lambda^{\frac1p}\|f\|_2, \tag{3.103}$$

This is a straightforward generalization of Lemma 2.2 in [5] to all complete manifold of bounded geometry and all pairs (p,q) satisfying (1.10). The proof relies on the local dyadic Strichartz estimates (3.98) along with the spectral theorem and functional calculus for multiplier operators. We skip the details here and refer to [5] for more details.

For each fixed j, if we use the microlocal pseudodifferential operators $A_{j,\ell}$ defined in (3.17), we can write

$$\psi_j(x)(\sigma_\lambda F)(t,x) = \sum_{\ell=1}^K \bigl(A_{j,\ell}\circ\sigma_\lambda\bigr)(F)(t,x) + R_jF(t,x). \tag{3.104}$$

where as in (3.16)

$$A_{j,\ell}(x,y),\; R_j(x,t;y,s)=0, \quad \text{if } x\notin B(x_j,\delta) \text{ or } \; y\notin B(x_j,3\delta/2). \tag{3.105}$$

As before, by fixing $c_1>0$ small enough in the symbol of $A_{j,\ell}$ operators, we have the uniform bounds

$$R_j(x,t;y,s) = O(\lambda^{-N}),\; N=1,2,... \tag{3.106}$$

As in the previous section, the microlocal operators $A_{j,\ell}$ will be useful in the local harmonic analysis arguments we shall describe later. Note also that by (3.2), (3.105) and (3.106) we also have

$$\|RF\|_{L^p_tL^q_x(M\times[0,T])} \le C_{p,q,M}\,\|F\|_{L^2_{t,x}(M\times\mathbb{R})},\; 1\le p\le q\le\infty,\; \text{if },\; R=\sum_j R_j. \tag{3.107}$$

Note that for fixed ℓ_0, if we let $A=\sum_j A_{j,\ell_0}$ as in (3.21), in view of (3.3), (3.103), (3.104) and (3.107), in order to prove (3.97), it suffices to prove that if all the sectional curvatures of M are nonpositive

$$\|A\sigma_\lambda S_\lambda f\|_{L^p_tL^q_x(M\times[0,T])} \le C\lambda^{\frac1p}\|f\|_2. \tag{3.108}$$

And if we consider the vector-valued operators

$$\mathcal{A}H(x,t) = (A_{1,\ell_0}H(x,t),\, A_{2,\ell_0}H(x,t),\dots) \tag{3.109}$$

and argue as in (3.25)-(3.27), (3.108) would be a consequence of

$$\|\mathcal{A}\sigma_\lambda S_\lambda f\|_{L^p_tL^q_x\ell^q_j(\mathbb{N}\times M\times[0,T])} \le C\lambda^{\frac1p}\|f\|_2. \tag{3.110}$$

The operators $\mathcal{A}\sigma_\lambda S_\lambda$ play the role of the $\tilde S_\lambda$ operators in [5] and [32]. As in the previous section, the vector-valued approach will allow us to only have to carry out the local bilinear harmonic analysis in individual coordinate patches coming from the geodesic normal coordinates in the balls $B(x_j,2\delta)$.

Now let's set up the height decomposition that we shall use, throughout this section, we assume

$$\|f\|_{L^2(M)} = 1. \tag{3.111}$$

Let us define vector-valued sets

$$\begin{aligned} A_+ &= \{(x,t,j) \in M \times [0,T] \times \mathbb{N} : \, |(\mathcal{A}\sigma_\lambda S_\lambda f)(x,t,j)| \ge \lambda^{\frac{n-1}{4}+\varepsilon_1}\} \\ A_- &= \{(x,t,j) \in M \times [0,T] \times \mathbb{N} : \, |(\mathcal{A}\sigma_\lambda S_\lambda f)(x,t,j)| < \lambda^{\frac{n-1}{4}+\varepsilon_1}\}. \end{aligned} \tag{3.112}$$

Recall here that

$$(\mathcal{A}\sigma_\lambda S_\lambda f)(x,t,j) = A_{j,\ell_0}\sigma_\lambda S_\lambda f(x,t). \tag{3.113}$$

Due to the numerology of the powers of λ arising, the splitting occurs at height $\lambda^{\frac{n-1}{4}+\varepsilon_1}$, with $\frac{n-1}{4}$ same as the previous section. Here $\varepsilon_1 > 0$ is a small constant that may depend on the dimension $n-1$. As we shall see later, we can take $\varepsilon_1 = \frac{1}{100}$ for $n-1 \ge 3$ while for $n-1 = 2$, the choice of ε_1 depends on the exponent q for *admissible* pairs (p,q), with $\varepsilon_1 \to 0$ as $q \to \infty$.

In order to prove (3.110) on the set A_+, we shall require the following lemma

Lemma 3.6. *Let $S_{t,\lambda}$ denote the operator $\eta(t/T)\beta(P/\lambda)e^{-it\lambda^{-1}\Delta_g}$. Then if M has non-positive sectional curvatures and $T = c_0 \log\lambda$ with $c_0 = c_0(M) > 0$ sufficiently small, we have for $\lambda \gg 1$*

$$\|S_{t,\lambda}S^*_{s,\lambda}\|_{L^1(M)\to L^\infty(M)} \le C\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}\exp(C_M|t-s|). \tag{3.114}$$

We shall postpone the proof of this lemma until the end of this section and first see how we can use it to prove (3.110) on the set A_+.

Proof of (3.110) *on the set A_+.* We first note that, by (3.103), (3.109) and (3.111), we have

$$\|\mathcal{A}\sigma_\lambda S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)} \le \|\mathcal{A}S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)} + CT^{\frac{1}{p}-\frac{1}{2}}\lambda^{\frac{1}{p}}.$$

Since $p \ge 2$ for (p,q) as in (1.10), (3.110) would follow from

$$\|\mathcal{A}S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)} \le C\lambda^{\frac{1}{p}} + \tfrac{1}{2}\|\mathcal{A}\sigma_\lambda S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)}. \tag{3.115}$$

To prove this, similar to what was done in [32], we choose $g = g(x,t,j)$ such that

$$\begin{aligned} \|g\|_{L^{p'}_t L^{q'}_x \ell^{q'}_j(A_+)} = 1 \ \text{ and } \ &\|\mathcal{A}S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)} \\ &= \sum_j \iint \mathcal{A}S_\lambda f(x,t,j) \cdot \overline{\big(\mathbf{1}_{A_+} \cdot g(x,t,j)\big)}\, dxdt. \end{aligned} \tag{3.116}$$

Then, since we are assuming that $\|f\|_2 = 1$, by the Schwarz inequality

(3.117)
$$\begin{aligned}\|\mathcal{A}S_\lambda f\|^2_{L^p_t L^q_x \ell^q_j(A_+)} &= \Big(\int f(x)\cdot\overline{\big(S^*_\lambda\mathcal{A}^*\big)(\mathbf{1}_{A_+}\cdot g)(x)}\,dx\Big)^2\\ &\le \int |S^*_\lambda\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)(x)|^2\,dx\\ &= \sum_j \iint \big(\mathcal{A}S_\lambda S^*_\lambda\mathcal{A}^*\big)(\mathbf{1}_{A_+}\cdot g)(x,t,j)\,\overline{(\mathbf{1}_{A_+}\cdot g)(x,t,j)}\,dxdt\\ &= \sum_j \iint \big(\mathcal{A}\circ L_\lambda\circ\mathcal{A}^*\big)(\mathbf{1}_{A_+}\cdot g)(x,t,j)\,\overline{(\mathbf{1}_{A_+}\cdot g)(x,t,j)}\,dxdt\\ &\quad+\sum_j \iint \big(\mathcal{A}\circ G_\lambda\circ\mathcal{A}^*\big)(\mathbf{1}_{A_+}\cdot g)(x,t,j)\,\overline{(\mathbf{1}_{A_+}\cdot g)(x,t,j)}\,dxdt\\ &= I + II,\end{aligned}$$

where L_λ is the integral operator with kernel equaling that of $S_{t,\lambda}S^*_{s,\lambda}$ if $|t-s|\le 1$ and 0 otherwise, i.e,

$$L_\lambda(x,t;y,s) = \begin{cases} S_{t,\lambda}S^*_{s,\lambda}(x,y), \text{ if } |t-s|\le 1,\\ 0 \text{ otherwise.}\end{cases} \tag{3.118}$$

Since $p\ge 2$, it is straightforward to see that (3.98) and (3.114) yield

$$\|L_\lambda\|_{L^{p'}_t L^{q'}_x \to L^p_t L^q_x} = O(\lambda^{\frac{2}{p}}). \tag{3.119}$$

If we use this, along with Hölder's inequality, (3.27) and (3.116), we obtain for the term I in (3.117)

$$\begin{aligned}|I| &\le \|\mathcal{A}L_\lambda\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^p_t L^q_x \ell^q_j}\cdot\|\mathbf{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x}\\ &\lesssim \|L_\lambda\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^p_t L^q_x \ell^q_j}\cdot\|\mathbf{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x \ell^{q'}_j}\\ &\lesssim \lambda^{\frac{2}{p}}\|\mathcal{A}^*(\mathbf{1}_{A_+}\cdot g)\|_{L^{p'}_t L^{q'}_x \ell^{q'}_j}\cdot\|\mathbf{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x \ell^{q'}_j}\\ &\lesssim \lambda^{\frac{2}{p}}\|g\|^2_{L^{p'}_t L^{q'}_x \ell^{q'}_j(A_+)} = \lambda^{\frac{2}{p}}.\end{aligned} \tag{3.120}$$

