COS598H Final Presentation: Stability Analysis in Swarms

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Outline

- Motivation: simulating flocks
- Gazi and Passino's model
- Global convergence
- Agentwise convergence

Background: modeling flocks

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Unfortunately…

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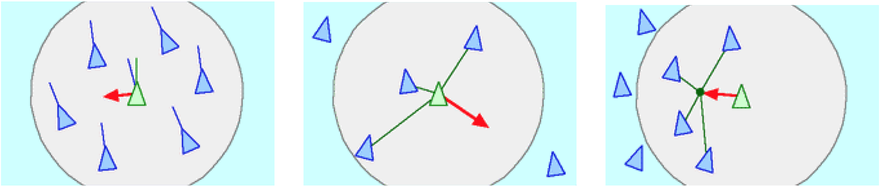
Three conditions enforced on individuals:

(a) **alignment*** – tendency to align parallel to avg. of proximal boids
(b) **separation** – tendency to avoid collision
(c) **cohesion** – tendency to avoid divergence

Background: modeling flocks (boids)



(a) alignment* (b) separation (c) cohesion

Image source: C. Reynolds

A model for global (b) & (c)

We'll see here a model that implements separation and cohesion
We'll then prove some results about it
Model due to Gazi & Passino, "Stability Analysis in Swarms," 2003

A model for global (b) & (c)

M agents, indexed by $i \in [M]$, with positions $\mathbf{x}^i \in \mathbf{R}^n$
$\mathbf{x}^i$ is implicitly $\mathbf{x}^i(t)$; the time-parameter is suppressed for convenience
All agents always know all others' positions, velocities with no delay
Agents are governed by the first-derivative relation

$$\dot{\mathbf{x}}^i = \sum_{j=1, \neq i}^{M} g(\mathbf{x}^i - \mathbf{x}^j)$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ implements pairwise separation/cohesion

The artificial social potential function g

We will consider the **artificial social potential function**

$$g(\mathbf{x}) := -\left(a - b\exp\left(-\frac{\|\mathbf{x}\|^2}{c}\right)\right)\mathbf{x}$$

Our first result about this model

Define the **center** in the obvious way:

$$\bar{\mathbf{x}} := \frac{1}{M} \sum_{i=1}^{M} \mathbf{x}^i$$

and the ith **offset** as

$$\mathbf{e}^i := \mathbf{x}^i - \bar{\mathbf{x}}$$

Lemma 1

$\bar{\mathbf{x}}$ is stationary.

Our first result about this model

Lemma 1

$\bar{\mathbf{x}}$ is stationary.

Proof.

$$\dot{\bar{\mathbf{x}}} = \frac{1}{M}\sum_{i=1}^{M}\dot{\mathbf{x}}^i = -\frac{1}{M}\sum_{i=1}^{M}\sum_{j=1,\neq i}^{M}\mathbf{v}_{ij}g_1(\mathbf{v}_{ij}) = 0$$

where $\mathbf{v}_{ij} := \mathbf{x}^i - \mathbf{x}^j$ and $g_1(\mathbf{v}) := \frac{\langle g(\mathbf{v}),\mathbf{v}\rangle}{\|\mathbf{v}\|^2}$. ■

Lyapunov function candidates

Given

$$\dot{x} = f(x) \tag{1}$$

for $f : \mathbf{R}^n \to \mathbf{R}^n$, consider $V \in \mathcal{C}^1(\mathbf{R}^n \to \mathbf{R})$ locally PSD on $U \subset \mathbf{R}^n$

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V is a **Lyapunov function candidate** (LFC)

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Free agent i at time t that is also farther than δ from $\bar{\mathbf{x}}$ has instantaneous motion towards $\bar{\mathbf{x}}$.

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Proof.

Manipulation of def. of $\bar{\mathbf{x}}$ gives

$$\dot{\mathbf{x}}^i = -aM\mathbf{e}^i + b \sum_{j=1,\neq i}^{M} \exp\left(-\frac{\|\mathbf{v}_{ij}\|^2}{c}\right) \mathbf{v}_{ij}$$

Using LFC $V_i := \frac{1}{2}\|\mathbf{e}^i\|^2$, bound

$$\dot{V}_i \leqslant -aM\|\mathbf{e}^i\|^2 + b\|\mathbf{e}^i\| \sum_{j=1,\neq i}^{M} \exp\left(-\frac{\|\mathbf{v}_{ij}\|^2}{c}\right) \|\mathbf{v}_{ij}\|$$

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Free agents

Lemma 2

Free agent i at time t that is also farther than δ from $\bar{\mathbf{x}}$ has instantaneous motion towards $\bar{\mathbf{x}}$.

