our setting: *working with random permutation matrices

def: $\tilde{S}_1^N, \dots, \tilde{S}_d^N$ i.i.d. random permutation matrices of dim N

$S_i^N := \tilde{S}_i^N |_{\{1_N\}^\perp}$ *throwing away trivial eigenvalue

$S^N = (S_1^N, \dots, S_d^N)$

↳ these are the terms in our sequence. what is limiting object?

formal def: F_d (free group of rank d), generators $g_1, \dots, g_d$

$\lambda : F_d \to B(\ell^2(F_d))$ via: $g \mapsto \lambda(g)$ s.t. $\lambda(g)\delta_h = \delta_{gh}$ where $\delta_w(x) = \begin{cases} 1 & x=w \\ 0 & \text{o.w.} \end{cases}$

then $s = (s_1, \dots, s_d)$ where $s_i = \lambda(g_i)$

recall: Tr, tr_N (normalized trace)

def: τ (∞-dim analog of tr_N): $\tau(a) = \langle \delta_e, a\delta_e \rangle$ on $B(\ell^2(F_d))$

architecture of argument:

· goal: show $P(\|X_N\| \geq \|X_F\| + \varepsilon) = o(1)$ (X_N ← $P(S^N)$, X_F ← $P(s)$)

↳ suffices to show $\mathbb{E}\,\mathrm{tr}_N\, \chi(X^N) = o(1/N)$ (χ ← smooth indicator for $[-\|X_F\|, \|X_F\|]^c$) *we'll show $\propto \frac{1}{N^2}$

· steps to do this:

① show $\forall h \in \mathcal{P}_q$ $\mathbb{E}\,\mathrm{tr}_N(h(X_N)) = \Psi_h(\frac{1}{N})$

$= \Psi_h(0) + \frac{1}{N}\Psi_h'(0) + O(\frac{1}{N^2})$ for some rational function Ψ_h

② extend this result to any smooth function

↳ let $\nu_0(h) = \Psi_h(0)$, $\nu_1(h) = \Psi_h'(0)$ we can think of these as linear functionals

③ bound $\mathrm{supp}\,\nu_1$

today we'll focus in on ①

two things we'll need to do here:

· show $\mathbb{E}\,\mathrm{tr}_N(h(X_N)) = \Psi_h(\frac{1}{N})$ for rational function Ψ_h *section 5

· analyze Taylor series *section 6 (+4)

↳ in particular, provide error bound on truncation

note: in general Ψ_h will be very complicated and difficult to understand

our approach will use relatively little about Ψ_h, except that it is rational, bounds on its denominator, and its behavior at $x = 1/N$

---

new for today:

how will we show rationality? show for monomials, then pass to polynomials bc of linearity

we start with the following:

lemma 5.1: let $w \in W_d$ be any word of length at most q and $N \geq q$. then: (→ need this for our counting method)

$$\mathbb{E}[\mathrm{tr}_N\, w(S^N)] = f_w(1/N)/g_q(1/N)$$

where f_w, g_q polynomials with $\deg f_w, \deg g_q \leq q(1+\log d)$ and:

$$g_q(x) = (1-x)^{d_1}(1-2x)^{d_2} \cdots (1-(q-1)x)^{d_{q-1}} \quad \text{where} \quad d_j = \min(d, \lfloor \tfrac{q}{j+1} \rfloor)$$

main idea of pf: use combinatorial interpretation of this quantity

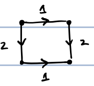
pf: notice $\mathbb{E}[\mathrm{Tr}\, w(S_1^N, \dots, S_d^N)] = \mathbb{E}[\mathrm{Tr}\, w(\tilde{S}_1^N, \dots, \tilde{S}_d^N)] - 1 = \underbrace{\text{exp \# of fixed points of } w(S^N)}_{\text{counts 1s on the diagonal ✓}} - 1$

this becomes a combinatorial problem - luckily one someone else has thought about!

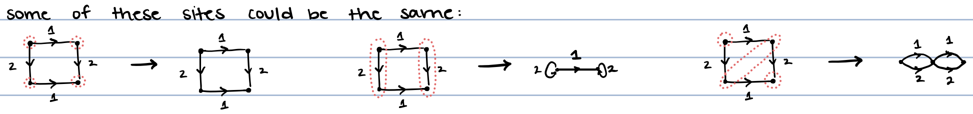
let's look at an example to see how we could think about this

ex: commutator: $w(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$ *WARNING: composing inverses from left to right - unconventional but fine

if we had fixed point, what could path look like? *universal graph



but some of these sites could be the same:

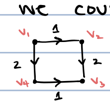


→ we actually won't care about #

and several other possibilities

each "excursion" will fall into one of these categories

now we count the # of inputs that could result in a specific category of excursion



category Γ

first label the vertices: $\binom{n}{v_\Gamma}$ v_Γ = # of vertices in Γ

now the possible inputs to induce this on chosen vertices:

$\prod_{j=1}^{d} (n - e_j^\Gamma)!$ where e_j^Γ = # of j edges in Γ

↳ each edge fixes action of σ on a distinct element using this framework we see:

$$\mathbb{E}[\mathrm{Tr}\, w(S_1^N, \ldots, S_d^N)] = \frac{1}{(N!)^d} \sum_{\Gamma \in Q_w} \binom{N}{v_\Gamma} \cdot \prod_{j=1}^{d} (N - e_j^\Gamma)!$$

$$= \sum_{\Gamma \in Q_w} \frac{N(N-1)\cdots(N-v_\Gamma+1)}{\prod_{j=1}^{d} N(N-1)\cdots(N-e_\Gamma^j+1)}$$

$$= \sum_{\Gamma \in Q_w} \left(\frac{1}{N}\right)^{e_\Gamma - v_\Gamma} \frac{\prod_{\ell=1}^{v_\Gamma-1}\left(1-\frac{\ell}{N}\right)}{\prod_{j=1}^{d}\prod_{\ell=1}^{e_\Gamma^j-1}\left(1-\frac{\ell}{N}\right)}$$

pull out a bunch of factors on N ⇝ rational function of $1/N$!

so we have:

$$\mathbb{E}[\mathrm{tr}_N\, w(S^N)] = -\frac{1}{N} + \sum_{\Gamma} \frac{\left(\frac{1}{N}\right)^{e_\Gamma - v_\Gamma+1}\prod_{\ell=1}^{v_\Gamma-1}\left(1-\frac{\ell}{N}\right)}{\prod_j\prod_\ell (1-\ell/N)}$$

* from we can clearly see $\mathbb{E}[\mathrm{tr}_N\, w(S^N)]$ is a rational function of $1/N$, but we want denom to be independent of w

we want the denominator to be independent of w (so also independent of Γ). effectively, we just find the common denominator of these terms. as a function of $1/N$, we'll have terms:

$$(1-x),\ (1-2x),\ \ldots,\ (1-(q-1)x) \qquad \text{since } e_\Gamma^j \le q$$

now we define the multiplicity of each term for a category Γ using the following parameter:

$$d_i^\Gamma = |\{1 \le j \le d : e_\Gamma^j \ge i+1\}| \le d_i = \min\{d, \lfloor \tfrac{q}{j+1} \rfloor\}$$ * since $d_i^\Gamma \ge d_{i-1}^\Gamma$ and $\sum e_\Gamma^i \le q$

since d_i is an upper bound on multiplicity and is independent of Γ, we use this to construct our common denominator:

$$g_q = (1-x)^{d_1}(1-2x)^{d_2}\cdots(1-(q-1)x)^{d_{q-1}}$$

then we can write:

$$\mathbb{E}[\mathrm{tr}_N\, w(S^N)] = -\frac{1}{N} + \sum_{\Gamma} \frac{\left(\frac{1}{N}\right)^{e_\Gamma - v_\Gamma+1}\prod_{\ell=1}^{v_\Gamma-1}\left(1-\frac{\ell}{N}\right)}{\prod_j\prod_\ell (1-\ell/N)}$$

$$= \left[-\frac{1}{N} g_q\!\left(\frac{1}{N}\right) + \sum_{\Gamma}\left(\frac{1}{N}\right)^{e_\Gamma - v_\Gamma+1}\prod_{\ell=1}^{v_\Gamma-1}\left(1-\frac{\ell}{N}\right)\cdot\prod_{i=1}^{q-1}\left(1-\frac{i}{N}\right)^{d_i - d_i^\Gamma}\right] \Big/ g_q(1/N)$$

to analyze the degrees:

$$\deg g_q = \sum_{i=1}^{q-1} d_i \le \int_1^q \min(d, q/x) = \int_1^{q/d} d + \int_{q/d}^{q} q/x \le q(1+\log d) - d$$

$$\deg f = e_\Gamma - v_\Gamma + 1 + v_\Gamma - 1 + \sum d_i - d_i^\Gamma = \underbrace{e_\Gamma - \sum d_i^\Gamma}_{+1 \text{ for each } e_i^\Gamma} + \sum d_i = \min(d,q) + \sum d_i = q(1+\log d)$$

immediately, by linearity this extends to $h \circ P$ for any polynomial h

* note since $g_{q-1} \mid g_q$, we can write Ψ_h with denominator g_q ✓

now we've shown Ψ_h is a rational function. we'd like to take a Taylor series and be able to truncate higher order terms, which means bounding the error