To estimate II, note that if we choose c_0 small enough so that if C_M is the constant in (3.114)

$$\exp(2C_M T)\le\lambda^{\varepsilon_1}, \quad \text{if } T = c_0\log\lambda \text{ and } \lambda\gg 1.$$

Then, since $\eta(t)=0$ for $|t|\ge 1$, it follows (3.114) that

$$\|G_\lambda\|_{L^1(M\times\mathbb{R})\to L^\infty(M\times\mathbb{R})}\le C\lambda^{\frac{n-1}{2}+\varepsilon_1}.$$

As a result, by Hölder's inequality (3.27) and (3.116), we can repeat the arguments to estimate I to see that

$$\begin{aligned}|II| &\le C\lambda^{\frac{n-1}{2}}\lambda^{\varepsilon_1}\|\mathbf{1}_{A_+}\cdot g\|^2_{L^1_{t,x}\ell^1_j}\le C\lambda^{\frac{n-1}{2}}\lambda^{\varepsilon_1}\|g\|^2_{L^{p'}_t L^{q'}_x \ell^{q'}_j}\cdot\|\mathbf{1}_{A_+}\|^2_{L^p_t L^q_x \ell^q_j}\\ &= C\lambda^{\frac{n-1}{2}}\lambda^{\varepsilon_1}\|\mathbf{1}_{A_+}\|^2_{L^p_t L^q_x \ell^q_j}.\end{aligned} \tag{3.121}$$

If we recall the definition of A_+ in (3.112), we can estimate the last factor:

$$\|\mathbf{1}_{A_+}\|^2_{L^p_t L^q_x \ell^q_j} \le \big(\lambda^{\frac{n-1}{4}+\varepsilon_1}\big)^{-2} \|\mathcal{A}\sigma_\lambda S_\lambda f\|^2_{L^p_t L^q_x \ell^q_j(A_+)}.$$

Therefore,

$$|II| \lesssim \lambda^{-\varepsilon_1} \|\mathcal{A}\sigma_\lambda S_\lambda f\|^2_{L^p_t L^q_x \ell^q_j(A_+)} \le \big(\tfrac12 \|\mathcal{A}\sigma_\lambda S_\lambda f\|_{L^p_t L^q_x \ell^q_j(A_+)}\big)^2,$$

assuming, as we may, that λ is large enough.

If we combine this bound with the earlier one, (3.120) for I, we conclude that (3.115) is valid, which completes the proof of (3.110) on the set A_+. □

Next, we shall give the proof of (3.110) on the set A_-. This requires the use of local bilinear harmonic analysis. Following the approach in the previous section, in view of (3.104), it suffices to carry out the analysis in geodesic normal coordinates of each individual balls $B(x_j, 2\delta)$, since our assumption of bounded geometry ensures bounded transition maps and uniform bounds on derivatives of the metric. Also note that since the case $p=\infty, q=2$ in (3.97) simply follows from spectral theorem, to prove (3.110) on the set A_-, for the remaining of this section, we shall assume

$$(p,q) = (2, \tfrac{2(n-1)}{n-3}) \text{ if } n\ge 4, \text{ or } (n-1)(\tfrac12 - \tfrac1q) = \tfrac2p,\ 4<q<\infty \text{ if } n=3. \tag{3.122}$$

The condition $q>4$ is equivalent to $q>p$ when $n-1=2$, this will allow us to simplify some of the calculations to follow.

To set up the second microlocalization needed for the Schrödinger setting, let us fix j in (3.15), as well as $\ell_0 \in \{1,\dots,K\}$ and consider the resulting pseudodifferential cutoff, A_{j,ℓ_0}, which is a summand in (3.104). Its symbol then satisfies the conditions in (3.18). The resulting geodesic normal coordinates on $B(x_j,2\delta)$ vanish at x_j. We may also assume that $\xi_{j,\ell_0} = (0,\dots,0,1)$. Since we are fixing j and ℓ_0 for now, analogous to [32], let us simplify the notation a bit by letting

$$\tilde\sigma_\lambda = A_{j,\ell_0}\sigma_\lambda, \tag{3.123}$$

The Q^θ_ν operators constructed in the last section provide "directional" microlocalization. We also need a "height" localization since the characteristics of the symbols of our scaled Schrödinger operators lie on paraboloids. The variable coefficient operators that we shall use are analogs of ones that are used in the study of Fourier restriction problems involving paraboloids.

To construct these, choose $b\in C^\infty_0(\mathbb{R})$ supported in $|s|\le 1$ satisfying $\sum_{-\infty}^{\infty} b(s-\ell)\equiv 1$. We then define the compound symbols $Q^\theta_\ell = Q^\theta_{j,\ell_0,\ell}$ and associated "height" operators by

$$\begin{aligned} Q^\theta_\ell(x,y,\xi) = \tilde{\tilde\psi}(y) b(\theta^{-1}\lambda^{-1}(p(x,\xi)-\lambda\kappa^\theta_\ell)),\ \kappa^\theta_\ell = 1+\theta\ell, \quad |\ell|\lesssim\theta^{-1}, \\ \text{and } Q^\theta_\ell h(x) = (2\pi)^{-(n-1)} \iint e^{i(x-y)\cdot\xi} Q^\theta_\ell(x,y,\xi)\, h(y)\, d\xi dy. \end{aligned} \tag{3.124}$$

Here $\tilde{\tilde\psi}\in C^\infty_0$ is supported in $|x|<2\delta$ which equals one when $|x|\le 3\delta/2$, as defined in (3.51).

Unlike in the earlier works [32, 5], the height operators here are defined in local coordinates and have cutoffs in y variable in order to avoid issues at infinity since M is not assumed to be compact. In the compact case, the analogous height operator can be simply defined using spectral multipliers, see e.g., [5, 32]. These operators microlocalize $p(x,\xi)$ to intervals of size $\approx \theta\lambda$ about "heights" $\lambda\kappa_\ell^\theta \approx \lambda$. By a simple integration by parts argument, if $Q_\ell^\theta(x,y)$ is the kernel of this operator then

$$Q_\ell^\theta(x,y) = O(\lambda^{-N})\,\forall\, N, \quad \text{if } \; d_g(x,y) \geq C_0\theta, \tag{3.125}$$

for a fixed constant C_0 if $\theta \in [\lambda^{-1/2+\varepsilon}, 1]$ with $\varepsilon > 0$.

For $\nu = (\nu', \ell) = (\theta k, \theta\ell) \in \theta\mathbb{Z}^{2(n-2)+1}$ we now define the cutoffs that we shall use:

$$Q_\nu^\theta = Q_{\nu'}^\theta \circ Q_\ell^\theta. \tag{3.126}$$

where $Q_{\nu'}^\theta$ are the directional microlocalization operators defined in (3.51). Both $Q_{\nu'}^\theta$ and Q_ℓ^θ operators here depend on our fixed j, ℓ_0, and as in (3.52), due to the way they are constructed, for small enough $\delta_0 > 0$ the principle symbol $q_\nu^\theta(x,y,\xi)$ of the Q_ν^θ operators satisfy

$$\begin{aligned} q_\nu^\theta(x,y,\xi) = q_\nu^\theta(z,y,\eta), \; (z,\eta) = \Phi_t(x,\xi), \qquad & \\ \text{if } \; \text{dist}\,((x,\xi), \text{supp}\, A_{j,\ell_0}) \leq \delta_0 \,\text{and} \;\; |t| \leq 2\delta_0. & \end{aligned} \tag{3.127}$$

The symbol of Q_ν^θ operators in (3.126) vanishes when either x or y is outside the 2δ-ball about the origin in our coordinates for Ω. By (3.101) (3.105) and (3.123), we can fix δ_1 in (3.4) small enough so that we also have, analogous to (2.39) in [32],

$$\begin{aligned} \tilde\sigma_\lambda = \sum_\nu \tilde\sigma_\lambda Q_\nu^{\theta_0} + R, \quad R = R_{\lambda,j,\ell_0}, \;\; \tilde\sigma_\lambda = A_{j,\ell_0}\sigma_\lambda, & \\ \text{where} \;\; R(x,t;y,s) = O(\lambda^{-N}), \,\forall\, N & \\ \text{and } R(x,t;y,s) = 0, \; \text{if } x \notin B(x_j, 2\delta) \text{ or } y \notin B(x_j, 2\delta), & \end{aligned} \tag{3.128}$$

with bounds for the remainder kernel independent of j. Here unlike in the previous section we take $\theta_0 = \lambda^{-\varepsilon_0}$ for some small constant ε_0 that we shall specify later, the choice of ε_0 depend on the dimension $n-1$.

