Gaussian Divisible Ensemble and Reduction to Main Theorem

1 Previous bound on m_N.

Idea: prove universality through Dyson Brownian motion. Compare empirical Green's function $m_N(z)$ with limiting Kesten McKay law, $m_d(z)$. This is done through Dyson brownian motion, which has three steps.

1. Optimal spectral rigidity: For any $\epsilon > 0$, it is the case that for $z = E + i\eta$ with $|E| \leqslant 2.01$ and, setting $\kappa = ||E| - 2|$, $N\eta\sqrt{\kappa+\eta} \geqslant N^{\epsilon}$,

$$|m_N - m_d| \leqslant N^{o_N(1)} \begin{cases} \frac{1}{N\eta} & |E| \leqslant 2 \\ \frac{1}{\sqrt{\kappa+\eta}}\left(\frac{1}{N\sqrt{\eta}} + \frac{1}{(N\eta)^2}\right) & |E| > 2. \end{cases} \tag{1.1}$$

   This is proven in the first paper.

2. Universality after adding a Gaussian component. [Landon-Yau '17] For any $\epsilon > 0$ and a GOE matrix X, $\lambda_2(H + \sqrt{t}X)$ has a Tracy-Widom distribution for $t \geqslant N^{-1/3+\epsilon}$.

3. Preservation of statistics after adding Gaussian component. $\lambda_2(H)$ and $\lambda_2(H + \sqrt{t}X)$ have the same distribution. This is the main result of the new paper..

The goal for today is to show how if we satisfy a given equation, this implies step 3. The way that we do this is through a sharper version of the proof of step 1, so let's recall how this is shown. We define

$$Q(z, \mathcal{G}) = \sum_{i \sim j} G_{jj}^{(j)}.$$

We want to show that Q satisfies the self-consistent equation satisfied by m_{sc}, so for $\ell = \epsilon \log_{d-1} N$, we take $\mathbb{T}_\ell = \mathcal{B}_\ell(o, \mathcal{T})$, for $\mathcal{T}$ the infinite $(d-1)$ rooted tree with root o. Then we set

$$Y_\ell(z, \Delta) = (-zI + H_{\mathbb{T}_\ell} - \Delta \Pi_{\partial \mathbb{T}_\ell})^{-1}_{oo}.$$

We can define $X_\ell(Q)$ to be the equivalent structure with the d-regular tree rather than the $(d-1)$-ary tree. Note that $m_{sc}(z) = Y_\ell(z, m_{sc}(z))$, $m_d(z) = X_\ell(z, m_{sc}(z))$. In the previous paper, the main result was our empirical statistics also satisfy these equations up to small error.

Proposition 1.1 (Theorem 3.3 simplified)**.** *For $p \geqslant 1$, $z \in \mathbf{D}$, $|E| \leqslant 2$,*

$$\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)|Q - Y(Q)|^p], \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)|m_N - X(Q)|^p] = N^{o_N(1)} \left(\frac{\sqrt{\kappa+\eta}}{N\eta}\right)^p.$$

There is a similar expression for $|E| \geqslant 2$, but for pedigogy I will stick to this case. We will now show how this gives (1.1).

For sufficiently large p, by Markov's inequality,

$$|Q - Y(Q)|, |m_N - X(Q)| \lesssim N^{o_N(1)} \frac{\sqrt{\kappa+\eta}}{N\eta}.$$

We now need to convert this into a statement about the Green's function. We have

$$\begin{aligned} Q - Y(Q) &= Q - m_{sc} + m_{sc} - Y(Q) \\ &= (1 - Y'(m_{sc}))(Q - m_{sc}) - \frac{Y''(m_{sc})}{2}(Q - m_{sc})^2 + O(|Q - m_{sc}|^3). \end{aligned}$$

This is a quadratic equation in $Q - m_{sc}$, through which we obtain that

$$Q - m_{sc} = O(\frac{Q - Y(Q)}{1 - Y'(m_{sc})}).$$

For this, we can find $Y'(m_{sc}) = m_{sc}^{2\ell+2}$. Moreover, from the textbook we know that $|1 - m_{sc}^2(z)| \asymp \sqrt{\kappa + \eta}$, meaning

$$Q - m_{sc} = O(\frac{N^{o_N(1)}}{N\eta}).$$

We then use a similar expansion as before, writing

$$m_N - m_d = m_N - X(Q) + X(Q) - m_d = O(\frac{N^{o_N(1)}}{N\eta}) - X'(m_{sc})(Q - m_{sc}) + O(|Q - m_{sc}|^2).$$

Thus, as $X'(m_{sc}) = \frac{d}{d-1} m_d^2 m_{sc}^{2\ell}$, and reducing the imaginary parts to their desired forms, we find (1.1).

