Optimal topological generators of U(1) and Short paths in SU(2)

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Topological generators

Say $S \subset G$ is a set of topological generators for the infinite group G.

Question: How good is a given set of topological generators?

Question: What does it mean for a set of topological generators to be "good"?

We'll work in $G \leqslant \mathrm{U}(n)$, a compact metric space.

Topological generators

Assign to each generator $s \in S$ a weight $w(s)$. The weight of a word is just the sum of its elements' weights.

Question: For each n, what is the radius of the largest ball in G whose translates by all words of weight $\leqslant n$ fail to cover G?

i.e., $$\sup \left\{ r \mid \exists x : \bigcup_{g \in S(n)} B_r(x) g \neq G \right\}$$

Loosely speaking, the "goodness" of S correlates with the asymptotics of that radius as $n \to \infty$ (smaller is better).

Topological generators

There are two intertwined questions that arise:

- Can we find particularly good topological generators?
- Given good topological generators, how well can we navigate?

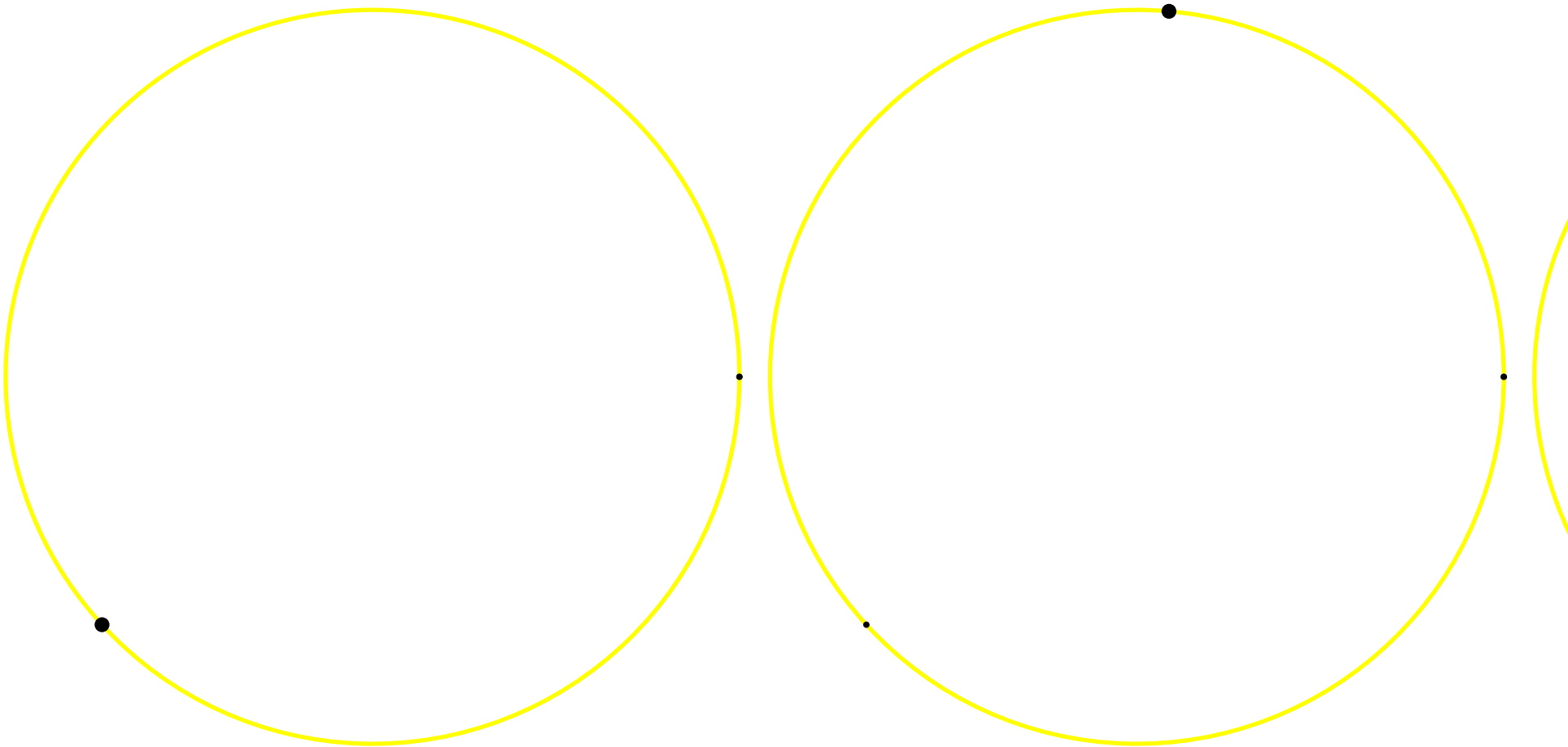
Outline

1. Optimal topological generators of U(1)
   - "Good" U(1)-generators
   - The "best" U(1)-generators

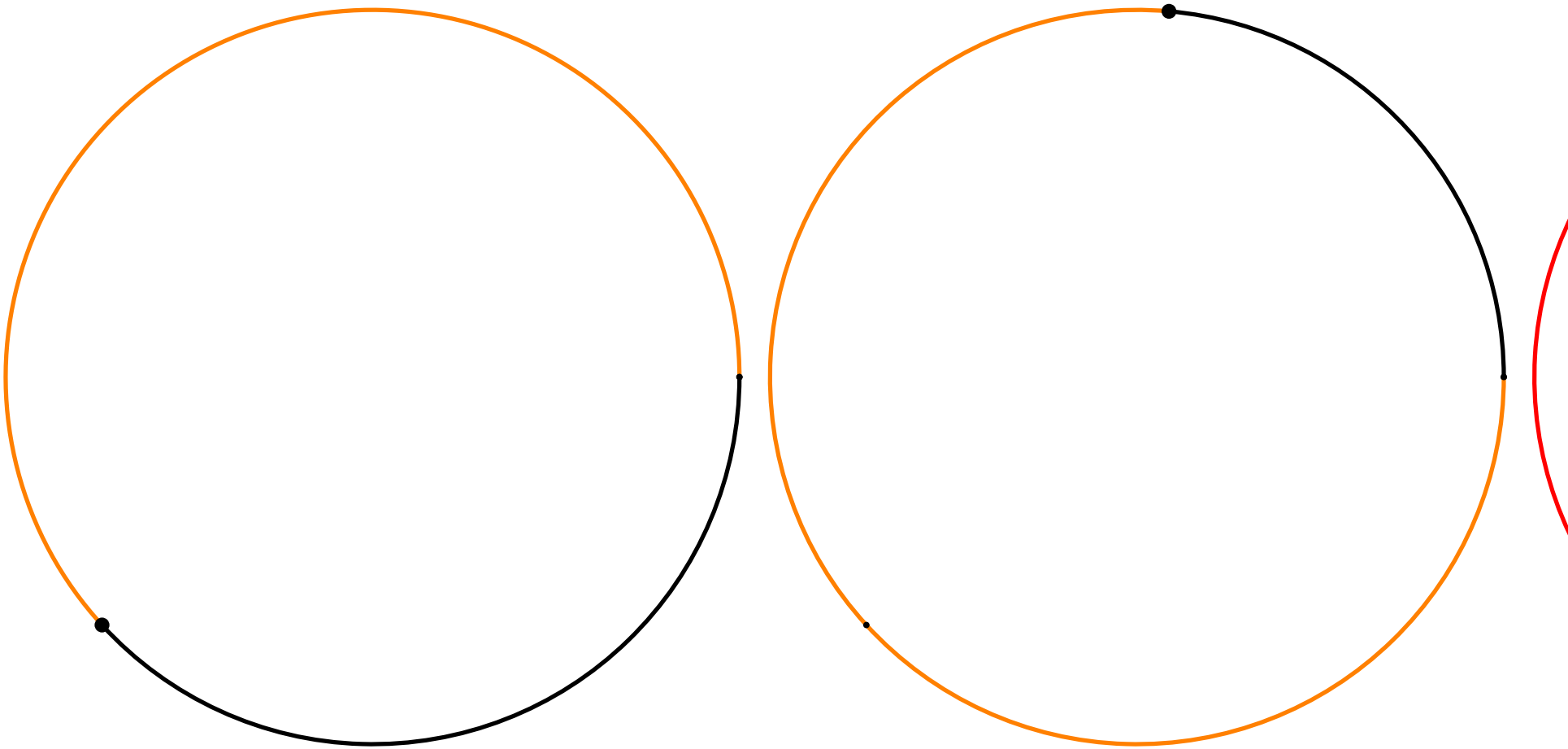
2. Short paths in SU(2)
   - Motivation
   - Context
   - Path-finding for one qubit

