goal: understand the spectrum of random d-regular graph on n vertices as $n \to \infty$

* in particular, for the Ramanujan graphs problem, we care about $\lambda_2, -\lambda_n$

what do we already know?

let A be the adjacency matrix of G with $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_2 \le \lambda_1 = d$ and let

$\lambda(G) = \max\{\lambda_2, -\lambda_n\}$

Alon-Boppana: $\lambda(G) \ge 2\sqrt{d-1} - o_n(1)$

Friedman: w.h.p. $\lambda(G) \le 2\sqrt{d-1} + o_n(1)$

we call a graph with $\lambda(G) \le 2\sqrt{d-1}$ Ramanujan – this is an optimal expander in a spectral sense by Alon-Boppana bound

longstanding open question:

do there exist infinite families of d-regular Ramanujan graphs for any d?

↳ true for certain d (LPS + others)

true for bipartite

so we don't have to keep writing $\sqrt{d-1}$, define:

$H = A/\sqrt{d-1}$

central question: how can we understand the edge statistic of $\sigma(H)$?

idea: frame this as a (complicated) random matrix model

model: draw uniformly at random an $n \times n$ symmetric 0-1 matrix with 0s on the diagonal and exactly d 1s in every row

issue: this is a very complicated model – entries have all sorts of ugly dependencies

instead of considering H, let's consider the following random matrix:

$H(t) = H + \sqrt{t}Z$ *mix of model that's hard to understand and a model that's well-understood

↳ $Z \sim$ constrained GOE, take standard GOE (symmetric with $X_{ij} \sim N(0,1)$, ind. entries) and project to subspace with $M\mathbf{1} = 0$

what can we say about $H(t)$?

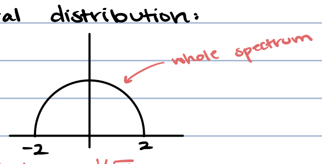
- if t is really small, $\sigma(H(t)) \overset{d}{\approx} \sigma(H)$
- if t is really large, $\sigma(H(t)) \overset{d}{\approx} \sigma(\sqrt{t}Z)$

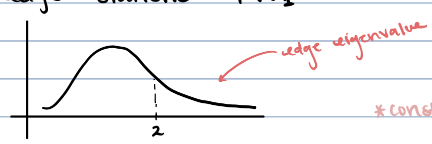
capturing interpolation between random d-reg graph model and GOE

(an aside)

GOE is very well understand

ex: spectral distribution: edge statistic: TW_1





*constrained GOE shares same limiting distributions

*scaled by $\sim 1/\sqrt{N}$

methods used to show this: detailed understanding of correlation functions

we need to know:

① for which t does $H + \sqrt{t}Z$ follow TW_1 at the edge?

② for which t does edge statistic of $H + \sqrt{t}Z$ look like edge statistic of H?

if we believe H follows TW, this isn't a miracle at all

~~by some miracle~~, is there a t that can do both at the edge?

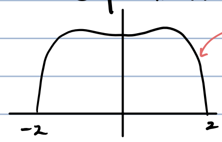
* in particular, we will need near-optimal bounds for these regimes of t

note: this is not an uncommon or novel argument to prove a random model has TW edge distribution

spoiler alert: what we'll find is ① holds for $t \ge N^{-1/3+o(1)}$ and ② holds for $t \le N^{-1/3+o(1)}$

for ①, this is known for H satisfying certain properties
in particular, we need to know (very precisely) what the bulk of the spectrum looks like

what's happening to eigenvalues in the bulk?
<u>McKay</u>: (1981) Kesten-McKay distribution: *also density of infinite d-reg tree



$\sigma(H)$ converges to this as $n \to \infty$

just knowing we're converging to KM isn't good enough — we need to understand how far off each eval is from its "expected" position, and we need it to be as tight as possible *optimal rigidity*

<u>def</u>: let γ_i be value s.t. $\int_{\gamma_i}^{2} \mu_d(x)\,dx = \frac{i - 1/2}{N-1}$ $2 \le i \le N$ where μ_d is Kesten-McKay distribution
↳ more intuitive definition: chop up KM dist. into $N-1$ chunks, each with mass $1/N-1$ then let γ_i be midpoints of each piece (N-1 bc we leave out trivial eval)

then if we have really good control over $|\lambda_i - \gamma_i|$, we can say how much noise we need to disrupt spectrum (idea is $\sigma(H)$ converges to KM, but we need some idea of how far off KM it is)
what's the best control we could hope for?
↳ width of buckets (otherwise limiting distribution isn't continuous!)

has to do w/ regularity - Landon and Yau (2017) * $\sqrt{N^{-2/3}}$ (?)
if we have this, then we can say for $t \gtrsim N^{-1/3}$, $\sigma(H(t))$ has TW dist. at edge
↳ not unique to KM: pg. 25: need "initial data sufficiently close to a nice profile"
* $\sqrt{\cdot}$ behavior at edge is important!

so for ①, we need to prove:
<u>thm</u>: $|\lambda_i - \gamma_i| \le N^{-2/3+\alpha}(\min\{i, N-i+1\})^{-1/3}$ for any $\alpha > 0$ *their first result*

for ②, we need to understand how edge statistic of $H(t)$ is evolving with t

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before we can further discuss <u>how</u> to do these things, we need to introduce some tools

what tools do we have to understand these distributions?
*effectively this paper is a tour de force of a lot of random matrix theory
<u>def</u>: $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$
μ_d : Kesten-McKay dist.
μ_{sc} : semi-circle dist.
↳ dist. of $\sigma(Z)$, so helpful to understand $\sigma(H + \sqrt{t}Z)$

