Recall Aldous's conjecture from Joao's talk.

Setup: $G([n], \binom{[n]}{2})$ a graph weighted by $c_{xy} \in \mathbb{R}_{\geq 0}$. (Let $c_{xx}=0$ always.)

- L^{RW} the unnormalized Laplacian of G,
  λ_1^{RW} is its least nontrivial eigenvalue.

- L^{IP} is the unnormalized Laplacian for the Cayley graph on S_n, where we identify $V=[n]$, and the generators are transpositions of the form $\{(x\ y) : xy \in V\}$.

  λ_1^{IP} is its least nontrivial eigenvalue.

Conjecture (Aldous): $\lambda_1^{RW} = \lambda_1^{IP}$.

Proved by Caputo, Liggett, & Richthammer.

Note: there was a question last week about laziness when recognizing RW as a special case of IP. CLR use a specific notion of "subprocess":

$$L_1(f \circ \pi) = (L_2 f) \circ \pi$$

$$f : S_2 \to \mathbb{R}, \quad \pi : S_1 \to S_2, \quad \sigma \mapsto \sigma(1).$$

$$L_1 = L^{IP}, \quad L_1 g(\sigma) = \sum_{xy} c_{xy}(g(\sigma) - g(\sigma^{xy}))$$

$$L_2 = L^{RW}, \quad L_2 f(x) = \sum_{y} c_{xy}(f(x) - f(y)).$$

Check the condition: $(L_2 f) \circ \pi(\sigma) = L_2 f(\sigma(1)) = \sum_y c_{\sigma(1)y}(f(\sigma(1)) - f(y))$

$$L_1(f \circ \pi)(\sigma) = \sum_{xy} c_{xy}(f(\pi(\sigma)) - f(\pi(\sigma^{xy})))$$

$$= \sum_{xy} c_{xy}(\underbrace{f(\sigma(1)) - f(\sigma^{xy}(1))}_{0 \text{ if } 1 \notin \{\sigma^{-1}(x), \sigma^{-1}(y)\}})$$

$$= \sum_y c_{\sigma(1)y}(f(\sigma(1)) - f(y))$$

In my view, this functions to account for IP's much larger state space. ($n!$ vs n.)

CLR's key technical tool is the octopus lemma:

for $c^{*,x}_{yz} = \dfrac{c_{xy}c_{yz}}{\sum_w c_{xw}}$ and $\nabla_{xy} : \mathbb{R}^{S_n} \longrightarrow \mathbb{R}^{S_n}$

$f \longmapsto (\eta \mapsto f(\eta^{xy}) - f(\eta))$,

and $\mu = US_n$, for any $x \in V$,

$$\sum_{y \in V} c_{xy}\, \mathbb{E}_\mu[(\nabla_{xy} f)^2] \geqslant \sum_{yz \in E} c^{*,x}_{yz}\, \mathbb{E}_\mu[(\nabla_{yz} f)^2].$$

(remember: if $x \in \{y,z\}$ then $c^{*,x}_{yz} = 0$)

Some example graphs.

First, some notation: let $M_n^{(x\ y)}$ be a matching on S^n connecting $\sigma \in S_n$ to $(x\ y)\sigma$ and let its Laplacian be $L_n^{(x\ y)}$.

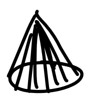
Let $x = n$ always.

- $G = K_n$. We compute $c_{yz}^{*,x} = 1/n-1$. So octopus tells us

$$\sum_{y<n} L_n^{(y\ n)} \succsim \frac{1}{n-1} \sum_{y<z<n} L_n^{(y\ z)} \qquad \text{(as operators)}$$

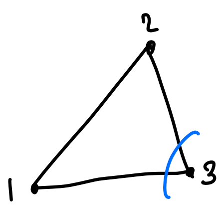
- $G = C_n$. Now $c_{y,z}^{*,x} = 1/2$, so

$$L_n^{(n-1\ n)} + L_n^{(1\ n)} \succsim \frac{1}{2} L_n^{(n-1\ 1)}.$$

- $G = Pyr_n$ (pyramid)  Again $c_{yz}^{*,x} = 1/n-1$

$$\sum_{y<n} L_n^{(y\ n)} \succsim \frac{1}{n-1} \sum_{y<z<n} L_n^{(y\ z)}$$

- $G = K_3$.