Let us now point out straightforward but useful properties of our operators. First, by (3.105), (3.128) and the support properties of $\tilde\psi$, $\tilde{\tilde\psi}$, we have

$$\begin{aligned} \tilde\sigma_\lambda Q_\nu^{\theta_0} H = \mathbf{1}_{B(x_j,2\delta)} \cdot \tilde\sigma_\lambda Q_\nu^{\theta_0}\big(\mathbf{1}_{B(x_j,2\delta)} \cdot H\big), \; Q_\nu^{\theta_0} = Q_{j,\ell_0,\nu}^{\theta_0} & \\ \text{and} \;\; RH = \mathbf{1}_{B(x_j,2\delta)} \cdot R\big(\mathbf{1}_{B(x_j,2\delta)} \cdot H\big), \; R = R_{\lambda,j,\ell_0}. & \end{aligned} \tag{3.129}$$

Also, we have the uniform bounds

$$\begin{aligned} \|Q_\nu^{\theta_0} h\|_{\ell_\nu^q L^q(M)} &\lesssim \|h\|_{L^q(M)}, \; 2 \leq q \leq \infty \\ \|\sum_{\nu'} (Q_\nu^{\theta_0})^* H(\nu', \cdot\,)\|_{L^p(M)} &\lesssim \|H\|_{\ell_{\nu'}^p L^p(M)}, \; 1 \leq p \leq 2. \end{aligned} \tag{3.130}$$

The second estimate follows via duality from the first. The first one is the analog of (2.42) in [32]. By interpolation, one just needs to verify that the estimate holds for the two endpoints, $p = 2$ and $p = \infty$. The former follows via an almost orthogonality argument, and the latter from the fact that for each x the symbols vanish outside of cubes of sidelength $\theta\lambda$ and $|\partial_\xi^\gamma Q_\nu^\theta(x,y,\xi)| = O((\lambda\theta)^{-|\gamma|})$, thus it is not hard to show we have the uniform bounds

$$\sup_{x\in B(x_j,2\delta)} \int_{B(x_j,2\delta)} |Q_\nu^{\theta_0}(x,y)|\,dy \le C.$$

Note that if we use (3.130), the support properties of the $Q_\nu^{\theta_0}$ operators and the finite overlap of the balls $\{B(x_j,2\delta)\}$ we obtain for our fixed $\ell_0 = 1,\dots,K$

$$\begin{aligned} &\big(\sum_{j,\nu} \|Q^{\theta_0}_{j,\ell_0,\nu} h\|^q_{L^q(M)}\big)^{1/q} \lesssim \|h\|_{L^q(M)},\ 2\le q\le\infty \\ &\|\sum_{j',\nu'} (Q^{\theta_0}_{j',\ell_0,\nu'})^* H(\nu',j',\,\cdot\,)\|_{L^p(M)} \lesssim \|H\|_{\ell^p_{\nu'}L^p(M)},\ 1\le p\le 2. \end{aligned} \tag{3.131}$$

In addition to this inequality and (3.103), we shall also require the following commutator bounds

$$\big\|(A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} - A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda)H\|_{L^p_tL^q_x(M\times[0,T])} \le C_q\lambda^{\frac1p-\frac12+2\varepsilon_0}\|H\|_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}, \tag{3.132}$$

assuming that δ, as well as δ_1 in (3.4) are fixed small enough.

To see this, if we use the auxiliary operator $\tilde A_{j,\ell_0}$ and Young's inequality as in the previous section, and apply Bernstein inequality in time, it suffices to show

$$\big\|(A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} - A_{j,\ell_0}Q^{\theta_0}_{j,\ell_0,\nu}\sigma_\lambda)H\|_{L^2_tL^2_x(M\times[0,T])} \le C_q\lambda^{-1+2\varepsilon_0}\|H\|_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}, \tag{3.133}$$

since $(n-1)(\frac12-\frac1q)+\frac12-\frac1p = \frac1p+\frac12$ for (p,q) satisfying (3.122).

This follows from the proof of (2.59) in [32] since, by (3.18), $A_{j,\ell_0}f$ vanishes outside $B(x_j,2\delta)$ and the two operators in (3.132) vanish when acting on functions vanishing on $B(x_j,2\delta)$. This allows one to prove (3.133), exactly as in [33], by just working in a coordinate chart ($B(x_j,2\delta)$ here) and, to obtain the inequality using (3.127) and Egorov's theorem related to the properties of the half wave operator e^{itP} in this local coordinate.

Next, as in [5] and [32], if $H = S_\lambda f$, we note that we can write for θ_0 and $\tilde\sigma_\lambda$ as in (3.123)

$$\big(\tilde\sigma_\lambda H\big)^2 = \sum_{\nu,\nu'} \big(\tilde\sigma_\lambda Q_\nu^{\theta_0}H\big)\cdot\big(\tilde\sigma_\lambda Q_{\nu'}^{\theta_0}H\big) + O(\lambda^{-N}\|H\|^2_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}),\ \forall N. \tag{3.134}$$

Recall that the $\nu = \theta_0\cdot\mathbb{Z}^{2(n-2)+1}$ index a $\lambda^{-\varepsilon_0}$-separated lattice in $\mathbb{R}^{2(n-2)+1}$. If we repeat the Whitney decomposition and the arguments in (3.62)-(3.69) as in the previous section. We can write

$$(\tilde\sigma_\lambda H)^2 = \Upsilon^{\text{diag}}_{j,\ell_0}(H) + \Upsilon^{\text{far}}_{j,\ell_0}(H) \tag{3.135}$$

when $n-1\ge 4$. And when $n-1=3$, we further decompose $\Upsilon^{\text{diag}}(H)$ and write

$$(\tilde\sigma_\lambda H)^4 \lesssim 2\overline{\Upsilon}^{\text{diag}}_{j,\ell_0}(H) + 2\overline{\Upsilon}^{\text{far}}_{j,\ell_0}(H) + 2(\Upsilon^{\text{far}}(H))^2. \tag{3.136}$$

Here the operators $\Upsilon^{\text{diag}}_{j,\ell_0}, \Upsilon^{\text{far}}_{j,\ell_0}, \overline{\Upsilon}^{\text{diag}}_{j,\ell_0}$ and $\overline{\Upsilon}^{\text{far}}_{j,\ell_0}$ are defined exactly in the same manner as in (3.63), (3.64) and (3.68), except that they now act on functions that also depend on the time variable. And we are treating the case $n-1=3$ separately as $\frac{2(n-1)}{n-3}=6$ when

$n-1=3$, which requires a slight modification when we use bilinear ideas from [47]. The remaining case $n-1=2$ is analogous to $n-1=3$, we shall briefly outline the necessary arguments in the end of this section.

We shall need the the following variant of Lemma 3.1 in [32]

Lemma 3.7. *Let $\theta_0=\lambda^{-\varepsilon_0}$ with $\lambda\gg 1$. If $n-1\ge 4$, $q_e=\frac{2(n-1)}{n-3}$ and $Q_\nu^{\theta_0}=Q^{\theta_0}_{j,\ell_0,\nu}$ as in* (3.126),

$$\begin{aligned}&\int\Big(\sum_j\int|\Upsilon^{diag}_{j,\ell_0}(H)|^{\frac{q_e}{2}}\,dx\Big)^{\frac{2}{q_e}}dt\\ &\le\int\Big(\sum_{j,\nu}\big\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}H\big\|^{q_e}_{L^{q_e}_x(B(x_j,2\delta))}\Big)^{\frac{2}{q_e}}dt+O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}).\end{aligned}\tag{3.137}$$

Additionally, for $n-1=3$ we have

$$\begin{aligned}&\int\Big(\sum_j\int|\overline{\Upsilon}^{diag}_{j,\ell_0}(H)|^{\frac{q_e}{4}}\,dx\Big)^{\frac{2}{q_e}}dt\\ &\le\int\Big(\sum_{j,\nu}\big\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}H\big\|^{q_e}_{L^{q_e}_x(B(x_j,2\delta))}\Big)^{\frac{2}{q_e}}dt+O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}).\end{aligned}\tag{3.138}$$

In the above and what follows $O(\lambda^{\mu-})$ denotes $O(\lambda^{\mu-\varepsilon})$ for some $\varepsilon>0$. As we shall see later in the proof, unlike Lemma 3.2 in the previous section, we can not fix j,ℓ_0 here. It is crucial that the $\ell_j^{\frac{q_e}{2}}$ norm is taken inside the dt integral. As in [32], since the Q^θ_ν operators are time independent, the main step in the proof of (3.137) is to show that for arbitrary $h_\nu,h_{\tilde\nu}$, which may depend on ν and $\tilde\nu$, we have

$$\begin{aligned}\Big\|\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}}Q^{\theta_0}_{\nu,j,\ell_0}h_\nu\cdot Q^{\theta_0}_{\tilde\nu,j,\ell_0}h_{\tilde\nu}\Big\|_{L^{q_e/2}_x}\le C\Big(&\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}}\|Q^{\theta_0}_{\nu,j,\ell_0}h_\nu\cdot Q^{\theta_0}_{\tilde\nu,j,\ell_0}h_{\tilde\nu}\|^{q_e/2}_{L^{q_e/2}_x}\Big)^{2/q_e}\\ &+O\big(\lambda^{-N}\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}}\|h_\nu\|_{L^1_x}\|h_{\tilde\nu}\|_{L^1_x}\big),\ \ \forall N.\end{aligned}\tag{3.139}$$

The constant C here is independent of j,ℓ_0. This result is analogous to (3.20) in [32] and follows from the same proof provided there. Similarly, the proof of (3.138) follows from a variant of (3.138) involving the product of four $Q^{\theta_0}_{j,\ell_0,\nu}$ operators.

If we fix δ as well as δ_1,δ_2 in (3.4) small enough, then we can use Lee's [38] bilinear oscillatory integral theorem and repeat the proof of Lemma 3.2 in [5] as well as the arguments in (3.38)-(3.42) of [32] for the case $n-1=3$ to obtain the following.

Lemma 3.8. *Let $\Upsilon^{far}_{j,\ell_0}(H),\overline{\Upsilon}^{far}_{j,\ell_0}(H)$ be as above with $\theta_0=\lambda^{-\varepsilon_0}$ and $q=\frac{2(n+2)}{n}$. Then for all $\varepsilon>0$ there is a $C_\varepsilon=C(\varepsilon,M)$ so that*

$$\int_M|\Upsilon^{far}_{j,\ell_0}(H)|^{q/2}\,dxdt\le C_\varepsilon\lambda^{1+\varepsilon}\left(\lambda^{1-\varepsilon_0}\right)^{\frac{n-1}{2}(q-\frac{2(n+1)}{n-1})}\|H\|^q_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}.\tag{3.140}$$

Similarly, for $n-1=3$ and $q=\frac{2(n+2)}{n}$,

$$\int_M|\overline{\Upsilon}^{far}_{j,\ell_0}(H)|^{\frac{q}{4}}\,dxdt\le C_\varepsilon\lambda^{1+\varepsilon}\left(\lambda^{1-\varepsilon_0}\right)^{\frac{n-1}{2}(q-\frac{2(n+1)}{n-1})}\|H\|^q_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}.\tag{3.141}$$

We now have collected the main ingredients that we need to prove the critical low height estimates.