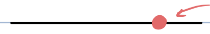
the main object we'll look at associated to these measures is their Stieltjes transform:
<u>def</u>: $m_N(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}$ $z \in \mathbb{C}^+$ *we actually end up studying a slightly different quantity, but we won't worry about that yet!
↳ related to resolvent, AKA Green's function
<u>def</u>: $G(z) = (H - z)^{-1}$
↳ $m_N(z) = \frac{1}{N}\mathrm{Tr}(G(z))$
<u>def</u>: $m_d(z) = \int \frac{\mu_d}{x - z}\,dx$
$m_{sc}(z) = \int \frac{\mu_{sc}}{x - z}\,dx$

we'll also define the analogous functions associated with $H + \sqrt{t}Z$:
<u>def</u>: $m_t(z) = \frac{1}{N}\sum_{\alpha=1}^{N} \frac{1}{\lambda_\alpha(t) - z}$ *empirical distribution of $H(t)$
<u>def</u>: $m_d(z, t) = \int \frac{\mu_t}{x - z}\,dx$ *<u>note</u>: μ_t can be written explicitly as a free convolution of μ_d and μ_{sc}

we will accomplish ① and ② by studying these functions for ① + ②, we're going to want really tight control over $|m_N(z) - m_d(z)|$

↳ very brief toy argument for why this helps us understand μ_N:

suppose μ_N always had an eval in a neighborhood somewhere:



← neighborhood where μ_N has eval

then for any z in this neighborhood $m_N(z)$ blows up

but if we can say $m_d(z)$ is small in this region (which we could since $m_d(z)$ is a deterministic object) and $|m_N(z) - m_d(z)|$ then this leads to a contradiction

* we will need much more sophisticated argument than this, but hopefully this gives some intuition that this is a good quantity to look at

what will our main tool to understand these functions be?

↳ loop equations (AKA Dyson-Schwinger) *cries*

from HMY: "at its core, the loop equation describes a recursive structure satisfied by the correlation functions of the system."

from Vadim: loop equations do not uniquely determine the distribution

the hope is that by studying loop equations, you can learn everything you want to know (or answer all questions you have)

ex: (nothing to do with graph model, this is not the loop eq. we'll study)

let $s_N(z)$ be Stieltjes transformation of empirical distribution of GOE

we know $s_N(z)$ should converge in some manner to $m_{sc}(z)$

$m_{sc}(z)$ satisfies:

$$m_{sc}^2 + z m_{sc} + 1 = 0$$

it's reasonable to assume $s_N(z)$ should roughly satisfy this. we can look at:

$$\left| \mathbb{E}[s_N^2 + z s_N + 1] \right| \lesssim \frac{\operatorname{Im}[s_N(z)]}{N \cdot \operatorname{Im}[z]}$$

in fact we can exactly identify the error:

$$\mathbb{E}\Big[\underbrace{s_N^2 + z s_N + 1}_{\text{self-consistent eq}} + \underbrace{\frac{\partial_z m_N}{N}}_{\text{error}}\Big] = 0$$

how has this helped us?

- say $s_N(z) = m_{sc}(z) + \Delta(z)$. combining the above, we see:

$$\mathbb{E}\ \Delta(z)^2 + \sqrt{(z-2)(z+2)}\ \Delta(z) + \frac{\partial_z s_N}{N}\Big] = 0$$

we've learned something about next term

* from Vadim: this is why they're called loop equations — you learn something about next term in some sort of expansion of $s_N(z)$ then plug this back in to learn about next term

how do we use loop eq. to prove optimal rigidity?

find tight bounds on higher moments:

$$\mathbb{E}|s_N^2 + z s_N + 1|^{2p}$$

using Markov ⇝ optimal estimate for $|s_N - m_{sc}|$ ⇝ optimal rigidity

for the GOE, this has all already been done, but this gives us framework for how to prove optimal rigidity:

- identify loop eq. for our target dist.
  - ↳ loop eq = self-consistent + error
- find tight bounds for higher moments of self-consistent eq.
- find bound on error (we'll just identify leading order term)

standard arguments [

- translate this to optimal bound on $|m_N(z) - m_d(z)|$ *Appendix C*
- translate this to optimal rigidity *pg. 144-145*

we will use the fact that μ_N comes from graph model to identify appropriate self-consistent equation, bound higher moments, and identify leading order error term

* this is novel part of this paper and where we will spend most of the time *

what self-consistent eq. will we use?

next time!

for ②, we need to understand how edge statistic is evolving with time, so we will take a time derivative and evaluate it near the edge

specifically we look at:

$$\partial_t \mathbb{E}\left[\prod_{1\le j\le p} N^{1/3}\left(m_t(z_j) - m_d(z_j,t)\right)\right] \quad \text{for } z_j \text{ near the edge}$$

↳ from Prop. 3.13

*we will actually see this is governed by the "microscopic" loop eq. - loop eq with z near edge

we'll integrate from 0 to $N^{-1/3+\alpha}$ and show the change is small