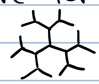
how can we understand Stieltjes transformations? in particular, how can we show m_N is close to m_d?
first, let's try to understand m_d better:

how can we characterize Kesten-McKay?
↳ spectral distribution for infinite d-regular tree
ex: 3-regular tree:



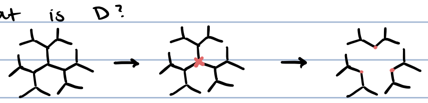
* Kesten McKay dist

what is m_d then?
↳ for empirical dist: $m_N(z) = \frac{1}{N}\mathrm{Tr}(G(z))$ ← "average" value of diagonal entry of $(H-z)^{-1}$
d-reg tree is vertex intransitive, so all diagonal entries are the same
how can we understand $(H-z)^{-1}_{ii}$?

Schur complement: $M = \begin{bmatrix} A-z & B \\ B^* & D-z \end{bmatrix}$ then $(M^{-1})_{11} = (A-z-B(D-z)^{-1}B^*)^{-1}$
↳ (1,1) block

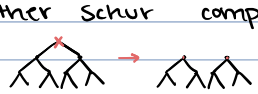
apply this with 1×1 block:
$$\begin{bmatrix} -z & 1/\sqrt{d-1} & 1/\sqrt{d-1} & 1/\sqrt{d-1} & 0 \cdots 0 \\ 1/\sqrt{d-1} & & & & \\ 1/\sqrt{d-1} & & & & \\ 1/\sqrt{d-1} & & D-z & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$
adj. matrix of subgraph induced when remove 1 vertex

what is D?



* D is adjacency matrix of this graph
← infinite rooted (d-1)-ary tree

now Schur complement tells us:
$$(M^{-1})_{11} = \left(-z - \frac{1}{d-1}[1\ 1\ 1\ 0 \cdots 0](D-z)^{-1}\begin{bmatrix}1\\1\\1\\0\\\vdots\\0\end{bmatrix}\right)^{-1}$$
↳ normalization
← bc of these vectors, we only care about what's happening at neighbors of removed vertex – i.e., root of (d-1)-ary tree
but all of these vertices are in different components! $\begin{bmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \end{bmatrix}^{-1} = \begin{bmatrix} D_1^{-1} & & \\ & D_2^{-1} & \\ & & D_3^{-1} \end{bmatrix}$
$$= \left(-z - \frac{d}{d-1}\cdot(D_1-z)^{-1}_{rr}\right)^{-1}$$
how can we figure out what $(D-z)^{-1}_{rr}$ is?
↳ another Schur complement:



* still infinite, rooted (d-1)-ary trees

$$(D_1-z)^{-1}_{rr} = \left(-z - \frac{d-1}{d-1}(D_1-z)^{-1}_{rr}\right)^{-1}$$
$$= \frac{1}{-z-(D_1-z)^{-1}_{rr}}$$
this is exactly satisfied by $m_{sc}(z)$: *can literally solve above + get m_{sc}
$m_{sc}(z) = \frac{1}{-z-m_{sc}(z)}$ } self-consistent equation

so we showed
- $m_d = \frac{1}{-z-\frac{d}{d-1}m_{sc}(z)}$ *KM can be defined via SC
- m_{sc} can be characterized as fixed point of: $\underbrace{m_{sc}(z) = \frac{1}{-z-m_{sc}(z)}}_{\text{self-consistent eq.}}$

idea: we can show something is close to semi-circular by showing it approximately satisfies: $Q(z) \approx \frac{1}{-z-Q(z)}$
then we can relate this quantity to m_d * this will be easier than directly working with m_d

but now we need a quantity other than $m_N(z)$ (since this → $m_d(z)$)
we do the most natural thing we could to get something related to graph model that is (hopefully) semi-circular — delete a vertex!
↳ in infinite case:

we do the same thing in graph:
- remove a vertex: $G^{(i)}$
- look at Green's function: $G^{(i)}(z)$
- consider what happens to $j \sim i$: $G^{(i)}_{jj}(z)$

def: $Q(z) := \frac{1}{Nd}\sum_{i \sim j} G^{(i)}_{jj}(z)$

now we have the following approach for optimal rigidity:

- show $|Q(z) - Y_\ell(Q(z), z)|$ is small
- show $|Q(z) - m_{sc}(z)|$ is small
- show $|m_N(z) - m_d(z)|$ is small

but all of these we need w.h.p., so we want:

- show $|Q(z) - Y_\ell(Q(z), z)|^{2p}$ is small
- show $|Q(z) - m_{sc}(z)|^{2p}$ is small
- show $|m_N(z) - m_d(z)|^{2p}$ is small

then we'll apply higher order Markov