Proof of (3.110) *on the set* A_-. Let us assume that $n-1 \geq 4$ and thus it suffices to consider $p=2, q=q_e=\frac{2(n-1)}{n-3}$. A main step in the proof of the A_- estimates then is to obtain the analog of (2.45) in [32]. We shall do so largely by repeating its proof, which we do so for the sake of completeness in order to note the small changes needed to take into account that, unlike (2.45) in [32], (3.110) here is a vector valued inequality. As noted before, we have taken this framework to help us exploit our assumption of bounded geometry, and, in particular, the fact that the doubles of the balls in our covering of M, $\{B(x_j, 2\delta)\}$, have uniformly finite overlap.

We first note that if $q=\frac{2(n+2)}{n}<q_e$, then by (3.113) and (3.135) for our fixed j, ℓ_0 we have for $H=S_\lambda f$

$$\begin{aligned}
&\big|\big(\mathcal{A}\sigma_\lambda(H)(x,t,j)\big)^2\big|^{q_e/2} = \big|\big(A_{j,\ell_0}(\sigma_\lambda H)(x,t,j)\big)^2\big|^{q_e/2}\\
&= |A_{j,\ell_0}(\sigma_\lambda H)(x,t)\cdot A_{j,\ell_0}(\sigma_\lambda H)(x,t)|^{\frac{q_e-q}{2}}\,\big|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(H)(x,t)+\Upsilon^{\mathrm{far}}_{j,\ell_0}(H)(x,t)\big|^{q/2}\\
&\leq |A_{j,\ell_0}(\sigma_\lambda H)(x,t)\cdot A_{j,\ell_0}(\sigma_\lambda H)(x,t)|^{\frac{q_e-q}{2}}\,2^{q/2}\big(|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(H)(x,t)|^{q/2}+|\Upsilon^{\mathrm{far}}_{j,\ell_0}(H)(x,t)|^{q/2}\big).
\end{aligned}$$

Thus if A_- is as in (3.112),

$$\begin{aligned}
&\|\mathcal{A}\sigma_\lambda H\|^2_{L^2_t L^{q_e}_x(A_-)} = \int\Big(\int\sum_j \mathbf{1}_{A_-}(x,t,j)\;|A_{j,\ell_0}(\sigma_\lambda H)\cdot A_{j,\ell_0}(\sigma_\lambda H)|(x,t)|^{q_e/2}\,dx\Big)^{\frac{2}{q_e}}dt\\
&\lesssim \int\Big(\sum_j\int\big[\mathbf{1}_{A_-}(x,t,j)\,|A_{j,\ell_0}(\sigma_\lambda H)(x)\cdot A_{j,\ell_0}(\sigma_\lambda H)(x)|^{\frac{q_e-q}{2}}\big]\,|\Upsilon^{\mathrm{diag}}_{j,\ell_0}(H)(x)|^{q/2}\Big)^{\frac{2}{q_e}}dt\\
&+\int\Big(\sum_j\int\big[\mathbf{1}_{A_-}(x,t,j)\,|A_{j,\ell_0}(\sigma_\lambda H)(x)\cdot A_{j,\ell_0}(\sigma_\lambda H)(x)|^{\frac{q_e-q}{2}}\big]\,|\Upsilon^{\mathrm{far}}_{j,\ell_0}(H)(x)|^{q/2}dx\Big)^{\frac{2}{q_e}}dt\\
&= C(I+II).
\end{aligned}$$

To estimate II, first note that by Hölder's inequality

(3.142)
$$\begin{aligned}
II &\lesssim \|\mathbf{1}_{A_-}(x,t,j)A_{j,\ell_0}(\sigma_\lambda H)(x,t)\|^{\frac{2(q_e-q)}{q_e}}_{L^\infty(A_-)}\cdot\Big(\int\Big(\sum_j\int|\Upsilon^{\mathrm{far}}_{j,\ell_0}(H)|^{\frac{q}{2}}dx\Big)^{\frac{2}{q_e}}dt\Big)\\
&\lesssim T^{1-\frac{2}{q_e}}\|\mathbf{1}_{A_-}(x,t,j)A_{j,\ell_0}(\sigma_\lambda H)(x,t)\|^{\frac{2(q_e-q)}{q_e}}_{L^\infty(A_-)}\cdot\Big(\sum_j\int\int|\Upsilon^{\mathrm{far}}_{j,\ell_0}(H)|^{\frac{q}{2}}dxdt\Big)^{\frac{2}{q_e}},
\end{aligned}$$

Recall that by (3.112) and (3.113),

$$|\mathbf{1}_{A_-}(x,t,j)A_{j,\ell_0}(\sigma_\lambda H)(x,t)|\lesssim\lambda^{\frac{n-1}{4}+\varepsilon_1}.$$

Thus by (3.140)

(3.143)
$$\begin{aligned}
II &\leq T^{1-\frac{2}{q_e}}\lambda^{(\frac{n-1}{4}+\varepsilon_1)(\frac{2(q_e-q)}{q_e})}\\
&\quad\cdot\Big(\lambda^{1+\varepsilon}\big(\lambda^{1-\varepsilon_0}\big)^{\frac{n-1}{2}(q-\frac{2(n+1)}{n-1})}\Big)^{\frac{2}{q_e}}\Big(\sum_j\|H\|^q_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}\Big)^{\frac{2}{q_e}}.
\end{aligned}$$

If we take $\varepsilon_0, \varepsilon_1$ and ε to be small enough, e.g., $\varepsilon = \varepsilon_1 = \frac{1}{100}$ and $\varepsilon_0 = \frac{1}{2n+2}$, it is straightforward to check that

$$II \lesssim \lambda^{1-}\Big(\sum_j \|H\|^q_{L^2_{t,x}(B(x_j,2\delta)\times\mathbb{R})}\Big)^{\frac{2}{q_e}} \lesssim \lambda^{1-}\|H\|^{\frac{2q}{q_e}}_{L^2_{t,x}} = O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}). \tag{3.144}$$

Here we also used the fact that $\|H\|^2_{L^2_{t,x}}$ dominates $\|H\|^{\frac{2q}{q_e}}_{L^2_{t,x}}$ since $q_e > q$ and $\|H\|_{L^2_{t,x}} \approx T$ since $H = S_\lambda f$, $\|f\|_2 = 1$ and $e^{-it\lambda^{-1}\Delta_g}$ is a unitary operator on L^2_x.

To control I, as in [32], we use Hölder's inequality followed by Young's inequality along with (3.137) to get

(3.145)

$$\begin{aligned}
I = &\int \Big(\sum_j \int \big[\mathbf{1}_{A_-}(x,t,j)\,|A_{j,\ell_0}(\sigma_\lambda H)(x)\cdot A_{j,\ell_0}(\sigma_\lambda H)(x)|^{\frac{q_e-q}{2}}\big]\,|\Upsilon^{\text{diag}}_{j,\ell_0}(H)(x)|^{q/2}\Big)^{\frac{2}{q_e}} dt \\
\le &\int \Big(\|\mathbf{1}_{A_-}\cdot \mathcal{A}\sigma_\lambda H\cdot \mathcal{A}\sigma_\lambda H\|^{\frac{q_e-q}{2}}_{\ell_j^{\frac{q_e}{2}} L_x^{\frac{q_e}{2}}}\,\big(\sum_j \int |\Upsilon^{\text{diag}}_{j,\ell_0}(H)|^{\frac{q_e}{2}}dx\big)^{\frac{q}{q_e}}\Big)^{\frac{2}{q_e}} dt \\
\le &\|\mathbf{1}_{A_-}\mathcal{A}\sigma_\lambda H\cdot \mathcal{A}\sigma_\lambda H\|^{\frac{q_e-q}{q_e}}_{L^1_t\ell_j^{\frac{q_e}{2}} L_x^{\frac{q_e}{2}}}\,\Big(\int\big(\sum_j \int |\Upsilon^{\text{diag}}_{j,\ell_0}(H)|^{\frac{q_e}{2}}dx\big)^{\frac{2}{q_e}}dt\Big)^{\frac{q}{q_e}} \\
\le &\tfrac{q_e-q}{q_e}\|\mathcal{A}\sigma_\lambda H\|^2_{L^2_t\ell^{q_e}_j L^{q_e}_x(A_-)} + \tfrac{q}{q_e}\Big(\int\big(\sum_j \int |\Upsilon^{\text{diag}}_{j,\ell_0}(H)|^{\frac{q_e}{2}}dx\big)^{\frac{2}{q_e}}dt\Big) \\
\le &\tfrac{q_e-q}{q_e}\|\mathcal{A}\sigma_\lambda H\|^2_{L^2_t\ell^{q_e}_j L^{q_e}_x(A_-)} + C\Big(\int\big(\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}H\|^{q_e}_{L^{q_e}_x(B(x_j,2\delta))}\big)^{\frac{2}{q_e}}dt\Big) \\
&\quad + O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}).
\end{aligned}$$

If we combine the above two estimate, we have the follow which is the analog of proposition 2.3 in [32].