Outline

Topological generators of U(1)



Topological generators of U(1)



Topological generators of U(1)

Say θ is an irrational angle. Here are some facts about this setting:

- θ generates a dense subgroup of S^1.
- There are at most three distinct distances marked off by any number of multiples of θ.
- θ cannot generate U(1) "too well."

The discrepancy function

Mark off the first m multiples of θ on S^1. Let $d_\theta(m)$ equal the length of the longest resulting arc.

Theorem (Graham–van Lint, 1966)

For any θ,

$$\limsup_{m\to\infty} m d_\theta(m) \geqslant 1 + \frac{2}{\sqrt{5}} \approx 1.89$$

with equality if and only if $\theta \asymp \varphi$.

Continued fractions notation

Every irrational number θ has an **(infinite) continued fraction**

$$\theta = [a_0, a_1, \dots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

with **convergents** denoted

$$[a_0, a_1, \dots, a_n] = \frac{h_n}{k_n} = \frac{a_n h_{n-1} + h_{n-2}}{a_n k_{n-1} + k_{n-2}}.$$

Define the useful values

$$x_n = x_n(\theta) = [a_{n+1}, \dots, a_1], \qquad \theta_n = [a_n, a_{n+1}, \dots],$$

the convention

$$[a_0, a_1, \dots, a_{n-1}, \dot{1}] = [a_0, a_1, \dots, a_{n-1}, 1, 1, 1, \dots],$$

and say $\mu \asymp \sigma$ when there exist $m, n \in \mathbb{N}$ for which $\mu_m = \sigma_n$. Fix the constants $\varphi = [\dot{1}] = \frac{1+\sqrt{5}}{2}$ and $\rho = 1 + \frac{2}{\sqrt{5}}$.

Continued fractions facts

Lemma (corollary from Hardy–Wright, Slater/Sós)

If $\alpha < a_{n+2}$ *and* $k_n + (\alpha+1)k_{n+1} - 1 \leqslant m \leqslant k_n + (\alpha+2)k_{n+1} - 2$, *then*

$$d_\theta(m) = \frac{\theta_{n+2} - \alpha}{k_n + k_{n+1}\theta_{n+2}}.$$

Proposition (Mozzochi, 2019)

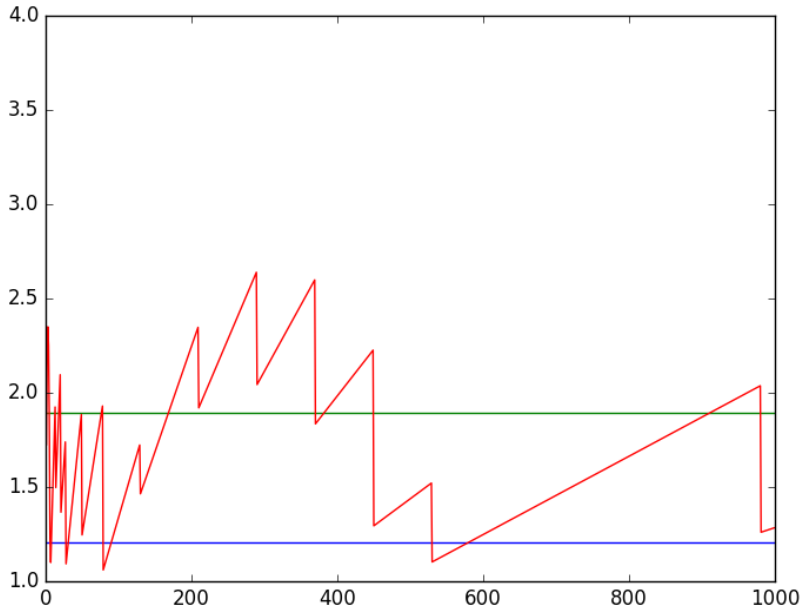
If $\theta \asymp \varphi$, $a_k = 1$ *for* $k \geqslant n$, *and* $k_n + k_{n+1} - 1 \leqslant m \leqslant k_n + 2k_{n+1} - 2$, *then*