Proof.

Using $\|\mathbf{v}_{ij}\| > \delta$ to bound the exponential term, we have

$$\dot{V}_i \leqslant -a\|\mathbf{e}^i\|^2 - (M-1)\left(a\|\mathbf{e}^i\| - b\delta\exp\left(-\frac{\delta^2}{c}\right)\right)\|\mathbf{e}^i\|$$

where the second term is NSD by choice of δ and assumption that $\|\mathbf{e}^i\| \geqslant \delta$; the inequality simplifies to

$$\dot{V}_i \leqslant -a\|\mathbf{e}^i\|^2 = -2aV_i.$$

i.e. we see that $\dot{\mathbf{x}}^i$ is in a direction such that $\|\mathbf{e}^i\|$ decreases. ∎

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$$\varepsilon := \frac{b}{a}\sqrt{\frac{c}{2e}}$$

and

$$\widehat{t} := -\frac{\ln \varepsilon}{a} + \frac{1}{a}\ln\left(2\max_{i\in[M]} V_i(0)\right)$$

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Theorem 1

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Proof.

Fix any $i \in [M]$. Recall:

$$\dot{V}_i \leqslant -aM\|\mathbf{e}^i\|^2 + b\|\mathbf{e}^i\| \sum_{j=1,\neq i}^{M} \exp\left(-\frac{\|\mathbf{v}_{ij}\|^2}{c}\right)\|\mathbf{v}_{ij}\|$$

The linear-exponential term obtains its maximum $\sqrt{\frac{c}{2e}}$ at $\|\mathbf{v}_{ij}\| = \sqrt{\frac{c}{2}}$ so $\dot{V}_i < 0$ if

$$\|\mathbf{e}^i\| > \frac{b(M-1)}{aM}\sqrt{\frac{c}{2e}} =: \varepsilon'$$

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Theorem 1

Each agent in the swarm will lie inside $B_\varepsilon(\bar{\mathbf{x}})$ by time $\widehat{t}$.

Proof.

Note that $\frac{M}{M-1}\varepsilon' = \varepsilon$, i.e. $\varepsilon' < \varepsilon$. Therefore $\mathbf{x}^i$ will not converge to anywhere outside $B_{\varepsilon'}(\bar{\mathbf{x}}) \subset B_\varepsilon(\bar{\mathbf{x}})$.

To compute finite-time convergence, recall:

$$\dot{V}_i \leqslant -2aV_i \implies V_i(t) \leqslant V_i(0)e^{-2at}$$

so $\|\mathbf{e}^i\| = \varepsilon$ by time

$$t_i \leqslant -\frac{1}{2a}\ln\left(\frac{\varepsilon^2}{2V_i(0)}\right)$$

which simplifies to the expression for $\widehat{t}$. ■

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Theorem 2

The swarm converges, as $t \to \infty$, to a value in Ω.

i.e. the swarm converges to a constant state

Convergence!

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The swarm converges, as $t \to \infty$, to a value in Ω.

Proof.

Consider the LFC (also an artificial potential function)

$$J(\vec{\mathbf{x}}) = \frac{1}{2} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left(a \|\mathbf{v}_{ij}\|^2 + bc \exp\left(-\frac{\|\mathbf{v}_{ij}\|^2}{c} \right) \right)$$

Since $\nabla_{\mathbf{x}^i} J(\vec{\mathbf{x}}) = -\dot{\mathbf{x}}^i$, at all times t:

$$\dot{J}(\vec{\mathbf{x}}) = -\sum_{i=1}^{M} \|\dot{\mathbf{x}}^i\|^2 \leqslant 0$$

By **LaSalle's invariance principle**, $\vec{\mathbf{x}} \to \{\vec{\mathbf{x}} : \dot{J}(\vec{\mathbf{x}}) = 0\}$. ∎

References

* V. Gazi and K. M. Passino. "Stability Analysis of Swarms." In *IEEE Transactions on Automatic Control* 48.4 (2003).
* C. Reynolds. "Flocks, Herds, and Schools: A Distributed Behavioral Model." In *Computer Graphics* 21.4 (1987).