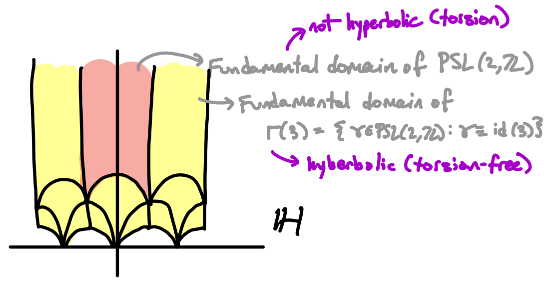
Operator of Interest: Laplace–Beltrami Δ_X, acting on fns $X \to \mathbb{C}$.

Δ_X has Rayleigh quotient $R_X(f) = \int_X |\nabla f|^2 dx \big/ \int_X |f|^2 dx$ w/ $f \equiv \text{const} \Rightarrow$ eigenvalue 0.

$$\lambda_1 = \inf_{\int_X f dx = 0} R_X(f)$$

Spaces of interest: quotient of free $\Gamma \leq PSL(2,\mathbb{R})$ acting on $\mathbb{H}$ on the left.



Main theorem: X a finite-area non-compact hyperbolic surface. X_n a unif. random n-cover of X. For any $\varepsilon > 0$, almost surely as $n \to \infty$:

$$\operatorname{spec} \Delta_{X_n} \cap [0, 1/4 - \varepsilon) = \operatorname{spec} \Delta_X \cap [0, 1/4 - \varepsilon)$$

(w/ same multiplicities)

Recall Friedman's theorem about random covers of graphs, proved in Bordenave–Collins.

There are also two corollaries, for the non-compact and closed cases:

- There is a sequence X_n of finite-area non-compact hyperbolic surfaces w/ $\chi(X_n) \xrightarrow{n} -\infty$ and $\inf(\operatorname{spec}\Delta_{X_n} \cap \mathbb{R}_+) \to 1/4$.

Proof is to take any X_1 w/ $\chi(X_1) = -1$. Ofal–Rosas give $\operatorname{spec}\Delta_{X_n} \cap [0, 1/4) = \{0\}$. Apply main thm. ✓

- There is a sequence X_n of closed hyperbolic surfaces w/ $g(X_n) \to \infty$ and $\lambda_1(X_n) \to 1/4$ optimal, via Huber.

Proof is to adapt the previous corollary to become closed, using Buser–Burger–Dodziuk / Brooks–Makover and the relation $\chi = 2 - 2\,\text{genus} - \#\text{cusps}$.

Construction

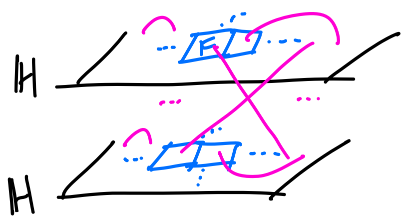
Γ acts on $\mathbb{H}$ via Mobius transformations. Take the "trivial" n-cover $\mathbb{H} \times [n]$.

For any homomorphism $\varphi : \Gamma \to S_n$, let Γ act via

$$\gamma(z, i) = (\gamma z, \varphi(\gamma) i)$$

and we quotient via this action to obtain X_φ.

Select generators $\langle \gamma_1, \dots, \gamma_d \rangle = \Gamma$. We randomize X_φ by picking $\sigma_1, \dots, \sigma_d \overset{iid}{\sim} U S_n$ and letting $\varphi : \gamma_i \longmapsto \sigma_i$, $1 \le i \le d$.



e.g. $n=2$, identify pairs of fundamental domains within / across $\mathbb{H}$s.

Some example Γs: $\Gamma(N)$, $N \geq 3$, of index $n^3 \prod_{p|n} 1 - 1/p^2$ in $SL(2,\mathbb{Z})$; $\left\langle \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ 2 & 1 \end{pmatrix} \right\rangle \cong \mathbb{F}_2$ of index 12 in $SL(2,\mathbb{Z})$.

X_φ is of course connected precisely when $\varphi(\Gamma)$ acts transitively on $[n]$.
(Dixon: two random elts. of S_n generate A_n or S_n a.s. in $n \to \infty$.)