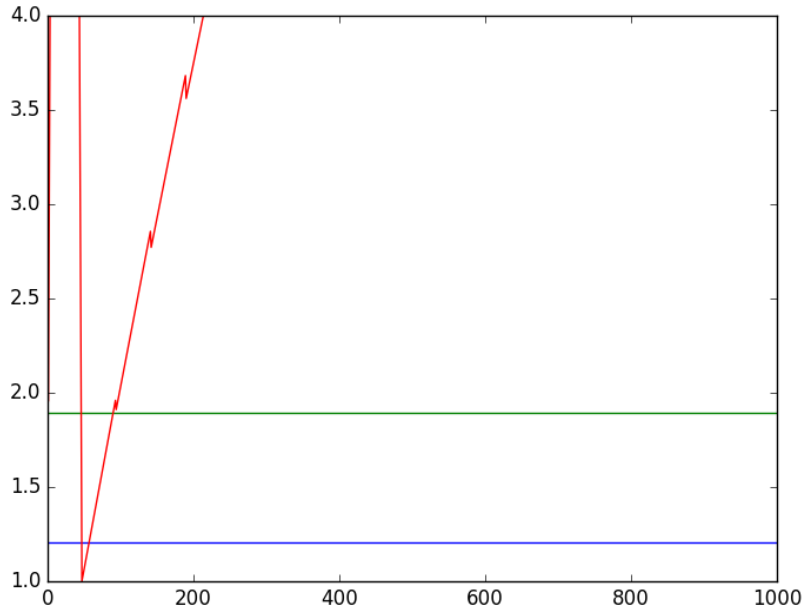
$$\max m d_\theta(m) = \frac{2x_n + 1 - \frac{1}{k_n}}{x_n + \varphi - 1}.$$

Outline

What we're looking for

Say $\theta \asymp \varphi$. When does $md_\theta(m)$ reach its lim sup, ρ, from below?





Put differently:

Question: When does there exist $M_\theta \in \mathbb{N}$ satisfying

$$m \geqslant M_\theta \implies md_\theta(m) \leqslant \rho?$$

Answer: Always!

Good U(1)-generators

Theorem

For all $\theta = [a_0, a_1, \ldots, a_N, \dot{1}] \asymp \varphi$, *there exists an* $M_\theta \in \mathbb{N}$ *satisfying*

$$m \geqslant M_\theta \implies m d_\theta(m) \leqslant \rho,$$

and an upper bound on M_θ *is computable.*

Proof outline. For each $n = N + d$, take m between $k_n + k_{n+1} - 1$ and $k_n + 2k_{n+1} - 2$, where

$$\max m d_\theta(m) = \frac{2x_n + 1 - \frac{1}{k_n}}{x_n + \varphi - 1}.$$

By bounding convergents, $m d_\theta(m) \leqslant \max m d_\theta(m) \leqslant \rho$ if

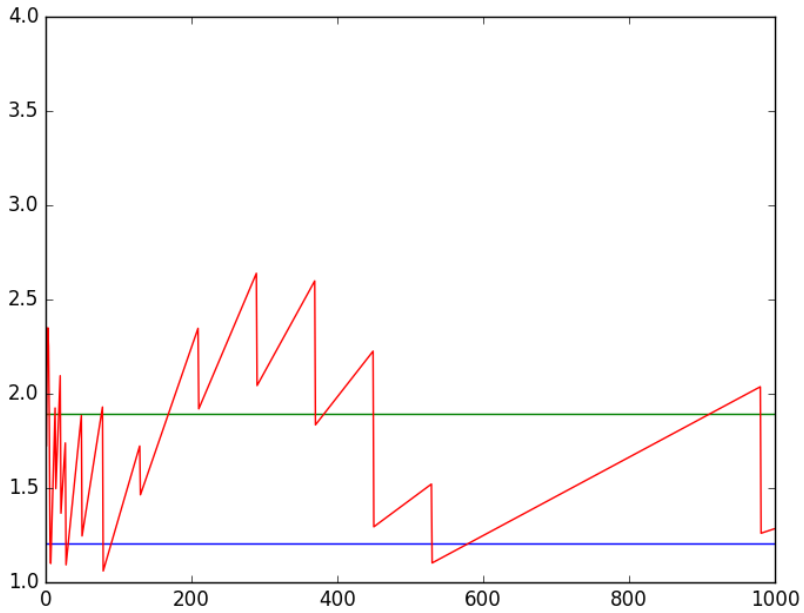
$$F_{d+1} > \frac{k_N + k_{N-1}}{5 + 2\sqrt{5}}$$