2 Gaussian Divisible Ensemble

We want to do something similar, so we need to consider all matrices formed during Dyson Brownian motion, this is known as the Gaussian divisible ensemble. We will consider $X \sim GOE(N)$, and $Z = (I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)X(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)$. We then set

$$H_t = H_0 + \sqrt{t}Z. \qquad \text{(Gaussian Divisible Ensemble)} \tag{2.1}$$

Our goal is to compare the spectral statistics at time 0 with those at time $t = N^{-1/3+\epsilon}$. To this end, we consider the change in the spectral edge as time develops. However, this is well studied, and we use the following lemma. For measure μ, we consider the free-convoluted measure $\mu_t := \mu \boxplus t^{-1/2}\mu_{sc}(t^{-1/2}\cdot)$, where μ_{sc} is the semicircle measure. It is known how these statistics develop.

Lemma 2.1 (Biane '97)**.** *Define $U_t := \{z \in \mathbb{C}^+ : \int \frac{1}{|x-z|^2} d\mu(x) < t^{-1}\}$. Then $z - ts_\mu(z)$ is a homeomorphism from $\overline{U}_t$ to $\mathbb{C}^+ \cup \mathbb{R}$ that is conformal on the interior. Moreover, for $z \in U_t$,*

$$s_\mu(z) = s_{\mu_t}(z - ts_\mu(z)).$$

A consequence of this lemma is that the right edge at time t is determined by ξ_t, the largest real z such that

$$\int \frac{1}{|x - z|^2} d\mu(x) = t^{-1}.$$

Specifically, for $\mathcal{A} := \frac{d(d-1)}{(d-2)^2}$, $\xi_t = 2 + \frac{\mathcal{A}^2 t^2}{4} + O(t^3)$, and

$$E_t = \xi_t - tm_d(\xi_t) = 2 + \frac{d-1}{d+2}t - \frac{\mathcal{A}^2}{4}t^2 + O(t^3).$$

We will therefore find a self-consistent equation for $m_t(z)$, which is the empirical Green's function at time t, and $m_d(z, t)$, which is the Stieltjes transform of μ_t. Then

$$Q_t := \frac{1}{Nd}\sum_{i \sim j} G_{jj}^{(i)}(z, t), \qquad Y_t(z) = Y_\ell(Q_t(z), z + tm_t(z)),$$

so in Y_t, we are converting the overall matrix back to time 0.

Given this, we satisfy a new equation, that is more accurate than the previous one. This is the main theorem of the new work.

3 Main results

Proposition 3.1 (Theorem 3.8)**.** *For sufficiently small $\delta > 0$, define $M = \{w : N^{-2/3-\delta} \leqslant \mathrm{Im}[w] \leqslant N^{-2/3+\delta}$, and $-N^{-2/3+\delta} \leqslant \mathrm{Im}[w] \leqslant N^{-2/3+\delta}\}$. We have, for any $0 \leqslant t \leqslant N^{2/3+\epsilon}$, $p \geqslant 0$, and $z = E_t + w$ and for $1 \leqslant j \leqslant p-1$ $z_j = E_t + w_j$ for $w \in M$*

$$\frac{\mathcal{A}^2}{\ell+1}\mathbb{E}\left[\mathbf{1}(\mathcal{G}\in\Omega)(Q_t(z)-Y_t(z))\prod_{j=1}^{p-1}(m_t(z_j)-m_d(z_j,t))\right]+\mathbb{E}\left[\mathbf{1}(\mathcal{G}\in\Omega)\frac{\partial_z m_t(z)}{N}\prod_{j=1}^{p-1}(m_t(z_j)-m_d(z_j,t))\right]$$
$$+\frac{2}{N^2}\sum_{j\in[p-1]}\mathbb{E}\left[\mathbf{1}(\mathcal{G}\in\Omega)\partial_{z_j}\left(\frac{m_t(z)-m_t(z_j)}{z-z_j}\right)\prod_{i\neq j}(m_t(z_i)-m_d(z_i,t))\right]=O\left(\frac{N^{(p+1)(2\delta-1/3)}}{(d-1)^{\ell/2}}\right).$$

A similar statement is true if we replace $(m_t(z)-m_d(z,t))$ with $Q_t - Y_t$. Note that the error here is smaller than $1/(N\eta)$, meaning it is sufficiently small to think about the distribution.

This level of tightness is sufficient for step 3 of DBM.