$c_{12}^{*,3} = 1/2$

$$L_3^{(13)} + L_3^{(23)} \succsim \frac{1}{2} L_3^{(12)}.$$

And this is tight! $\dim \operatorname{Ker}\left(L_3^{(13)} + L_3^{(23)} - \frac{1}{2} L_3^{(12)}\right) = 3$.

Though the structure of this Kernel is somewhat opaque to me.

How important is it that μ be uniform?

At least a little bit. Consider the situation of a point mass @ the identity:

$\mu = \delta_e$. $X = n$, $G = K_n$

LHS: $\sum_{y<n} (f((y\ n)) - f(e))^2 \overset{?}{\geq} \sum_{y<z<n} \frac{1}{n-1} (f((y\ z)) - f(e))^2$

If $f(e) = f((y\ n)) = 0 \quad \forall y < n$,

$f((y\ z)) = 1 \quad \forall\ y < z < n$.

Then $LHS = 0$, $RHS = \binom{n-1}{2} \cdot \frac{1}{n-1} > 0$

Say $f(\eta)$ only tracks $\eta(1)$. i.e., $f(\eta) = g(\eta(1))$.

Then, $\nabla_{xy} f(\eta) = \begin{cases} 0, & \eta(1) \notin \{x,y\} \\ f(y) - f(\eta(1)), & x = \eta(1) \end{cases}$, so

$$E_\mu\left[(\nabla_{xy} f(\eta))^2\right] = \frac{2}{n}\,(g(x)-g(y))^2.$$

This setting is actually a consequence of:

$$\sum_{y<n} c_{yn}\,(g(y)-g(n))^2 = \sum_{yz\in E} c^{*,n}_{yz}\,(g(y)-g(z))^2 + \frac{1}{\sum_y c_{yn}}\,(Lg(n))^2$$

Proof: $\quad Lg(n) = \sum_y c_{yn}\,(g(n)-g(y))$

so $(Lg(x))^2 = \sum_y \sum_z c_{yn}\,c_{zn}\,(g(n)-g(y))(g(n)-g(z))$

so $\frac{1}{\sum_y c_{yn}}(Lg(x))^2 = \sum_y \sum_z c^{*,n}_{yz}\,(g(n)-g(y))(g(n)-g(z))$

$\quad = 2\sum_{y<z} c^{*,n}_{yz}\,(g(n)-g(y))(g(n)-g(z))$

Meanwhile, $\sum_{yz\in E} c^{*,n}_{yz}\,(g(y)-g(z))^2 = \sum_{y<z} c^{*,n}_{yz}\,(g(y)-g(z))^2$

So the RHS is $\sum_{yz\in E} c^{*,n}_{yz}\left((g(n)-g(y))^2 + (g(n)-g(z))^2\right)$

Each y shows up n times, so

$$= \sum_y (g(n)-g(y))^2 \underbrace{\sum_z c^{*,n}_{yz}}_{c_{yn}}$$

To get the case $f(\eta) = g(\eta(1))$, just drop the last term.

So, there, octopus is tight when $Lg(n) = 0$, i.e. g is harmonic @ x

(and not nec. anywhere else).

Sketch of a proof (using representation theory)

Consider the group $G = S_n$. We will work w/ its group algebra $\mathbb{C} S_n$.

For a rep ρ of G, we define its action on $\mathbb{C}S_n$ by linearizing its action on V. i.e., $\rho: G \longrightarrow \operatorname{End} V$ gives $\rho: \mathbb{C}G \longrightarrow \operatorname{End}$

Important reps.:

- $\mathbb{1} \circlearrowright \mathbb{C}$, $\mathbb{1}(g) = 1 \ \forall g \in G$ "trivial". $\dim = 1$
- $\mathbb{L} \circlearrowright \mathbb{C}G$, $\mathbb{L}(g)(h) = gh$ "left-regular" $\dim = n!$
- $\mathbb{D}_n \circlearrowright \mathbb{C}^n$, $\mathbb{D}_n(\sigma)(x_i)_{i=1}^n = (x_{\sigma(i)})_{i=1}^n$ "defining" $\dim = n$.

