Quantum Ergodicity on Regular Graphs

Zachary Stier

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In this note, I will present Anantharaman's 'ergodic-theoretic' "Proof 3" (§5) of quantum ergodicity for diagonal operators on regular graphs (theorem statement below). This proof joins two others in reproving the result of Anantharaman–Le Masson. The objective here is to somewhat streamline the presentation of the result through reorganization, slightly simplified notation, and easier proofs due to the cleaner nature of the relevant operators. Please do let me know if you spot any mistakes!

Fix q. Let $\mathbb{T}$ be the $(q+1)$-regular tree. We consider sequences $(G_n(V_n, E_n))_{n\in\mathbb{N}}$ of $(q+1)$-regular non-bipartite graphs arising as finite quotients of $\mathbb{T}$, with $\#V_n$ increasing. d_n is the shortest-path metric on G_n. $\mathcal{A}_G$ is the *unnormalized* adjacency matrix of G. For $x \in V_n$, we say that $r_n(x)$ is the greatest integer R such that the induced subgraph $G_n[\{y \in V_n : d_n(x,y) \leqslant R\}]$ is isomorphic to the $(q+1)$-regular tree of depth R (having leaves at depth R). Say that $r_n = r(G_n) = \min_{x\in V_n} r_n(x)$ is G_n's *injectivity radius*, since the isomorphism with the finite tree gives rise to an injective map $G_n[\{y \in V_n : d_n(x,y) \leqslant r_n\}] \hookrightarrow \mathbb{T}$ for all $x \in V_n$.

Definition 1 (family of expanders)**.** $(G_n)_{n\in\mathbb{N}}$ is a *family of expanders (EXP)* if the spectral gap of the normalized adjacency matrix is bounded below by some constant.

Definition 2 (Benjamini–Schramm convergence)**.** $(G_n)_{n\in\mathbb{N}}$ *converges in the sense of Benjamini–Schramm (BSC)* if for all injectivity radii r, the proportion of V_n of injectivity radius up to r limits in n to 0.

Theorem 3 (quantum ergodicity)

Suppose $(G_n)_{n\in\mathbb{N}}$ satisfies **EXP** and **BSC**. Suppose $\mathcal{A}_{G_n}$ has eigenpairs $\left(\lambda_j^{(n)}, \psi_j^{(n)}\right)$. Consider the vector $\boldsymbol{w}_n \in \mathbb{C}^{V_n}$, $\|\boldsymbol{w}_n\|_\infty \leqslant 1$. Put μ_n as the average of $\boldsymbol{w}_n$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j\in[n]} \left| \sum_{x\in V_n} \boldsymbol{w}_n(x) \psi_j^{(n)}(x)^2 - \mu_n \right|^2 = 0.$$

1 Miscellaneous preliminaries

All balls of radius k in $\mathbb{T}$ have the same size, $(q+1)q^{k-1}$. Call this quantity $v(k)$.

Let $\Gamma_n < \operatorname{Aut}\mathbb{T}$ induce the quotient graph $\Gamma_n\backslash\mathbb{T} = G_n$. Let $\pi_n : V(\mathbb{T}) \to V_n$ be the quotient map, and let $\tilde{V}_n$ be a fixed choice of lifts of V_n under π_n. (In general, elements and subsets of the tree cover will be denoted with tildes.)

Definition 4. Let $\mathcal{H}^{(n)}$ consist of the Γ_n-invariant linear operators on $V(\mathbb{T})$ which are ℓ^2 with respect to the inner product

$$\langle K_1, K_2\rangle_{\mathcal{H}^{(n)}} := \frac{1}{\#V_n}\sum_{\substack{\tilde{x}\in\tilde{V}_n\\ \tilde{y}\in V(\mathbb{T})}} \overline{K_1(\tilde{x},\tilde{y})}K_2(\tilde{x},\tilde{y}).$$

Introduce also the decomposition

$$\mathcal{H}^{(n)} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_k^{(n)}$$

for $\mathcal{H}_k^{(n)}$ those $K \in \mathcal{H}^{(n)}$ such that if $d_n(\tilde{x},\tilde{y}) \neq k$ then $K(\tilde{x},\tilde{y}) = 0$.