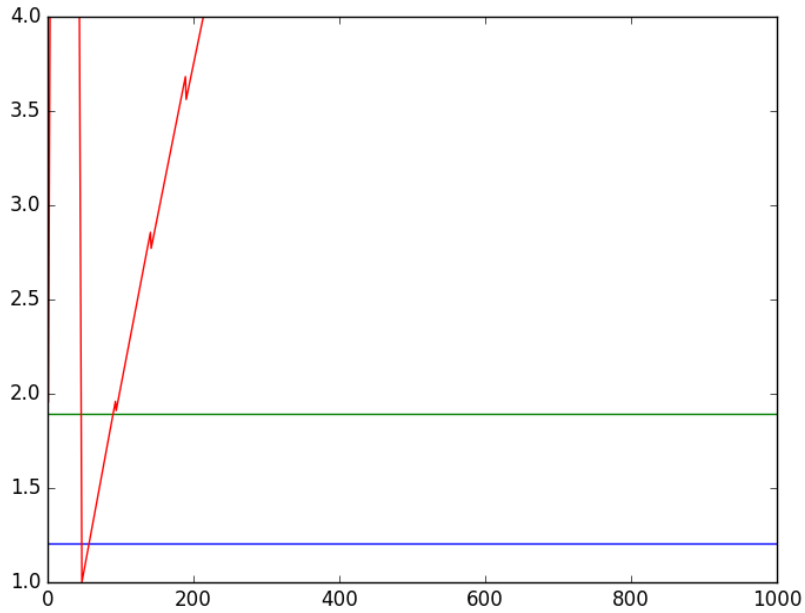
which obviously holds for all d, hence all m, beyond a certain value. ■

Outline

How can we do better?

Recall that $md_\theta(m)$ can behave strangely for small m.





The natural question then is which values of θ avoid this behavior.

Theorem (Sarnak's golden mean conjecture; Mozzochi, 2019)

For all $m \in \mathbb{N}$,

$$md_\varphi(m) \leqslant \rho.$$

The main result of this paper

Theorem

There are eight distinct generators of U(1) *with that property.*

We call this set $\mathcal{S}$ and $\mathcal{S}/\bowtie$ has these representatives:

	1	2	3	4	5	exact	num. val.
η_7	2	1	1	1	i	$\frac{3-\sqrt{5}}{2}$	0.381...
η_6	2	1	2	1	i	$\frac{25-\sqrt{5}}{62}$	0.367...
η_8	2	2	1	1	i	$\frac{7+\sqrt{5}}{22}$	0.419...
η_4	3	1	1	1	i	$\frac{5-\sqrt{5}}{10}$	0.276...
η_5	3	2	1	1	i	$\frac{9+\sqrt{5}}{38}$	0.295...
η_2	4	1	1	1	i	$\frac{7-\sqrt{5}}{22}$	0.216...
η_3	4	2	1	1	i	$\frac{11+\sqrt{5}}{58}$	0.228...
η_1	5	2	1	1	i	$\frac{13+\sqrt{5}}{82}$	0.185...

Proof of the theorem I

Proof outline. Let $\theta = [a_0, a_1, \ldots, a_N, \dot{1}]$ where $a_N \geqslant 2$. We first argue that $N < 6$. Take $m = k_N + 2k_{N+1} - 1$, where

$$md_\theta(m) = \varphi \frac{2k_{N+1} + k_N - 1}{\theta_{N+1}k_N + k_{N-1}}$$

and $md_\theta(m) \leqslant \rho$ is equivalent to

$$((2-\rho)a_N + 1 + \rho - \rho\varphi)k_N + (2-\rho)k_{N-1} \leqslant 1$$

which is false when $k_N \geqslant 13$ and $k_{N-1} \geqslant 8$ as is the case with $N \geqslant 6$.