$f : X \to \mathbb{C}$ lifts to $\tilde{f} : X_\varphi \to \mathbb{C}$ in the natural way, so that

$$\operatorname{spec} \Delta_X \subset \operatorname{spec} \Delta_{X_\varphi} \text{ as multisets.}$$

Let $\mathcal{L}^2_{new}(X_\varphi) = \{ f \in \mathcal{L}^2(X_\varphi) : \langle f, \tilde{g} \rangle = 0 \ \forall g \in \mathcal{L}^2(X) \}$

(since we already know all of the eigenfunctions/eigenvalues arising from lifting $\mathcal{L}^2(X)$)

(By taking $g \in \mathcal{L}^2(X)$ localized @ $z \in X$, we see that $f \in \mathcal{L}^2_{new}(X_\varphi)$ is mean-0 on z's fiber, for a.e. z)

Importantly, $\mathcal{L}^2(X_\varphi) \cong \widetilde{\mathcal{L}^2(X)} \oplus \mathcal{L}^2_{new}(X_\varphi)$.

Approach to main result WTS $\Delta_{X_\varphi}|_{\mathcal{L}^2_{new}(X_\varphi)}$ has no spectrum in $[0, 1/4 - \varepsilon)$.

If, given $0 \le s \le 1$, $\exists R_{X_\varphi}(s)$ s.t. $(\Delta_{X_\varphi} - s(1-s)) R_{X_\varphi}(s) = \mathrm{id}_{\mathcal{L}^2_{new}(X_\varphi)}$
then Δ_{X_φ} could not possibly have $s(1-s) \in \operatorname{spec} \Delta_{X_\varphi}|_{\mathcal{L}^2_{new}(X_\varphi)}$.

So, we aim to show this for $\frac{1}{2} + \sqrt{\varepsilon} \le s \le 1$; the values taken by $s(1-s)$ are $[0, 1/4 - \varepsilon]$.
Proceed by finding (talk #2.) $\mathbb{M}_\varphi(s)$, $\mathbb{L}_\varphi(s)$ obeying

$$(\Delta_{X_\varphi} - s(1-s)) \mathbb{M}_\varphi(s) = \mathrm{id} + \mathbb{L}_\varphi(s), \qquad \|\mathbb{L}_\varphi(s)\|_{\mathcal{L}^2_{new}(X_\varphi)} < 1$$

so that $R_{X_\varphi}(s) = \mathbb{M}_\varphi(s)(\mathrm{id} + \mathbb{L}_\varphi(s))^{-1}$. Will also use $\mathbb{L}_\varphi$'s compactness.
$\mathbb{M}_\varphi$ is a "parametrix" — approx. sol'n to a PDE. It and $\mathbb{L}_\varphi$ will have cusp and interior parts.

relatively straightforward much work

In particular, we will conjugate $\mathbb{L}^{int}_\varphi(s)$ to $\sum_{\gamma \in \Gamma} a_\gamma(s) \otimes \varphi(\gamma)$
where
- the sum is finitely-supported
- $a_\gamma(s) \in \mathrm{End}(\mathcal{L}^2(F))$ compact (fundamental domain of Γ on $\mathbb{H}$)
- $\varphi(\gamma)$ viewed as $\in \mathrm{End}(\mathrm{mean}\, 0(\mathbb{C}^{[n]}))$

so that by Bordenave-Collins,

$$\|\mathbb{L}^{int}_\varphi(s)\| = \Big\| \sum_{\gamma \in \Gamma} a_\gamma(s) \otimes \varphi(\gamma) \Big\|_{\mathcal{L}^2(F) \otimes \mathrm{mean}\,0(\mathbb{C}^{[n]})}$$

☆ important here that Γ be free, which is why we are in a non-compact setting

$$\le \Big\| \sum_{\gamma \in \Gamma} a_\gamma(s) \otimes \rho_\infty(\gamma) \Big\|_{\mathcal{L}^2(F) \otimes \ell^2(\Gamma)} + \varepsilon' \quad \text{[BC]} \ ☆$$

(right-regular representation.)

for desired ε' ($= 1/20$) and a.s. for n sufficiently large (depending on ε')

☆ Since the a_γ are compact, we approximate them by finite-rank operators and then apply [BC].

Then, the norm term in ☆ can be made to be arbitrarily small. This is done via a "range cutoff parameter" T for which $\|\mathbb{L}^{(T)}_{\mathbb{H}}(s)\| \le \tilde{C} T e^{(\frac{1}{2} - s)T}$, so this explains both why we can take $\le 1/5$ and why $s = 1/2$ will not be attainable. [HM, Lemma 5.2].

The last major step is essentially a ε-net on $s \in [1/2 + \sqrt{\varepsilon}, 1]$:

$\mathbb{L}_\varphi^{int}(s)$ is cts. in s, i.e.

$$\| \mathbb{L}_\varphi^{int}(s_1) - \mathbb{L}_\varphi^{int}(s_2) \| \leq \# \operatorname{supp} a_\gamma \cdot \sup_{\gamma \in \Gamma} \| a_\gamma(s_1) - a_\gamma(s_2) \|_{\mathcal{L}^2(F)}$$
$$\leq \# \operatorname{supp} a_\gamma \cdot C \cdot |s_1 - s_2|$$

so take a finite subset of $[1/2 + \sqrt{\varepsilon}, 1]$ appropriately spaced.