$\left(\mathcal{H}_k^{(n)}\right)_{k\in\mathbb{N}}$ is clearly a decomposition of $\mathcal{H}^{(n)}$ into orthogonal subspaces.

When we are handed a test function w_n, we realize it as a Γ_n-invariant diagonal operator W_n (an element of $\mathcal{H}^{(n)}$) by putting $W_n(\gamma\tilde{x},\gamma\tilde{x}) = w_n(\pi_n(\tilde{x}))$ for all $\tilde{x} \in \tilde{V}_n$ and $\gamma \in \Gamma_n$.

When we wish for $K \in \mathcal{H}^{(n)}$ to act on $\ell^2(V_n)$, we put

$$K_G(x,y) := \sum_{\gamma\in\Gamma} K(\tilde{x},\gamma\tilde{y})$$

for fixed lifts $\tilde{x} \in \pi_n^{-1}(\tilde{x})$ and $\tilde{y} \in \pi_n^{-1}(\tilde{y})$.

In general we will consider only one graph at a time (arising from some n) and as such will drop the sub-/superscripts $_n$ and $^{(n)}$ as appropriate.

Definition 5. For $K \in \mathcal{H}$, we write $\mathcal{C}(K) := [\mathcal{A}_{\mathbb{T}}, K]$.

Lemma 6
$\mathcal{C}$ is a bounded operator.

This is immediate from the definition, since $\|\mathcal{A}_{\mathbb{T}}\| = q+1$.

Definition 7 (Hilbert–Schmidt norm)**.** For $K \in \mathcal{H}$, let

$$\|K\|_{\mathrm{HS}}^2 := \frac{1}{\#V}\sum_{\tilde{x}\in\tilde{V}}\left|\sum_{\tilde{y}\in V(\mathbb{T})} K(\tilde{x},\tilde{y})\right|^2.$$

That is, $\|\cdot\|_{\mathrm{HS}}$ is the matrix norm after K_G "collapses" K to be a functional on $V_n \times V_n$.

2 Reduction to mean-zero

As alluded to earlier, pick some n and drop subsequent reference to it.

We first establish that it suffices to consider only $\boldsymbol{w}$ with $\mu = 0$. Indeed, if $\boldsymbol{w}'$ has mean μ and induces W', then $\langle \psi_j, W'_G \psi_j \rangle - \mu = \langle \psi_j, (W'_G - \mu \mathbb{I}_V)\psi_j \rangle$, where $\mathbb{I}_V$ is the identity operator. Thus it suffices to prove for $W'' := W' - \mu \mathbb{I}_V$ that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j\in[n]} \left| \langle \psi_j, W''_G \psi_j \rangle \right|^2 = 0. \tag{1}$$

The condition that the largest diagonal entry lies in $\mathbb{D}$ could in principle be violated, but replacing now W'' by $W := \frac{1}{2}W''$ (and then at the very end multiplying back by 2) resolves the concern.

3 Case of mean-zero

We are motivated by (1) to study the following:

Definition 8 (quantum variance). Suppose $\mathcal{A}_G$'s eigenpairs are (λ_j, ψ_j). Then the *quantum variance* of $K \in \mathcal{H}$ is

$$\mathrm{QVar}(K) := \frac{1}{\#V} \sum_{j\in[\#V]} \left| \langle \psi_j, K_G \psi_j \rangle \right|^2.$$

Lemma 9

For all $K \in \mathcal{H}$:

(a) $\mathrm{QVar}(K) \leqslant \|K\|_{\mathrm{HS}}^2$;

(b) $\mathrm{QVar}(\mathcal{C}(K)) = 0$;

(c) if $\mathrm{QVar}(K') = 0$ then for any K, $\mathrm{QVar}(K + K') = \mathrm{QVar}(K)$.