Each group term $w \in \mathbb{C}G$ & rep. $\rho \circlearrowright V$ gives a "Laplacian" operator

$$\Delta_G(w,\rho) := \mathbb{1}(w) - \rho(w) \in \operatorname{End} V .$$

Let $T_n \subset S_n$ be the set of transpositions. If $\operatorname{supp} w \subset S_n$, and

$$w = \sum_{(x\,y)\in T_n} w_{xy} (x\ y) \qquad \text{where } w_{xy} \geq 0, \text{ then}$$

$$\Delta_{S_n}(w, \mathbb{D}_n) = L^{RW} \qquad \text{and} \qquad \Delta_{S_n}(w, \mathbb{L}) = L^{IP} \ !$$

So to compare for Aldous's conj, suffices to compare these group/rep. Laplacians.

Neat facts: • if $\rho \cong \rho_1 \oplus \rho_2$ then $\Delta_G(w,\rho) \cong \Delta_G(w,\rho_1) \oplus \Delta_G(w,\rho_2)$

So, if we say $\Psi_G(w,\rho) := \inf(\operatorname{spec} \Delta_G(w,\rho) \cap \mathbb{R}_{>0})$,

then $\Psi_G(w,\rho) = \min\{\Psi_G(w,\rho_1), \Psi_G(w,\rho_2)\}$.

• let $\operatorname{irr} G$ be the collection of irreps. Then, $\mathbb{L} \cong \bigoplus_{\rho \in \operatorname{irr} G} \rho^{\oplus \dim \rho}$.

Thus, $\lambda_1^{IP} = \min_{\substack{\rho \in \operatorname{irr} G \\ \rho \neq \mathbb{1}}} \Psi_{S_n}(w,\rho)$

and the content of the end result will be that this result is achieved by

$\mathbb{D}_n \setminus \mathbb{1}$!

not standard notation

$\left(\text{as in, } D_n = \mathbb{1} \oplus \tau_n \text{ , and we claim that } \tau_n = \underset{\substack{\rho \in \mathrm{irr}\, G \\ \rho \neq \mathbb{1}}}{\mathrm{argmin}}\ \Psi_{S_n}(w, \rho) \right)$

One direction is very easily handled (in CLR), so we just want to show that $\quad \Psi_{S_n}(w, \tau_n) \leq \min_{\substack{\rho \in \mathrm{irr}\, G \\ \rho \neq \mathbb{1}}} \Psi_{S_n}(w, \rho)$. ✪

Joao's talk already discussed using the octopus lemma as an input to prove the conjecture, so I will focus on restating & proving the lemma.

Cesi reduces ✪ to showing this octopus lemma:

$$\Delta_{S_n}(\omega - \theta(\omega), \rho) \succeq 0 \quad \forall\, \rho \in \mathrm{irr}\, S_n \quad \text{where}$$

$$w = \sum_{(xy) \in T_n} w_{xy}\,(x\ y) \in \mathbb{C} T_n, \quad w_{xy} \geq 0, \text{ and } \theta(w) = \sum_{(xy) \in T_{n-1}} \left(w_{xy} + \frac{w_{xn} w_{yn}}{\sum_{i<n} w_{in}} \right) (x\ y).$$

We can readily check that the original octopus lemma amounts to

$$\Delta_{S_n}\Big(\sum_{(yn) \in T_n} w_{yn}\,(y\ n)\ ,\ \mathbb{L} \Big) \geq \Delta_{S_n}\Big(\sum_{(xy) \in T_{n-1}} \frac{w_{xn} w_{yn}}{\sum w_{in}} (x\ y)\ ,\ \mathbb{L} \Big)$$

$\mathbb{1}(\omega)$

or: $\quad \Delta_{S_n}(w - \theta(w), \mathbb{L}) \geq 0$

or, any $\mathbb{1} \neq \rho \in \mathrm{irr}\, S_n$.

