Thus far, we have reduced the problem down to comparisons of Green's f'ns, as an input to edge comparison results for the Gaussian divisible ensemble (H_t).

Last week, Theo told us about how to prove such a comparison via

<u>Corollary 3.3</u> (HMY, May 2024): $\mathbb{E}\left[|Q(z) - Y_\ell(Q(z), z)|^p\right] \leq N^{\varepsilon}\left(\frac{\sqrt{\kappa+\eta}}{N\eta}\right)^p$ if G is tangle-free (whp),

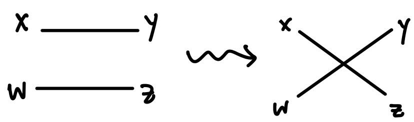
to be used in Markov's inequality and w/ Taylor expansions.

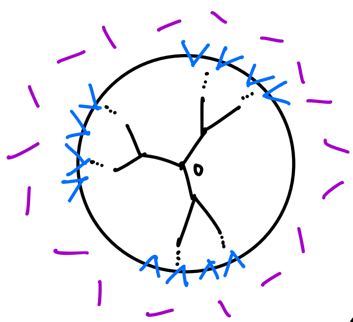
Plan for the next 3 weeks: prove Cor. 3.3; §4 today, §5 next week (Zhongkai), §6 after (Izzy), and on 5/6 discuss what else is needed to get the full strength of the December paper.

We begin by introducing the main combinatorial technique: <u>switching</u>

For directed edges $\vec{xy}, \vec{wz} \in \vec{E}$, the switching is the new pair $\vec{xz}, \vec{wy} \notin \vec{E}$.

Doing so creates a new graph $\tilde{G}$ from G.





Then, for $o \in V(G)$ w/ depth-ℓ tree-like nbhd, for each $\vec{\ell_\alpha a_\alpha}$ where $\ell_\alpha \in B_\ell(o)$ and $a_\alpha \notin B_\ell(o)$, we switch with randomly chosen $\vec{b_\alpha c_\alpha} \in G \setminus B_\ell(o)$ to create $\tilde{G}$.

$(1 \leq \alpha \leq d(d-1)^\ell)$

Pick $c < 1$ and let $R = \frac{c}{4}\log_{d-1} N$.

We say that α is <u>switchable</u> if $B_{R/4}(\{a_\alpha, b_\alpha, c_\alpha\}, G \cup \{a_\alpha b_\alpha\} \setminus B_\ell(o))$ is a tree, and if $\text{dist}(\{a_\alpha, b_\alpha, c_\alpha\}, \{a_\beta, b_\beta, c_\beta\}) > R/4 \;\forall \beta$.

<u>Lemma 2.10</u>: w.p. $\geq 1 - 1/N^{1-2c}$, all α are switchable.

Important fact: resampling always preserves the law of the random graph G.

$\boxed{\text{Approach:}}$ resample to write the entries $G^{(i)}_{0,0}$ in terms of entries of $\tilde{G}$.

Many terms will immediately be negligible, but some will take the form $\tilde{G}^{(b_\alpha)}_{c_\alpha, c_\alpha}$, which means we can then iterate, rooted @ c_α.

In particular, iterating the resampling introduces new terms which either have a longer string of small terms, or have a factor of $\frac{1}{(d-1)^{\ell/2}} = \frac{1}{N^{c/256}}$. Once the string of small terms is long enough, we are done, so we are guaranteed to only need a controlled number of iterations $(256/c)$.

Details: A <u>generic term</u> takes the form of a product of

- any number of
$$\left.\begin{array}{l}
- (d-1)^{2\ell}(G_{ss'} - P_{ss'}) \\
- (d-1)^{2\ell}(G^{(b)}_{c,c} - Q) \\
- (d-1)^{2\ell}\, G^{(b,b')}_{c,c'} \\
- (d-1)^{2\ell}\, G_{ss'} \\
- G_{u,v} \\
- 1/G_{u,u} \\
- 1 - \partial_1 Y_\ell(Q(z), z) \\
- Q - Y_\ell \\
- (d-1)^{2\ell}(Q - m_{sc})
\end{array}\right\} \text{or complex conjugates}$$
- at least two "special" terms $(d-1)^{2\ell} G_{ss'}$
- $\#\ 1 - \partial_1 Y_\ell = \#$ special terms
- $\#$ special terms $+\ \#\ Q - Y_\ell(Q) + \#\ \overline{Q - Y_\ell(Q)} = 2p-1$

Defs: $Q(z,G) = \frac{1}{Nd} \sum_{ij \in \vec{E}(G)} G^{(i)}_{jj}(z)$

$Y_\ell(\Delta(z), z) = \left(-z + H(\mathbb{T}_\ell) - \frac{1}{d-1}\Delta(z)(d \cdot \mathrm{id}_{\mathbb{T}} - \deg \mathbb{T})\right)^{-1}_{0,0}$

where $\mathbb{T}$ is the depth-ℓ $(d-1)$-ary tree rooted @ o.