Proof. (a) Notice that $\|K\|_{\mathrm{HS}}^2 = \mathrm{Tr}(K_G^\dagger K_G) = \mathrm{Tr}(U^\dagger K_G^\dagger K_G U) = \mathrm{Tr}\Big((U^\dagger K_G U)^\dagger (U^\dagger K_G U) \Big) = \|U^\dagger K_G U\|_{\mathrm{HS}}^2$. In particular, when U is the change of basis matrix to $\{\psi_j\}_{j\in[\#V]}$, $\|U^\dagger K_G U\|_{\mathrm{HS}}^2 = \frac{1}{\#V} \sum_{i,j\in[\#V]} \left| \langle \psi_i, K_G \psi_j \rangle \right|^2 \geqslant \frac{1}{\#V} \sum_{j\in[\#V]} \left| \langle \psi_j, K_G \psi_j \rangle \right|^2 = \mathrm{QVar}(K)$.

(b) We have $\psi_j^\top \mathcal{A}_G K_G \psi_j - \psi_j^\top K_G \mathcal{A}_G \psi_j = (\lambda_j \psi_j)^\top K_G \psi_j - \psi_j^\top K_G (\lambda_j \psi_j) = 0$ for all $j \in [n]$.

(c) The hypothesis implies that for all j, $\psi_j^\top K'_G \psi_j = 0$ so $\psi_j^\top (K_G + K'_G)\psi_j = \psi_j^\top K_G \psi_j$. ■

By boundedness of $\mathcal{C}$, in

$$\frac{1}{T} \int_0^T \sum_{j\leqslant M} \frac{(it)^j}{j!} \mathcal{C}^j(W) \, \mathrm{d}t$$

the sum and the integral commute. Hence it equals

$$\sum_{j\leqslant M}\frac{(iT)^j}{(j+1)!}\mathcal{C}^j(W) \tag{2}$$

which is an M-term approximation to $\frac{1}{T}\int_0^T \exp(it\mathcal{C})(W)\,\mathrm{d}t$. However, only the $j=0$ term in (2) survives taking QVar, by **Lemma 12**(b,c), i.e.

$$\mathrm{QVar}(W)=\mathrm{QVar}\left(\frac{1}{T}\int_0^T\sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j(W)\,\mathrm{d}t\right). \tag{3}$$

We will use the bound $\mathrm{QVar}(K)\leqslant\|K\|_{\mathrm{HS}}^2$ and subsequently **Proposition 11** to bound the quantum variance.

Recall that the sequence of graphs are all β-spectral expanders (**BSC** says that there exists some such $\beta>0$).

Lemma 10

For all W with $\langle \boldsymbol{w},\mathbf{1}\rangle=0$,

$$\left\|\frac{1}{T}\int_0^T\exp(it\mathcal{C})(W)\,\mathrm{d}t\right\|_{\mathcal{H}}\leqslant 2\sqrt{\frac{2}{T}}\frac{1}{\sqrt[4]{(q+1)\beta}}\|W\|_{\mathcal{H}}.$$

Proof. WLOG let $\|W\|_{\mathcal{H}}=1$. (If $\|W\|_{\mathcal{H}}=0$ then the result is trivial, and otherwise all computations are homogeneous in scalars.)