Proposition 3.9. *Fix a complete $n-1 \ge 2$ dimensional Riemannian manifold (M,g) of bounded geometry and assume that* (3.111) *is valid. If $H = S_\lambda f$ is as in* (3.96), *(p,q) satisfies* (3.122) *and $\varepsilon_0, \varepsilon_1$ in the definition of A_- and θ_0 are small enough, we have*

$$\|\mathcal{A}\sigma_\lambda H\|_{L^p_t L^q_x \ell^q_j(A_-)} \lesssim \Big(\int\big(\sum_{j,\nu}\|A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}H\|^q_{L^q_x(B(x_j,2\delta))}\big)^{\frac{2}{q}}dt\Big)^{\frac12} + \lambda^{\frac1p -}. \tag{3.146}$$

The case $n-1 \ge 4$ in (3.146) directly follows from the above estimates for I and II. One can similarly use (3.138) and (3.141) and modify the arguments in [32] to handle the case when $n-1=3$.

The arguments for $n-1=2$ and general (p,q) in (3.122) is similar to the case $n-1=3$. Recall that when $n-1=3$, $q_e = \frac{2(n-1)}{n-3} = 6 \in [2^2, 2^3]$. As a result, an additional round of Whitney decomposition is needed for $(\Upsilon^{\text{diag}})^2$ in order to get the desired estimate (3.138) in Lemma 3.7. When $n-1=2$, q can be arbitrary large, if $q \in [2^{k+1}, 2^{k+2}]$ for some $k \in \mathbb{N}^+$, then one can repeat the arguments for $n-1=3$ in [32] k times, the resulting diagonal term will involve a product of 2^{k+1} terms of involving $A_{j,\ell_0}\sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu}H$ and will satisfy the analog of (3.138) with $q_e/4$ replaced by $q/2^k$. Each iteration of

Whitney decomposition also generates off-diagonal terms, which can be treated using bilinear oscillatory integral estimates. However, as $q \to \infty$, unlike (3.143), we need to take ε_0 and ε_1 to be small enough depending on q, instead of some fixed small constant.

Thus to prove (3.110), it remains to control the first term on the right side of (3.146). By (3.132) along with the fact that $\ell^2 \subset \ell^q$ if $q \geq 2$, we have

(3.147)
$$\begin{aligned}
&\Big(\int \big(\sum_{j,\nu} \big\| A_{j,\ell_0} \sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} H \big\|^q_{L^q_x(B(x_j,2\delta))} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\lesssim \Big(\int \big(\sum_{j,\nu} \big\| A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu} \sigma_\lambda H \big\|^q_{L^q_x(B(x_j,2\delta))} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\quad + \Big(\int \big(\sum_{j,\nu} \big\| (A_{j,\ell_0} \sigma_\lambda Q^{\theta_0}_{j,\ell_0,\nu} - A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu}) \sigma_\lambda H \big\|^q_{L^q_x(B(x_j,2\delta))} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\lesssim \Big(\int \sum_{j,\nu} \big\| A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu} \sigma_\lambda H \big\|^q_{L^q_x(B(x_j,2\delta))} \big)^{\frac{2}{q}} dt \Big)^{\frac12} + \lambda^{\frac1p - \frac12 + 2\varepsilon_0} \Big(\sum_\nu \| H \|^2_{L^2_{t,x}} \Big)^{\frac12}.
\end{aligned}$$

Since the number of choices of ν is $O(\lambda^{(2n-3)\varepsilon_0})$ and H is independent of ν, the second term in the right is dominated by $\lambda^{(n-\frac32)\varepsilon_0} \| H \|_{L^2_{t,x}}$. Thus if we choose $\varepsilon_0 < \frac{1}{2n+1}$, the second term on the right side of (3.147) is $O(\lambda^{\frac1p -})$.

Next recall that $H = S_\lambda f$ and $\|f\|_2 = 1$, if we use (3.20), (3.131), followed by (3.103), we can control the term in the right as follows

$$\begin{aligned}
&\Big(\int \sum_{j,\nu} \big\| A_{j,\ell_0} Q^{\theta_0}_{j,\ell_0,\nu} \sigma_\lambda S_\lambda f \big\|^q_{L^q_x(B(x_j,2\delta))} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\leq \Big(\int \sum_{j,\nu} \big\| Q^{\theta_0}_{j,\ell_0,\nu} \sigma_\lambda S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\leq \Big(\int \sum_{j,\nu} \big\| Q^{\theta_0}_{j,\ell_0,\nu} S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} + \Big(\int \sum_{j,\nu} \big\| Q^{\theta_0}_{j,\ell_0,\nu} (I-\sigma_\lambda) S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\leq \Big(\int \sum_{j,\nu} \big\| Q^{\theta_0}_{j,\ell_0,\nu} S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} + \Big(\int \big\| (I-\sigma_\lambda) S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} \\
&\leq \Big(\int \sum_{j,\nu} \big\| Q^{\theta_0}_{j,\ell_0,\nu} S_\lambda f \big\|^q_{L^q_x} \big)^{\frac{2}{q}} dt \Big)^{\frac12} + \lambda^{\frac1p} T^{\frac1p - \frac12}.
\end{aligned} \tag{3.148}$$

If we combine (3.146) and the preceding two inequalities we conclude that we would obtain (3.110) and consequently finish the proof of the estimates in Theorem 1.3 if, for (p,q) as in (3.122) and T as in (3.96), we could show that

$$Uf(t,x,j,\nu) = (Q^{\theta_0}_{j,\ell_0,\nu} S_\lambda f)(x,t), \tag{3.149}$$

satisfies

$$\|Uf\|_{L^p_t \ell^q_j \ell^q_\nu L^q_x} \lesssim \lambda^{\frac1q} \|f\|_{L^2(M)}. \tag{3.150}$$

We shall require the following lemma

Lemma 3.10. *Fix t, j, ℓ_0, ν, let $K_{t,\lambda}$ denote the operator*

$$\eta(t/T)Q^{\theta_0}_{j,\ell_0,\nu}\beta(P/\lambda)e^{-it\lambda^{-1}\Delta_g}.$$

Then if (M,g) is a complete manifold of bounded geometry all of whose sectional curvatures are nonpositive and $T = c_0 \log\lambda$ is fixed with $c_0 = c_0(M) > 0$ sufficiently small, we have for $\lambda \gg 1$

$$\|K_{t,\lambda}K^*_{s,\lambda}\|_{L^1(M)\to L^\infty(M)} \le C\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}. \tag{3.151}$$

We shall postpone the proof of this lemma until the end of this section and first see how we can use it to prove (3.150). By applying the abstract theorem of Keel-Tao [37] and a simple rescaling argument, we would have (3.150) if

$$\|Uf(t,\cdot)\|_{\ell^2_j\ell^2_\nu L^2_x} \le C\|f\|_{L^2_x}, \tag{3.152}$$

and

$$\|U(t)U^*(s)G\|_{\ell^\infty_j\ell^\infty_\nu L^\infty_x} \le C\lambda^{\frac{n-1}{2}}\,|t-s|^{-\frac{n-1}{2}}\,\|G\|_{\ell^1_j\ell^1_\nu L^1_x}, \tag{3.153}$$

with

$$\begin{aligned}\big(U(t)U^*(s)G\big)(x,j,\nu) &= \\ &= \eta(t/T)\sum_{j',\nu'}\eta(s/T)\Big[\big(Q^{\theta_0}_{j,\ell_0,\nu}e^{-i(t-s)\lambda^{-1}\Delta_g}(Q^{\theta_0}_{j',\ell_0,\nu'})^*\big)G(\,\cdot\,,j',\nu')\Big](x).,\end{aligned} \tag{3.154}$$

It is not hard to check that (3.152) follows from (3.131) with $p = 2$ and the fact that $e^{-it\lambda^{-1}\Delta_g}$ is unitary, and (3.153) follows from the estimate (3.151). □

3.3. Kernel estimates

Let us start out by proving the bounds in Lemmas 3.1 and 3.5 that were used to prove the spectral projection estimates in Theorem 1.5.

Proof of Lemma 3.1. We first note that since P is nonnegative, if we replace e^{-itP} in (3.35) with e^{itP}, then, by (1.4), the resulting operator maps $L^1(M) \to L^\infty(M)$ with norm $O(\lambda^{-N})\,\forall N$. Thus, by Euler's formula, if

$$\tilde G_\lambda(x,y) = \int_{-\infty}^{\infty}(1-a(t))\,T^{-1}\hat\Psi(t/T)\,\big(\cos t\sqrt{-\Delta_g}\big)(x,y)\,dt, \tag{3.155}$$

it suffices to show that, under the assumptions of Lemma 3.1, we have

$$\tilde G_\lambda(x,y) = O(\lambda^{\frac{n-1}{2}}\exp(C_M T)), \tag{3.156}$$

assuming that $T = c_0\log\lambda$, with $c_0 = c_0(M) > 0$ sufficiently small.

