last time, proved the following:

<u>thm 6.1</u>: fix self-adjoint non-commutative polynomial P of deg q_0 and let $K = \|P\|_{C^*(F_d)}$. then

$\exists$ a linear functional ν_i on $\mathcal{P}$ $\forall i \in \mathbb{Z}_+$ s.t $\forall N, m, q \in \mathbb{N}$ and $h \in \mathcal{P}_q$:

$$\left| \mathbb{E}[\mathrm{tr}_N h(P(S^N, S^{N*}))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i} \right| \leq \frac{(4qq_0(1+\log d))^{4m}}{N^m} \|h\|_{C^0[-K,K]}$$

*we only proved this for sufficiently large N (based on q), but with a few algebraic tricks, easily extended to all N

our eventual goal: show $\mathbb{E}\,\mathrm{tr}_N \chi(X_n) = o(1/N)$

three steps to do this:

① extend linear functionals ν_i from $\mathcal{P}$ to C^∞

② show extended linear functions can still be used to approximate $\mathbb{E}\,\mathrm{tr}\, h(X_N)$ for $h \in C^\infty$

③ evaluate expression for $\mathbb{E}\mathrm{tr}_N \chi(X_N)$ *reduces to evaluating supp ν_1 (for $m=2$) — David will cover this

↳ for $m=2$: $|\mathbb{E}[\mathrm{tr}_N \chi(X_N)] - \cancel{\tau(\chi(X_F))}^{\,0} - \frac{1}{N}\nu_1(\chi)| \leq O(\frac{1}{N^2})$

<u>thm 7.1</u>: P, q_0, K same as in thm 6.1. then $\exists$ compactly supported distributions ν_i $\forall i \in \mathbb{Z}^+$ s.t:

$$\left| \mathbb{E}[\mathrm{tr}_N(h(P(S^N, S^{N*}))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i} \right| \lesssim \frac{(4q_0(1+\log d))^{4m}}{N^m} \beta_* \|f^{(4m+1)}\|_{L^\beta[0,2\pi]}$$

*$\lesssim$ hiding constant coming from analytic lemma

$\forall N, m \in \mathbb{N}$, $\beta > 1$, $h \in C^\infty(\mathbb{R})$ where $f(\theta) := h(K\cos\theta)$ and $1/\beta_* = 1 - 1/\beta$

pf: idea: · extend ν_i to $h \in C^\infty$

· use thm 6.1 to show above for $h \in \mathcal{P}$ *main hurdle

· extend to all $h \in C^\infty$ by density

what does "compactly supported distribution" mean?

<u>def</u>: linear functional ν on $C^\infty(\mathbb{R})$ is a compactly supported distribution if $\exists c, K > 0$ and $m \in \mathbb{Z}_+$ s.t:

$$|\nu(h)| \leq c\|h\|_{C^m[-K,K]} \quad \forall h \in C^\infty(\mathbb{R})$$

why does this matter to us? it will be helpful for:

· understanding supp ν

· extending ν_i

we aren't going to go too heavily into the details on extending ν_i except to say:

· using our work last time, we can show ν_i is compactly supported on $[-K,K]$ for $h \in \mathcal{P}$

· we can think of ν_i compactly supported on $[-K,K]$ as ν_i only "cares" about function's behavior on $[-K,K]$ *if $f = g$ on $[-K,K]$ then $\nu(f) = \nu(g)$

· since $\mathcal{P}$ are dense on $C^\infty([-K,K])$, there is a natural extension of ν_i to C^∞, and we should expect ν_i to only "care" about $[-K,K]$

*details in lemma 4.7

now we turn to showing claimed error bound in thm 7.1.

from thm 6.1, for $h \in \mathcal{P}_q$ we have:

$$\left| \mathbb{E}[\mathrm{tr}_N h(P(S^N, S^{N*}))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i} \right| \leq \frac{(4qq_0(1+\log d))^{4m}}{N^m} \|h\|_{C^0[-K,K]}$$

this bound is q dependent though — our first step is to try to remove q dependence (and hopefully show desired bound for $h \in \mathcal{P}$).

we still want to use thm 6.1, but instead of applying it directly to h, we'll apply it to Chebyshev polynomials + relate h to Chebyshev polynomials

we are only concerned with the behavior of h on $[-K,K]$. we can express h as:

$$h(x) = \sum_{j=0}^{q} a_j T_j(K^{-1}x)$$ since $\{T_0(K^{-1}\cdot), \dots, T_q(K^{-1}\cdot)\}$ form a basis for $\mathcal{P}^q[-K,K]$

now we might try to apply thm 6.1 to just $T_i(K^{-1}\cdot)$. by Δ inequality, we have:

$$\left| \mathbb{E}[\mathrm{tr}_N h(P(S^N, S^{N*}))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i} \right| \leq \sum_{j=0}^{q} \frac{(4jq_0(1+\log d))^{4m}}{N^m} \|T_j\|_{C^0[-K,K]} = \frac{(4q_0(1+\log d))^{4m}}{N^m} \sum_{j=1}^{q} j^{4m} |a_j|$$ *dropped $j=0$ ✓

how can we understand this sum?

since $T_j(\cos\theta) = \cos(j\theta)$, we have:

$$h(K\cos\theta) = \sum_{j=0}^{q} a_j T_j(\cos\theta) = \sum_{j=0}^{q} a_j \cos(j\theta)$$ *a Fourier series!

the following gives us a way to understand these $\{a_j\}$:

<u>lemma 4.4:</u> $h \in \mathcal{P}_q$, $f(\theta) := h(K\cos\theta)$ then:

$|a_0| \leq \|h\|_{C^0[-K,K]}$ and $\sum_{j=1}^{q} j^m |a_j| \lesssim \beta_* \|f^{(m+1)}\|_{L^\beta[0,2\pi]}$

$\forall m \in \mathbb{Z}^+$ and $\beta > 1$, β_* s.t. $1/\beta_* = 1 - 1/\beta$

* classical Fourier analytic result – couldn't find pf, but uses some sort of Hölder

applying this for $h \in \mathcal{P}$ we have: $\left|\mathbb{E}[\mathrm{tr}_N(h(P(S^N, S^{N*})))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i}\right| \lesssim \frac{(4q_0(1+\log d))^{4m}}{N^m} \beta_* \|f^{(4m+1)}\|_{L^\beta[0,2\pi]}$

then by density of $\mathcal{P}$ in $C^\infty([-K,K])$, we're done.