<u>thm 6.1:</u> fix self-adjoint non-commutative polynomial P of deg q_0 and let $K = \|P\|_{C^*(F_d)}$. then $\exists$ a linear functional ν_i on $\mathcal{P}$ $\forall i \in \mathbb{Z}_+$ s.t. $\forall N, m, q \in \mathbb{N}$ and $h \in \mathcal{P}_q$:

$$\left|\mathbb{E}[\mathrm{tr}_N\, h(P(S^N, S^{N*}))] - \sum_{i=0}^{m-1} \frac{\nu_i(h)}{N^i}\right| \le \frac{(4qq_0(1+\log d))^{3+m}}{N^m} \|h\|_{C^0[-K,K]}$$

* they prove a slightly more general statement (allowing different kinds of P), but proof is largely the same
* break out h and P because we want to extend to smooth h

pf: $\mathbb{E}[\mathrm{tr}_N\, h(P(S^N, S^{N*}))]$ is a linear combination of words of S_N, so by above, for sufficiently large N:

↳ we will only show for large N

$$\mathbb{E}[\mathrm{tr}_N\, h(P(S^N, S^{N*}))] = \Psi_h(1/N)$$

we want to relate this to the limiting quantity, so we do a Taylor expansion around $x = 0$.

by Taylor's thm:

$$\left|\mathbb{E}[\mathrm{tr}_N\, h(P(S^N, S^{N*}))] - \sum_{k=0}^{m-1} \frac{\Psi_h^{(k)}(0)}{k!}\left(\frac{1}{N}\right)^k\right| \le \frac{\|\Psi_h^{(m)}\|_{C^0[0,1/N]}}{m!} \frac{1}{N^m}$$

we define:

$$\nu_k(h) = \frac{\Psi_h^{(k)}(0)}{k!}$$ * since $\Psi_{h+\lambda g} = \Psi_h + \lambda \Psi_g$, this is a linear functional *linear functional were useful for extension

now we need to bound $\|\Psi_h^{(m)}\|_{C^0[0,1/N]}$

we've come to a purely analytical question: how can we understand higher derivatives of rational functions?

our key tool:

<u>Markov brother's inequality:</u> for $p \in \mathcal{P}_q$, $a > 0$, $m \in \mathbb{N}$ we have: $\|p^{(m)}\|_{C^0[0,a]} \le \|T_q^{(m)}\| \cdot \|p\|_{C^0[0,a]}$

← Chebyshev polynomial

* several pfs of this exist using various techniques
* for a slightly weaker result, one very clean approach:
  - express p in terms of Chebyshev polynomial
  - apply Δ ineq + bounds on coef w.r.t. $\|p\|$
  - use bounds on $\|T_k'\|$ ✓

we want to use this, but we want to apply it to a rational function

obviously if g was constant then above would hold, and it turns out if g is ~ constant, we can find a bound: ← g is ~ constant

<u>rational Markov:</u> $r = f/g$, $f, g \in \mathcal{P}_q$, $a > 0$, $m \in \mathbb{N}$. suppose $c := \sup_{x,y \in [0,a]} \left|\frac{g(x)}{g(y)}\right| < \infty$. then:

$$\|r^{(m)}\|_{C^0[0,a]} \leq m! \left(\frac{5cq^2}{a}\right)^m \|r\|_{C^0[0,a]}$$

<u>pf:</u> <u>idea</u>: apply pdt rule to $f = rg$, then apply Markov brother's

inductively applying the pdt rule to $f = rg$, we can derive:

$$r^{(m)} g = f^{(m)} - \sum_{k=1}^{m} \binom{m}{k} r^{(m-k)} g^{(k)}$$

we want to isolate $r^{(m)}$ and look at the norm:

$$r^{(m)} = \frac{f^{(m)}}{g} - \sum_{k=1}^{m} \binom{m}{k} r^{(m-k)} \frac{g^{(k)}}{g}$$

$$\|r^{(m)}\| \leq \left\|\frac{f^{(m)}}{g}\right\| + \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \left\|\frac{g^{(k)}}{g}\right\|$$ * by Δ ineq where $\|\cdot\| = \|\cdot\|_{C^0[0,a]}$

we need to deal with $\|\cdot\|$ of a quotient — recall we have: $c := \sup_{x,y \in [0,a]} \left|\frac{g(x)}{g(y)}\right|$ so for any h:

$$\left\|\frac{h}{g}\right\| \leq \frac{\|h\|}{\inf|g|} \leq c\frac{\|h\|}{\|g\|}$$

applying above + Markov brother's to the $\frac{f^{(m)}}{g}$ and $\frac{g^{(k)}}{g}$ terms yields:

$$\begin{aligned}
\|r^{(m)}\| &\leq c\left[\frac{\|f^{(m)}\|}{\|g\|} + \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \frac{\|g^{(k)}\|}{\|g\|}\right] \\
&\leq \frac{c}{\|g\|}\left[\|T_q^{(m)}\| \cdot \|f\| + \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \cdot \|T_q^{(k)}\| \cdot \|g\|\right] \\
&= c\left[\|T_q^{(m)}\| \cdot \frac{\|f\|}{\|g\|} + \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \cdot \|T_q^{(k)}\|\right] \\
&\leq c\left[\|T_q^{(m)}\| \cdot \|r\| + \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \cdot \|T_q^{(k)}\|\right] \\
&\leq 2c \sum_{k=1}^{m} \binom{m}{k} \|r^{(m-k)}\| \cdot \|T_q^{(k)}\|
\end{aligned}$$

now we have an expression relating $\|r^{(m)}\|$ to lower derivatives. we prove the desired statement by induction.
this reduces to just analyzing a sum (omitted)

this allows us to reduce the problem to bounding $\|\psi_n\|_{C^0[0,1/M]}$, if we can show g_q is approximately constant. since we know exactly what g_q is, we can see this easily:

by the def of g_q $\forall x \in [0, 1/N]$ for N sufficiently large:

$$(1-qx)^{q(1+\log d)-1} \leq g_q(x) \leq 1 \quad \rightsquigarrow \quad e^{-1} \leq g_q(x) \leq 1 \quad ✓$$

now we need to bound $\|\psi_n\|_{C^0[0,1/N]}$. by definition:

$$\psi_n(1/N) = \mathbb{E}\, \mathrm{tr}_N(h(X_N))$$ * $X_n = P(S^N)$

and by functional calculus this gives us:

$$|\psi_n(1/N)| \leq \|h\|_{C^0[-K,K]}$$ * $\|P\| \leq K$

we have control of ψ_n on discrete set of points — the following helps us extend to the full norm:

<u>lemma 4.2</u>: (interpolation) $p \in \mathcal{P}_q$. fix a subset $I \subseteq [0,a]$ s.t. $\delta := \sup_{x \in [0,a]} \inf_{y \in I} |x-y|$ satisfies $4q^2\delta \leq a$. then: (measure of how fine mesh is)

$$\|p\|_{C^0[0,a]} \leq 2 \sup_{x \in I} |p(x)|$$

<u>pf:</u> $\forall y \in [0,a]$:

$$\begin{aligned}
|p(y)| &\leq \sup_{x \in I} |p(x)| + \delta \|p'\|_{C^0[0,a]} \\
&\leq \sup_{x \in I} |p(x)| + \frac{a}{4q^2} \|T_q'\| \cdot \|p\|_{C^0[0,a]} \quad \text{by Markov brother's} \\
&\leq \sup_{x \in I} |p(x)| + \frac{1}{2} \|p\|_{C^0[0,a]}
\end{aligned}$$

$$\Rightarrow \|p\|_{C^0[0,a]} \leq 2 \sup_{x \in I} |p(x)| \quad ✓$$

using this lemma, for M sufficiently large (so mesh is fine enough) on $[0, 1/M]$ we have: ↦ $M = 2(qq_0(1+\log d))^2$

$$|f_n(1/N)| \leq |\psi_n(1/N)| \leq \|h\|_{C^0[-K,K]}$$ * on mesh, since $g_q(1/N) \leq 1$

$$\|f_n\|_{C_0[0,1/M]} \leq 2\|h\|_{C^0[-K,K]}$$ * by interpolation lemma

$$\Rightarrow \|\psi_n\|_{C^0[0,1/M]} \leq 2e\|h\|_{C^0[-K,K]}$$ * using l.b. on $g_q(x)$

applying bound from rational Markov, we are done.

<u>details:</u> letting $M = 2(qq_0(1+\log d))^2$ and recalling $\deg f \leq qq_0(1+\log d)$ we have:

$$\begin{aligned}
\frac{\|\psi^{(m)}\|_{C^0[0,1/M]}}{m!} &\leq (5M\,(qq_0(1+\log d))^2 e)^m \cdot \|\psi\|_{C^0[0,1/M]} && \text{* rational Markov} \\
&\leq (10e\,(qq_0(1+\log d))^4)^m \; 2e\,\|h\|_{C^0[-K,K]} && \text{* interpolation lemma} \\
&\leq (4qq_0(1+\log d))^{4m} \cdot \|h\|_{C^0[-K,K]} && \text{* } (10e)^m 2e \leq 4^{4m} \;\; \forall m \geq 1
\end{aligned}$$

for $N \geq M$, this bound holds, so the claim is shown ✓