To prove this, we can use the arguments of Bérard [3]. Indeed, if we use the covering map coming from the exponential map $\kappa = \exp_x : T_xM \simeq \mathbb{R}^n \to M$ at x, then κ is a covering map and $(\mathbb{R}^n, \tilde g)$ $\kappa^* g = \tilde g$, is the universal cover. Like (M,g), all of the sectional curvatures of $(\mathbb{R}^n,\tilde g)$ are nonpositive. As in (2.102) above, let Γ be the associated deck transformations and choose a Dirichlet domain D associated with the origin, which is in the lift of x. If $\tilde x, \tilde y$ are the lifts to D of $x, y \in M$, we have the formula

$$\big(\cos t\sqrt{-\Delta_g}\big)(x,y) = \sum_{\alpha\in\Gamma}\big(\cos t\sqrt{-\Delta_{\tilde g}}\big)(\tilde x,\alpha(\tilde y)). \tag{3.157}$$

As a result,

$$\tilde{G}_\lambda(x,y) = \sum_{\alpha\in\Gamma} \int_{-\infty}^{\infty} (1-a(t))\, T^{-1}\hat{\Psi}(t/T)\, \bigl(\cos t\sqrt{-\Delta_{\tilde{g}}}\bigr)(\tilde{x},\alpha(\tilde{y}))\, dt. \tag{3.158}$$

To use this formula, we first note that, by (3.4), $\hat{\Psi}(s)=0$ if $|s|>2$, which means that the integrands in (3.158) vanishes for $|t|>2T$. Also, by finite propagation speed for the wave operator,

$$\bigl(\cos t\sqrt{-\Delta_{\tilde{g}}}\bigr)(\tilde{x},\tilde{z}) = 0 \text{ if } \ d_{\tilde{g}}(\tilde{x},\tilde{z}) > |t|,$$

and so each of the summands in (3.158)

$$K_\alpha(\tilde{x},\tilde{y}) = \int_{-\infty}^{\infty} (1-a(t))\, T^{-1}\hat{\Psi}(t/T)\, \bigl(\cos t\sqrt{-\Delta_{\tilde{g}}}\bigr)(\tilde{x},\alpha(\tilde{y}))\, dt = 0, \qquad \text{if } \ d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y})) > 2T = 2c_0\log\lambda. \tag{3.159}$$

Furthermore, if $c_0>0$ here is small enough then since $(\mathbb{R}^n,\tilde{g})$ is of bounded geometry and all of its sectional curvatures are nonpositive, as in [3], [42, §3.6], one can use the Hadamard parametrix and stationary phase arguments to see that for T as above one has the uniform bounds

$$K_\alpha(\tilde{x},\tilde{y}) = O(\lambda^{\frac{n-1}{2}}). \tag{3.160}$$

As a result, we would obtain the bound (3.156) if we could verify that there are $O(\exp(C_MT))$ nonzero summands in (3.158) for T as above. To do this, we let $r = r_{\mathrm{Inj}}(M)/4$. Then if $B_{\tilde{g}}(\tilde{z},r)$ is the geodesic ball in $(\mathbb{R}^n,\tilde{g})$ with center $\tilde{z}$ and radius r, we must have

$$B_{\tilde{g}}(\alpha(\tilde{y}),r)\cap B_{\tilde{g}}(\alpha'(\tilde{y}),r) = \emptyset \text{ if } \ \alpha\neq\alpha', \text{ and } \ \alpha,\alpha'\in\Gamma. \tag{3.161}$$

Note that, by the above, in order for $K_\alpha(\tilde{x},\tilde{y})$ to be nonzero we must also have that $B_{\tilde{g}}(\alpha(\tilde{y}),r)\subset B_{\tilde{g}}(\tilde{x},2T+r)$. Additionally, the volume of $B_{\tilde{g}}(\tilde{z},r)$ must be $O(1)$ due to the fact that $(\mathbb{R}^n,\tilde{g})$ is of bounded geometry. Similarly, since the sectional curvatures of $(\mathbb{R}^n,\tilde{g})$ must be bounded below, by standard volume comparison theorems (see e.g., [19]) the volume of $B_{\tilde{g}}(\tilde{x},2T+r)$ must be $O(\exp(C_MT))$, assuming, as we may that $T>r$. These two crude volume estimates along with (3.161) yield the above claim about the number of nonzero summands in (3.159), which finishes the proof. □

Proof of Lemma 3.5. In view of the first estimate in (3.55) for $q=\infty$, we can use Euler's formula as above to see that we would have (3.86) if we could show that for $T=c_0\log\lambda$ with $c_0>0$ sufficiently small we have for $\lambda\gg 1$

(3.162)

$$Q^{\theta_0}_{j,\ell_0,\nu}\tilde{G}_{\lambda,N}(x,y) = \int_{-\infty}^{\infty} (1-a(t))\, T^{-1}\hat{\Psi}(t/T)\, \beta(|t|/N)\, Q^{\theta_0}_{j,\ell_0,\nu}\bigl(\cos t\sqrt{-\Delta_g}\bigr)(x,y)\, dt = O(T^{-1}\lambda^{\frac{n-1}{2}}N^{1-\frac{n-1}{2}}),$$

assuming that the sectional curvatures of (M,g) are nonpositive.

We can use (3.157) to write

$$Q^{\theta_0}_{j,\ell_0,\nu}\tilde{G}_{\lambda,N}(x,y)=\sum_{\alpha\in\Gamma}K^{j,\ell_0,\nu}_\alpha(\tilde{x},\tilde{y}),\ \text{where} \tag{3.163}$$

$$K^{j,\ell_0,\nu}_\alpha(\tilde{x},\tilde{y})=\int_{-\infty}^{\infty}(1-a(t))T^{-1}\hat{\Psi}(t/T)\beta(|t|/N)Q^{\theta_0}_{j,\ell_0,\nu}\big(\cos t\sqrt{-\Delta_{\tilde{g}}}\big)(\tilde{x},\alpha(\tilde{y}))\,dt,$$

abusing notation a bit here by letting $Q^{\theta_0}_{j,\ell_0,\nu}$ here denote the lift of the operator on (M,g) to $(\mathbb{R}^n,\tilde{g})$ via the covering map.

Since the integrand in (3.163) vanishes when $|t|\notin(N/2,2N)$ one can use the Hadamard parametrix along with (3.55) to see that, by the arguments in [6],

$$K^{j,\ell_0,\nu}_\alpha(\tilde{x},\tilde{y})=\begin{cases}O(\lambda^{\frac{n-1}{2}}N^{-\frac{n-1}{2}})\ \text{ if }\ d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\in[N/4,4N]\\ O(\lambda^{-m})\ \forall\, m\in\mathbb{N}\ \text{otherwise,}\end{cases} \tag{3.164}$$

if $T=c_0\log\lambda$ with $c_0>0$ sufficiently small.

This, by itself will not yield (3.86). For this, let $\tilde{\gamma}=\tilde{\gamma}_{j,\ell_0,\nu}\subset\mathbb{R}^n$ be the geodesic through the origin of the lift of the geodesic $\gamma_{j,\ell_0,\nu}\subset M$ associated with $Q^{\theta_0}_{j,\ell_0,\nu}$. Then the arguments in [6] also yield that if $T=c_0\log\lambda$ with $c_0>0$ small enough one has

$$K^{j,\ell_0,\nu}_\alpha(\tilde{x},\tilde{y})=O(\lambda^{-m})\ \forall\, m\in\mathbb{N}\ \text{if}\ \ d_{\tilde{g}}(\tilde{\gamma},\alpha(\tilde{y}))\ge C_0, \tag{3.165}$$

for some fixed $C_0=C_0(M)$. Since we can also use the volume counting arguments in [6] to see that that number of $\alpha\in\Gamma$ for which $d_{\tilde{g}}(\tilde{x},\alpha(\tilde{y}))\in[N/4,4N]$ and $d_{\tilde{g}}(\tilde{\gamma},\alpha(\tilde{y}))\le C_0$ is $O(N)$, we obtain (3.86) from (3.163), (3.164) and (3.165).

If we assume that the sectional curvatures of (M,g), and hence $(\mathbb{R}^n,\tilde{g})$, are pinched below zero as in (3.87), then we have much more favorable dispersive estimates for the main term in the Hadamard parametrix, as noted in [4] and [33]. This leads to the improvement of the first part of (3.164) under this curvature assumption:

$$K^{j,\ell_0,\nu}_\alpha(\tilde{x},\tilde{y})=O_m(\lambda^{\frac{n-1}{2}}N^{-m})\ \forall\, m\in\mathbb{N}. \tag{3.166}$$

By using this along with the above arguments, we obtain the other estimate, (3.87), in Lemma 3.5. ☐

Now we shall prove the bounds that were used for the Strichartz estimates.

Proof of Lemma 3.6. To prove (3.114), we shall mostly follow the proof of Proposition 4.1 in [5] as well as the ideas in the proof of Lemma 3.1 above. Note that for fixed t and s, $\beta^2(P/\lambda)e^{-i(t-s)\lambda^{-1}\Delta_g}=\beta^2(P/\lambda)e^{i(t-s)\lambda^{-1}P^2}$ is the Fourier multiplier operator on M with

$$m(\lambda,t-s;\tau)=\beta^2(|\tau|/\lambda)e^{i(t-s)\lambda^{-1}\tau^2}. \tag{3.167}$$

We have extended m to be an even function of τ so that we can write

$$\beta^2(P/\lambda)e^{-i(t-s)\lambda^{-1}\Delta_g}=(2\pi)^{-1}\int_{-\infty}^{\infty}\hat{m}(\lambda,t-s;r)\,\cos r\sqrt{-\Delta_g}\,dr, \tag{3.168}$$

where

$$\hat{m}(\lambda,t-s;r)=\int_{-\infty}^{\infty}e^{-i\tau r}\beta^2(|\tau|/\lambda)\,e^{i(t-s)\lambda^{-1}\tau^2}\,d\tau. \tag{3.169}$$

We note that, by a simple integration by parts argument,

$$\partial_r^k \hat{m}(\lambda, t-s; r) = O(\lambda^{-N}(1+|r|)^{-N})\,\forall N, \qquad \text{if } |t-s|\le 2^j, \text{ and } |r|\ge C_0 2^j, \; j=0,1,2,\dots, \tag{3.170}$$

with C_0 fixed large enough. Since $\beta(|\tau|/\lambda)=0$ if $|\tau|\notin[\lambda/4,2\lambda]$ one may take $C_0=100$, as we shall do.