Proof of the theorem II

Proof outline, cont. Now we argue that there exist finite bounds on each a_n, $n \in [5]$ by finding necessary conditions for $\theta \in \mathcal{S}$. Take m between $k_{n-1} + k_n - 1 \leqslant m \leqslant k_{n-1} + 2k_n - 2$, where

$$\max m d_\theta(m) = \frac{k_{n-1} + 2k_n - 1}{k_{n-2} + k_{n-1}\theta}$$

which manipulates to

$$a_n \leqslant \frac{\rho - 1}{2 - \rho} + 1 + \frac{1}{(2 - \rho)F_n}$$

hence using numerical values gives

$$a_1 \leqslant 18, \qquad a_2 \leqslant 18, \qquad a_3 \leqslant 14, \qquad a_4 \leqslant 12, \qquad a_5 \leqslant 11.$$

Proof of the theorem III

Proof outline, cont. How can we check the remaining $18 \times 18 \times 14 \times 12 \times 11 \approx 600000$ irrationals? We return to the intervals for each n given by Slater/Sós, where we compute a local maximum for n up to $29 = N + d$. 29 arises from the k-terms of θ being bounded by those of

$$[0, 18, 18, 14, 12, 11, \dot{1}],$$

at which point $d = 24$ is guaranteed to be the point where θ approaches from below. (Recall the previous Theorem.) ■

Is there a bias? alternation?

Chebyshev and Littlewood lead us to wonder about the asymptotic relevance of each generator.

On a bounded interval $[M]$, let

$$w(M) = \operatorname*{argmin}_{\theta \in \mathcal{S}} \max_{m \in [M]} m d_m(\theta).$$

Does each $[\theta] \in \mathcal{S}$ arise as $w(M)$ infinitely often? with positive density?