(Green's f'n for the tree, w/ boundary weights.)

$\mathcal{E}_r$ is the set of generic terms with exactly r of

- $(d-1)^{2\ell}(G^{(a)}_{c,c} - Q)$
- $(d-1)^{2\ell}\, G^{(b,b')}_{c,c'}$
- $(d-1)^{2\ell}\,(G_{s,s'} - P_{s,s'})$
- $(d-1)^{2\ell}(Q - m_{sc})$

HY established that these are small. ($1/N^c$)

Proposition 3.10: if we have a generic term $R_{\vec{i}} = (G^{(i)}_{0,0} - Y_\ell(Q))R'_{\vec{i}} \in \mathcal{E}_r$, then $\frac{1}{Z}\sum_{\vec{i}} \mathbb{E}[(\tilde{G}^{(i)}_{0,0} - Y_\ell(Q))R'_{\vec{i}}(\tilde{G})]$ is a weighted sum of terms

$$\frac{1}{(d-1)^{\ell/2}} \frac{1}{Z'} \sum_{\vec{i}'} \mathbb{E}[R_{\vec{i}'}(G)], \quad R_{\vec{i}'} \in \mathcal{E}_r$$

and

$$\frac{1}{Z'} \sum_{\vec{i}'} \mathbb{E}[R_{\vec{i}'}(G)], \quad R_{\vec{i}'} \in \mathcal{E}_{r'}, \quad r' > r$$

where $(\tilde{G}^{(i)}_{0,0} - Y_\ell(Q))R'_{\vec{i}}(\tilde{G})$ arises from resampling $R_{\vec{i}}(G)$ (Proposition 3.9)

all w/ manageable error $\mathbb{E}[\Psi_p]$

($\vec{i}$ is a choice of vertices at which to evaluate the generic element)

How can we get to Cor. 3.3?

$$\mathbb{E}\left[(Q - Y_\ell(Q))^p\, \overline{(Q - Y_\ell(Q))}^p\right] = \frac{1}{Nd}\sum_{0,i} \mathbb{E}\left[(G^{(i)}_{0,0} - Y_\ell(Q))(Q - Y_\ell(Q))^{p-1}\, \overline{(Q - Y_\ell(Q))}^p\right]$$

$$= \frac{1}{Nd}\sum \mathbb{E}\left[\mathbb{1}\cdot (G^{(i)}_{0,0} - Y_\ell(Q))(Q - Y_\ell(Q))^{p-1}\, \overline{(Q - Y_\ell(Q))}^p\right]$$

$$+ O\left(\frac{1}{N^{1-2c}}\right)\mathbb{E}\left[|Q - Y_\ell(Q)|^{p-1}\right]$$

$\hookrightarrow \mathbb{P}$[no vert. near o has depth-R tree] times $|G^{(i)}_{0,0} - Y_\ell(Q)| = O(1)$

inductively bound by error term

looks like a generic term

indicator for the event that there <u>is</u> a good vert. nearby

So aside from the error, we have a term as in Props. 3.9/3.10.

For z at the critical scale (tacitly assumed always),

each term acquires the extra factor(s) of order $1/N^{c/256}$

so after $256/c$ iterations, the terms

are bounded by the error $O(\mathbb{E}[\Psi_p])$

↳ relating terms to this qty. is the focus of §5.

Towards Props. 3.9/3.10 (§6):

Another def'n:

$$P(G, z, \Delta(z)) = \left(-z + H(G) - \tfrac{1}{d-1}\Delta(z)(d\cdot \mathrm{id} - \deg G)\right)^{-1} \quad \text{(more general form of } Y\text{)}$$

$$|P(G, z, \Delta(z))_{i,i}| = \Theta(1), \quad |P(G, z, \Delta(z))_{i,j}| \lesssim \left(\frac{|m_{sc}(z)|}{\sqrt{d-1}}\right)^{\mathrm{dist}(i,j)}$$

if $|\Delta(z) - m_{sc}(z)| \ll 1/\operatorname{diam} G$. (from HY)

↳ "Spectrum of ..."

Recall the resolvent identity,

$$(A+B)^{-1} - A^{-1} = \sum_{k\in\mathbb{N}} (-BA^{-1})^k$$

We're going to use the Woodbury formula to get a <u>new</u> way to understand the difference of Green's functions (resolvents) for a resampling, at the level of the matrices (rather than entrywise as we will eventually need — cf generic terms).