For a Borel set $E\subset\mathbb{R}$, let μ^E be the spectral projector associated to E which takes as input any self-adjoint operator on (not in) $\mathcal{H}$. In particular, if $E_1:=[-\delta,\delta]$ and $E_2:=\mathbb{R}\smallsetminus E_1$, then putting $W_j:=\mu^{E_j}(\mathcal{C})W$, we get $W=W_1+W_2$ and $\langle W_1,W_2\rangle_{\mathcal{H}}=0$ so that $\|W_1\|_{\mathcal{H}}^2+\|W_2\|_{\mathcal{H}}^2=1$, in particular $\|W_j\|_{\mathcal{H}}\leqslant 1$. Then, we readily compute

$$\left\|\frac{1}{T}\int_0^T\exp(it\mathcal{C})W_2\,\mathrm{d}t\right\|_{\mathcal{H}}\leqslant\frac{1}{T}\left|\int_0^T\exp(it\delta)\,\mathrm{d}t\right|=\frac{|\exp(i\delta T)-1|}{\delta T}\leqslant\frac{2}{\delta T}$$

where, thinking of the operators as coming from the discrete spectrum, we have that W_2 is the weighted sum of "eigenvectors" of $\mathcal{C}$ which appear in W, and $\exp(it\mathcal{C})$ stretches the λ-eigenspace by $\exp(it\lambda)$; the eigenspaces are pairwise orthogonal, and have initial ℓ^2-sum ($\mathcal{H}$-norm) of 1, so we sum these but then bound by the most extreme case, that is, $\lambda=\delta$. This accounts for the first inequality; the rest is just calculus.

We note now that $\ker\mathcal{C}\cap\mathcal{H}_0=\mathbb{C}\mathbb{I}_V$, i.e. $W\in(\ker\mathcal{C})^\perp$. W_1 must also be diagonal, since the spectral projector will preserve $\mathcal{C}$'s 0 eigenspace and its complement. Again because of the projector's action, $\|\mathcal{C}(W_1)\|_{\mathcal{H}}\leqslant\delta$. We aim to upper bound $\|W_1\|_{\mathcal{H}}=\|\boldsymbol{w}_1\|_2$, supposing W_1's diagonal is specified by $\boldsymbol{w}_1\in\mathbb{C}^V$. The key is to note that $\|\mathcal{C}(W_1)\|_{\mathcal{H}}^2=\boldsymbol{w}_1^\dagger L\boldsymbol{w}_1\geqslant(q+1)\beta\|\boldsymbol{w}_1\|_2^2$, where L is the unnormalized Laplacian of G, which we assumed to be a β-spectral expander. Hence $\|\boldsymbol{w}_1\|_2\leqslant\frac{\delta}{\sqrt{(q+1)\beta}}$. In evaluating the norm of the integral, we note the unitarity of $\exp(it\mathcal{C})$ and so bound the integrand by these constants (independent of t) to obtain

$$\left\|\frac{1}{T}\int_0^T\exp(it\mathcal{C})W_1\,\mathrm{d}t\right\|_{\mathcal{H}}\leqslant\frac{\delta}{\sqrt{(q+1)\beta}}.$$

Finally, apply the triangle inequality:

$$\left\|\frac{1}{T}\int_0^T \exp(it\mathcal{C})W\,\mathrm{d}t\right\|_{\mathcal{H}} \leqslant \frac{\delta}{\sqrt{(q+1)\beta}} + \frac{2}{\delta T}$$

and select δ to minimize the right-hand side, obtaining the claimed result. ■

Let $T = T(\varepsilon)$ satisfy, from **Lemma 10**,

$$\left\|\frac{1}{T}\int_0^T \exp(it\mathcal{C})(W)\,\mathrm{d}t\right\|_{\mathcal{H}} \leqslant \varepsilon\|W\|_{\mathcal{H}}. \tag{4}$$

Let $M = M(\varepsilon)$ satisfy, from $\mathcal{C}$'s boundedness,

$$\left\|\exp(it\mathcal{C}) - \sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j\right\|_{\mathcal{H}\to\mathcal{H}} \leqslant \varepsilon \tag{5}$$

on $t\in[0,T]$. Applying the triangle inequality to $\|\cdot\|_{\mathcal{H}}$, (4) and (5),

$$\left\|\frac{1}{T}\int_0^T \sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j(W)\,\mathrm{d}t\right\|_{\mathcal{H}} \leqslant 2\varepsilon\|W\|_{\mathcal{H}}. \tag{6}$$

In light of **Lemma 9**(a), we want to relate $\|\cdot\|_{\mathrm{HS}}$ and $\|\cdot\|_{\mathcal{H}}$.