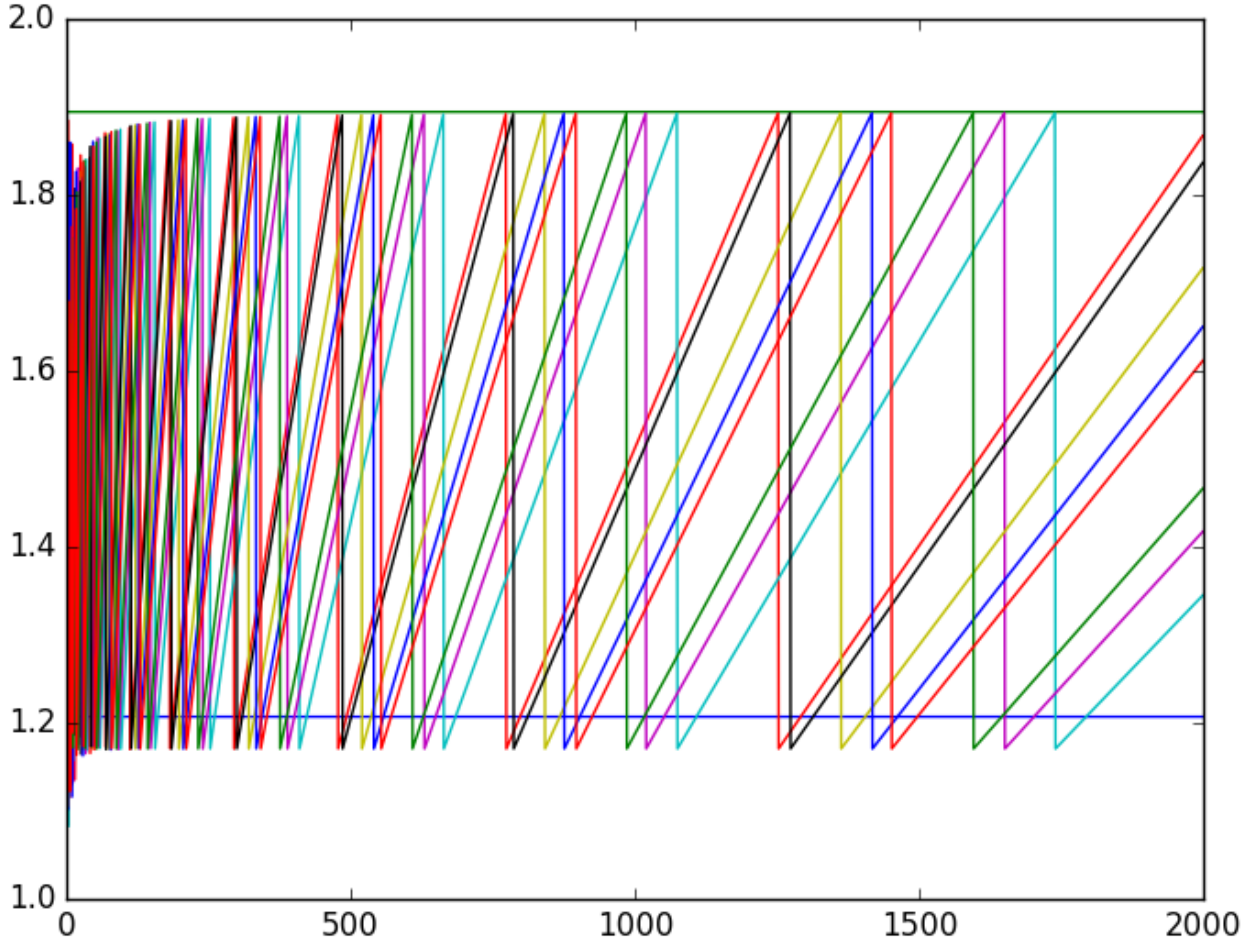
Theorem

Let $W_\theta(M)$ be the number of times for $m \in [M]$ that $[\theta] = w(m)$. Then,

$$\liminf_{M \to \infty} \frac{W_\theta(M)}{M} \geqslant 2.7\%$$

for all $\theta \in \mathcal{S}$.

Empirical evidence



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Why unitary groups?

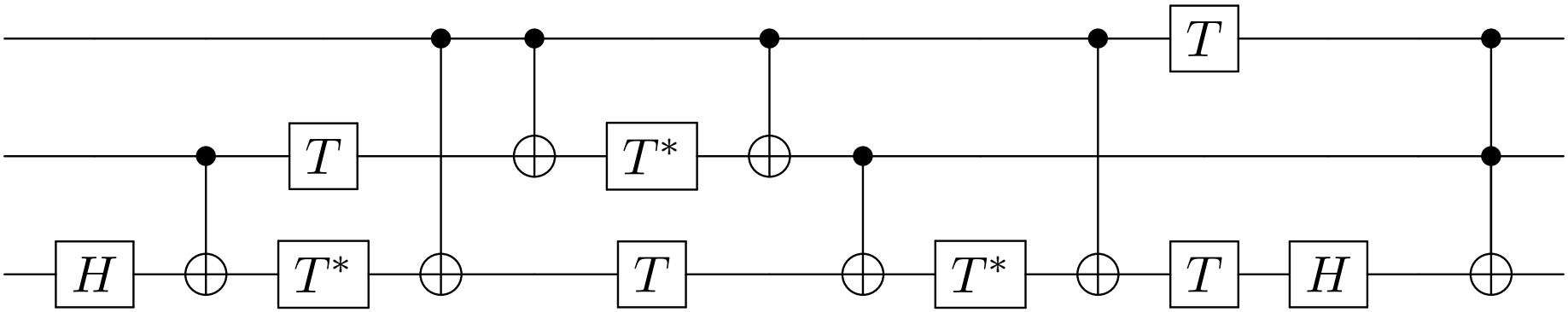
Single qubits are norm-1 $\mathbb{C}^2$ vectors equivalent under scalar multiplication by $\mathbb{C}^\times$. Qubits in general are inside $(\mathbb{C}^2)^{\otimes n}$. Linear operators sending qubits to qubits (i.e., gates) are unitary matrices.

Any n-qubit gate can be synthesized with one-qubit gates and some copies of a fixed two-cubit gate.