To use this fix an even function $a\in C_0^\infty(\mathbb{R})$ satisfying

$$a(r)=1,\; |r|\le 100 \quad \text{and} \quad a(r)=0 \text{ if } \; |r|\ge 200.$$

Then if we let

$$\tilde{S}_{\lambda,j}(t,s)(P) = (2\pi)^{-1}\int a(2^{-j}r)\hat{m}(\lambda,t-s,r)\cos rP\,dr \tag{3.171}$$

we have the symbol $F_{\lambda,j}(\tau)$ of the multiplier operator

$$F_{\lambda,j}(P) = \tilde{S}_{\lambda,j}(t,s)(P) - \beta^2(P/\lambda)e^{i(t-s)\lambda^{-1}P^2}$$

is $O(\lambda^{-N_1}(1+\tau)^{-N_2})\;\;\forall N_1,N_2$ if $|t-s|\le 2^j$. Thus by (1.4) we have

$$\|F_{\lambda,j}(P)\|_{L^1(M)\to L^\infty(M)} \lesssim 1, \;\text{ if } \;|t-s|\le 2^j.$$

Consequently, if we let $\tilde{S}_{\lambda,j}(x,t;y,s)$ denote the kernel of the multiplier operator $\tilde{S}_{\lambda,j}(t,s)(P)$, we would have (3.114) if we could show that

$$|\tilde{S}_{\lambda,j}(x,t;y,s)| \le \lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}\exp(C2^j), \quad \text{if } \;|t-s|\le 2^j \qquad \text{with } \; j=0,1,2,\dots \;\text{ and } \;\; 2^j\le c_0\log\lambda \tag{3.172}$$

with $c_0=c_0(M)$ fixed small enough.

To prove (3.172), as in the proof of Lemma 3.1, we shall use the Hadamard parametrix and the Cartan-Hadamard theorem to lift the calculations that will be needed up to the universal cover $(\mathbb{R}^{n-1},\tilde{g})$ of (M,g). Let Γ be the associated deck transformations and choose a Dirichlet domain D associated with the origin. If $\tilde{x},\tilde{y}$ are the lifts to D of $x,y\in M$, by (2.102) if we set

$$K_{\lambda,j}(\tilde{x},t;\tilde{y},s) = (2\pi)^{-1}\int a(2^{-j}r)\hat{m}(\lambda,t-s;r)\left(\cos r\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})\,dr, \tag{3.173}$$

we have the formula

$$\tilde{S}_{\lambda,j}(x,t;y,s) = \sum_{\alpha\in\Gamma} K_{\lambda,j}(\tilde{x},t;\alpha(\tilde{y}),s). \tag{3.174}$$

Also, by Huygen's principle and the support properties of a, we have that

$$K_{\lambda,j}(\tilde{x},\tilde{y}) = 0 \text{ if } \; d_{\tilde{g}}(\tilde{x},\tilde{y})\le C_1 2^j \tag{3.175}$$

for a uniform constant C_1. Based on this, if we argue as in the proof of Lemma 3.1 using (3.161) along with simple volume estimates related to the bounded geometry assumption, it is not hard to show that the number of non-zero summands on the right side of (3.174) is $O(\exp(C2^j))$. As a result, we would obtain (3.172) if we could show that

$$|K_{\lambda,j}(\tilde{x},t;\tilde{y},s)| \le C\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}, \quad \text{if } |t-s|\le 2^j \quad \text{with } j=0,1,2,\dots,\ 2^j \le c_0\log\lambda. \tag{3.176}$$

As in the previous section, to prove (3.176), we can use the Hadamard parametrix for $\partial_r^2 - \Delta_{\tilde{g}}$ since $(\mathbb{R}^{n-1},\tilde{g})$ is a Riemannian manifold without conjugate points, i.e., its injectivity radius is infinite. More explicitly for $\tilde{x}\in D$, $\tilde{y}\in\mathbb{R}^{n-1}$ and $|r|>0$

$$\big(\cos r\sqrt{-\Delta_{\tilde{g}}}\big)(\tilde{x},\tilde{y}) = \sum_{\nu=0}^{N} w_\nu(\tilde{x},\tilde{y})W_\nu(r,\tilde{x},\tilde{y}) + R_N(r,\tilde{x},\tilde{y}) \tag{3.177}$$

where w_ν, W_ν and R_N satisfies (2.75)-(2.79).

By (3.177), it suffices to see that if we replace $(\cos r\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\tilde{y})$ in (3.173) by each of the terms in the right side of (3.177) then each such expression will satisfy the bounds in (3.176).

Let us start with the contribution of the main term in the Hadamard parametrix which is the $\nu=0$ term in (3.177). In view of (2.75) and (2.78) it would give rise to these bounds if

$$(2\pi)^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\cos(r|\xi|)\,a(2^{-j}r)\,\hat{m}(\lambda,t-s;r)\,drd\xi = O(\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}) \quad \text{when} \quad |t-s|\le 2^j. \tag{3.178}$$

However, by (3.167) and (3.170) and the support properties of a,

$$\begin{aligned}(2\pi)^{-1}&\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\cos(r|\xi|)\,a(2^{-j}r)\,\hat{m}(\lambda,t-s;r)\,drd\xi\\ &= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\cos(r|\xi|)\hat{m}(\lambda,t-s;r)\,drd\xi + O(\lambda^{-N})\\ &= \int_{\mathbb{R}^{n-1}} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\beta^2(|\xi|/\lambda)e^{i(t-s)\lambda^{-1}|\xi|^2}\,d\xi + O(\lambda^{-N}).\end{aligned} \tag{3.179}$$

A simple stationary phase argument shows that the last integral is $O(\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}})$, and so we conclude that the main term in the Hadamard parametrix leads to the desired bounds.

Similarly, one can use stationary phase to show that if $|t-s|\le 2^j$

$$\begin{aligned}(2\pi)^{-1}&\iint e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}e^{\pm ir|\xi|}\alpha_\nu(|\xi|)\,a(2^{-j}r)\hat{m}(\lambda,t-s;r)\,drd\xi\\ &= \int_{\mathbb{R}^{n-1}} e^{id_{\tilde{g}}(\tilde{x},\tilde{y})\xi_1}\beta^2(|\xi|/\lambda)e^{i(t-s)\lambda^{-1}|\xi|^2}\,\alpha_\nu(|\xi|)\,d\xi + O(\lambda^{-N})\\ &= O(\lambda^{\frac{n-1}{2}-\nu}|t-s|^{-\frac{n-1}{2}}).\end{aligned} \tag{3.180}$$

Note that by (3.175) we may assume that $d_{\tilde{g}}(\tilde{x},\tilde{y})\le Cc_0\log\lambda$. So by (3.177) and (2.79), if we choose c_0 small enough, the contributions from the higher order terms would be $O(\lambda^{\frac{n-1}{2}-\frac{1}{2}}|t-s|^{-\frac{n-1}{2}})$.

We also need to see that the remainder term in (3.177) leads to the bounds

$$\begin{aligned}(3.181)\quad &\int_{-\infty}^{\infty} a(2^{-j}r)\hat{m}(\lambda,t-s;r)R(r,\tilde{x},\tilde{y})\,dr \\ &= \int_{-\infty}^{\infty} \beta^2(|\tau|/\lambda)e^{i(t-s)\lambda^{-1}\tau^2}\,\big[a(2^{-j}\,\cdot\,)R(\,\cdot\,,\tilde{x},\tilde{y})\big]\hat{}\,(\tau)\,d\tau = O(\lambda^{-N}),\ \forall N.\end{aligned}$$

Since we are assuming that $d_{\tilde{g}}(\tilde{x},\tilde{y}) \le Cc_0 \log\lambda$, by (2.77) and support properties of α, the last factor in the integral in the right, which is the Fourier transform of $r \to a(r)R(r,\tilde{x},\tilde{y})$, is $O(|\tau|^{-N}\exp(CNc_0\log\lambda))$. So, by the support properties of β, the last integral in (3.181) is $O(\lambda^{-N})$ if we fix c_0 small enough. □

Proof of Lemma 3.10. If we use the second part of (3.130), it suffices to show

$$(3.182)\quad \|\eta(t/T)\eta(s/T)Q^{\theta_0}_{j,\ell_0,\nu}\beta^2(P/\lambda)e^{-i(t-s)\lambda^{-1}\Delta_g}\|_{L^1(M)\to L^\infty(M)} \le C\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}}.$$

Recall that by the first part of (3.130), we have $\|Q^{\theta_0}_{j,\ell_0,\nu}\|_{L^\infty(M)\to L^\infty(M)} \le C$. Thus if we repeat the arguments in the proof of Lemma 3.6 above, it suffices to show

$$\begin{aligned}(3.183)\quad |\sum_{\alpha\in\Gamma} K_{\lambda,j}(\tilde{x},t;\alpha(\tilde{y}),s))| \le C\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}},\quad &\text{if } |t-s|\le 2^j \\ &\text{with } j=0,1,2,\ldots,\ 2^j\le c_0\log\lambda.\end{aligned}$$

where

$$(3.184)\quad K_{\lambda,j}(\tilde{x},t;\tilde{y},s) = (2\pi)^{-1}\int a(2^{-j}r)\hat{m}(\lambda,t-s;r)\left(Q^{\theta_0}_{j,\ell_0,\nu}\circ \cos r\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x},\tilde{y})\,dr,$$

and

$$(3.185)\quad K_{\lambda,j}(\tilde{x},\tilde{y}) = 0 \text{ if } \ d_{\tilde{g}}(\tilde{x},\tilde{y}) \ge C_1 2^j$$

for a uniform constant C_1. As in (3.174), the number of non-zero summands on the right side of (3.183) is $O(\exp(C2^j))$.