Proposition 11

Suppose $K\in\mathcal{H}$ is supported at up to distance k, that is, if $d(\tilde{x},\tilde{y}) > k$ then $K(\tilde{x},\tilde{y}) = 0$. If $k < r(G)$, then $\|K\|_{\mathrm{HS}} = \|K\|_{\mathcal{H}}$. In general,

$$\|K\|_{\mathrm{HS}}^2 \leqslant \|K\|_{\mathcal{H}}^2 + v(k)^2\|K\|_\infty^2\frac{\#\{x\in V : r(x)\leqslant k\}}{\#V}.$$

Proof. For the first case, we know that

$$\|K\|_{\mathrm{HS}}^2 = \frac{1}{\#V}\sum_{\tilde{x},\tilde{y}\in\tilde{V}}\left|\sum_{\gamma\in\Gamma}K(\tilde{x},\gamma\tilde{y})\right|^2. \tag{7}$$

By the definition of r, if $k < r(G)$ then $\sum_{\gamma\in\Gamma}K(\tilde{x},\gamma\tilde{y}) = K(\tilde{x},\gamma_0\tilde{y})$ for some $\gamma_0\in\Gamma$ and so $K(\tilde{x},\gamma\tilde{y}) = 0$ for $\gamma\in\Gamma\setminus\gamma_0$, i.e. since all but one term in the sum is 0,

$$\left|\sum_{\gamma\in\Gamma}K(\tilde{x},\gamma\tilde{y})\right|^2 = |K(\tilde{x},\gamma_0\tilde{y})|^2 = \sum_{\gamma\in\Gamma}|K(\tilde{x},\gamma\tilde{y})|^2. \tag{8}$$

Substituting (8) into (7) recovers $\|K\|_{\mathcal{H}}^2$.

For the general case, if $\rho(\pi(\tilde{x})) > k$ then this work still holds. Collect such vertices into R_k^c, i.e. $R_k := \{\tilde{x}\in\tilde{V} : r(\pi(\tilde{x}))\leqslant k\}$ (of course R_k is in bijection via π with

$\{x \in V : r(x) \leqslant k\}$). Then,

$$\|K\|_{\mathrm{HS}}^2 = \frac{1}{\#V}\left(\sum_{\tilde{x}\in R_k^c}\sum_{\tilde{y}\in\tilde{V}}\sum_{\gamma\in\Gamma}|K(\tilde{x},\gamma\tilde{y})|^2 + \sum_{\tilde{x}\in R_k}\sum_{\tilde{y}\in\tilde{V}}\left|\sum_{\gamma\in\Gamma}K(\tilde{x},\gamma\tilde{y})\right|^2\right)$$

$$\leqslant \|K\|_{\mathcal{H}}^2 + \frac{1}{\#V}\sum_{\tilde{x}\in R_k}(\#B_k(\tilde{x})\|K\|_\infty)^2 \tag{9}$$

$$= \|K\|_{\mathcal{H}}^2 + \frac{\#R_k}{\#V}(v(k)\|K\|_\infty)^2, \tag{10}$$

(9) by the triangle inequality and noting that $K(\tilde{x},\gamma\tilde{y})$ for such $\tilde{x}$ is nonzero only if $\gamma\tilde{y}$ lies in $B_k(\tilde{x})$; and (10) since all balls in $\mathbb{T}$ have the same size $v(k)$. ■