Woodbury formula: for any matrices A, U, C, V, w/ relevant invertibility conditions:

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$$

(Wikipedia has a nice proof)

$\hookrightarrow$ U, V vectors $\rightsquigarrow$ Sherman–Morrison.

Let $\tilde{H} - H = UCU^T$. We will use:

Note: $\tilde{P}^{-1} - P^{-1} = \tilde{H} - H$

where P is P on the forest of edges involved in the switching.

① $$\tilde{G} - G = (\tilde{H}-z)^{-1} - (H-z)^{-1}$$
$$= (H - z + UCU^T)^{-1} - (H-z)^{-1}$$
$$= -GU(C^{-1} + U^T G U)^{-1} U^T G$$

② $$\tilde{P} - P = -PU(C^{-1} + U^T P U)^{-1} U^T P$$

Let $H(uv)$ be the norm. adj. mat. of the edge uv ($\mathrm{rank}\, H(uv) = 2$) and

$$\xi_\alpha = H(l_\alpha c_\alpha) + H(b_\alpha a_\alpha) - H(l_\alpha a_\alpha) - H(b_\alpha c_\alpha) \quad \text{and}$$
$$F = (\tilde{H} - H) + (\tilde{H}-H)\tilde{P}(\tilde{H}-H)$$
$$= \sum_\alpha \xi_\alpha + \sum_{\alpha,\beta} \xi_\alpha \tilde{P} \xi_\beta .$$

<u>Lemma 4.3</u>: $$\tilde{G} - G = \sum_{k\in\mathbb{N}} GF((G-P)F)^k G$$

<u>Proof</u> (elementary, but this result is crucial):

$$P^{-1}\tilde{P}P^{-1} - P^{-1} = P^{-1} - \tilde{P}^{-1} + (P^{-1}-\tilde{P}^{-1})\tilde{P}(P^{-1}-\tilde{P}^{-1})$$

② //

$$-U(C^{-1}+U^TPU)U^T = F.$$

① $\Rightarrow$ $$\tilde{G} - G = -GU(C^{-1} + U^TPU + U^T(G-P)U)^{-1}U^TG$$

$$\overset{\text{(R1)}}{=} -GU(C^{-1}+U^TPU)^{-1} \sum_{k\in\mathbb{N}} (-1)^k \left(U^T(G-P)U(C^{-1}+U^TPU)^{-1}\right)^k U^TG \quad \checkmark$$

So, how can we use this fact?

ex: Lemma 4.5 ① : $\tilde{G}_{s,w} - G_{s,w} = \sum_{\vec{s}} \gamma_{\vec{s}} U_{\vec{s}} + O(1/N^2)$

where $\vec{s} \in (\{ \ell_\alpha, a_\alpha, b_\alpha, c_\alpha : 1 \le \alpha \le d(d-1)^\ell \})^{2k}$, $k \le J$

and $U_{\vec{s}} = (d-1)^{2\ell k} G_{ss_1} \underbrace{(G_{s_2s_3} - P_{s_2s_1})}_{\text{bounded in Theorem 2.4 (Huang-Yau)}} (G_{s_4s_5} - P_{s_4s_3}) \cdots G_{s_{2k}w}$

and $\sum_{\vec{s}} \gamma_{\vec{s}} \in O(\text{poly } \ell)$.

Proof idea: Bound entries $|F_{t_\alpha, t'_{\alpha'}}| \lesssim (d-1)^{-\text{dist}(\ell_\alpha, \ell_{\alpha'})}$
for label $t \in \{\ell, a, b, c\}$ using def F
and HY's bounds on $\tilde{P}_{ij}$.

Then, $GF((G-P)F)^{k-1}G = ((G-P)F)^k G + PF((G-P)F)^{k-1} G$

and we will take $\gamma_{\vec{s}}$ of the form $F_{s_1s_2} F_{s_3s_4} \cdots (d-1)^{-2k\ell}$
(possibly w/ P_{ss_1} in front)

and $\sum_{\vec{s}} (d-1)^{-2k\ell} \prod (d-1)^{-\text{dist}(\ell_{\alpha_i}, \ell_{\alpha_{i+1}})}$

$$\le \left(\frac{4}{(d-1)^{2\ell}}\right)^k \prod_j \sum_{\alpha_{j-1}, \alpha_j} (d-1)^{-\text{dist}} = O(\text{poly } \ell)$$

Takeaway: from Woodbury-type bound, reduce to analysis of better-understood Green's function differences.

This section of the paper carries this out for several more types of differences (some only requiring Schur complements, but many needing Lemma 4.3)

(overflow, delivered 4/22)

Why does the Woodbury-type result (Proposition 4.3) converge?

