NOTES

We discuss the work of Hide–Magee [HM23] on near optimal spectral gap of hyperbolic surfaces.

The key is to construct the inverse of $\Delta - s(1-s)$ on a (noncompact) finite area hyperbolic surface $\Gamma\backslash\mathbb{H}$. On the hyperbolic plane, the Schwartz kernel of $R_{\mathbb{H}}(s) = (\Delta - s(1-s))^{-1}$ is given by

$$R_{\mathbb{H}}(s;x,y) = \frac{1}{4\pi}\int_0^1 \frac{t^{s-1}(1-t)^{s-1}}{(\sigma - t)^s}dt, \quad \sigma = \cosh^2\left(\frac{r}{2}\right), \quad r = d_{\mathbb{H}}(x,y).$$

This is bounded on $L^2(\mathbb{H})$ when $s > 1/2$. Note the Schwartz kernel of $R_{\mathbb{H}}(s)$ only depends on r. This is a manifestation that Δ commutes with the group action by $PSL(2,\mathbb{R})$.

In general, if an operator has Schwartz kernel of the form $k_0(d_{\mathbb{H}}(x,y))$, it is given by a Fourier multiplier for $\Delta f = (\xi^2 + 1/4)f$.

$$h(\xi) = \sqrt{2}\int_{-\infty}^{\infty} e^{i\xi u}\int_{|u|}^{\infty}\frac{k_0(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}}d\rho du.$$

Thus

$$\|K\|_{L^2\to L^2} = \sup_{\xi \geqslant 0}|h(\xi)|.$$

Now we want to descend this resolvent to the quotient $\Gamma\backslash\mathbb{H}$. This does not work unless $s > 1$ such that one have $R(s,r) \leqslant Ce^{-sr}$. We need to cut off. Let χ_0 be a cutoff such that $\chi_0(x) = 1$ from $(-\infty, 0]$ and vanishes for $x \in [1,\infty)$. Let $\chi_T(x) = \chi_0(x - T)$ and $R_{\mathbb{H}}^{(T)}(s,r) = \chi_T(r)R_{\mathbb{H}}(s,r)$. In polar coordinates,

$$\Delta_{\mathbb{H}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r}\frac{\partial}{\partial r} - \frac{1}{\sinh^2 r}\frac{\partial}{\partial \theta}.$$

Therefore,

$$[\Delta_x, R_{\mathbb{H}}^{(T)}(s,r)] = -\partial_r^2\chi_T - \partial_r\chi_T\partial_r - \frac{1}{\tanh r}\partial_r\chi_T.$$

So

$$(\Delta_{\mathbb{H}} - s(1-s))R_{\mathbb{H}}^{(T)} - I = L_{\mathbb{H}}^{(T)}(s,r) = -\left(\partial_r^2\chi_T + \frac{1}{\tanh r}\partial_r\chi_T\right)R_{\mathbb{H}}(s,r) - \partial_r\chi_T\partial_r R_{\mathbb{H}}(s,r)$$

and

$$|L_{\mathbb{H}}^{(T)}(s,r)| \leqslant Ce^{-sr}.$$

Lemma 0.1. $\|L_{\mathbb{H}}^{(T)}(s,r)\|_{L^2\to L^2} \leqslant CTe^{(s-1/2)T}$.

Proof. It suffices to estimate

$$\begin{aligned}
&\sqrt{2}\left|\int_{-\infty}^{\infty}\int_{|u|}^{\infty}\frac{L_{\mathbb{H}}^{(T)}(s,r)\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(u)}}drdu\right| \\
&\leqslant 2\sqrt{2}\left|\int_0^{T+1}e^{i\xi u}\int_{\max(|u|,T)}^{T+1}\frac{|L_{\mathbb{H}}^{(T)}(s,r)|\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(u)}}drdu\right| \\
&\leqslant Ce^{-sT}\int_0^{T+1}\int_{\cosh\max(|u|,T)}^{\cosh(T+1)}\frac{1}{\sqrt{y-\cosh(u)}}dydu \\
&\leqslant CTe^{(s-1/2)T}.
\end{aligned}$$

□

We will pretend that $\Gamma\backslash\mathbb{H}$ is compact. In fact, when there is a cusp, it suffices to deal with the cusp part separately, which can be explicitly analyzed. The paper considers the cusp case because Bordenave—Collins requires the fundamental group to be a free group.

Let F be a Dirichlet fundamental domain, i.e.

$$F = \bigcap_{\gamma\in\Gamma\backslash\{id\}}\{z\in\mathbb{H} : d(o,z) < d(z,\gamma o)\}.$$

For $f \in C_\phi^\infty(\mathbb{H}; V_n^0)$, we write

$$L_{\mathbb{H},n}^{(T)}(s)f(x) = \sum_{\gamma\in\Gamma}\int_{y\in F}L_{\mathbb{H},n}^{(T)}(s;\gamma x,y)\rho_\phi(\gamma^{-1})f(y)dy.$$

Recall

$$L_\phi^2(\mathbb{H}; V_n^0) \cong L^2(F)\otimes V_n^0.$$

Under this isomorphism, we have

$$L_\phi^{(T)}(s,r) \cong \sum_{\gamma\in\Gamma}a_\gamma^{(T)}(s)\otimes\rho_\phi(\gamma^{-1}), \quad a_\gamma^{(T)}(s;x,y) = L_{\mathbb{H}}^{(T)}(s;\gamma x,y).$$

The operator for the regular representation ρ_∞ is

$$L_{\mathbb{H}}^{(T)}(s,r) \cong \sum_{\gamma\in\Gamma}a_\gamma^{(T)}(s)\otimes\rho_\infty(\gamma^{-1}), \quad a_\gamma^{(T)}(s;x,y) = L_{\mathbb{H}}^{(T)}(s;\gamma x,y).$$

Since $a_\gamma^{(T)}(s)$ is compact, we can choose finite rank operators $b_\gamma^{(T)}(s)$ such that

$$\|a_\gamma^{(T)}(s) - b_\gamma^{(T)}(s)\|_{L^2(F)\to L^2(F)} \leqslant \epsilon(T).$$

Applying Bordenave—Collins, we know a.a.s. as $n \to \infty$,

$$\left\| \sum_{\gamma \in \Gamma} b_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1}) \right\| \leqslant \left\| \sum_{\gamma \in \Gamma} b_\gamma^{(T)}(s) \otimes \rho_\infty(\gamma^{-1}) \right\| + \epsilon.$$

Therefore,

$$\left\| \sum_{\gamma \in \Gamma} a_\gamma^{(T)}(s) \otimes \rho_\phi(\gamma^{-1}) \right\| \leqslant \left\| \sum_{\gamma \in \Gamma} a_\gamma^{(T)}(s) \otimes \rho_\infty(\gamma^{-1}) \right\| + \epsilon'.$$

The right hand side is small because of Lemma 0.1.

We have to make the proof uniform for $s \in [s_0, 1]$. This uses the fact that

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(F) \to L^2(F)} \leqslant C(T)|s_1 - s_2|.$$

References

[HM23] W. Hide and M. Magee, *Near optimal spectral gaps for hyperbolic surfaces*, Annals of Mathematics **198** (2023), no. 2, 791–824.