Let $\Gamma(G) = \{ w \in \mathbb{C}G^{s.a.} : \Delta_G(w, \mathbb{L}) \succeq 0 \}$ where $\omega^* = \sum_{g \in G} \overline{\omega_g}\, g^{-1}$ and $\mathbb{C}G^{s.a.} = \{ w \in \mathbb{C}G : \omega = \omega^* \}$.

Then $\Gamma(G)$ is a real cone. We will use this momentarily. First, we argue that if $\omega^2 \in \Gamma(G)$ then $w \in \Gamma(G)$. Clearly $w \in \Gamma(G)$ iff $\mathbb{1}(\omega) \geq \|\rho(\omega)\| \quad \forall\, \rho \in \mathrm{irr}\, G$. Also, since ρ is a homomorphism, $\rho(w^2) = \rho(\omega)^2$, so <u>if</u> $w^2 \in \Gamma(G)$ then

$$\begin{array}{ccc} \mathbb{1}(w^2) & \geq & \|\rho(w^2)\| \\ \| & & \| \\ \mathbb{1}(\omega)^2 & & \|\rho(\omega)^2\| = \|\rho(\omega)\|^2 \end{array}$$

and thus $\|\rho(\omega)\| \leq \mathbb{1}(\omega)$, as desired.

For convenience, let $x_i = w_{in}$

Call $\hat{\omega} = \sum_i x_i \cdot (w - \theta(w)) = \sum_i x_i \cdot \sum_i x_i\,(i\ n) - \frac{1}{2}\sum_{i \neq j} x_i x_j\,(i\ j)$.

$$= \sum_i x_i^2 (i\ n) + \sum_{i<j} x_i x_j \big((i\ n) + (j\ n) - (i\ j) \big).$$

Since $\operatorname{supp} \hat{\omega} \subseteq T_n$, $\hat{\omega}^2 = \sum_\mu \alpha_\mu^{(n)} \vec{x}^{\mu}$ where $\alpha_\mu^{(n)} \in \mathbb{C}S_n$ and $\mu \in \mathbb{N}^{n-1}$, $\|\mu\|_1 = 4$.

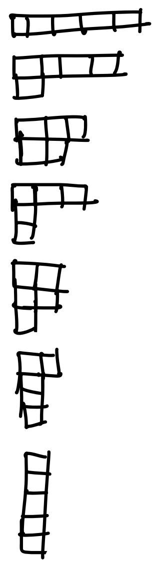
i.e., break $\hat{\omega}$ into its x-monomials.

if $\alpha_\mu^{(n)} \in \Gamma(G)\ \forall \mu \Rightarrow \hat{\omega}^2 \in \Gamma(G)$ since $\vec{x}^{\mu} \geq 0$

$\Rightarrow \hat{\omega} \in \Gamma(G)$ and the octopus is proved.

The remainder comes down to a finite calculation for each $\alpha_\mu^{(n)}$:

each permutation that arises is, up to similarity, in $\mathbb{C}S_5$. Thus it suffices to calculate w/ S_5, which has only



these irreps.

This omitted calculation finishes the proof of the octopus lemma.

One last comment on the octopus lemma to highlight its use in Cesi: his induction takes the form $\forall\, w \in \mathbb{C}T_n^{(+)}$,

$$\Psi_{S_n}(w, \mathbb{D}_n) \leq \Psi_{S_{n-1}}(\theta(w), \mathbb{D}_{n-1}) \quad \circledast$$

which I think is especially clean, also terminating at an accessible point $n=2$.

Indeed, θ is chosen to satisfy general conditions to make the induction work (where for each n there is to be a set $\mathcal{A}_n \subseteq \mathbb{C}G$ with $\theta : \mathcal{A}_n \longrightarrow \mathcal{A}_{n-1}$ and obeying $\circledast$)

though it appears that θ arose in CLR from taking the Schur complement; I'm not sure how θ might arise purely rep.-theoretically.