Proposition 3.2 (Proposition 3.13)**.** *For any $p \geqslant 1$, $z = E_t + w$ for $w \in M$,*

$$\partial_t\mathbb{E}\left[\mathbf{1}(\mathcal{G}\in\Omega)\prod_{j=1}^{p}N^{1/3}(m_t(z_j)-m_d(z_j,t))\right]\lesssim\frac{N^{2(p+2)\delta}N^{1/3}}{(d-1)^{\ell/2}}.$$

Thus, if $\ell \gg 2(p+3)\delta\log_{d-1}N$,

$$\mathbb{E}\left[\mathbf{1}(\mathcal{G}\in\Omega)\prod_{j=1}^{p}N^{1/3}(m_t(z_j)-m_d(z_j,t))\right]\Bigg|_{t=0}^{N^{-1/3+\delta}}=O((d-1)^{-\ell/4}).$$

The rest of the talk is dedicated towards showing that Proposition 3.1 implies Proposition 3.2. The first step to this proof is to consider the change in Stieltjes transforms visa vi Dyson Brownian motion. We have Dyson's original equation

$$\mathrm{d}\lambda_i=\sqrt{\frac{2}{N}}\mathrm{d}B_i(t)+\frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_j-\lambda_i}.$$

Therefore, by Itô's lemma,

$$\mathrm{d}m_t(z)=-\sqrt{\frac{2}{N^3}}\sum_i\frac{\mathrm{d}B_i(t)}{(\lambda_i-z)^2}+\frac{1}{2}\partial_z(m_t^2(z)+\frac{\partial_z m_t(z)}{N})\mathrm{d}t.$$

We can create a similar equation for $m_d(z,t)$, which is simpler as it is deterministic.

$$\mathrm{d}m(z,t)=\frac{1}{2}\partial_z(m_t^2(z))\mathrm{d}t.$$

Taking the difference immediately gives a function $\mathrm{d}(m_t(z)-m_d(z,t)$, however we will slightly alter this by making z a function of t as well, writing it as $z = E_t + w$, for w fixed. Using the fact that

$\partial_t E_t = -m_d(E_t, t)$, we can write

$$\mathrm{d}(m_t(z) - m_d(z,t)) = -\sqrt{\frac{2}{N^3}} \sum_i \frac{\mathrm{d}B_i(t)}{(\lambda_i - z)^2} + \frac{1}{2}\partial_z(F_t(z))\mathrm{d}t \tag{3.1}$$

for

$$F_t(z) := (m_t(z) - m_d(z,t))^2 + 2(m_d(z,t) - m_d(E_t,t))(m_t(z) - m_d(z,t)) + \frac{\partial_z m_t(z)}{N}.$$

We now will rewrite $F_t(z)$ as a function of the self-consistent equation.

Claim 3.3 (Lemma 3.14)**.**

$$F_t(z) = \frac{\mathcal{A}}{\ell+1}(Q_t - Y_t) + \frac{\partial_t m_t(z)}{N} + O(N^{-5/6+\epsilon}). \tag{3.2}$$

This is proven doing a similar similar expansion as we did to show (1.1). The only difference is that we expand around $z = 2$ as well, where we know $1 - m_{sc}(z)^2 = 2\sqrt{z-2} + O(|z-2|)$, and we expand $\sqrt{z_t - 2}$ as $\sqrt{\xi_t - 2} + \sqrt{z - E_t} + O(t\sqrt{|z - E_t|} + t^2)$.

Proof of Proposition 3.2 assuming Proposition 3.1. By (3.1) and Itô's Lemma, we have

$$\begin{aligned} &\partial_t \mathbb{E}\left[\mathbf{1}(\mathcal{G} \in \Omega) \prod_{j=1}^{p} (m_t(z_j) - m_d(z_j,t))\right] \\ &= \frac{1}{2}\sum_{i=1}^{p} \mathbb{E}\left[\mathbf{1}(\mathcal{G} \in \Omega)\partial_{z_i} F_t(z_i) \prod_{j \neq i} ((m_t(z_j) - m_d(z_j,t)))\right] \\ &+ \sum_{j \neq i} \mathbb{E}\left[\frac{\mathbf{1}(\mathcal{G} \in \Omega)}{N^3}\left(\sum_{\alpha \in [N]} \frac{1}{(\lambda_\alpha - z_i)^2(\lambda_\alpha - z_j)^2}\right) \prod_{k \neq i,j} (m_t(z_k) - m_d(z_k,t))\right]. \end{aligned} \tag{3.3}$$

We can write all of this as a derivative in ∂_z. We have

$$\begin{aligned} (3.3) = \sum_i \partial_{z_i} \mathbb{E}\Big[\mathbf{1}(\mathcal{G} \in \Omega)\Big(F_t(z_i) \prod_{j \neq i} (m_t(z_j) - m_d(z_j,t)) \\ + \sum_{j \neq i} \frac{2}{N^3} \partial_{z_j} \frac{m_t(z) - m_t(z_j)}{z - z_j} \prod_{k \neq i,j} (m_t(z_k) - m_d(z_k,t))\Big)\Big]. \end{aligned} \tag{3.4}$$

We then use Cauchy's integral formula. Take C to be a ball of radius $N^{-2/3-\delta}/10$ around z_i. Then by Proposition 3.1,

$$N^{p/3}|(3.4)| \lesssim \oint_C \frac{N^{(p+1)(2\delta)-1/3}|\mathrm{d}z|}{(d-1)^{\ell/2}|z - z_i|^2} \lesssim \frac{N^{2(p+2)\delta+1/3}}{(d-1)^{\ell/2}} \tag{3.5}$$

as desired. □