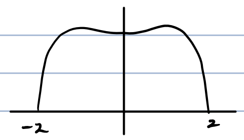
overall goal: understand distribution of λ_2, λ_n of a random d-regular graph
↳ specifically show it has TW_1 dist.

in talk 1, we reduced the problem to showing:
① $H + \sqrt{t} Z$ has TW at edge for $t \geq N^{-1/3+c}$ w.h.p.
② $H + \sqrt{t} Z$ has same edge statistic as H at $t \leq N^{-1/3+c}$ w.h.p.

in talk 2, we showed optimal rigidity of $H \Rightarrow$ ①

so how can we show optimal rigidity? and how can we show ②?
↳ studying Stieltjes transforms through loop eq.

recall:



Kesten - McKay distribution
↳ this is limiting distribution for random d-regular graph on n vertices as $n \to \infty$

our main tool for understanding distributions: Stieltjes transform

def: $m_d(z) = \int \frac{\mu_d}{x-z}\,dx$ where μ_d is KM density

def: $m_N(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z}$ * this is a random object

both defined for $z \in \mathbb{C}^+$

↳ related to resolvent, AKA Green's function

def: $G(z) = (H-z)^{-1}$ * Green's function *
↳ $m_N(z) = \frac{1}{N}\mathrm{Tr}(G(z))$

how can studying Stieltjes transform tell us about locations of evals?
suppose we want to know about the empirical distribution near some point $E \in \mathbb{R}$
let $z = E + i\eta$. then:

$$\begin{aligned} \mathrm{Im}\, m_N(z) &= \mathrm{Im}\,\frac{1}{N}\mathrm{Tr}\,G(z) \\ &= \mathrm{Im}\,\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - E - i\eta} \\ &= \frac{1}{N}\,\mathrm{Im}\sum_{i=1}^{N}\frac{\lambda_i - E + i\eta}{(\lambda_i - E)^2+\eta^2} \\ &= \frac{1}{N}\sum_{i=1}^{N}\frac{\eta}{(\lambda_i-E)^2+\eta^2} \\ &\approx \frac{1}{N\eta}\sum_{i=1}^{N}\mathbb{1}_{|\lambda_i - E|\leq \eta} \end{aligned}$$

*smoothed out empirical measure — can formalize with ∞

note: $0 \leq (\lambda_i - E)^2 \leq \eta^2 \Leftrightarrow \eta^2 \leq (\lambda_i - E)^2 + \eta^2 \leq 2\eta^2 \Leftrightarrow \frac{1}{2\eta} \leq \frac{\eta}{(\lambda_i - E)^2+\eta^2} \leq \frac{1}{\eta}$

↳ hiding some constants

so by studying $m_N(z)$ we can learn about $\mathbb{P}(|\lambda_i - E| \leq \eta)$

we call η the spectral resolution around E
↳ we will need control down to $\eta \approx N^{-1+\varepsilon}$ (as seen below)

thm they show:

thm: $\forall \varepsilon > 0$ and $z = E + i\eta$ with $|E| \leq 2-\varepsilon$, $\underbrace{\eta \geq N^{-1+\varepsilon}}_{\text{spectral res.}}$: $\quad |m_N - m_d| \lesssim \frac{N^{\varepsilon}}{N\eta}$ w.h.p.

claim: this is sufficient to prove optimal rigidity.

pf: * this is a "standard argument" in this area, so they don't prove this in full generality (only at edge)

straightforward argument at edge: * to show $\lambda_2 \leq 2 + N^{-2/3+\alpha}$

- if we're close enough to edge, $\mathrm{Im}[m_d(z)]$ will be small
- using above + $|m_N - m_d|$ bound $\rightsquigarrow$ $\mathrm{Im}[m_N(z)]$ should be small
- but if $\exists \lambda_i$ close to edge, $\mathrm{Im}[m_N(z)]$ won't be small ↯

$\Rightarrow \lambda_2 \leq 2 + N^{-2/3+\alpha}$ w.h.p.

for in the bulk:

* I will sketch the argument for semi-circular (we will at some point need to work directly with the density function, so s.c. will be more straightforward but key ideas are still there)

reference: Benaych-Georges + Knowles lectures on the local semicircle law for Wigner matrices
Erdös + Yau, Dynamical approach to RMT

we want to show:

$$|s_N(z) - m_{sc}(z)| \leq \frac{N^{\varepsilon}}{N\eta} \quad \text{for } z = E + i\eta,\ \eta \geq N^{-1+\varepsilon} \text{ w.h.p.} \Rightarrow \text{optimal rigidity}$$

step 1: ⊛ $\Rightarrow$ w.h.p. $|\rho_N(I) - \mu_{sc}(I)| \leq O(N^{-1})$ for any interval I

key tool for going from Stieltjes transform $\Rightarrow$ measure:

Helffer - Sjöstrand formula: $g \in C^2(\mathbb{C})$ with compact support. then for $\lambda \in \mathbb{C}$:

$$g(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial} g(z)}{\lambda - z} \, dz \quad \text{where} \quad \bar{\partial} = \tfrac{1}{2}(\partial_x + i\partial_y)$$

*pf: appendix C of Erdős + Yau, contour integral + Green's formula + Cauchy's thm

how will we use this?

- fix I, let f be smoothed indicator of I *smoothness depends on η
- use HS to express $\int f(y)(\rho_N - \mu_{sc})(dy)$ w.r.t. $s_N - m_{sc}$
- bound $\int f(y)(\rho_N - \mu_{sc})(dy)$ using $|s_N - m_{sc}|$ bound *these bounds depend on η
- show $\rho_N(I) - \mu_{sc}(I) \approx \int f(y)(\rho_N - \mu_{sc})(dy)$

assume $i \leq N/2$ ($i > N/2$ is analogous)

step 2: use above $\Rightarrow$ $|\lambda_i - \gamma_i| \leq CN^{\varepsilon} N^{-2/3} i^{-1/3}$

by definition of γ_i we have:

$$0 = \rho_N((-\infty, \lambda_i]) - \mu_{sc}((-\infty, \gamma_i]) = \underbrace{\rho_N((-\infty, \lambda_i]) - \mu_{sc}((-\infty, \lambda_i])}_{\text{same interval}} + \int_{\lambda_i}^{\gamma_i} \underbrace{d\mu_{sc}(x)}_{\text{density of measure } \mu_{sc}} dx$$

by step 1 then:

$$\left| \int_{\lambda_i}^{\gamma_i} d\mu_{sc}(x) dx \right| \leq \frac{1}{N} \quad \text{w.h.p.}$$

using $|s_N - m_{sc}|$ bound + form of $d\mu_{sc}$, argue $d\mu_{sc}(\lambda_i) \approx d\mu_{sc}(\gamma_i)$. then:

$$|\lambda_i - \gamma_i| \, d\mu_{sc}(\gamma_i) \leq \frac{1}{N} \quad \text{w.h.p.}$$

then we use density of s.c. to say:

$$\gamma_i + 2 \approx \left(\frac{i}{N}\right)^{2/3}$$

$$d\mu_{sc}(\gamma_i) \approx \left(\frac{i}{N}\right)^{1/3}$$

then we'd have:

$$|\lambda_i - \gamma_i| \leq N^{-1} \cdot \left(\frac{i}{N}\right)^{-1/3} = N^{-2/3} \cdot i^{-1/3} \quad \text{w.h.p. as desired}$$

upshot: studying $m_N(z)$ can help us prove optimal rigidity