What can we do if we only have a finite set of one-qubit gates?

Exact and approximate synthesis

Say we only have $CNOT = I_2 \oplus i \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $H = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $T = R_{\pi/4}$.
We can exactly get some nice circuits.



However, all we can say is that $\langle H, T \rangle$ is dense in $SU(2)$.

Outline

Previous results I

Theorem (Ross–Selinger, 2016)

There is a $\operatorname{poly}\log\frac{1}{\varepsilon}$ *algorithm to factor almost any diagonal element of* $\mathrm{SU}(2)$ *in* $\langle H, T\rangle$ *to precision* ε *using at most* $(3 + o(1))\log_2\frac{1}{\varepsilon}$ *copies of* T*.*

Theorem (Parzanchevski–Sarnak, 2018)

Suppose S *is a finite set topologically generating* $\mathrm{SU}(2)$ *whose entries are in* $\mathbb{Z}[i, \sqrt{p}]$*, where* $\mathcal{O}_{\mathbb{Q}[\sqrt{p}]}$ *is a UFD. Then Ross–Selinger's algorithm holds with* S *to give total path length at most* $(3 + o(1))\log_p\frac{1}{\varepsilon}$*.*

Previous results II

Theorem (Parzanchevski–Sarnak, 2018)

Taking S and p as before, almost any element of $\mathrm{SU}(2)$ may be approximated to within ε by decomposing using Euler angles, obtaining total path length at most $(9 + o(1)) \log_p \frac{1}{\varepsilon}$.

This is the result that we resolve to improve. To do so, we seek inspiration from the study of LPS Ramanujan graphs $X^{p,q}$.

Theorem (Carvalho Pinto–Petit, 2018; Sardari, 2019)

There is a Ross–Selinger analogue for exact factorization in $X^{p,q}$ of length at most $(3 + o(1)) \log_p q$ almost always.

Theorem (Carvalho Pinto–Petit, 2018)

There is an algorithm for exact factorization in $X^{p,q}$ of length at most $(7 + o(1)) \log_p q$ almost always.

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Setup and roadmap

Implicitly, "approximating x within ε" has meant finding $y \in \mathrm{SU}(2)$ where

$$\varepsilon > d(x, y) = 1 - \frac{1}{2} \left| \operatorname{tr} x^* y \right| .$$

We work with "V-gates": set $\Gamma = \langle s_1, s_2, s_3 \rangle$, $S = \{s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}\}$,

$$s_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1+2i & \\ & 1+2i \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, \quad s_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

We approximate given $g \in \mathrm{SU}(2)$ as $\delta_1 \gamma \delta_2$ where δ_i are diagonal and $\gamma \in \Gamma$ has short factorization.

Theorem

There is a $\operatorname{poly} \log \frac{1}{\varepsilon}$ *algorithm to approximate almost any element of* $\mathrm{SU}(2)$ *in path length at most* $(7 + o(1)) \log_5 \frac{1}{\varepsilon}$.

The approximation algorithm

Given $g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, we seek $\gamma = \frac{1}{5^{\frac{1}{2}k}} \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix} \in \mathrm{SU}(2)$ (since then $\gamma \in \Gamma$) with $|x_0 + x_1 i|$ near $|\alpha| < \sqrt{1 - \varepsilon_0^2}$.

Solve in $\mathbb{Z}$ with minimal k:

$$\left| (x_0^2 + x_1^2) - |\alpha|^2 \, 5^k \right| \leqslant |\alpha| \, \varepsilon 5^k,$$
$$(x_0^2 + x_1^2) + (x_2^2 + x_3^2) = 5^k.$$

We heuristically expect this to be solved when $5^k \varepsilon \in O(1)$, so γ's factorization is of length about $\log_5 \frac{1}{\varepsilon}$.
Using technical results, appropriate diagonals δ_1 and δ_2 are easily computable such that we can piece together the approximation of g: $d(g, \delta_1 \gamma \delta_2) < \frac{4}{\varepsilon_0} \varepsilon$ and δ_1 and δ_2 are each approximable to within ε in length $3 \log_5 \frac{1}{\varepsilon}$, for a total length of $7 \log_5 \frac{1}{\varepsilon}$.

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