NOTES

This note is about Buser's paper [Bus78] relating cubic graphs to compact Riemann surfaces.

A cubic graph is a graph such that each vertex has degree 3. Let $\mathcal{G}_n$ be the set of cubic graphs with $2n$ vertices. The partition number of a cubic graph G is defined as

$$p(G) = \min \frac{(V^-, V^+))}{|V^-|}$$

where the minimum is taken over a partition of the vertices V of G into $V^- \cup V^+$ with $V^- \cap V^+ = \varnothing$ and $|V^-| \leqslant |V^+|$. Here (V^-, V^+) is the set of edges having one endpoint in V^- and another in V^+. I think this is also called the Cheeger constant. Cheeger's inequality says

$$\frac{p(G)^2}{6} \leqslant 3 - \lambda_2(G) \leqslant 2p(G).$$

This implies that the spectral gap $3 - \lambda_2(G)$ being small is equivalent to $p(G)$ being small.

A compact hyperbolic surface M is a compact smooth Riemannian manifold with constant negative curvature -1. We will only consider orientable surfaces. Their topology are classified by the genus $g \geqslant 2$. But for a fixed genus there are many hyperbolic surfaces. Indeed there is a continuous moduli space $\mathcal{M}_g$ of real dimension $6g-6$. People are interested in the spectral gap of the Laplacian $-\Delta_g$, which has eigenvalues

$$0 < \lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \to \infty.$$

There is a Cheeger inequality which works on any compact manifold:

$$\lambda_1(-\Delta_g) \geqslant \frac{h(M)^2}{4}.$$

Buser proved the following theorem.

Theorem 1. *The following statements are equivalent.*

- $\lim\limits_{g\to\infty} \sup_{M\in\mathcal{M}_g} \lambda_1(M) = 0$;
- $\lim\limits_{n\to\infty} \max_{G\in\mathcal{G}_n} p(G)=0$.

Nowadays we know both are false, due to the existence of expander graphs (and Ramanujan graphs and almost Ramanujan surfaces).

0.1. **From graph to surface.** Suppose we have a graph with big $p(G)$, we would like to construct a compact hyperbolic surface with a big first eigenvalue. The construction of the surface is given by gluing pairs of pants together. Cheeger's inequality $\lambda_1(-\Delta) \geqslant h^2/4$ gives a lower bound on the first eigenvalue $\lambda_1(-\Delta)$. So it suffices to find a surface with a big Cheeger constant.

We will fix a given pair of pants whose boundary geodesics are short and the distances between the boundary geodesics are long. Then we form the surface M_G by gluing the pairs of pants according to the graph G, where a pair of pants corresponds to a vertex and connecting boundary geodesics corresponds to an edge. Suppose we have a partition of the surface $M = M^- \cup M^+$. Since the distances between boundary geodesics are long, it is most efficient to cut using the boundary geodesics. By analyzing a few cases, we may assume we the best cut is essentially given by the boundary geodesics (i.e. this will not increase the Cheeger constant too much). Then the conclusion follows from the assumption on the Cheeger constant of the graph G.

0.2. **From surface to graph.** Assume that $p(G)$ is small when $|G|$ is large. We would like to show $\lambda_1(-\Delta)$ is small. We recall the minimax principle:

$$\lambda_1(-\Delta) = \min_{\int f dx = 0} \frac{\int |\nabla f|^2 dx}{\int |f|^2 dx}.$$

Now we encode the surface using a graph. Buser showed that any compact hyperbolic surface can be written as the union of trigons:

(1) a simply connected geodesic triangle;
(2) or a doubly connected region with one boundary component being a closed geodesic, and another component being a broken geodesic with two break points;

such that the side length is $\leqslant \log 4$ and the area is between $\pi - 6\arcsin(\sqrt{2}/3)$ and $2\pi/3$. From the surface we can then associate a graph G, by mapping each trigon to a vertex and each boundary geodesic to an edge. Since $p(G)$ is small, we can divide the graph G into two parts V^- and V^+ so that the boundary length $|(V^-, V^+)|$ is short. We can then consider the corresponding division of the surface into two parts M^- and M^+ along the boundary geodesics. We consider the function

$$\varphi^{\pm}(x) = \begin{cases} \min\left(\frac{1}{\epsilon}\operatorname{dist}(q, \partial M^-), 1\right), & x \in M^{\pm}, \\ 0, & x \in M^{\mp} \end{cases}$$

and $f = \varphi^+ - \alpha\varphi^-$ so that $\int f dx = 0$. For $\epsilon > 0$ sufficiently small (Buser took $\epsilon = 1/4$):

$$\lambda_1(-\Delta) \leqslant \frac{\int |\nabla f|^2 dx}{\int |f|^2 dx} \lesssim \frac{1}{\epsilon}\frac{|\partial M^-|}{|M|} \lesssim \frac{1}{\epsilon} p(G).$$

The key property here is that the geometry of the trigons we choose are uniformly bounded (we use in the last step).

References

[Bus78] P. Buser, *Cubic graphs and the first eigenvalue of a Riemann surface*, Math. Z. **162** (1978), no. 1, 87–99, ISSN: 0025-5874,1432-1823, DOI: *10.1007/BF01437826*, URL: *https://doi.org/10.1007/BF01437826*.