Theorem 2.4 (Huang-Yau) gives that entries of $G-P$ are at most $1/N^{\delta}$ for $\delta > 9/64$, so $\|G-P\|_F^2 \le (d(d-1)^{\ell})^2/N^{2\delta} = \frac{1}{\mathrm{poly}\,N}$.

Meanwhile, F has entries indexed by label $\mathcal{L} \in \{a,b,c,\ell\}$, $\alpha \le d(d-1)^{\ell}$, and an imediate calculation is that F's entries are $\lesssim (d-1)^{-\mathrm{dist}(\ell_a, \ell_{a'})}$

so some are a constant but most decay

i.e. $\|F\|_F = O(1)$. ✓

$p=1$ instance:

Want to bound $\mathbb{E}[\,|Q-Y_\ell(Q)|^2]$

$$= \mathbb{E}\left[(Q-Y_\ell(Q))(\overline{Q-Y_\ell(Q)})\right]$$

$$= \frac{1}{Nd}\sum_{o,i}\mathbb{E}\left[(G^{(i)}_{o,o}-Y_\ell(Q))\,\overline{Q-Y_\ell(Q)}\right]$$

$$= \frac{1}{Nd}\sum_{o,i}\mathbb{E}\left[(G^{(i)}_{o,o}-Y_\ell(Q))\,\overline{Q-Y_\ell(Q)}\cdot \mathcal{X}[\exists v: \mathrm{dist}(v,o)<\ell,\ v \text{ has } R \text{ tree nbhd}]\right]$$

$$+ \frac{1}{Nd}\underbrace{\mathbb{P}[\text{no such } v]}_{\substack{O(1/N^{1+c})\\ \text{(Prop 2.2)}}}\underbrace{(\text{unif bound on } G^{(i)}_{o,o}-Y_\ell(Q))}_{O(1)}\,\mathbb{E}\left[\overline{Q-Y_\ell(Q)}\right]$$

Def of Ω: true for all but $\sim d(d-1)^{\ell}N^{c}$ verts (prop. 2.2)

$$\begin{array}{c} R_{\vec{io}} \\ \parallel \\ R_{o,i} \end{array} = (G^{(i)}_{o,o}-Y_\ell(Q))\,(\overline{Q-Y_\ell(Q)}) \rightarrow R'_{\vec{io}}$$

↳ Generic w.r.t. G's subgraph $\mathcal{F}_0$ induced by $\vec{o}\vec{v} = \vec{i}_0$

So,

essentially, # of copies of the free $\mathcal{F}_0$ in G.

by def of Ψ_p

$$= \frac{1}{Z_{\mathcal{F}_0}} \sum_{\vec{i}_0} \mathbb{E}[R_{\vec{i}_0} \chi] + O(\mathbb{E}[\Psi_p])$$

Apply Proposition 3.9: (proved next week)

$$= \frac{1}{Z_{\mathcal{F}_0}} \sum_{\vec{i}_0} \mathbb{E}[\chi (\tilde{G}^{(i)}_{0,0} - Y_\ell(Q)) R'_{\vec{i}_0}] + \overbrace{O(\mathbb{E}[\Psi_p])}^{\text{new + old}}$$

↓

Analyze this w/ Prop. 3.10, i.e.

use Prop. 4.1 on the $\tilde{G}^{(i)}_{0,0} - Y_\ell(Q)$ term to write it as a bounded sum and use Lemmas 4.1, 4.2, 4.5, 4.6 on the terms forming $R'_{\vec{i}_0}$. This pushes attn. to terms on the boundary of o's treelike nbhd, eg.

how Lemma 4.1 (i) states

$$\tilde{G}^{(i)}_{0,0} - Y_\ell(Q) = \frac{c_1}{(d-1)^\ell} \sum_{\substack{\alpha \text{ far} \\ \text{from } i}} \left(G^{(b_\alpha)}_{c_\alpha, c_\alpha} - Q \right) + \frac{c_2}{(d-1)^\ell} \sum_{\substack{\alpha, \beta \\ \text{far from } i}} G^{(b_\alpha, b_\beta)}_{c_\alpha, c_\beta}$$

$$+ \sum_{\vec{\alpha}} \frac{c_{\vec{\alpha}}}{(d-1)^{2\ell}} \prod G^{(b_{\alpha_i} b_{\alpha_{i+1}})}_{c_{\alpha_i}, c_{\alpha_{i+1}}} \cdot (d-1)^{2\ell}$$

and $\sum c_{\vec{\alpha}} + c_1 + c_2 = \text{poly } \ell$.