In light of **Proposition 11**, we obtain, for any $\varepsilon > 0$ and appropriate M and T depending on ε:

$$\begin{aligned}
\mathrm{QVar}(W) &= \mathrm{QVar}\left(\frac{1}{T}\int_0^T\sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j(W)\,\mathrm{d}t\right) && \text{restatement of (3)}\\
&\leqslant \left\|\frac{1}{T}\int_0^T\sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j(W)\,\mathrm{d}t\right\|_{\mathrm{HS}}^2 && \text{by Lemma 9(a)}\\
&\leqslant \left\|\frac{1}{T}\int_0^T\sum_{j\leqslant M}\frac{(it)^j}{j!}\mathcal{C}^j(W)\,\mathrm{d}t\right\|_{\mathcal{H}}^2 + v(M)^2M\frac{\#\{x\in V: r(x)\leqslant M\}}{\#V} && \text{by Proposition 11}\\
&\leqslant 4\varepsilon^2\|W\|_{\mathcal{H}}^2 + v(M)^2M\frac{\#\{x\in V: r(x)\leqslant M\}}{\#V} && \text{by (6)}\\
&\leqslant 4\varepsilon^2 + v(M)^2M\frac{\#\{x\in V: r(x)\leqslant M\}}{\#V}. && \text{by def'n } \|\cdot\|_{\mathrm{HS}} \text{ and } \|W\|_\infty \leqslant 1
\end{aligned} \tag{11}$$

To be perfectly explicit about the convergence: given $\tilde{\varepsilon}$, we must identify n for which $\mathrm{QVar}(W_m) < \tilde{\varepsilon}$ for $m > n$. Let $\varepsilon = \sqrt{\frac{\tilde{\varepsilon}}{8}}$. By **BSC** applied to $r = M$, there exists n such that

$$\frac{\#\{x\in V: r_n(x)\leqslant M\}}{\#V_n} < \frac{\tilde{\varepsilon}}{2v(M)^2M}.$$

By the reduction in §2, QED.

Remark 12. It is worth noting the only appearances of the properties we assume of our graphs.

- Regularity is used in constructing the universal cover $\mathbb{T}$.
- **BSC** is only used in the conclusion (but is teed up by **Proposition 11**).
- **EXP** is only used in **Lemma 10**.
- Mean-zero is also only used in **Lemma 10**, in appealing to the Rayleigh quotient.

4 Making the bound explicit

This amounts to writing down an adequate $M(\varepsilon)$ from (5), for use in (11). One can check that (for instance) $M = (2q+3)T + \log_T \frac{1}{\varepsilon}$ suffices, by Taylor's theorem. Then, picking

$$T(\varepsilon) = \left(\frac{2\sqrt{2}}{\varepsilon \sqrt[4]{(q+1)\beta}} \right)^2 = \frac{8}{\varepsilon^2 \sqrt{(q+1)\beta}}$$

in order to explicitly obtain M, if we have a handle on how $\#\{x \in V_n : r_n(x) \leqslant R\}$ grows in n we can select appropriate n.

5 Brief remarks on the other proofs

5.1 "Proof 2 ("ultra-short")"

The objective in this proof is again to obtain an expression of the form (11), here by multiplying the operator (in general not necessarily diagonal, though the proof above is also done in the paper for general W) by carefully-constructed operators arising from a decomposition of $\mathcal{C}$ (and the observation, along the lines of **Proposition 9**(c), that if $\mathrm{QVar}(K + K') = 0$ then $\mathrm{QVar}(K) = \mathrm{QVar}(K')$, and then exploiting **Proposition 11** and the structure of the relevant adjoint operators.

5.2 "Proof 4 ("nonbacktracking")"

A two-part talk by Anantharaman on this proof can be found here and here. The core idea, as in the main proof presented here, is to introduce "complications" (in the case above, the integral of an exponential action). Here the quantum variance is considered for *arcs* in the tree, and in the quantum variance we modify W by multiplying on either side by the *nonbacktracking* adjacency matrix. Then, we bound the quantum variance by passing up to analysis on the tree, as before discarding a vanishing collection of short cycles, and otherwise treating paths as pairs of start- and endpoints, allowing us to treat operators on paths of any length as lying in the same Hilbert space of operators. The nonbacktracking quantum variance is then connected back to the undirected case.