If we repeat the arguments in (3.178)-(3.181), it suffices to replace $\cos r\sqrt{-\Delta_{\tilde{g}}}$ by the main term in the Hadamard parametrix as the higher order terms and remainder term will contribute errors of $O(\lambda^{\frac{n-1}{2}-\frac{1}{2}})$ as long as we choose c_0 small enough as above. Thus the proof of Lemma 3.10 would be complete if we can show that

$$\begin{aligned}(3.186)\quad &(2\pi)^{-2n-1}\sum_{\alpha\in\Gamma}\int_{-\infty}^{\infty}\iiint e^{i((\tilde{x}-\tilde{z})\cdot\eta + d_{\tilde{g}}(\tilde{z},\alpha(\tilde{y}))\xi_1)}Q^{\theta_0}_{j,\ell_0,\nu}(\tilde{x},\tilde{z},\eta) \\ &\qquad\qquad \cdot\cos(r|\xi|)\,a(2^{-j}r)\,\hat{m}(\lambda,t-s;r)\,drd\eta d\tilde{y}d\xi \\ &= O(\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}})\quad \text{when}\quad |t-s|\le 2^j.\end{aligned}$$

As in (3.179), by (3.167) and (3.170) and the support properties of a, each term in the summand of (3.186) can be simplified as

$$(2\pi)^{-2n-2}\int e^{i((\tilde{x}-\tilde{z})\cdot\eta + d_{\tilde{g}}(\tilde{z},\alpha(\tilde{y}))}Q^{\theta_0}_{j,\ell_0,\nu}(\tilde{x},\tilde{z},\eta)\beta^2(|\xi|/\lambda)e^{i(t-s)\lambda^{-1}|\xi|^2}\,d\eta d\tilde{y}d\xi + O(\lambda^{-N}).$$

Here $Q^{\theta_0}_{j,\ell_0,\nu}(\tilde{x},\tilde{z},\eta)$ is the symbol for the operator $Q^{\theta_0}_{j,\ell_0,\nu}$.

By a simple stationary phase argument, the last integral is $O(\lambda^{\frac{n-1}{2}}|t-s|^{-\frac{n-1}{2}})$. On the other hand, recall that as in (3.126) $Q^\theta_{j,\ell_0,\nu} = Q^\theta_{j,\ell_0,\nu'} \circ Q^\theta_{j,\ell_0,\ell}$. If we let $\tilde\gamma = \tilde\gamma_{j,\ell_0,\nu} \subset \mathbb{R}^n$ be the geodesic through the origin of the lift of the geodesic $\gamma_{j,\ell_0,\nu} \subset M$ associated with $Q^{\theta_0}_{j,\ell_0,\nu'}$ and $\kappa^{\theta_0}_\ell$ as in the definition of $Q^{\theta_0}_{j,\ell_0,\ell}$ in (3.124), then one can follow the arguments in the proof of Proposition 4.2 in [5] to see that the last integral is $O(\lambda^{-m})\ \forall\, m \in \mathbb{N}$ unless

$$d_{\tilde g}(\tilde\gamma, \alpha(\tilde y)) \le C_0, \tag{3.187}$$

and

$$d_{\tilde g}(\tilde x, \alpha(\tilde y)) \in \big[2|t-s|(\kappa^{\theta_0}_\ell - C_0\lambda^{-\varepsilon_0}),\, 2|t-s|(\kappa^{\theta_0}_\ell + C_0\lambda^{-\varepsilon_0})\big] \tag{3.188}$$

for some fixed $C_0 = C_0(M)$ and $\theta_0 = \lambda^{-\varepsilon_0}$. By (3.161) with simple volume counting arguments, one can see that number of $\alpha \in \Gamma$ for which (3.187) and (3.188) hold is $O(1)$. This finishes the proof of Lemma 3.10. □

4. **Littlewood-Paley estimates.**

Lemma 4.1. *Let $\beta \in C_0^\infty(1/2, 2)$ with $\sum_{k=-\infty}^{\infty} \beta(s/2^k) = 1$, and define $\beta_k(s) = \beta(s/2^k)$, $\beta_0(s) = \sum_{k\le 0} \beta(s/2^k)$. If (M, g) is a complete manifold of bounded geometry, we have for $2 \le q < \infty$*

$$\|u\|_{L^q(M)} \lesssim \|Su\|_{L^q(M)} + \|u\|_{L^2(M)}, \tag{4.1}$$

where $Su = \Big(\sum_{k\ge 0} |\beta_k(P)u|^2\Big)^{\frac12}$ with $P = \sqrt{-\Delta_g}$.

Lemma 4.1 is a generalization of Bouclet [9, Theorem 1.3] to complete manifolds of bounded geometry, and its proof mostly follows from the same arguments there. For the sake of completeness, we provide the detailed proof below.

On compact manifolds, the above estimate holds without the additional term $\|u\|_{L^2(M)}$. However, on non-compact manifolds, the estimate may fail without this term, since otherwise it would imply the L^q boundness of the multiplier operator $\beta_k(P)$. See [1] for a discussion in the context of hyperbolic spaces.

By Minkowski's integral inequality, (4.1) implies

$$\|u\|_{L^q(M)} \lesssim \Bigg(\sum_{k\ge 0} \|\beta_k(P)u\|^2_{L^q(M)}\Bigg)^{\frac12} + \|u\|_{L^2(M)}. \tag{4.2}$$

This combined with Theorem 1.3 and L^2 orthogonality yield Corollary 1.4.

To prove Lemma 4.1, we shall require the following

Lemma 4.2. *For $k \ge 1$, we can write $\beta_k(P) = B_k + C_k$ with*

$$\|\sum_{k\ge 0} a_k B_k u\|_{L^q(M)} \lesssim \|u\|_{L^q(M)}, \ \text{ if } a_k = \pm 1\ \forall k \text{ and } 1 < q < \infty. \tag{4.3}$$

And for $q \ge 2$,

$$\|C_k u\|_{L^q(M)} \lesssim_N 2^{-Nk} \|u\|_{L^2(M)}. \tag{4.4}$$

Proof. We can extend $\beta \in C_0^\infty(1/2, 2)$ to an even function by letting $\beta(s) = \beta(|s|)$. For $\delta < r_{\text{Inj}}(M)/2$, we can fix $\rho \in C_0^\infty$ satisfying $\rho(t) = 1$, $|t| \le \delta/2$ and $\rho(t) = 0$, $|t| \ge \delta$, and define

$$\begin{aligned}\beta_k(P) &= (2\pi)^{-1} \int \hat{\beta}_k(t) \cos tP dt \\ &= (2\pi)^{-1} \int \rho(t)\hat{\beta}_k(t) \cos tP dt + (2\pi)^{-1} \int (1-\rho(t))\hat{\beta}_k(t) \cos tP dt \\ &= B_k + C_k.\end{aligned} \tag{4.5}$$

It is not hard to check that the symbol of C_k is $O((1+|\tau|+2^k)^{-N})$, thus (4.4) follows from Sobolev estimates. To prove (4.3), we cover M by geodesic balls of radius δ. Using the finite propagation speed property of the wave propagator $\cos tP$ and locally finite property of the covering, we can reduce the calculations needed to a fixed geodesic ball. Then, (4.3) follows from standard arguments using the Hadamard parametrix for $\cos tP$. □

Proof of Lemma 4.1. Let us denote $S_B = \left(\sum_{k\ge 0} |B_k u|^2\right)^{\frac{1}{2}}$. Note that by using a standard argument using Rademacher functions(see e.g.,[43, § 0]), (4.3) implies the following square function estimate

$$\|S_B u\|_{L^q(M)} \lesssim \|u\|_{L^q(M)}, \ 1 < q < \infty. \tag{4.6}$$

Since $\beta_{k_1}(P)\beta_{k_2}(P) \equiv 0$ if $|k_1 - k_2| \ge 2$, we have

$$\begin{aligned}\int_M u_1\bar{u}_2 dx \lesssim\ & \|S_B u_1\|_{L^q(M)}\|S_B u_2\|_{L^{q'}(M)} + \|u_2\|_{L^{q'}(M)} \\ &\times \Big(\sum_{\{k_1,k_2\ge 0, |k_1-k_2|\le 1\}} \|B_{k_1}C_{k_2}u_1\|_{L^q(M)} + \|C_{k_1}B_{k_2}u_1\|_{L^q(M)} + \|C_{k_1}C_{k_2}u_1\|_{L^q(M)}\Big)\end{aligned} \tag{4.7}$$

By (4.3), we have $\|B_k\|_{L^q(M)\to L^q(M)} \lesssim 1$. If we combine this with (4.6) and (4.4), it is not hard to show that

$$\int_M u_1\bar{u}_2 dx \lesssim \|u_2\|_{L^{q'}(M)}(\|S_B u_1\|_{L^q(M)} + \|u_1\|_{L^2(M)}). \tag{4.8}$$

This implies

$$\|u\|_{L^q(M)} \lesssim \|S_B u\|_{L^q(M)} + \|u\|_{L^2(M)}. \tag{4.9}$$

To replace S_B by S, let us define $S_C = \left(\sum_{k\ge 0} |C_k u|^2\right)^{\frac{1}{2}}$, then

$$\begin{aligned}\|S_B u\|_{L^q(M)} &\le \|Su\|_{L^q(M)} + \|S_C u\|_{L^q(M)} \\ &\le \|Su\|_{L^q(M)} + \sum_{k\ge 0} \|C_k u\|_{L^q(M)} \\ &\le \|Su\|_{L^q(M)} + \|u\|_{L^2(M)}.\end{aligned} \tag{4.10}$$

In the last inequality we used (4.4). This finished the proof of Lemma 4.1 □

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