This means that there is uniformity in making n "large enough" across all relevant s values since this only needs to be imposed across a finite set.

$\mathbb{L}_\varphi^{cusp}$ is handled separately, but can also be made arbitrarily small, to allow

$(\mathrm{id} + \mathbb{L}_\varphi^{int}(s) + \mathbb{L}_\varphi^{cusp}(s))^{-1}$

Let's take a moment to discuss why the operator $\mathbb{L}$ has its form compatible w/ Bordenave-Collins.

general form: for $f : \mathbb{H} \to V_0^n$ smooth (mean 0 ($\mathbb{C}^{[n]}$)), $f \in \mathcal{L}^2(F)$, $f(\gamma z) = \rho_\varphi(\gamma) f(z)$, $x \in \mathbb{H}$,

$$\mathbb{L}(s)[f](x) = \int_{\mathbb{H}} \mathbb{L}(s; x, y) f(y) \, d\mathbb{H}(y)$$

$$= \sum_{\gamma \in \Gamma} \int_F \mathbb{L}(s; \gamma x, y) \rho_\varphi(\gamma^{-1}) f(y) \, d\mathbb{H}(y).$$

We have an isomorphism $\mathcal{L}^2_\varphi(\mathbb{H}; V_0^n) \cong \mathcal{L}^2(F) \otimes V_0^n$ (complete w.r.t. $\|\cdot\|_{\mathcal{L}^2(F)}$)

via $\mathcal{U} : \quad f \mapsto \sum_i \langle f|_F, e_i \rangle \otimes e_i$

(i.e. as a function, $\mathcal{U}f(z) = \sum_i \langle f(z), e_i \rangle e_i$ for $z \in F$)

and this is agnostic to the choice of onb. $\{e_i\}$ of V_0^n.

(This will be important shortly!)

We then consider $\mathcal{U} \mathbb{L} \mathcal{U}^{-1}$.

This calculation is not explicitly worked in the paper, so I include it for completeness.

Consider some $f \overset{\mathcal{U}}{\mapsto} \sum_i \langle f|_F, e_i \rangle \otimes e_i = g$.

$$\mathcal{U} \mathbb{L} \mathcal{U}^{-1} g = \mathcal{U} \mathbb{L} f$$

$$= \mathcal{U} \sum_\gamma \int_F \mathbb{L}(s; \gamma \cdot, y) \rho_\varphi(\gamma^{-1}) f(y) \, d\mathbb{H}(y)$$

$$= \sum_\gamma \int_F \mathcal{U}\left(\mathbb{L}(s; \gamma \cdot, y) \rho_\varphi(\gamma^{-1}) f(y)\right) d\mathbb{H}(y)$$

$$= \sum_\gamma \int_F \sum_i \langle \mathbb{L}(s; \gamma \cdot, y) \rho_\varphi(\gamma^{-1}) f(y), e_i \rangle \otimes e_i \, d\mathbb{H}(y)$$

$$= \sum_\gamma \int_F \sum_i \langle \mathbb{L}(s; \gamma \cdot, y) f(y), \rho_\varphi(\gamma) e_i \rangle \otimes e_i \, d\mathbb{H}(y)$$

in here, change $\{e_i\}$ for $\{e_i' = \rho_\varphi(\gamma^{-1}) e_i\}$

$$= \sum_\gamma \sum_i \int_F \mathbb{L}(s; \gamma \cdot, y) \langle f(y), e_i \rangle \, d\mathbb{H}(y) \otimes \rho_\varphi(\gamma^{-1}) e_i$$

and if we define $a_\gamma(s)[f](x) = \int_F \mathbb{L}(s; \gamma x, y) f(y) \, d\mathbb{H}(y)$

then this is exactly $\left(\sum_\gamma a_\gamma(s) \otimes \rho_\varphi(\gamma^{-1})\right) g$ as desired.

Construction-specific details give that the a_γ are Lipschitz (i.e. using $\mathbb{L}$ in §5)

A similar conjugation is done to the representation sum $\sum_\gamma a_\gamma(s) \otimes \rho_\infty(\gamma^{-1})$

using $\mathcal{L}^2(F) \otimes \ell^2(\Gamma) \cong \mathcal{L}^2(\mathbb{H})$

$$f \otimes \delta_\gamma \mapsto f \circ \gamma^{-1}$$

this is 0 on $\mathbb{H} \setminus \gamma F$